ROB 201 — 02 April 2025

Summary: For $f(t)$ right-sided ($f(t)=0$, $t<0$), and p.w. cont., its Laplace transform is

$$F(s) := \mathcal{L}\{f(t)\} := \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt \quad \text{for } s \in ROC =$$

$$= \left\{ s = a + i\omega \in \mathbb{C} \;\middle|\; \int_{0^-}^{\infty} |f(t)|\, e^{-at}\, dt < \infty \right\}.$$

Fact: If $f'(t)$ has a Laplace transform, then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-) \leftarrow$$ important for I.C.'s of ODEs.

Function $f(t), t \geq 0$	Laplace Transform $F(s)$	ROC (can be ignored)
$\delta(t)$	1	all s
$u_{\text{stp}}(t)$	$\frac{1}{s}$	real$(s) > 0$
$t^n\, u_{\text{stp}}(t)$	$\frac{n!}{s^{n+1}}$	real$(s) > 0$
$e^{at}\, u_{\text{stp}}(t)$	$\frac{1}{s-a}$	real$(s - a) > 0$
$t^n e^{at}\, u_{\text{stp}}(t)$	$\frac{n!}{(s-a)^{n+1}}$	real$(s - a) > 0$
$\sin(\omega t)\, u_{\text{stp}}(t)$	$\frac{\omega}{s^2+\omega^2}$	real$(s) > 0$
$\cos(\omega t)\, u_{\text{stp}}(t)$	$\frac{s}{s^2+\omega^2}$	real$(s) > 0$
$e^{at}\sin(\omega t)\, u_{\text{stp}}(t)$	$\frac{\omega}{(s-a)^2+\omega^2}$	real$(s - a) > 0$
$e^{at}\cos(\omega t)\, u_{\text{stp}}(t)$	$\frac{s-a}{(s-a)^2+\omega^2}$	real$(s - a) > 0$
$t\sin(\omega t)\, u_{\text{stp}}(t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	real$(s) > 0$
$t\cos(\omega t)\, u_{\text{stp}}(t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	real$(s) > 0$
$f(t-b)\cdot u_{\text{stp}}(t-b), b \geq 0$	$e^{-bs}F(s)$	ROC of $F(s)$

Time Domain	Laplace Domain
$f'(t)$	$sF(s) - f(0^-)$
$f''(t)$	$s^2F(s) - sf(0^-) - f'(0^-)$
$f'''(t)$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
$\vdots$	$\vdots$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
$f^{(n)}(t)$	$s^nF(s)$ when all initial conditions are zero.

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{(f'(t))'\} =$$
$$= s\mathcal{L}\{f'(t)\} - f'(0^-)$$
$$= s[sF(s) - f(0^-)] - f'(0^-)$$
$$= s^2F(s) - s\, f(0^-) - f'(0^-), \text{ etc.}$$

$$\dot{y} + 3y = e^{-3t}u_s(t), \quad y(0^-) = 4$$

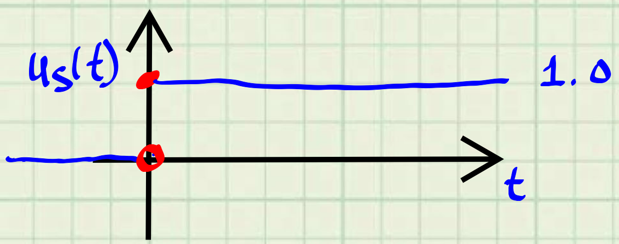
$$\Updownarrow$$

$$(sY(s) - 4) + 3Y(s) = \frac{1}{s+3}$$

$$(s+3)Y(s) - 4 = \frac{1}{s+3}$$

$$(s+3)Y(s) = 4 + \frac{1}{s+3}$$

$$Y(s) = \frac{4}{s+3} + \frac{1}{(s+3)^2} \longleftrightarrow y(t) = 4e^{-3t}u_s(t) + te^{-3t}u_s(t)$$



Today: Inverse Laplace transforms

Key Method:

PFE = Partial Fraction Expansion

$$F(s) = \frac{s+2}{(s+3)(s+4)} \overset{?}{\longleftrightarrow} \mathcal{L}^{-1}\{F(s)\} = f(t)$$

$$= \frac{k_1}{s+3} + \frac{k_2}{s+4}$$

Note

$$F(s)(s+3) = k_1 \frac{\cancel{s+3}}{\cancel{s+3}} + k_2 \frac{s+3}{s+4}$$

$$\| $$

$$\frac{(s+2)\cancel{(s+3)}}{(s+4)\cancel{(s+3)}}$$

$$k_1 = \lim_{s \to -3} (s+3)F(s) = \lim_{s \to -3} \frac{\cancel{(s+3)}(s+2)}{\cancel{(s+3)}(s+4)}$$

