ROB 103: See message on Piazza

Recap: $f: \mathbb{R}^m \to \mathbb{R}^n$

Partial derivative $\frac{\partial f}{\partial x_i}(x_0) \approx \frac{f(x_0 + h e_i) - f(x_0 - h e_i)}{2h}$

compute "slopes" one variable at a time $\quad \frac{\partial f}{\partial x_i}(x_0) \approx \frac{f(x_0 + h e_i) - f(x_0)}{h}$

Jacobian: $\frac{\partial f}{\partial x}(x_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x_0) & \frac{\partial f}{\partial x_2}(x_0) & \cdots & \frac{\partial f}{\partial x_m}(x_0) \end{bmatrix}$

= $n \times m$ matrix

If $f: \mathbb{R}^m \to \mathbb{R}$ (that is, $n=1$), then

$$\frac{\partial f}{\partial x}(x_0) = \nabla f(x_0)$$

Back to $f: \mathbb{R}^m \to \mathbb{R}^n$

$\Updownarrow$

$$f(x) = f(x_1, \dots, x_m) = \begin{bmatrix} f_1(x_1, \dots, x_m) \\ f_2(x_1, \dots, x_m) \\ \vdots \\ f_n(x_1, \dots, x_m) \end{bmatrix}$$

$$\frac{\partial f}{\partial x_i}(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_i}(x_0) \\ \vdots \\ \frac{\partial f_n}{\partial x_i}(x_0) \end{bmatrix}$$

Linear Approximation about a point x_0

$$f(x) \approx \underbrace{f(x_0)}_{n\times 1} + \underbrace{\frac{\partial f}{\partial x}(x_0)}_{n\times m} \underbrace{(x - x_0)}_{m\times 1}$$

Compute the linear approx. of

$$f(x_1,x_2,x_3) = \begin{bmatrix} x_1 \, x_2 \, x_3 \\ \log(2+\cos(x_1)) + (x_2)^{x_1} \\ \dfrac{x_1 \, x_3}{1+(x_2)^2} \end{bmatrix}$$

at the point $x_0 = \begin{bmatrix} \pi \\ 1.0 \\ 2.0 \end{bmatrix}$

See Julia demo! ☺ for the Jacobian

$$A = \frac{\partial f}{\partial x}(x_0) = \begin{bmatrix} 2 & 6.28 & 3.14 \\ 0 & 3.14 & 0 \\ 1 & -3.14 & 1.57 \end{bmatrix}$$

$$f(x_0) = \begin{bmatrix} 2\pi \\ 1 \\ \pi/5 \end{bmatrix}$$

$$f_{lin}(x) = \begin{bmatrix} 2\pi \\ 1 \\ \pi/5 \end{bmatrix} + A\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} \pi \\ 1 \\ 2 \end{bmatrix}\right)$$

Check out the green call-out box that covers our journey from slopes, to derivatives to gradients, and culminating in Jacobians.

Roots of $f: \mathbb{R}^n \to \mathbb{R}^n$

Newton-Raphson Algorithm

Let x_k be our current estimate of a root of $f(x) = 0$.

- We linearize f about x_k

$$f(x) \approx f(x_k) + \frac{\partial f}{\partial x}(x_k)\,(x - x_k)$$

We approximately zero $f(x)$ by exactly zeroing the linear approx.

$$f(x_{k+1}) \approx 0 \iff 0 = f(x_k) + \frac{\partial f}{\partial x}(x_k)\left(x_{k+1} - x_k\right)$$

If $\det\left(\frac{\partial f}{\partial x}(x_k)\right) \neq 0$, we can solve for x_{k+1}

$$\frac{\partial f}{\partial x}(x_k)\left(x_{k+1} - x_k\right) = -f(x_k) \qquad (*)$$

$$x_{k+1} - x_k = -\left[\frac{\partial f}{\partial x}(x_k)\right]^{-1} f(x_k)$$

$$\boxed{x_{k+1} = x_k - \left[\frac{\partial f}{\partial x}(x_k)\right]^{-1} f(x_k)}$$

This is the standard presentation of Newton Raphson

In ROB 101, we try to avoid computing matrix inverses!

Define $\Delta x_k := x_{k+1} - x_k$

so that $x_{k+1} = \underset{\text{current value}}{x_k} + \underbrace{\Delta x_k}_{\text{update}}$

We have from $(*)$

$$\underbrace{\frac{\partial f}{\partial x}(x_k)}_{A} \underbrace{\Delta x_k}_{x} = \underbrace{-f(x_k)}_{b}$$

While $\|f(x_k)\| > tol$

$\frac{\partial f}{\partial x}(x_k) = Q_k \cdot R_k$

\# $R_k \Delta x_k = -Q_k^T f(x_k)$

$\Delta x_k = \text{BackSub}(R_k, -Q_k^T f(x_k))$

\# $x_{k+1} = x_k + \Delta x_k$ (math)

$x_k = x_k + \Delta x_k$ (Julia)

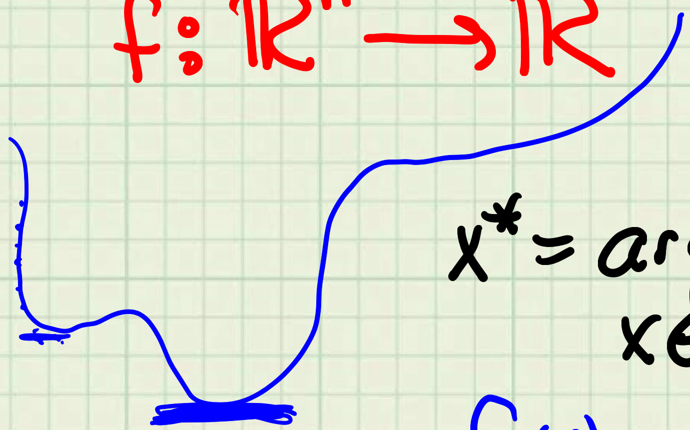
END

End of Chapter 10.

Chapter 11: Optimization

find minima of functions

$f: \mathbb{R}^n \to \mathbb{R}$



$$x^* = \arg\min_{x \in \mathbb{R}^n} f(x)$$

$$f(x) = \underbrace{\|Ax - b\|^2}_{\text{least squares}}$$