Recitation 2 Drill Problems

Tuesday, September 15, 2020 3:43 PM

Review

System of equations as $Ax = b$

PROBLEM:

$$y + 2z = 5 \quad (1)$$
$$2y + 3z = 6 \quad (2)$$

Step 1: Order the equations and stack the vector of unknowns. (x)

Step 2: Move the constants to the RHS and stack them in order. (b)

Step 3: Find the coefficients of your unknowns to make the L.H.S. = Ax (A)

Case I

$$x = \begin{bmatrix} y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad \det(A) = 3 - 4 = -1 \neq 0$$

Unique Solution

Case II

$x = \begin{bmatrix} y \\ z \end{bmatrix}$, Equations are in reverse order

$$2y + 3z = 6 \quad (1)$$
$$y + 2z = 5 \quad (2)$$

$$b = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \quad \det(A) = 4 - 3 = 1 \neq 0$$

Unique Solution

Case III

$x = \begin{bmatrix} z \\ y \end{bmatrix}$, Original Set of Equation

$$y + 2z = 5 \ (1)$$
$$2y + 3z = 6 \ (2)$$

$$b = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad \det(A) = 4-3 \neq 0$$

Case IV

$x = \begin{bmatrix} z \\ y \end{bmatrix}$, New set of equations,

$2y + 3z = 6 \; (1)$

$y + 2z = 5 \; (2)$

$$b = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \quad \det(A) = 3-4 = -1 \neq 0.$$

PROBLEM:

$$y = 3z + 2 \longleftrightarrow \underbrace{y - 3z}_{\text{variables}} = \underbrace{2}_{\text{constants}}$$

$$z + y + 3 = 0 \longleftrightarrow y + z = -3$$

Rearrange terms on both sides,

$$x = \begin{bmatrix} y \\ z \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}, \quad \det(A) = 1(1) - (-3)(1)$$

$$= 4 \neq 0$$

unique solution.

ROB 101 - Computational Linear Algebra
Recitation #2

Tribhi Kathuria

Sept 8, 2020

1 Solution of Linear System of Equations

In solving $Ax = b$, we exploit the form of A for finding solutions using Forward and Backward Substitution. However, for us to do that A must a Triangular Matrix.

1.1 Triangular matrices

Fact: For any Triangular Matrix A, det(A) is the product of all elements on its diagonal.

1.1.1 Upper Triangular Matrices

A matrix, A is upper Triangular, if and only if, for every element a_{ij} in the matrix, i.e. the element at i^{th} row and j^{th} column, the following condition holds true:
$\boxed{a_{ij} = 0 \text{ if } i > j}$
Example:

$$\begin{bmatrix} 2 & 3 & 7 \\ 0 & 2 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

a_{32}, $i=3, j=2$.

a_{31}, $i=3, j=1$

$i>j$; $a_{ij}=0$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & & & \\ \vdots & a_{32} & & a_{nn} \end{bmatrix}$$

Method of Solution:

Backward Substitution

1.1.2 Lower Triangular Matrices

A matrix, A is Lower Triangular, if and only if, for every element a_{ij} in the matrix, i.e. the element at i^{th} row and j^{th} column, the following condition holds true:
$a_{ij} = 0$ if $j > i$
Example:

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

a_{23}, $i=2, j=3$

Method of Solution:

Forward Substitution

Build a matrix of coefficients, A, and check if the Matrix is Triangular or not from the following system of equations:

1.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 24 \\ 14 \end{bmatrix}$$

$$2x_1 + 2x_2 = 24$$
$$2x_2 = 14$$

could be any thing

$A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ $\quad det(A) = 2(2) - 2(0) = 4$

Using Fact 1: $det(A) = a_{11} \cdot a_{22}$
$= 2(2) = 4$

Remark: Unique Solution, A is upper triangular

2.

$$2x_1 + 3x_2 + 4x_3 = 23$$
$$6x_1 + 7x_2 = 22$$
$$9x_1 = 18$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 23 \\ 22 \\ 18 \end{bmatrix}$$

REORDER, $x := \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$

"Missing Coefficients"

$A = \begin{bmatrix} 4 & 3 & 2 \\ 0 & 7 & 6 \\ 0 & 0 & 9 \end{bmatrix}$ $\quad det(A) = a_{11} a_{22} a_{33}$

$= 4(7)(9) \neq 0$ Unique Solu^n

Remark:

3.

$$2x_1 + 2x_2 + 3x_3 + 4x_4 = 20$$
$$5x_2 + 6x_3 + 7x_4 = 34$$
$$8x_3 + 9x_4 = 25$$
$$10x_4 = 10$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad b = \begin{bmatrix} 20 \\ 34 \\ 25 \\ 10 \end{bmatrix}$$

$A = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ $\quad det(A) = a_{11} a_{22} a_{33} a_{44}$

$= 2(5)(8)(10) \neq 0$ Unique Solu^n

Remark: Upper Triangular

4.