$$= \frac{s+2}{s+4}\Bigg|_{s=-3} = \frac{-1}{+1} = -1$$

$$k_2 = \lim_{s \to -4} (s+4)F(s) = \lim_{s \to -4} \frac{\cancel{(s+4)}(s+2)}{(s+3)\cancel{(s+4)}} =$$

$$= \frac{s+2}{s+3}\Bigg|_{s=-4} = \frac{2}{1} = 2$$

$$F(s) = \frac{-1}{s+3} + \frac{2}{s+4} \longleftrightarrow (-1)e^{-3t}u_s(t) + 2e^{-4t}u_s(t)$$

$$\frac{1}{s+a} \longleftrightarrow e^{-at}u_s(t)$$

$$\frac{s+2}{(s+3)(s+4)} = \frac{k_1}{s+3} + \frac{k_2}{s+4} = \frac{k_1(s+4) + k_2(s+3)}{(s+3)(s+4)}$$

Equate numerators

$$s+2 = s(k_1 + k_2) + (4k_1 + 3k_2)$$

$$\left.\begin{aligned} k_1 + k_2 &= 1 \\ 4k_1 + 3k_2 &= 2 \end{aligned}\right\}$$ Solve simultaneous equations

General Result

$$F(s) = \frac{N(s)}{D(s)}$$ have real coefficients

$\deg(N(s)) < \deg(D(s))$

Let $r_1, r_2, \ldots, r_n$ be the roots of $D(s)$ and distinct

$$F(s) = \frac{k_1}{s-r_1} + \frac{k_2}{s-r_2} + \cdots + \frac{k_n}{s-r_n}$$

then $$k_i = \lim_{s \to r_i} (s-r_i)F(s)$$

$$= \left. \frac{N(s)\,\cancel{(s-r_i)}}{(s-r_1)\cdots\cancel{(s-r_i)}\cdots(s-r_n)} \right|_{s=r_i}$$

$$\frac{1}{s-r_i} \longleftrightarrow e^{r_i t}\,u_s(t)$$

$$f(t) = \sum_{i=1}^{n} e^{r_i t}\,u_s(t)$$ will always be real

Fact: Iff r_i is real, k_i is real

Fact: If $r_{i+1} = r_i^*$, then $k_{i+1} = k_i^*$

Note: r_i^* = complex conjugate

Example $F(s) = \dfrac{3}{(s+1)(s+2+3i)(s+2-3i)}$

$r_1 = -1$

$r_2 = -2-3i$

$r_3 = -2+3i = r_2^*$

$k_1 = \frac{3}{10}$

$k_2 = -\frac{3}{20} - \frac{i}{20}$

$k_3 = -\frac{3}{20} + \frac{i}{20} = k_2^*$

Using Laplace Transforms to Solve ODEs

Warning Will Robinson!

$$\ddot{y} + 5\dot{y} + 6y = u + 7\dot{u}$$

where $u(t) = e^{-t} u_s(t)$ and $y(0^-) = 2$, $\dot{y}(0^-) = 3$

Warning: $\dot{u}(0^-) = 0$, etc because $u(t)$ is assumed right-sided

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{\dot{y}(t)\} = sY(s) - y(0^-)$$

$$\mathcal{L}\{\ddot{y}(t)\} = \underbrace{s^2 Y(s) - s y(0^-)}_{s\mathcal{L}\{\dot{y}\}} \underbrace{- \dot{y}(0^-)}_{-\dot{y}(0^-)}$$

$$\mathcal{L}\{u(t)\} = U(s)$$

$$\mathcal{L}\{\dot{u}(t)\} = sU(s)$$

[no $u(0^-)$ to subtract off]

$$\underbrace{[s^2 Y(s) - s y(0^-) - \dot{y}(0^-)]}_{\ddot{y}(t)} + 5\underbrace{[sY(s) - y(0^-)]}_{\dot{y}(t)} +$$

$$6\underbrace{Y(s)}_{y(t)} = \underbrace{U(s)}_{u(t)} + 7\underbrace{sU(s)}_{\dot{u}(t)}$$

Group Terms, Move the I.C.s to the Right also

$$[s^2 + 5s + 6]Y(s) = [1 + 7s]U(s) + (s+5)y(0^-) + \dot{y}(0^-)$$

Divide both sides by $s^2 + 5s + 6$

$$Y(s) = \underbrace{\frac{[7s+1]}{(s^2+5s+6)}U(s)}_{\substack{\text{Forced Response} \\ \{\text{Indep. of I.C.}\}}} + \underbrace{\frac{(s+5)y(0^-) + \dot{y}(0^-)}{s^2+5s+6}}_{\substack{\text{Initial Cond. Response} \\ \{\text{Indep of input}\}}}$$

The total response being the sum of the Forced Response and the Initial

Condition Response is the

Superposition Principle!

