ROB 201 - Calculus for the Modern Engineer
HW #4

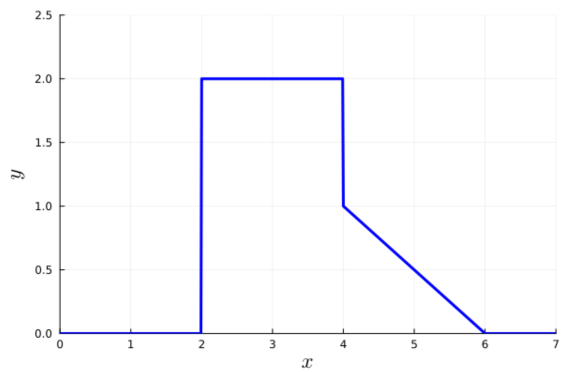
Prof. Grizzle

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete.

1. (a) Create a "Cheat Sheet" for Chapters 3 and 4 of the textbook. Here is an example from ROB 101.
   (b) Note any material where you found the explanation confusing or difficult to master.

2. Solids of revolution.

   (a) Sketch BY HAND the solid of revolution corresponding to rotating the image below about the $\boldsymbol{y}$**-axis**. Don't worry: we're not looking for art-school-quality sketches.



   (b) Compute the volume of the solid of revolution defined by the two functions
   - $f : [0, 2] \to \mathbb{R}$ by $f(x) = 2x$
   - $g : [0, 2] \to \mathbb{R}$ by $g(x) = x$,

   when rotated about the $\boldsymbol{y}$**-axis**. You are not obliged to make a sketch of any kind, though usually, at least sketching the area out in the (x, y)-plane helps to set up the problem correctly. Show your work.

   In each case, pay attention to the axis of rotation. Parts (a) and (b) are not related. To be exceedingly pedantic, when setting up the problem in part (b), ignore the sketch in part (a); instead, use the given functions.

3. For each of the following sets or functions, compute the maximum value if it exists. If it does not exist, briefly state why, and then compute the supremum. See the hints for example solutions.

   (a) $A := \left\{ x \in \mathbb{R} \;\middle|\; \frac{x^2+2}{x^2+1} \geq 1.5 \right\}$

   (b) $B := \left\{ x \in \mathbb{R} \;\middle|\; \frac{1}{x^3} < 8 \right\}$

   (c) $f : [0, \infty) \to \mathbb{R}$ by $f(x) = 6 - (x-3)^2$

   (d) $g : [0, \infty) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x = k\pi,\ k \in \{0, 1, 2, \ldots\} \\ \cos(x) & \text{otherwise} \end{cases}$$

4. Evaluate the following limits using any method you wish, and then provide a brief (ten words or less) description of your reasoning.

   (a) $\lim_{x \to \frac{\pi}{2}^-} \cos(\sin(x))$

   (b) $\lim_{x \to 3^-} \dfrac{x^2 - 9}{x - 3}$

   (c) $\lim_{x \to 0^-} e^{\frac{1}{x^3}}$

5. In each case below, provide ONE example of a function that meets the stated properties. Provide a few hints as to why your answer is correct. Those hints could be in the form of text, a plot, or both. Note the domain and codomain are specified for each function. You can choose any range you wish as long as it is contained in the specified codomain.

   (a) $f : [0, 1] \to [2, 4]$ has exactly **three** points where it is discontinuous.

   (b) $f : (-1, 0) \to \mathbb{R}$ is continuous, and satisfies $\lim_{x \to -1^+} f(x) = 3.0$ and $\lim_{x \to 0^-} f(x) = +\infty$.

   (c) $f : \mathbb{R} \to \mathbb{R}$ is bounded, is piecewise continuous on $(-\infty, 0]$, and continuous on $(0, \infty)$.

6. Let $f(x)$ be a differentiable function, and define

$$h(x) := \frac{1}{f(x)}.$$

   Task: At points where $f(x) \neq 0$, derive a formula for $h'(x)$ using the **chain rule**. **You may not use the quotient rule.**

   Instructions: Follow the method used in the textbook to derive the chain rule. Write clearly, define any notation you introduce, and explain your reasoning using complete sentences.

Hints

Prob. 1 Nothing to add.

Prob. 2 For part (b), you have to decide between the **disc-washer method** or the **shell method**. You do not have to make any sketches, though, of course, that may help you.

