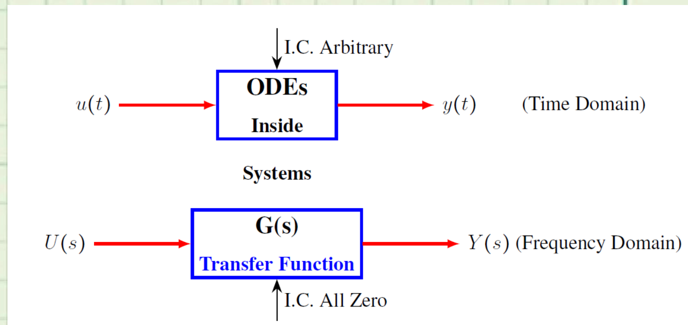
Summary: Transfer Functions map inputs to outputs. $\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_0 u + b_1\dot{u} + b_2\ddot{u}$



$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{Y(s)}{U(s)}$$

ALL I.C.s zero

$$Y(s) = G(s) \cdot U(s)$$

Transfer Function

$$\dot{x} = Ax + bu \qquad x(0^-) = 0 \quad \Rightarrow \quad G(s) = c(sI - A)^{-1} b$$
$$y = cx$$

BIBO STABILITY of $G(s) = \frac{N(s)}{D(s)}$, no common factors, $\deg(N(s)) \leq \deg(D(s))$ iff all poles (roots of $D(s)$) negative real parts. Zeros (roots of $N(s)$) do not PLAY a ROLE in stability (stay tuned)

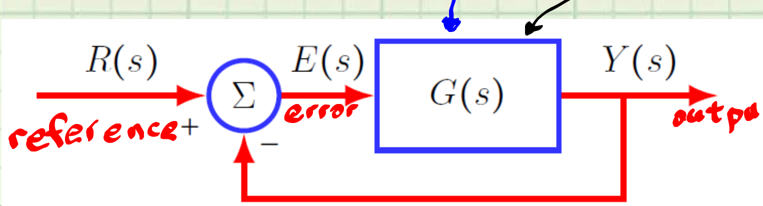
$s^2 + a_1 s + a_0 = 0$ has all roots w/ neg. real parts if, and only if $a_1 > 0$, $a_0 > 0$

$s^3 + a_2 s^2 + a_1 s + a_0 = 0$ has all roots w/ neg. real parts if, and only if $a_2 > 0$, $a_0 > 0$, & $a_1 - \frac{a_0}{a_2} > 0$

Unity Feedback System

open-loop transfer function

also called forward path t.f.



Graphical Representation of Linear Equations

$$\left.\begin{aligned} E(s) &= R(s) - Y(s) \\ Y(s) &= G(s)\cdot E(s) \end{aligned}\right\} \Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} \;\Leftrightarrow\; Y(s) = \frac{G(s)}{1+G(s)} R(s)$$

Example: $G(s) = \dfrac{s+2}{s^2-1}$ unstable $\left[s^2 + \underset{a_1}{0}\cdot s + \underset{a_0}{(-1)}\right]$

$$\frac{G(s)}{1+G(s)} = \frac{\frac{s+2}{s^2-1}}{1+\frac{s+2}{s^2-1}} \frac{(s^2-1)}{(s^2-1)} = \underbrace{\frac{s+2}{s^2+s+1}}_{\text{BIBO stable}}$$

Can go the other way

$G(s) = \dfrac{-3}{s^2+s+1}$ BIBO

$$\frac{G(s)}{1+G(s)} = \frac{-3}{1+\frac{-3}{s^2+s+1}} = \frac{-3}{s^2+s-2}$$ Unstable

---

Today:

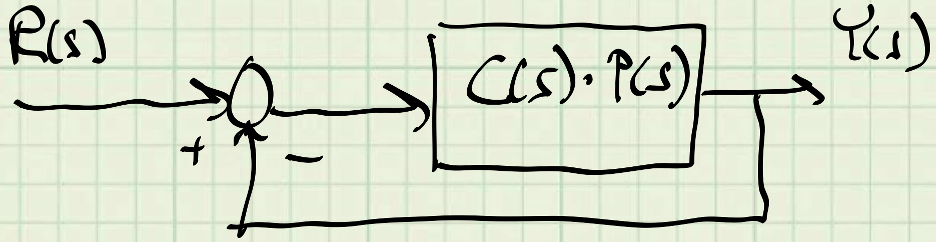
- Proportional Derivative Control
- How to "tune" it

- Can do most of Project 3 after today
- No feedback control on the Final Exam

Cascade Control Architecture

Plant is the generic term for the

system that is to be controlled.



$$G(s) = C(s)\cdot P(s) \Longrightarrow \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)}$$

Proportional Derivative Controller [PD]

