Review/Summary: Let A be an $n \times n$ matrix.

The Following Are Equivalent (TFAE):

(a) • For every $n \times 1$ vector b, the system of linear equations $Ax = b$ has a unique solution.

(b) • $\det(A) \neq 0$

(c) • There exists an $n \times n$ matrix, which we denote by A^{-1}, that satisfies

$$A \cdot A^{-1} = A^{-1} \cdot A = I_{n \times n}.$$

Moreover, $(A \cdot A^{-1} = I_{n \times n}) \iff (A^{-1} \cdot A = I_{n \times n})$

Remark: TFAE means: (a) $\iff$ (b) $\iff$ (c)

In other symbols:

$$\boxed{x = A^{-1} b}$$

$$\begin{tikzcd}
 & (a) \arrow[dl, Rightarrow] & \\
(b) \arrow[rr, Rightarrow] & & (c) \arrow[ul, Rightarrow]
\end{tikzcd}$$

Today: Theory vs. Practice
Start the vector space, $\mathbb{R}^n$ ☺

$$A^{-1}A = \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & & e_n \end{array} \\ \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & 1 \end{bmatrix}_{n\times n} \end{array} = AA^{-1}$$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-th entry}$$

<u>Fact:</u> Computing A^{-1} is the same thing as solving $\underline{Ax = e_i}$, for $i = 1, 2, \ldots, n$.

<u>Exercise</u>

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Solve $Ax = e_i$, $i = 1, 2, 3$ and then place the solutions as the colums of $B = [sol_1 \ sol_2 \ sol_3]$

Check $A \cdot B = B \cdot A = I_{3\times3}$.

① A^{-1} is computationally intensive and is a very inefficient means to solve $Ax=b$.

② Theory $\det(A) \neq 0 \iff A^{-1}$ exists

Practice, even $\det(A)$ fairly far from zero can be misleading: $\det(A)$ is an unreliable friend!

$\det(I_{n\times n}) = 1.0$

Example 2 Consider a 3×3 matrix

$$A = \begin{bmatrix} 100.0000 & 90.0000 & -49.0000 \\ 90.0000 & 81.0010 & 5.4900 \\ 100.0000 & 90.0010 & 59.0100 \end{bmatrix}. \tag{2}$$

We compute the determinant and check that it is not close to zero. Indeed, $\det(A) = 0.90100$ (correct to more than ten decimal points[1]), and then bravely, we use Julia to compute the inverse, yielding

$$A^{-1} = \begin{bmatrix} -178.9000 & -10,789.0666 & 9,889.0666 \\ 198.7791 & 11,987.7913 & -10,987.981 \\ -110.9878 & -0.1110 & 0.1110 \end{bmatrix}.$$

[1] $\det(A) - 0.9010 = -1.7 \times 10^{-11}$.

Question: How do you often obtain big numbers from "medium" numbers?

Answer: Divide by tiny numbers

As a contrast, we compute $A = L \cdot U$, the LU factorization (without permutation), yielding

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.9 & 1.0 & 0.0 \\ 1.0 & 1.0 & 1.0 \end{bmatrix} \qquad U = \begin{bmatrix} 100.000 & 90.000 & -49.000 \\ 0.000 & \boxed{-0.001} & 99.000 \\ 0.000 & 0.000 & 9.010 \end{bmatrix}.$$

We see that U has a small number on the diagonal, and hence, if we do back substitution to solve $Ux = y$, for example, as part of solving

$$Ax = b \iff L \cdot Ux = b \iff (Ly = b \text{ and } Ux = y)$$

we know in advance that we'll be dividing by -0.001.

Theoretical Linear Algebra!

Learning Objectives

- Instead of working with individual vectors, we will work with a collection of vectors.
- Our first encounter with some of the essential concepts in Linear Algebra that go beyond systems of equations.

Outcomes

- Vectors as n-tuples of real numbers
- $\mathbb{R}^n$ as the collection of all n-tuples of real numbers
- Linear combinations of vectors
- Linear independence of vectors
- Relation of these concepts to the existence and uniqueness of solutions to $Ax = b$.
- How the LU Factorization makes it very straightforward to check the linear independence of a set of vectors, and how it makes it reasonably straightforward to check if one vector is a linear combination of a set of vectors.

Def. An n-tuple is an ordered list of n numbers, $(x_1, x_2, \dots, x_n)$. In Julia, it is any one-dimensional array.

We use the symbol $\mathbb{R}$ denote the real numbers

$$\mathbb{R} \longleftrightarrow (-\infty, \infty)$$

Recall ∞ it is not a number. It is a concept.

Def.

$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n \}$

All n-tuples of real numbers.

We identify $\mathbb{R}^n$ with the set of column vectors of length n.

$$\mathbb{R}^n = \{ (x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \} \longleftrightarrow \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \; x_i \in \mathbb{R} \right\}$$

For us, $x \in \mathbb{R}^n \longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, x_i \in \mathbb{R}$

Elements of $\mathbb{R}^n$ are column vectors. We simply call them **VECTORS**.

Remark: We could have chosen to identify $\mathbb{R}^n$ with row vectors.

First consequence of our choice:
If A is an $n \times m$ matrix, then its columns are vectors in $\mathbb{R}^n$. Conversely, given a finite set of vectors in $\mathbb{R}^n$, we can "stack" them together and form a matrix!

$A = n \times m \longleftrightarrow$ A set of m vectors in $\mathbb{R}^n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & & a_{2m} \\ & & & \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix} = \begin{bmatrix} a_1^{col} & a_2^{col} & \cdots & a_m^{col} \end{bmatrix}$$

$$\longleftrightarrow \quad a_j^{col} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \in \mathbb{R}^n$$

Two vectors x and $y \in \mathbb{R}^n$ are equal $\Longleftrightarrow$ $x_i = y_i$ for $1 \le i \le n$.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \iff \begin{matrix} x_1 = y_1 \\ x_2 = y_2 \\ \vdots \\ x_n = y_n \end{matrix}$$

We define vector addition by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \underline{+} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

We define scalar times vector multiplication by, if $a \in \mathbb{R}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, then

$$a \cdot x := a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a x_1 \\ a x_2 \\ \vdots \\ a x_n \end{bmatrix}$$ (no surprise)

Let's recall properties of $\mathbb{R}$

$$\left.\begin{matrix} a+b = b+a \\ a \cdot b = b \cdot a \end{matrix}\right\} \text{ commutative}$$

$$(a+b)+c = a+(b+c)$$
$$(a\cdot b)\cdot c = a\cdot(b\cdot c)$$
} associative

$$a\cdot(b+c) = a\cdot b + a\cdot c$$
} distributive

Similar Properties for Vectors

$u, v, w \in \mathbb{R}^n$

$$(u+v)+w = u+(v+w)$$

$a \in \mathbb{R}$

$$a\cdot(u+v) = a\cdot u + a\cdot v$$

$$(a+b)\cdot u = a\cdot u + b\cdot u$$

$b \in \mathbb{R}$

$\boxed{\text{Stop Here}}$