Prob. 3

- $C := \{x \in \mathbb{R} \mid \frac{2x-1}{x+1} \leq 1.0\}$

  Ans. $x^* := \max(C) = 2.0$

  We first note that $\frac{2x-1}{x+1}\Big|_{x=0} = -1$ and $\frac{2x-1}{x+1}\Big|_{x=5} = \frac{9}{6} = 1.5$, and thus there likely exists a point in between where the function equals 1.0. To find out, we solve

  $$\frac{2x-1}{x+1} = 1$$
  $$\Downarrow$$
  $$2x - 1 = x + 1$$
  $$\Downarrow$$
  $$x = 2$$

  We note that because the set is defined with a less than or equal to sign, $2 \in C$. Is 2 the largest element?

  We next note that for $x > 2$, $\frac{2x-1}{x+1} > 1$ and hence $x \notin C$. Thus, if $y \in C$, then $y \leq 2$, proving that $x^* = 2$ is the maximum value in the set.

  It's not required, but if you really want to show that $x > 2 \implies \frac{2x-1}{x+1} > 1$, write $x = 2 + \delta$ for $\delta > 0$. Then,

  $$\frac{2x-1}{x+1}\Big|_{x=2+\delta} = \frac{4 + 2\delta - 1}{2\delta + 1} = \frac{3 + 2\delta}{3 + \delta} = 1 + \frac{\delta}{3+\delta} > 1$$

  for $\delta > 0$, and hence $2 + \delta \notin C$.

- $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \frac{4+(x-5)^4}{1+e^{-x}}$

  Ans. The function is unbounded and hence does not have a maximum. $x^* := \sup_{x \in [0,\infty)} f(x) = \infty$

  From Chapter 2, we have a result on the limits of products and ratios, shown in the box below. Applying this result to our problem we have

  - $f(x) = \frac{4+(x-5)^4}{1+e^{-x}} =: \frac{\mathrm{num}(x)}{\mathrm{den}(x)}$
  - $\lim_{x\to\infty} \mathrm{den}(x) = \lim_{x\to\infty} 1 + e^{-x} = 1$
  - $\lim_{x\to\infty} \mathrm{num}(x) = \lim_{x\to\infty} 4 + (x-5)^4 = \infty$

  and hence by the Proposition, the result is established!

Proposition 2.44: Limits of Products and Ratios

Suppose that $g : (0, \infty) \to \mathbb{R}$ has a limit at infinity of one, i.e., $\lim_{x\to\infty} g(x) = 1.0$. Then for any function $f : (0, \infty) \to \mathbb{R}$,

$$\begin{aligned} \lim_{x\to\infty} f(x)\cdot g(x) &= \lim_{x\to\infty} f(x), \text{ and} \\ \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} f(x). \end{aligned} \tag{2.31}$$

In particular,

(a) if $\lim_{x\to\infty} f(x) = L$, for $L \in \mathbb{R}$, then $\lim_{x\to\infty} f(x)\cdot g(x) = L$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$;

113

(b) if $\lim_{x\to\infty} f(x) = \pm\infty$, then $\lim_{x\to\infty} f(x)\cdot g(x) = \pm\infty$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \pm\infty$; and

(c) if $\lim_{x\to\infty} f(x)$ does not exist, then $\lim_{x\to\infty} f(x)\cdot g(x)$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ do not exist.

Note: If $\lim_{x\to\infty} g(x) = M \neq 1$, and $M \neq 0$, then $\frac{g(x)}{M}$ has limit one. Hence, using $f(x)\cdot g(x) = (Mf(x))\cdot\frac{g(x)}{M}$,and $\frac{f(x)}{g(x)} = \frac{\frac{1}{M}f(x)}{\frac{1}{M}g(x)}$ reduces the general case of the problem to the given (special) conditions.

If you remember (2.31)**, then the special cases follow immediately.**

Prob. 4 Nothing to add.

Prob. 5 KISS is a good policy.

Prob. 6 Start by recognizing that $h(x) = \frac{1}{f(x)}$ is the composition of two functions:

$$h(x) = g(f(x)) \quad \text{where} \quad g(u) = \frac{1}{u}.$$

Now use the chain rule to differentiate. Remember: if $g(u) = u^{-1}$, then $g'(u) = -\frac{1}{u^2}$. The rest is substitution and simplification.