Derivative Gain

$$C(s) = K_p + K_D s \longleftrightarrow K_p + K_D \frac{d}{dt}$$

Proportional Gain

$$U(s) = C(s)E(s) = (K_p + K_D s)E(s)$$

$$u(t) = \left(K_p + K_D \frac{d}{dt}\right) e(t) = K_p e(t) + K_D \dot{e}(t)$$

Example: $P(s) = \frac{1}{s^2 - 2s + 1}$ unstable

We select a PD controller

$C(s) = K_p + K_D s$

$$\frac{Y(s)}{R(s)} = \frac{C(s)\,P(s)}{1 + C(s)\,P(s)} = \frac{(K_p + K_D s)\cdot \frac{1}{s^2 - 2s + 1}}{1 + (K_p + K_D s)\cdot \frac{1}{s^2 - 2s + 1}}$$

$$= \frac{K_p + K_D s}{s^2 - 2s + 1 + (K_p + K_D s)}$$

$$= \frac{K_p + K_D s}{s^2 + (K_D - 2)s + (1 + K_p)}$$

Find values for K_p and K_d so that the closed-loop is BIBO stable

$$K_D - 2 > 0 \iff K_D > 2$$

$$1 + K_p > 0 \iff K_p > -1$$

Question: Which values to choose for K_p and K_D ???

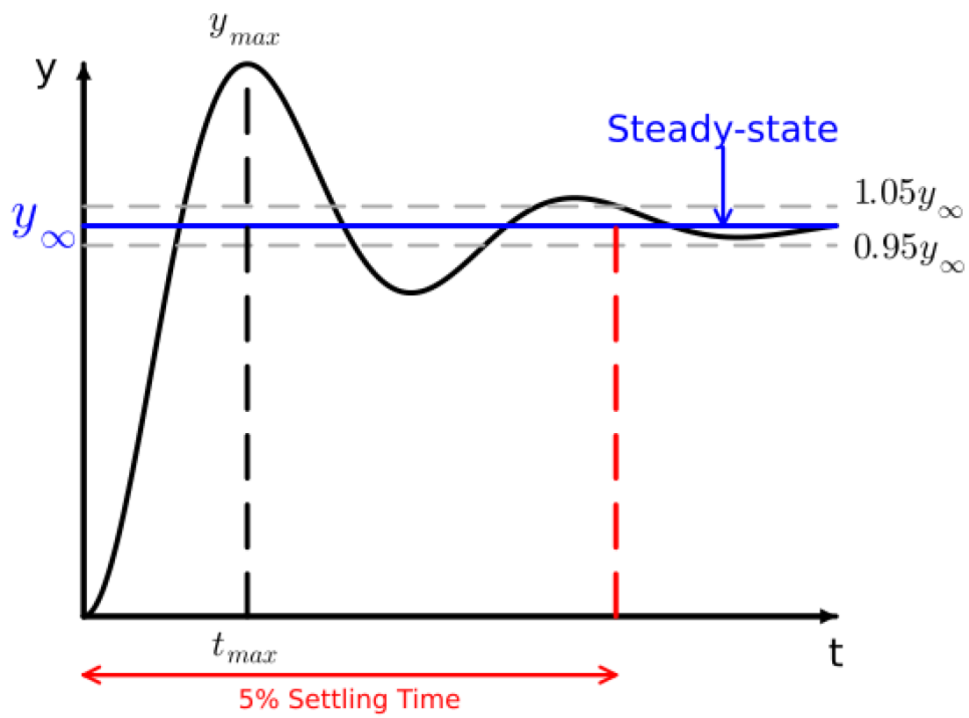


Figure 10.8: Typical step response of an underdamped system. The 5% settling time is the time it takes for the system to enter and then remain within the interval $[0.95y_\infty, 1.05y_\infty]$.

