2020 Sept 21 ROB 101

Reminder: (a) Proj #1 due Oct. 01, 11:59 PM EST
(b) HW always has drill prob. + julia

Review: Simplified LU Factorization

$\boxed{\text{Given:}}$ $A = n \times n$ matrix, seek

$$L = \begin{bmatrix} \ddots & 0 \\ & \ddots \end{bmatrix}_{n \times n} \quad \text{and} \quad U = \begin{bmatrix} \ddots & \\ 0 & \ddots \end{bmatrix} \quad \text{such that}$$

$$A = L \cdot U$$

Why Important: Reduces solution of $Ax = b$ to two simpler problems

$$Ax = b \iff L \cdot \underbrace{(Ux)}_{y} = b \iff \begin{cases} Ly = b & \text{(forward)} \\ \& \\ Ux = y & \text{(back)} \end{cases}$$

Scales to huge problems!

Summary of Friday follows:

[but in an even more algorithmic form]

Given $M = [n \times n]$

Initialize: Temp = copy(M)

L = [empty]

U = [empty]

```
for k = 1:n
        R = Temp[k, :]'        or  R = Temp[k:k; :]
        C = Temp[:, k]
        pivot = C[k]
    if pivot ≠ 0

            C = C ./ pivot     or  C = (1/pivot) * C

            L = [L  C]
            U = [U; R]

            Temp = Temp - C*R
    else
        break    (exit the for loop and
                  stop )
    end
end
```

row

column

scalar

1.0 in k-th entry

$$Temp = \begin{bmatrix} -2 & -4 & -6 \\ -2 & 1 & -4 \\ -2 & 11 & -4 \end{bmatrix}_{3\times3} \Rightarrow n=3$$

$k=1$ $\quad R = [-2 \quad -4 \quad -6]$

$$C = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}, \text{ pivot} = -2 \Rightarrow C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C * R = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [-2 \quad -4 \quad -6] = \begin{bmatrix} -2 & -4 & -6 \\ -2 & -4 & -6 \\ -2 & -4 & -6 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad U = [-2 \quad -4 \quad -6]$$

$$Temp = Temp - C * R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 15 & 2 \end{bmatrix}$$

$k=2$

$$R = [0 \quad 5 \quad 2]$$

$$C = \begin{bmatrix} 0 \\ 5 \\ 15 \end{bmatrix}, \text{ pivot} = 5 \Rightarrow C = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$C * R = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 15 & 6 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \qquad U = \begin{bmatrix} -2 & -4 & -6 \\ 0 & 5 & 2 \end{bmatrix}$$

$$Temp = Temp - C * R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

k=3

$$R = \begin{bmatrix} 0 & 0 & -4 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}, \text{pivot} = -4 \Rightarrow C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} -2 & -4 & -6 \\ 0 & 5 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\therefore \quad M = L * U$$

$$\begin{bmatrix} -2 & 4 & -6 \\ -2 & 1 & -4 \\ -2 & 11 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & -4 & -6 \\ 0 & 5 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

<u>Done</u>

Today: • New Example
• Permutation Matrices

Find a solution to $Ax = b$

for $A = \begin{bmatrix} 2 & -1 \\ 6 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

using our full "pipeline" [C.S. speak for process]

$$A \longrightarrow L \cdot U$$

$$Ax = b \longrightarrow \begin{cases} Ly = b \\ Ux = y \end{cases}$$

k=1 $\quad \text{Temp} = \begin{bmatrix} 2 & -1 \\ 6 & 2 \end{bmatrix}$

$R = \begin{bmatrix} 2 & -1 \end{bmatrix}$ $\qquad U = \begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} 2 & -1 \end{bmatrix}$

$C = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \frac{1}{2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\qquad L = \begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

k=2 $\quad \text{Temp} = \text{Temp} - C * R$

$$= \begin{bmatrix} 2 & -1 \\ 6 & 2 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \end{bmatrix}}_{\begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$R = \begin{bmatrix} 0 & 5 \end{bmatrix}$ $\qquad U = \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix}$

$$C = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \frac{1}{5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$Temp = Temp - C * R$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 5 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} L & C \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} U \\ R \end{bmatrix}$$

We have $\begin{bmatrix} 2 & -1 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix}$

seek to solve $Ax = b$

$$L(\underbrace{U\,x}_{y}) = b \iff \begin{matrix} Ly = b \\ Ux = y \end{matrix}$$

$Ly = b$ $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\Rightarrow$ forward substitution

$y_1 = 3$

$y_2 = 4 - 3y_1 = 4 - 3(3) = -5$

$y = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$Ux = y \Leftrightarrow \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

back substitution (bottom up)

$5x_2 = -5$

$\therefore x_2 = -1$

$2x_1 = 3 + x_2$

$= 3 + (-1) = 2$

$x_1 = 1$

$\therefore x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Seems long and tedious by

hand, but it programs very well and scales to huge problems!

What happens when our current algorithm breaks?

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$\underline{k=1}$

$$R = [0 \quad 1]$$

$$C = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{0} \quad \text{not defined}$$

$M \rightarrow$ swap the rows

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Matrix view of swapping rows is a PERMUTATION

Matrix ∇_0

Define identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n} = \text{ONE on the diagonal}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$I \cdot B = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 5 & 6 \end{bmatrix}$$

$$= B$$

Relation to Permutation matrices

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \longrightarrow \begin{bmatrix}1\\3\\2\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} = P$$

$$P \cdot B = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1&2\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}3&4\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}5&6\end{bmatrix}$$

$$= \begin{bmatrix}1&2\\0&0\\0&0\end{bmatrix} + \begin{bmatrix}0&0\\0&0\\3&4\end{bmatrix} + \begin{bmatrix}0&0\\5&6\\0&0\end{bmatrix}$$

$$= \begin{bmatrix}1&2\\5&6\\3&4\end{bmatrix}$$

Relation to LU Factorization

General Result Given A

an $n \times n$ matrix, there exist a permutation matrix P, a Lower Triangular Matrix L, and an Upper Triangular Matrix U such that

$$\boxed{P \cdot A = L \cdot U}$$

Solve $Ax = b$ knowing that $P \cdot A = L \cdot U$?

$$Ax = b \Rightarrow P \cdot A x = P \cdot b \Rightarrow$$

$$L \cdot U x = P \cdot b \Rightarrow$$

$$\boxed{\begin{cases} Ly = P \cdot b \\ Ux = y \end{cases}}$$

When is $P \neq I$?

$\boxed{\text{Case 1}}$ example Temp $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 2 & 2 \end{bmatrix}$

$R = [0 \; 0 \; 3]$

$C = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \frac{1}{0}$ = Problem

Case 2 : Not another one ..