Solutions HW 04

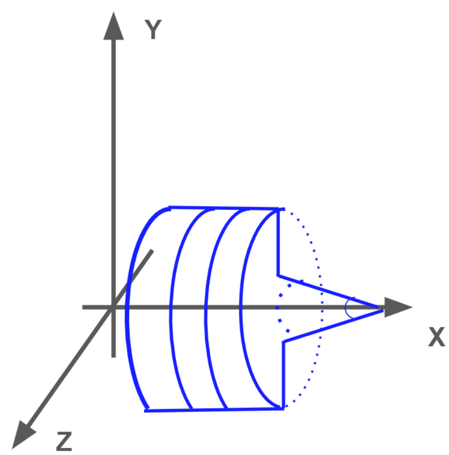
Prob. 1 (a) Included at the end of the solution set

(b) These will vary by person, but some of the more challenging topics may have been:

i. Chap 03: Why some functions are Riemann integrable and others not
ii. Chap 03: How to handle integrals when the limits of integration are in a funny order
iii. Chap 03: How to integrate a function that has been shifted or scaled
iv. Chap 03: Center of mass
v. Chap 03: Moment of inertia
vi. Chap 04: Epsilon-delta definition of limits
vii. Chap 04: Formal definition of continuous functions
viii. Chap 04: Supremum and infimum of sets and functions

Prob. 2

(a)

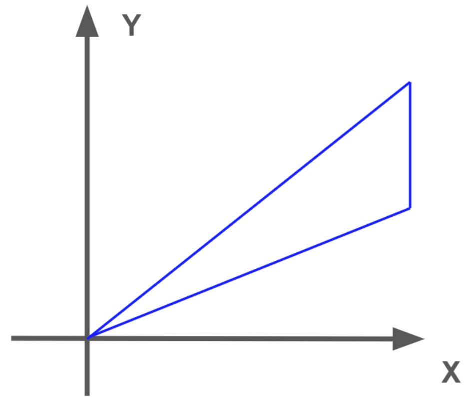


(b) The two functions

- $f : [0, 2] \to \mathbb{R}$ by $f(x) = 2x$
- $g : [0, 2] \to \mathbb{R}$ by $g(x) = x$,

lie totally in the first quadrant. Hence, we can apply the Proposition on Solids of Revolution (given below).

Ans. $V_{\text{y-axis}} = \boxed{\dfrac{16\pi}{3}}$



The shell method tells us to compute:

$$V_{\text{y-axis}} := \int_a^b 2\pi x \cdot (f(x) - g(x))\ dx$$

For our functions:

$$f(x) - g(x) = 2x - x = x$$

and the limits of integration are $a = 0$, $b = 2$. Thus,

$$V_{\text{y-axis}} = \int_0^2 2\pi x \cdot x\, dx = \int_0^2 2\pi x^2\, dx$$

$$= 2\pi \int_0^2 x^2\, dx = 2\pi \cdot \left[\frac{x^3}{3}\right]_0^2$$

$$= 2\pi \cdot \frac{8}{3} = \boxed{\frac{16\pi}{3}}$$

Prob. 3

(a) $A := \left\{x \in \mathbb{R} \;\middle|\; \frac{x^2+2}{x^2+1} \geq 1.5\right\}$ **Ans.** $\boxed{\max(A) = 1}$

Solution: Since the denominator $x^2 + 1 > 0$ for all x, we can multiply both sides without changing the inequality,

$$\frac{x^2+2}{x^2+1} \geq 1.5 \quad \Longleftrightarrow \quad x^2 + 2 \geq 1.5(x^2+1)$$

$$x^2 + 2 \geq 1.5x^2 + 1.5 \quad \Rightarrow \quad 0.5 \geq 0.5x^2 \quad \Rightarrow \quad x^2 \leq 1$$

So:

$$A = \left\{x \in \mathbb{R} \;\middle|\; x^2 \geq 1\right\} = [-1, 1] \quad \Rightarrow \quad \boxed{\max(A) = 1}$$

(b) $B := \left\{x \in \mathbb{R} \;\middle|\; \frac{1}{x^3} < 8\right\}$ **Ans.** $\boxed{\sup(B) = \infty}$

Solution: First, observe that for $x > 0$,

$$\frac{1}{x^3} < 8 \quad \Longleftrightarrow \quad x^3 > \frac{1}{8} \quad \Longleftrightarrow \quad x > \frac{1}{2}.$$

For $x = 0$, $\frac{1}{x^3}$ is undefined. Finally, for $x < 0$, $x^3 < 0$, so $\frac{1}{x^3} < 0 < 8$ — thus all negative values of x satisfy the inequality Therefore, the solution set is:

$$B = (-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$$

This set is unbounded from the right and thus

$$\boxed{\sup(B) = \infty}$$

(c) $f(x) = 6 - (x-3)^2$, with domain $x \in [0, \infty)$ **Ans.** $\boxed{\max f = 6 \text{ at } x = 3}$