Key Qualitative Features

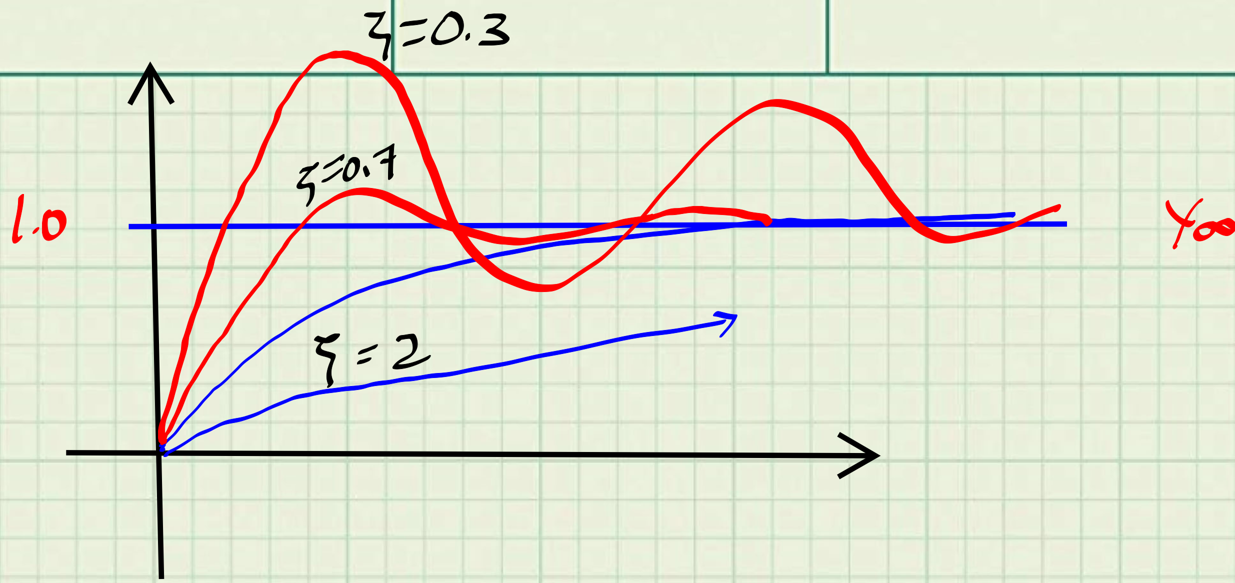
- % overshoot := $\frac{y_{\max} - y_\infty}{y_\infty} \times 100\%$
- 5% Settling time := $\min_{T>0} \{|y(t) - y_\infty| \leq 0.05 y_\infty \text{ for all } t \geq T\}$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

standard second-order

ζ = damping ratio

ω_n = undamped natural freq.



$$\zeta \quad \leftrightarrow \quad \text{percent overshoot} \quad = \quad 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\ \%$$

$$\omega_n \quad \leftrightarrow \quad 5\%\ \text{settling time} \quad \approx \quad \frac{3}{\zeta\omega_n}.$$

5%

$T_s = \frac{3}{\zeta\omega_n}$

$\omega_n = \frac{3}{\zeta \cdot T_s}$

; $\zeta \to 1$ yields no overshoot. Common numbers to keep in
ntrol system designers use the exact formula for the damp
g will be required in the end no matter what because the m

ζ	$\approx$ Overshoot	True Overshoot
1.0	0 %	0.00 %
0.9	0 %	0.15 %
0.8	2 %	1.5 %
0.7	5 %	4.6 %
0.6	10 %	9.5 %
0.5	15 %	16.3 %

$$G_{CL} \approx \frac{(K_P + K_D s)\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K_D\left(s+\frac{K_P}{K_D}\right)\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \longleftrightarrow y(t) = \left(1 - e^{at}\sin(\omega t+\theta)\right)$$

$\zeta = 0.7,\ T_s = 3$

$$G_{CL}(s) = \frac{K_D(s+z)\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad z = \frac{K_P}{K_D}$$

PD adds a zero in the numerator

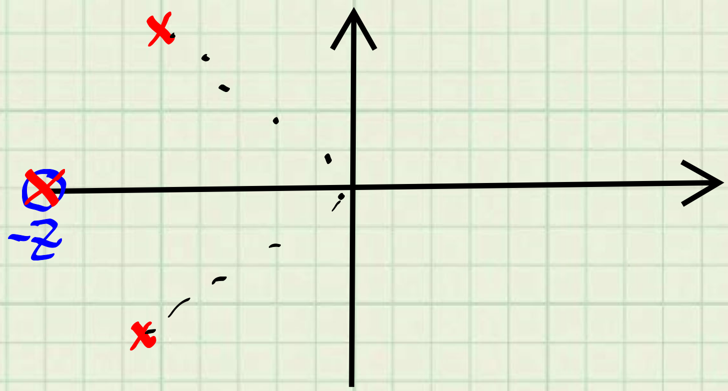
$$y_{CL} \longleftrightarrow K_D(s+z)\cdot y(t) = K_D\left(\frac{d}{dt}+z\right)y(t) =$$

$$= zK_D\, y(t) + \underbrace{K_D \frac{d}{dt} y(t)}$$

$$\frac{d}{dt}\left(1 - e^{at}\sin(\omega t+\theta)\right) = \boxed{\begin{array}{l} -a e^{at}\sin(\omega t+\theta) \\ - e^{at}\cdot\omega \sin(\omega t+\theta) \end{array}}$$

Additional Oscillations

Left Half Plane



Can we remove the deleterious effects of the zero added by the PD controller?