(superstition)

Allows one to divide and conquer

Let's do this in code!

1. Why Are Initial Conditions Taken at 0^- in ODEs?

- ODEs describe physical systems that exist before we start analyzing them. These systems have a history, even if we begin our observations at $t = 0$. - For example, a mass-spring system may already be oscillating before $t = 0$, or an electrical circuit may have stored energy in capacitors and inductors before a switch is flipped. - When applying the Laplace Transform to an ODE, we transform derivatives using the formula:

$$\mathcal{L}\left\{\frac{d}{dt}y(t)\right\} = sY(s) - y(0^-).$$

The term $y(0^-)$ accounts for the system's initial state just before $t = 0$, ensuring continuity.

2. Why Are Signal Inputs Defined at 0^+?

- Signals, such as step functions or impulse functions, represent external forcing functions. Unlike system states, they do not have a "past"; they are applied at a specific moment. - The standard unit step function is defined as:

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

This definition assumes that the signal is turned on at $t = 0^+$. - The Laplace Transform of a step function is:

$$\mathcal{L}\{u(t)\} = \frac{1}{s}.$$

Since signals are defined at 0^+, there is no need to consider past values.

One more PFE by hand

$$\frac{2s+13}{(s+2)(s+3)} = \frac{k_1}{s+2} + \frac{k_2}{s+3}$$

Solve for k_1 and k_2

$$k_1 = \left.\frac{(2s+13)\cancel{(s+2)}}{\cancel{(s+2)}(s+3)}\right|_{s=-2} = 9$$

$$k_2 = \left.\frac{(2s+13)\cancel{(s+3)}}{(s+2)\cancel{(s+3)}}\right|_{s=-3} = -7$$

$$Y(s) = \frac{9}{s+2} - \frac{7}{s+3}$$

$$y(t) = 9e^{-2t}u_s(t) - 7e^{-3t}u_s(t)$$

Engineering Practice

1) If we need the time domain function, we solve the ODE using a numerical solver.

2) We usually prefer qualitative insight over complicated expressions

$$\dddot{y} - \ddot{y} + 2y = \sin(t)\, u_s(t), \quad y(0^-) = \dot{y}(0^-) = \ddot{y}(0^-) = 0$$

$$\underbrace{(s^3 - s^2 + 2)}_{(s+1)(s-1-i)(s-1+i)} Y(s) = \frac{1}{s^2+1} = \frac{1}{(s+i)(s-i)}$$

$$Y(s) = \frac{1}{(s^2+1)(s^3-s^2+2)}$$

$$= \frac{k_1}{s+i} + \frac{k_1^*}{s-i} + \frac{k_3}{s-1-i} + \frac{k_3^*}{s-1+i} + \frac{k_4}{s+1}$$

$$\left.\begin{array}{l} \dfrac{1}{s+i} \longleftrightarrow e^{-it} u_s(t) \\[2ex] \dfrac{1}{s-i} \longleftrightarrow e^{+it} u_s(t) \end{array}\right\} \begin{array}{l} \text{oscillations} \\ e^{it} = \cos(t) + i\sin(t) \end{array}$$

$$\left.\begin{array}{l} \dfrac{1}{s-1-i} \longleftrightarrow e^{(1+i)t} u_s(t) \\[2ex] \dfrac{1}{s-1+i} \longleftrightarrow e^{(1-i)t} u_s(t) \end{array}\right\} \begin{array}{l} \text{Exploding} \\ \text{Oscillations} \end{array}$$

$$\dfrac{1}{s+1} \longleftrightarrow e^{-t} u_s(t) \; \Big\} \; \text{Decaying Exponentially fast}$$

Qualitative Message

$Re\{r_i\} < 0$ decaying exponentials (w or w/o oscillations)

$Re\{r_i\} > 0$ exploding exponentials (w or w/o oscillations)

$Re\{r_i\} = 0$ constant or oscillations

parallels

$\dot{x} = Ax$ $\lambda_i(A)$

Can we apply Laplace transforms to State-Variable Models?

$$\dot{x}_1 = -5x_1 + x_2 + 7u \qquad x_1(0^-) = 2$$

$$\dot{x}_2 = -6x_1 + u \qquad x_2(0^-) = 13$$

$$y = x_1 \qquad u(t) = e^{-t} u_s(t)$$

$$sX_1(s) - x_1(0^-) = -5X_1(s) + X_2(s) + 7U(s)$$

$$sX_2(s) - x_2(0^-) = -6X_1(s) + U(s)$$

$$Y(s) = X_1(s)$$

Plug in $x_1(0^-) = 2$, $x_2(0^-) = 13$

$U(s) = \frac{1}{s+1}$ and solve

for $Y(s)$!

Straight forward Algebra....

Monday the vector/matrix

view of $\dot{x} = Ax + bu$

$y = cx$