Solution: This is a concave-down parabola with vertex at $x = 3$. Since:

$$(x-3)^2 \geq 0, \quad \text{with equality at } x = 3,$$

we get:

$$f(3) = 6 - 0 = 6 \quad \Rightarrow \quad \boxed{\max f = 6}$$

(d) $g : [0, \infty) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x = k\pi, \quad k \in \{0, 1, 2, \ldots\} \\ \cos(x) & \text{otherwise} \end{cases}$$

Ans. $\boxed{\sup g = 1 \quad \text{and no maximum exists}}$

Solution: The function agrees with $\cos(x)$ except at countably many isolated points: $x = k\pi$ for $k \in \mathbb{Z}$, where it is reset to 0. In particular:

$$g(0) = 0, \quad g(\pi) = 0, \quad \text{etc.}$$

However, for values close to (but not equal to) these points, $\cos(x)$ comes arbitrarily close to 1 — for example:

$$\lim_{x \to 2\pi^-} g(x) = \cos(2\pi^-) \approx 1$$

Since $g(x) < 1$ for all x, but values of $g(x)$ can get arbitrarily close to 1, we conclude:

$$\text{No maximum exists, but } \boxed{\sup g = 1}$$

Prob. 4 (a) $\displaystyle\lim_{x \to \frac{\pi}{2}^-} \cos(\sin(x))$ **Ans.** $\boxed{\cos(1)} \approx \boxed{0.5403}$

Reason: Composition of continuous functions.

Both $\cos(x)$ and $\sin(x)$ are continuous on $(0, \pi)$, so their composition is continuous. Thus,

$$\lim_{x \to \frac{\pi}{2}^-} \cos(\sin(x)) = \cos\left(\lim_{x \to \frac{\pi}{2}^-} \sin(x)\right) = \cos(1).$$

(b) $\displaystyle\lim_{x \to 3^-} \frac{x^2 - 9}{x - 3}$ **Ans.** $\boxed{6}$

Reason: Cancel factor; evaluate simplified expression.

Note:

$$x^2 - 9 = (x-3)(x+3), \quad \text{so} \quad \frac{x^2 - 9}{x - 3} = x + 3 \quad \text{for } x \neq 3.$$

Therefore,

$$\lim_{x \to 3^-} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3^-} (x + 3) = 6.$$

(c) $\displaystyle\lim_{x \to 0^-} e^{\frac{1}{x^3}}$ **Ans.** $\boxed{0}$

Reason: Exponent tends to $-\infty$.

As $x \to 0^-$, we have $x < 0$, so $x^3 < 0$ and $\frac{1}{x^3} \to -\infty$. Then,

$$e^{\frac{1}{x^3}} \to e^{-\infty} = 0.$$

Prob. 5

(a) **Ans.** $f(x) = \begin{cases} 2+x & \text{if } x \neq \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 3.1 & \text{if } x = \frac{1}{4} \\ 3.5 & \text{if } x = \frac{1}{2} \\ 3.9 & \text{if } x = \frac{3}{4} \end{cases}$

Explanation: This function is defined on $[0,1]$, and equal to the continuous function $2+x$ except at three points, where we assign different values within the codomain $[2,4]$. It is discontinuous exactly at those three points by design.

(b) **Ans.** $f(x) = \frac{-3}{x}$

Explanation: The function is continuous on $(-1,0)$ and satisfies:

$$\lim_{x\to -1^+} f(x) = \frac{-3}{-1} = 3, \qquad \lim_{x\to 0^-} f(x) = +\infty.$$

This is exactly what the problem requires.

(c) **Ans.** **(i)** $f(x) = 0$ for all $x \in \mathbb{R}$

Explanation: This function is constant and therefore continuous everywhere. Since continuous functions are automatically piecewise continuous, the criteria are trivially satisfied. Don't let fancy words fool you!

(ii) A more interesting example:

$$f(x) = \begin{cases} -1, & x = -2 \\ \cos(x), & \text{otherwise} \end{cases}$$

Explanation: This function has a removable discontinuity at $x = -2$, and is otherwise continuous. On $(-\infty, 0]$, it's piecewise continuous (with just one jump). On $(0,\infty)$, $\cos(x)$ is continuous. The function is bounded and meets all requirements.

Prob. 6 We are asked to compute the derivative of $h(x) = \frac{1}{f(x)}$ using the **chain rule**.

Step 1: Recognize the composition.

We write $h(x)$ as a composition:

$$h(x) = g(f(x)) \quad \text{where} \quad g(u) = \frac{1}{u} = u^{-1}.$$

Step 2: Differentiate using the chain rule.