Bottom line, we can cancel a stable zero with a stable pole

Missing Ingredients

- Segway is a NL model.

$$\dot{x} = \underline{f(x)} + b\,\underline{u} \qquad x = \begin{bmatrix} \theta \\ \varphi \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix}$$

$$x_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{equilibrium}$$

$$f(x_e) = 0$$

$A = 4\times4$

$$f(x_e + \delta x) \approx \underbrace{\cancel{f(x_e)}^{\,0} + \overbrace{J_f(x_e)} \cdot \delta x}_{\text{Linearization of } f}$$

$$\boxed{\delta\dot{x} = A\,\delta x + bu}$$

Linear Approx of the ODE

Segway also comes from the Robot Equations

$$q = \begin{bmatrix} \theta \\ \varphi \end{bmatrix} \quad \begin{matrix} \text{lean angle} \\ \text{wheel angle} \end{matrix}$$

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \Gamma \quad \leftarrow \text{external motor torque}$$

The linearized robot equations are

$$\delta q = q - q_e$$

$$\delta\dot{q} = \dot{q} - 0$$

$$\delta\ddot{q} = \ddot{q} - 0$$

$$\boxed{D(q_e)\,\delta\ddot{q} + \frac{\partial G}{\partial q}(q_e)\cdot\delta q = \Gamma}$$

$$\delta x = \begin{bmatrix} \delta q \\ \delta\dot{q} \end{bmatrix} = \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

$$\delta\dot{x} = \begin{bmatrix} \delta\dot{q} \\ \delta\ddot{q} \end{bmatrix} = \begin{bmatrix} \delta x_2 \\ D(q_e) \backslash \left[-\frac{\partial G}{\partial q}(q_e)\,\delta x_1 + \Gamma \right] \end{bmatrix}$$

$$= \begin{bmatrix} 0_{2\times 2} & I_2 \\ A_{21} & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + bu$$

$$A_{21} = D(q_e) \backslash \left(-\frac{\partial G}{\partial q}(q_e) \right)$$

$$b = \begin{bmatrix} 0_{2\times 1} \\ D(q_c) \backslash \Gamma \end{bmatrix}$$

Linear Models $\rightarrow$ Transfer Functions

$\rightarrow$ PD Controller $\rightarrow$ Pre-compensator

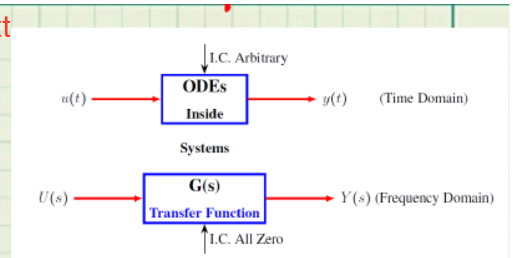
Beautiful balancing controller for the Segway/BallBot

Lecture 24

Text

Summary: Transfer

Functions map inputs to outputs. $\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_0 u + b_1\dot{u} + b_2\ddot{u}$



$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{Y(s)}{U(s)}$$

$$Y(s) = G(s) \cdot U(s)$$

Transfer Function

ALL I.C.s zero

$\dot{x} = Ax + bu$

$y = cx$

$x(0^-) = 0 \Rightarrow G(s) = c(sI - A)^{-1} b$

BIBO STABILITY of $G(s) = \frac{N(s)}{D(s)}$ no common factors, $\deg(N(s)) \le \deg(D(s))$ iff all poles (roots of $D(s)$) negative real parts. Zeros (roots of $N(s)$) do not PLAY a ROLE in stability (stay tuned)

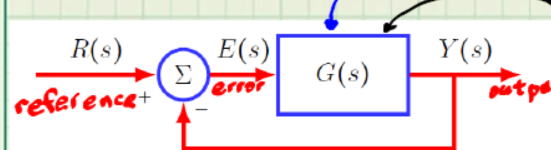
$s^2 + a_1 s + a_0 = 0$ has all roots w/ neg. real parts if, and only if $a_1 > 0$, $a_0 > 0$

$s^3 + a_2 s^2 + a_1 s + a_0 = 0$ has all roots w/ neg. real parts if, and only if $a_2 > 0$, $a_0 > 0$, & $a_1 - \frac{a_0}{a_2} > 0$

Unity Feedback System

open-loop transfer function

also called forward path t.f.



Graphical Representation of Linear Equations

$$\left.\begin{aligned} E(s) &= R(s) - Y(s) \\ Y(s) &= G(s)\cdot E(s) \end{aligned}\right\} \Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} \quad \Updownarrow \quad Y(s) = \frac{G(s)}{1+G(s)} R(s)$$

Example: $G(s) = \dfrac{s+2}{s^2-1}$ unstable $\left[s^2 + \underset{a_1}{\underset{\uparrow}{0}}\cdot s + \underset{a_0}{\underset{\uparrow}{(-1)}}\right]$

$$\frac{G(s)}{1+G(s)} = \frac{\frac{s+2}{s^2-1}}{1+\frac{s+2}{s^2-1}} \frac{(s^2-1)}{(s^2-1)} = \underbrace{\frac{s+2}{s^2+s+1}}_{\text{BIBO stable}}$$

Can go the other way

$G(s) = \dfrac{-3}{s^2+s+1}$ BIBO

$$\frac{G(s)}{1+G(s)} = \frac{-3}{1+\frac{-3}{s^2+s+1}} = \frac{-3}{s^2+s-2} \quad \text{Unstable}$$