$$2x_1 = 4$$
$$8x_1 + 3x_2 = 28$$
$$2x_1 + 3x_2 + x_3 = 22$$
$$2x_1 + 2x_2 + 3x_3 + 2x_4 = 46$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 28 \\ 22 \\ 46 \end{bmatrix}$$

$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 3 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 2 & 2 & 3 & 2 \end{bmatrix}$ $\quad det(A) = 2(3)(1)(2) \neq 0$ Unique Solu^n

Remark: Lower Triangular

1.2 Forward Substitution

$$2x_1 = 4$$
$$8x_1 + 3x_2 = 28$$
$$2x_1 + 3x_2 + x_3 = 22$$
$$2x_1 + 2x_2 + 3x_3 + 2x_4 = 46$$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 3 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 2 & 2 & 3 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 28 \\ 22 \\ 46 \end{bmatrix}$$

Answer

$$2x_1 = 4 \Rightarrow x_1 = 2 \qquad x_1 = b_1/a_{11}$$

$$8x_1 + 3x_2 = 28 \Rightarrow 8(2) + 3x_2 = 28$$

$$\Rightarrow x_2 = \frac{28 - 8(2)}{3} \qquad x_2 = \frac{b_2 - a_{21}(x_1)}{a_{22}}$$

$$= 4$$

$$2x_1 + 3x_2 + x_3 = 22 \Rightarrow 2(2) + 3(4) + x_3 = 22$$

$$\Rightarrow x_3 = \frac{22 - 2(2) - 3(4)}{1} = 6$$

$$x_3 = \frac{b_3 - a_{31}(x_1) - a_{32}(x_2)}{a_{33}}$$

$$2x_1 + 2x_2 + 3x_3 + 2x_4 = 46 \Rightarrow 2(2) + 2(4) + 3(6) + 2x_4 = 46$$

$$\Rightarrow x_4 = \frac{46 - 2(2) - 2(4) - 3(6)}{2}$$

$$= 8$$

1.3 Backward Substitution

$$A = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 + 4x_4 &= 20 \\ 5x_2 + 6x_3 + 7x_4 &= 34 \\ 8x_3 + 9x_4 &= 25 \\ 10x_4 &= 10 \end{aligned}$$

$x_4 = \frac{b_4}{a_{44}}$, $x_3 = \frac{b_3 - a_{34}x_4}{a_{33}}$

$x_2 = \frac{b_2 - a_{23}x_3 - a_{24}x_4}{a_{22}}$

$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4}{a_{11}}$

$$b = \begin{bmatrix} 20 \\ 34 \\ 25 \\ 10 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Using the General Solution (See Appendix)

$$x_4 = \frac{b_4}{a_{44}} = \frac{10}{10} = 1$$

$$x_3 = \frac{25 - 9(1)}{8} = 2$$

$$x_2 = \frac{34 - 6(2) - 7(1)}{5} = 3$$

$$x_1 = \frac{20 - 2(3) - 3(2) - 4(1)}{2} = 2$$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$$

2 Appendix

Review: The general form of a lower triangular system with a non-zero determinant is

$$\begin{aligned} a_{11}x_1 &= b_1 \quad (a_{11} \neq 0) \\ a_{21}x_1 + a_{22}x_2 &= b_2 \quad (a_{22} \neq 0) \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \quad (a_{nn} \neq 0) \end{aligned} \tag{1}$$

and the solution proceeds from top to bottom, like this

$$\begin{aligned} x_1 &= \frac{b_1}{a_{11}} \quad (a_{11} \neq 0) \\ x_2 &= \frac{b_2 - a_{21}x_1}{a_{22}} \quad (a_{22} \neq 0) \\ \vdots &= \vdots \\ x_n &= \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}}{a_{nn}} \quad (a_{nn} \neq 0). \end{aligned} \tag{2}$$

Similarly, The general form of an upper triangular system with a non-zero determinant is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \quad (a_{11} \neq 0) \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \quad (a_{22} \neq 0) \\ a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \quad (a_{33} \neq 0) \\ \vdots &= \vdots \\ a_{nn}x_n &= b_n \quad (a_{nn} \neq 0), \end{aligned} \tag{3}$$

and the solution proceeds from bottom to top, like this,

$$\begin{aligned} x_1 &= \frac{b_1 - a_{12}x_2 - \cdots - a_{1n}x_n}{a_{11}} && (a_{11} \neq 0) \\ x_2 &= \frac{b_2 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}} && (a_{22} \neq 0) \\ \vdots &= \vdots && \vdots \\ x_{n-1} &= \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} && (a_{n-1,n-1} \neq 0) \\ x_n &= \frac{b_n}{a_{nn}} && (a_{nn} \neq 0), \end{aligned} \tag{4}$$

HINT:
Put $n=4$ for Backward Subs Problem

$$x_4 = \frac{b_4}{a_{44}}, \quad x_3 = \frac{b_3 - a_{34}x_4}{a_{33}}$$

$$x_2 = \frac{b_2 - a_{22}x_3 - a_{24}x_4}{a_{22}}$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4}{a_{11}}$$