We apply the chain rule:

$$h'(x) = g'(f(x)) \cdot f'(x).$$

Step 3: Compute $g'(u)$.

Since $g(u) = u^{-1}$, we have:

$$g'(u) = -u^{-2} = -\frac{1}{u^2}.$$

Step 4: Substitute back into the expression for $h'(x)$:

$$h'(x) = -\frac{1}{(f(x))^2} \cdot f'(x).$$

Final Answer:

$$\boxed{h'(x) = -\frac{f'(x)}{(f(x))^2}}.$$

This is the quotient rule applied to the special case $\frac{1}{f(x)}$, but derived here only via the chain rule, as requested.

ROB 201 CHAPTER 3 STUDY GUIDE

Riemann Integral

Definite Integral to functions defined over a closed interval $[a,b]$.

- Partition of Interval: $\Delta x = \frac{\overbrace{b-a}^{\text{interval}}}{\underbrace{n}_{\text{subintervals}}}$ $\quad x_i := a+(i-1)\Delta x$ for $i = 0, 1, 2, \dots, n$

$\boxed{\text{Ex: } f(x) = x^2 \text{ over interval } [1,3] \text{ using } n = 4 \text{ subintervals}}$

① Interval and Subintervals
- Interval: $[1, 3]$ (a, b)
- Subintervals 4 ($n = 4$)

② Calculate Δx:

$\Delta x = \frac{3-1}{4} = 0.5$

③ Define Partition Points x_i:
- $x_i := 1+(i-1)\cdot 0.5 \implies$
  - $x_1 = 1+(1-1)\cdot 0.5 = 1$
  - $x_2 = 1+(2-1)\cdot 0.5 = 1.5$
  - $x_3 = 1+(3-1)\cdot 0.5 = 2$
  - $x_4 = 1+(4-1)\cdot 0.5 = 2.5$

④ Compute $f(x_i)$ at each x_i:

$f(x) = x^2$
- $f(x_1) = (1)^2 = 1$
- $f(x_2) = (1.5)^2 = 2.25$
- $f(x_3) = (2)^2 = 4$
- $f(x_4) = (2.5)^2 = 6.25$

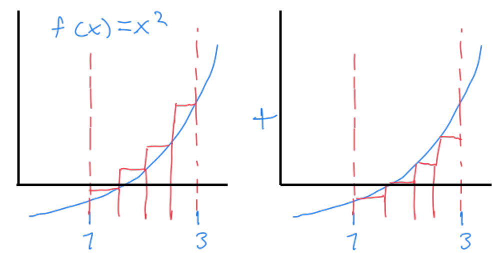
⑤ Calculate Riemann Sum S_n:

$S_n = \sum_{i=1}^{4} f(x_i)\cdot \Delta x$

$S_4 = f(x_1)\cdot 0.5 + f(x_2)\cdot 0.5 + f(x_3)\cdot 0.5 + f(x_4)\cdot 0.5$

$S_4 = 1\cdot 0.5 + 2.25\cdot 0.5 + 4\cdot 0.5 + 6.25 \cdot 0.5$

$\boxed{S_4 \approx 6.75}$



On the graph of $f(x) = x^2$, in bounds $[1, 3]$ the Riemann sum S_n combines the lower and upper sums

Non Riemann Integral Functions:

① Function w/ infinite discontinuity
② unbounded on interval
③ oscillate Infinitely
④ Interval Extends to Infinity

Integral Notation

$\int x^n dx = \frac{x^{n+1}}{n+1} + \underbrace{C}_{\text{For indefinite}}$ $\qquad$ Ex: $\int x^2 dx = \frac{x^3}{3} + C$

Integral Notation of a function: $\int_a^b f(x)\,dx := \begin{cases} \lim_{n\to\infty} \text{Area}_n^{\text{low}} \\ \lim_{n\to\infty} \text{Area}_n^{\text{up}} \end{cases}$

Ex: $\int_1^3 x^2 dx \Leftarrow$ Definite Integral

Ex: $\int_1^x x^2 dx \Leftarrow$ Indefinite Integral

- $\int_a^b f(x)dx$ when $a > b \implies \int_a^b f(x)\,dx := -\int_b^a f(x)\,dx$

Ex: $\int_5^2 x^2 dx = -\int_2^5 x^2 dx$

Shift Property

Evaluate definite Integral

$\int_0^1 (x-2)^2 dx = \int_{0-2}^{1-2} x^2 d\,$

$= \int_{-2}^{-1} x^2 dx = \frac{x^3}{3}\Big|_{-2}^{-1}$

$= \frac{(-1)^3}{3} - \frac{(-2)^3}{3} = -\frac{1}{3} + \frac{8}{3} = \boxed{2\tfrac{1}{3}}$