Today: **There is a handout at the end of the lecture**

- Proportional Derivative (PD) Control
- How to tune a PD controller
- **Note:** Can complete most of Project 3 after today
- **Note:** No feedback control on the Final Exam

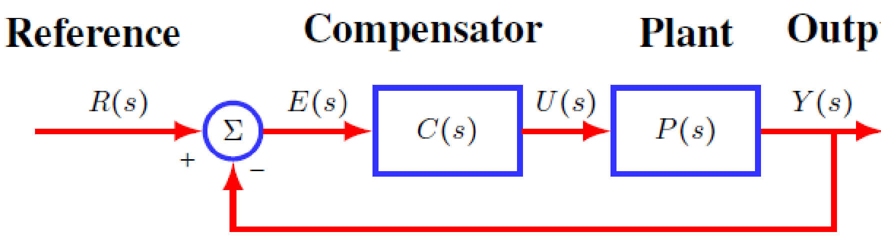


Figure 2.6: Cascade control system configuration, where $C(s)$ is the transfer function of the compensator and $P(s)$ is the transfer function of the plant. Their product, $G(s) = C(s)\,P(s)$ is the overall forward-path transfer function.

Plant is the generic term for the system to be controlled.

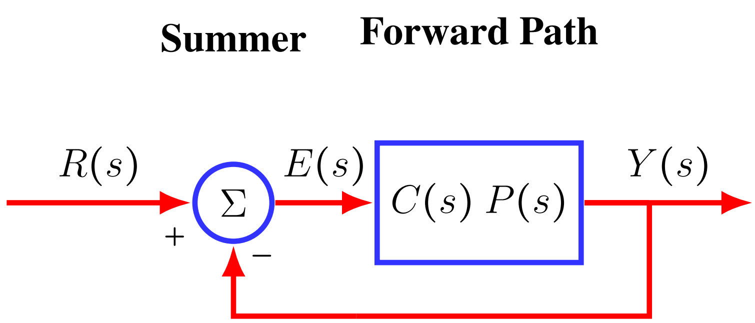


Figure 1: Equivalent Unity Feedback System.

Hence,

$$Y(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} R(s) \iff \frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)}$$

Proportional Derivative Controller (PD)

Derivative Gain

$$C(s) = K_P + K_D\, s \iff K_P + K_D \frac{d}{dt}$$

Proportional Gain

It follows that

$$
\begin{aligned}
U(s) &= C(s)E(s) = (K_P + K_D s)E(s) \\
u(t) &= (K_P + K_D \frac{d}{dt})e(t) = K_P e(t) + K_D \frac{de(t)}{dt}
\end{aligned}
$$

Example: $P(s) = \frac{1}{s^2-2s+1}$ **Unstable**

$$
\begin{aligned}
\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1+C(s)P(s)} &= \frac{(K_p + K_d s)\,\frac{1}{s^2-2s+1}}{1+(K_p+K_d s)\,\frac{1}{s^2-2s+1}} \\
&= \frac{K_p + K_d s}{s^2 - 2s + 1 + (K_p + K_d s)} \\
&= \frac{K_p + K_d s}{s^2 + (K_d - 2)s + (1 + K_p)}
\end{aligned}
$$

$$
\text{BIBO Stable} \iff \begin{array}{l} K_d - 2 > 0 \\ 1 + K_p > 0 \end{array}
$$

$$
\iff \begin{array}{l} K_d > 2 \\ K_p > -1 \end{array}
$$

Stability analysis provides the set of values yielding closed-loop stability. How do we select “good” values within this set?

(Below) Typical step response of a BIBO stable 2nd-order closed-loop system

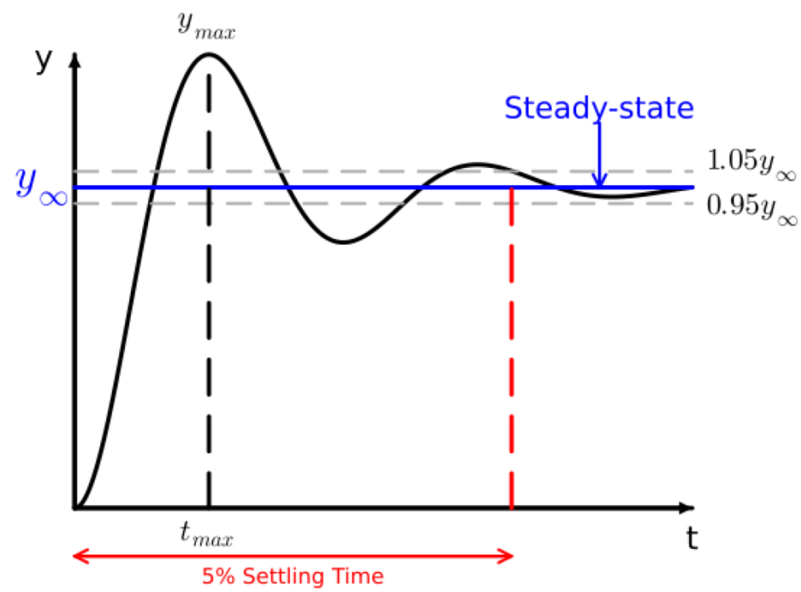


Figure 10.8: Typical step response of an underdamped system. The 5% settling time is the time it takes for the system to enter and then remain within the interval $[0.95y_{\infty}, 1.05y_{\infty}]$.