Scaling Property

$\int_0^1 (2x)^2 dx = 4\int_0^1 x^2 dx = 4\frac{x^3}{3}\Big|_0^1 = 4\left(\frac{1}{3}-0\right) = \frac{4}{3}$

$\int_0^1 (2x)^2 dx = \frac{1}{2}\int_0^2 x^2 dx = \frac{1}{2}\frac{x^3}{3}\Big|_0^2$

$= \frac{1}{2}\left(\frac{8}{3} - 0\right) = \boxed{\frac{4}{3}}$

One Sided Limits

- $f(x)$ is continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$

Ex: $f(x) = \begin{cases} x^2 & x < 1 \\ 2x-1 & x \geq 1 \end{cases}$ Is $f(x)$ continuous at $x = 1$?

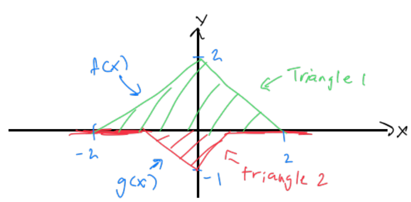
- left hand limit: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$
- right hand limit: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x-1) = 2\cdot 1 - 1 = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$ ✓

continuous at $x = 1$ ✓

Area between two functions

Ex: $f(x) = \begin{cases} 2-|x| & |x| \leq 2 \\ 0 & \text{o.w.} \end{cases}$ $\qquad g(x) = \begin{cases} |x|-1 & |x| \leq 1 \\ 0 & \text{o.w.} \end{cases}$



- Area $= \underbrace{\frac{1}{2}\cdot 4\cdot 2}_{1} + \underbrace{\frac{1}{2}\cdot 2\cdot 1}_{2} = 5.0$

Simpson's Rule

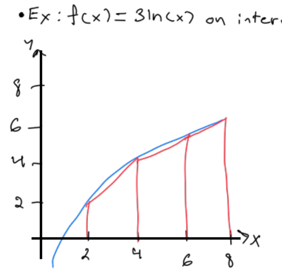
Uses Parabolas to approximate area

$\alpha \cdot \frac{(\Delta x)^3}{12} + \gamma \cdot \Delta x$

Trapezoid Rule

$\lim_{n\to\infty} \sum_{i=1}^{n} 0.5 \cdot \frac{f(x_i) + f(x_{i+1})}{x_{i+1} - x_i}$

- Ex: $f(x) = 3\ln(x)$ on interval $[2, 8]$.



use trapezoids for closer approximation

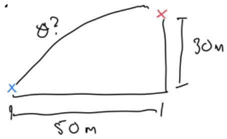
☐ Balistic Motion

$$p(t) \underset{\text{Integral}}{\overset{\text{Derivative}}{\rightleftarrows}} v(t) \underset{\text{Integral}}{\overset{\text{Derivative}}{\rightleftarrows}} a(t)$$

position, velocity, acceleration

☐ $v(t) = v_0 + \int_{t_0}^{t} a(\tau)\, d\tau$

Ex:



- $m = 140g$
- $v_0 = 15 m/s$
- $g = 9.81 m/s^2$

$$\underbrace{\begin{bmatrix} v_x(0) \\ v_y(0) \end{bmatrix}}_{\text{start velocity}} = \begin{bmatrix} 15\cos(\theta) \\ 15\sin(\theta) \end{bmatrix}$$

$$p(T) = \begin{bmatrix} 15\cos(\theta) \cdot T \\ 15\sin(\theta) \cdot T - \frac{1}{2}(0.14)(9.81)T^2 \end{bmatrix} = \begin{bmatrix} 50 \\ 30 \end{bmatrix}$$

$$T = \frac{50}{15\cos(\theta)} \Rightarrow 30 = 50\tan(\theta) - 0.6867\left(\frac{10}{3\cos\theta}\right)^2$$

↳ rad ≃ 0.5404 $\theta \simeq 41$

Robot Links

Key Concepts

1. **Newton's Second Law:**
   - Formula: $F = ma$
   - Describes the motion of a single point mass.
2. **Robot Link as a Body:**
   - A link is composed of infinitely many point masses.
   - Using calculus, we treat the link as a continuous collection of points to understand its properties.
3. **Total Mass:**
   - The mass of a link is an integral over its volume.
   - Formula: $M_T = \rho \cdot V = \rho \cdot h \cdot A$
   - For a function $f(x)$ representing the upper boundary and $g(x)$ the lower boundary, the total mass M_T is given by:
     $M_T = \rho \cdot h \cdot \int_{x_{min}}^{x_{max}} (f(x) - g(x))\, dx$
4. **Center of Mass:**
   - The center of mass is the point where the total mass of the system is balanced.
   - For discrete point masses, the x-coordinate of the center of mass x_c is:
     $x_c = \frac{\sum_{i=1}^{N} m_i \cdot x_i}{\sum_{i=1}^{N} m_i}$
   - This formula represents a weighted average of the positions of the point masses, where each position is weighted by its corresponding mass.

Center of Mass for Point Masses

To find the center of mass for a system of discrete point masses:

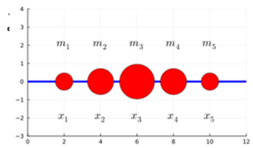
1. **Total Mass:**
   $M_T = \sum_{i=1}^{N} m_i$
2. **Total x-Moment about the Origin:**
   $\text{Total x-Moment} = \sum_{i=1}^{N} m_i \cdot x_i$
3. **x-coordinate of the Center of Mass:**
   $x_c = \frac{\text{Total x-Moment}}{M_T} = \frac{\sum_{i=1}^{N} m_i \cdot x_i}{\sum_{i=1}^{N} m_i}$

Notation:

Notation

- A: Total area of a link.
- h: Thickness of a link.
- V: Total volume of a link ($V = A \cdot h$).
- ρ: Material density of a link.
- M_T: Total mass of a link ($M_T = \rho \cdot V = \rho \cdot h \cdot A$).

Ex:



$(m_1, m_2, m_3, m_4, m_5) = (2, 3, 4, 3, 2)$

$(x_1, x_2, x_3, x_4, x_5) = (2, 4, 6, 8, 10)$

$M_T = 14$ x-moment $= m_1 \cdot x_1 + m_2 \cdot x_2 + m_3 \cdot x_3 + m_4 \cdot x_4 + m_5 \cdot x_5 = 84$

$$\boxed{x_c = \frac{84}{14} = 6 = x_3}$$

Simpson's Rule

☐ uses Parabolas to approximate area

$$\alpha \cdot \frac{(\Delta x)^3}{12} + \gamma \cdot \Delta x$$

ROB 201 CHAPTER 4 STUDY GUIDE

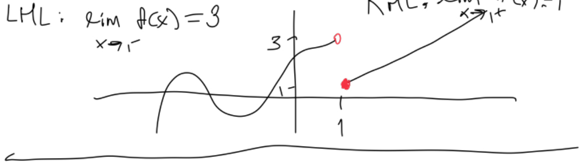
Left and Right Limits

- Left hand limit: $\lim_{x \to x_0^-} f(x)$
- Right hand limit: $\lim_{x \to x_0^+} f(x)$

Ex:

LHL: $\lim_{x \to 1^-} f(x) = 3$

RHL: $\lim_{x \to 1^+} f(x) = 1$



Ex: $f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$ $\lim_{x \to 1^-} f(x) = 2(1)+1 = 3$; $\lim_{x \to 1^+} f(x) = 1^2 = 1$

Continuity

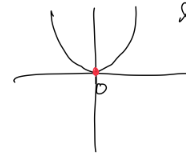
$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$

$\Downarrow$

$\lim_{x \to x_0} f(x)$

- A function is continuous at $x = x_0$ if $\Rightarrow$
  1. $\lim_{x \to x_0} f(x)$ exists
  2. $f(x_0)$ is defined
  3. $\lim_{x \to x_0} f(x) = f(x_0)$

Ex: $f(x) = x^2$



$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ $\rightarrow$ function continuous

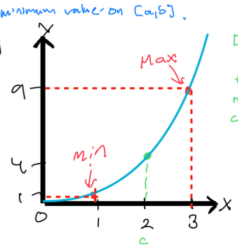
Function continuous on both the range and everywhere

Key Properties

- If f is continuous on $[a, b]$ and k is between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = k$ (Intermediate Value Theorem)
- Extreme Value Theorem: if f is continuous on a closed interval $[a, b]$, then f attains a maximum and a minimum value on $[a, b]$.

Ex: For $f(x) = x^2$ on $[1, 3]$

- $f(x)$ is continuous



- Since the function ranges from $f(x) = 1$ to $f(x) = 3$ on $[1, 3]$, there must be a point where $c = 2$ b/c $a \leq c \leq b$.