Key Qualitative Features

$$\% \text{ overshoot} := \frac{y_{\max} - y_{\infty}}{y_{\infty}} \times 100\%$$

$$5\% \text{ Settling time} := \min_{T>0} \{|y(t) - y_{\infty}| \le 0.05 y_{\infty} \text{ for all } t \ge T\}$$

Standard Second-order System: $G(s) := \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

- ω_n is called the **undamped natural frequency**
- ζ is called the **damping ratio**

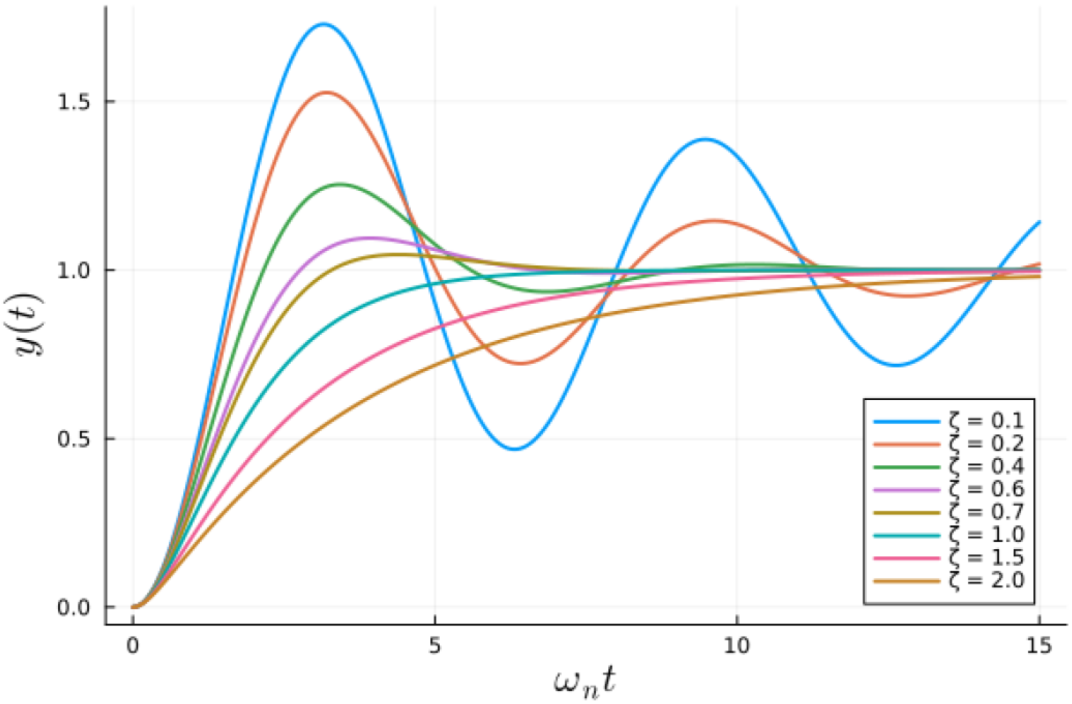


Figure 10.10: Step response of a standard second-order system, $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$. There is overshoot for $0 < \zeta < 1$, while there is no overshoot for $\zeta \geq 1$. There is a trade-off between fast response and overshoot. **Note that the x-axis is scaled and equal to $\omega_n t$.**

$$5\% \ \ T_s \approx \frac{3}{\zeta\omega_n}$$

Choose parameters to set the overshoot and the settling time:

- Choose ζ to set overshoot
- Then solve for ω_n to meet the 5%-settling time: $\omega_n = \frac{3}{\zeta T_s}$

$$\zeta \leftrightarrow \text{percent overshoot} = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\ \%$$
$$\omega_n \leftrightarrow \text{5\% settling time} \approx \frac{3}{\zeta\omega_n}.$$

; $\zeta \to 1$ yields no overshoot. Common numbers to keep in
ntrol system designers use the exact formula for the damp
g will be required in the end no matter what because the m

ζ	$\approx$ Overshoot	True Overshoot
1.0	0 %	0.00 %
0.9	0 %	0.15 %
0.8	2 %	1.5 %
0.7	5 %	4.6 %
0.6	10 %	9.5 %
0.5	15 %	16.3 %

Do Demo 3 in Chapter 10 Cell 6 Transient Response of an Open-loop Second-order Transfer Function. Getting a feel for the transient characteristics of a system via step responses: u(t) = unit step function. Use the formulas that relate the PD parameters to ω_n and ζ.