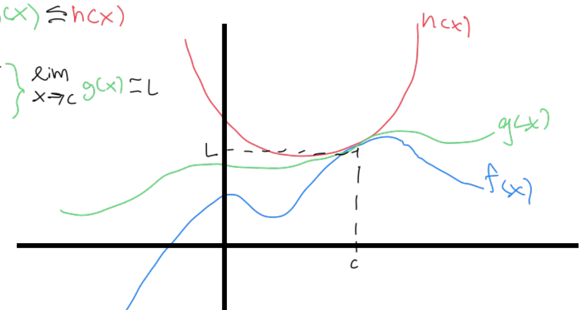
Squeeze Theorem

- $f(x) \leq g(x) \leq h(x)$

Ex:

- $\lim_{x \to c} f(x) = L$
- $\lim_{x \to c} h(x) = L$

$\Big\}$ $\lim_{x \to c} g(x) = L$

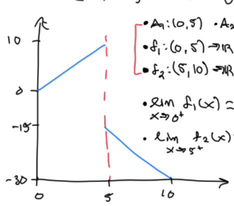


Piecewise Continuity

- A function is piecewise continuous if it is continuous on each piece of its domain, with only a finite number of discontinuities.

Ex:

$f : [0, 10] \to \mathbb{R}$ by $f(x) = \begin{cases} 2x & 0 \leq x < 5 \\ -3x & 5 \leq x \leq 10 \end{cases}$



- $A_1 : (0, 5)$ • $A_2 (5, 10)$
- $f_1 : (0, 5) \to \mathbb{R}$ $f_1(x) = 2x$
- $f_2 : (5, 10) \to \mathbb{R}$ $f_2(x) = -3x$

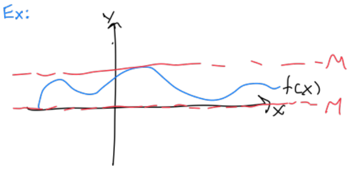
$\Big\}$ Continuous

- $\lim_{x \to 0^+} f_1(x) = 0 = f(0)$
- $\lim_{x \to 5^-} f_1(x) = 10$
- $\lim_{x \to 5^+} f_2(x) = -15$
- $\lim_{x \to 10^-} f_2(x) = -30 = f(10)$

Unbounded VS Bounded

- f is bounded from below if there exists $M > -\infty$ such that $f(x) \geq M$ for all $x \in A$.

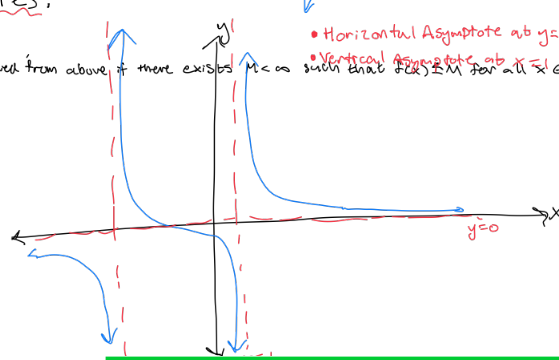
Ex:



Unbounded $\Rightarrow$ If

- Asymptotes:

- f is bounded from above if there exists $M < \infty$ such that $f(x) \leq M$ for all $x \in A$.



- Horizontal Asymptote at $y = 0$
- Vertical Asymptote at $x = 1$, $x = -3$

Max/Min Element +Sup/Inf Values

Ex: $A := [3, 4]$ $B := (3, 4)$ (Max/Min Element)

- $\text{Min}\{A\} = 3$
- $\text{Max}\{A\} = 4$
- $\text{Min}\{B\}$ DNE
- $\text{Max}\{B\} = 4$

- Infimum and Supremum (greatest / least upper Bound)

$A := [3, 4]$ $B := (3, 4)$

- $\inf\{A\} = 3$
- $\sup\{A\} = 4$
- $\inf\{B\} = 3$
- $\sup\{B\} = 4$

Proof of Prop. 4.18 (Common Continuous Functions

We'll prove a few of the dozen or more functions listed in Prop. 4.18: the following functions are continuous for all $x \in \mathbb{R}$ unless indicated otherwise:

(a) Monomials: $f(x) = x^k$

(b) Polynomials: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

(c) Sine: $f(x) = \sin(x)$

(d) Natural Exponential: $f(x) = e^x$

(e) Natural Logarithm (continuous for $x > 0$): $f(x) = \ln(x)$

(f) Power (continuous for $x > 0$ and $y \in \mathbb{R}$): $f(x) = x^y$

(g) Square Root (continuous for $x \geq 0$): $f(x) = \sqrt{x}$