Effect of the zero from the PD controller

$P(s) = \frac{k_0}{s^2+a_1 s+a_0}$

$C(s) = K_P + K_D s = K_D \underbrace{\left(s + \frac{K_P}{K_D}\right)}_{(s+z)} = K_P \underbrace{\left(\frac{K_D}{K_P}s + 1\right)}_{(\frac{s}{z}+1)}$ has a zero at $s = -z$.

The closed-loop system is then

$$
\begin{aligned}
G_{cl}(s) &= \frac{C(s)P(s)}{1+C(s)P(s)} \\
&= k_0 \frac{K_P+K_D s}{s^2+(a_1+K_D k_0)s+(a_0+K_P k_0)} \\
&= k_0 \frac{K_P(\frac{s}{z}+1)}{s^2+(a_1+K_D k_0)s+(a_0+K_P k_0)} \quad \text{where } z := \frac{K_P}{K_D}
\end{aligned}
$$

This shows two things:

- The transfer function has a “zero“ at $s = -z = -\frac{K_P}{K_D}$
- The “characteristic equation (denominator of the transfer function)“ is

$$
s^2 + \underbrace{(a_1 + K_D k_0)}_{2\zeta\omega_n} s + \underbrace{(a_0 + K_P k_0)}_{\omega_n^2} = 0.
$$

A tiny bit of Algebra yields

$$
\begin{aligned}
K_P &= \frac{\omega_n^2 - a_0}{k_0} \\
K_D &= \frac{2\zeta\omega_n - a_1}{k_0}.
\end{aligned}
$$

Coming back to the zero, how does it affect the system’s response?

Vocabulary: When zero is negative, we sometimes call it a “stable zero”, and when the zero is positive, an “unstable zero”.

$$\widetilde{G}(s) = \frac{\left(\frac{1}{z}s + 1\right)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \implies \text{zero at } \; s = -z$$

$z > 0$ stable zero

$z < 0$ unstable zero

z "smallish"(aka, close to the origin) $\implies \dfrac{1}{z}$ Big

Time Domain View of a Zero

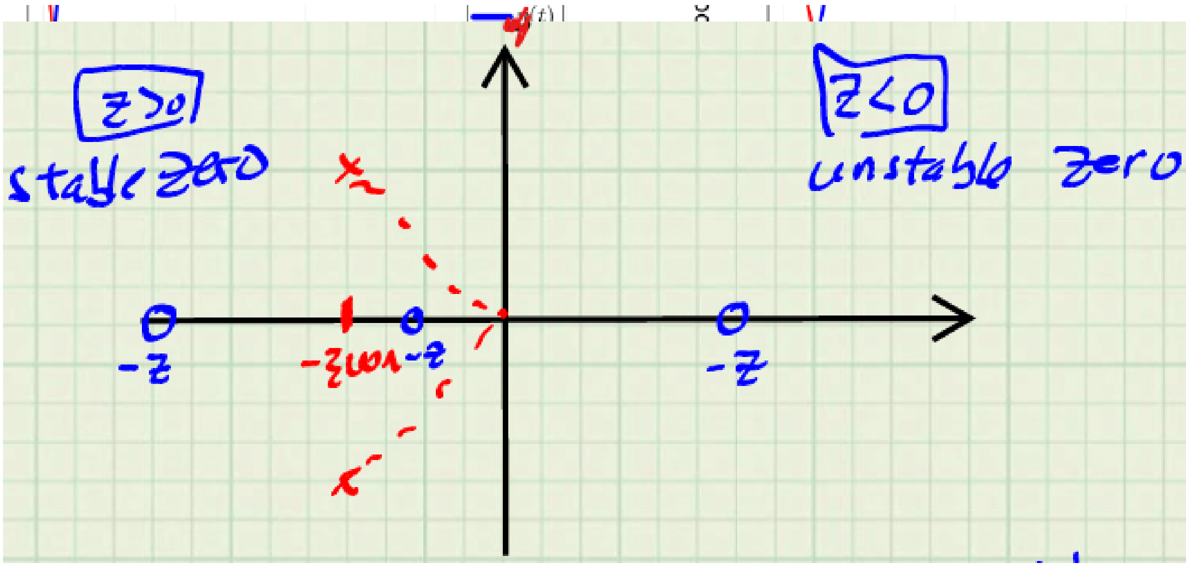
Write $\widetilde{Y}(s) = Y(s) + \frac{1}{z}s\,Y(s)$, where $\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$. Then,

$$\tilde{y}(t) = y(t) + \frac{1}{z}\frac{d}{dt}y(t)$$

$$y(t) = 1 - e^{-at}\sin(\omega t + \theta)$$

$$\frac{d}{dt}y(t) = ae^{-at}\sin(\omega t + \theta) - e^{-at}\cos(\omega t + \theta)\omega$$

$$\tilde{y}(t) = \underbrace{\left[1 - e^{-at}\sin(\omega t + \theta)\right]}_{y(t)} + \underbrace{\left[\frac{a}{z}e^{-at}\sin(\omega t + \theta) - \frac{\omega}{z}e^{-at}\cos(\omega t + \theta)\right]}_{\text{additional oscillations and overshoot}}$$

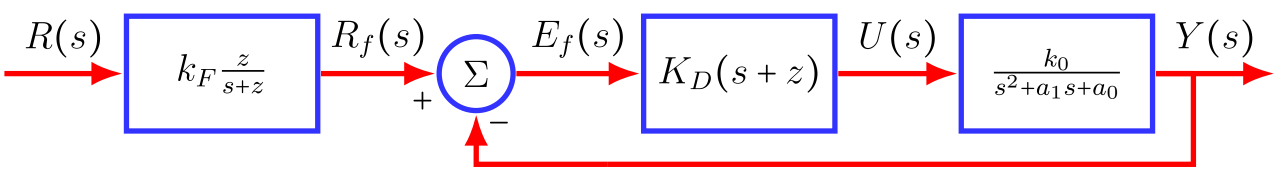


Zeros increase the overshoot. When zeros are "unstable", they can cause "undershoot".

Can we compensate for this additional overshoot?

Sometimes! **When the zero is in the left half plane (aka, stable zero)**, we can cancel it with a stable pole via a **pre-compensator**!

Pre-Compensator **PD** $K_D s + K_P$ **Plant**



$$\frac{Y(s)}{R(s)} = \left(k_F \frac{z}{s+z}\right)\left(k_0 \frac{K_D(s+z)}{s^2+(a_1+K_D k_0)s+(a_0+K_P k_0)}\right)$$

$$= (k_F\, z\, k_0\, K_D)\left(\frac{1}{s^2+(a_1+K_D k_0)s+(a_0+K_P k_0)}\right)$$

Canceling the zero is only allowed when it is in the left-half plane. It returns us to a standard second-order system where the overshoot and settling time are easily related to ζ and ω_n.

Fact: "DC Gain" equals one if, and only if, $\frac{k_F\ z\ k_0\ K_D}{a_0+K_P k_0} = 1$. Solving for k_F yields

$$k_F = \frac{a_0 + k_0\ K_P}{z\ k_0\ K_D} = \frac{a_0 + k_0\ K_P}{k_0\ K_P} = \frac{1 + \frac{k_0}{a_0}\ K_P}{\frac{k_0}{a_0}\ K_P} = \frac{1 + C(0)\ P(0)}{C(0)\ P(0)}$$

because $z = \frac{k_P}{k_D}$, $C(0) = K_P$, and $P(0) = \frac{k_0}{a_0}$.

With this value for k_F, $r(t) = u_s(t) \implies \lim_{t\to\infty} y(t) = 1.0$

Do Demo 4 in Chapter 10 Cell 8 Tuning the PD gains for stability and a nice transient response can be hard without a pre-compensator. We try with and without!

If time permits: Intuition on PD Controllers

From our understanding of linearization about a point

$$e(t) \approx \underbrace{e(t_0)}_{\text{current error}} + \underbrace{\dot{e}(t)(t-t_0)}_{\text{predicted error, or anticipated error}}$$

$$e(t+\Delta t) \approx e(t_0) + \dot{e}(t_0)\cdot\Delta t$$

where Δt is the look-ahead time. Have students imagine driving a car through a hole in the car floor, or show the videos below.

1. Controlling Self Driving Cars

2. How to Turn - Vision (the most important thing) serves as a metaphor for the look-ahead (prediction into the future) provided by a PD controller, as shown below.

The derivative term allows prediction, as shown below

$$
\begin{aligned}
u(t) &= K_p e(t) + K_d \dot{e}(t) \\
&= K_p \left[e(t) + \frac{K_d}{K_p} \dot{e}(t) \right] \quad \frac{K_d}{K_p} \text{ is like } \Delta t \\
&= K_p \left[e(t) + \dot{e}(t) \frac{K_d}{K_p} \right] \quad \text{the look-ahead time}
\end{aligned}
$$

Handout on Tuning PD Controllers for Second-order Systems

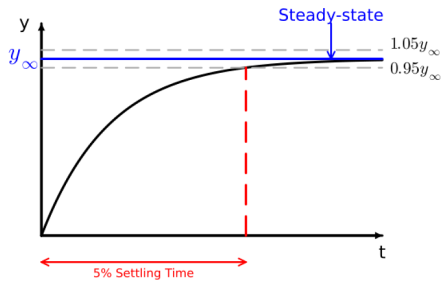


Figure 10.9: **(First-order Systems:)** The step response of a first-order system $\frac{k_0}{\tau s+1}$ has 10% – 90% rise time approximately equal to 2τ, no overshoot, and its 5% settling time is approximately 3τ. The peak value is the same as y_∞, meaning it is not achieved for any finite value of t.

Figure 2: Typical Overdamped Step Response

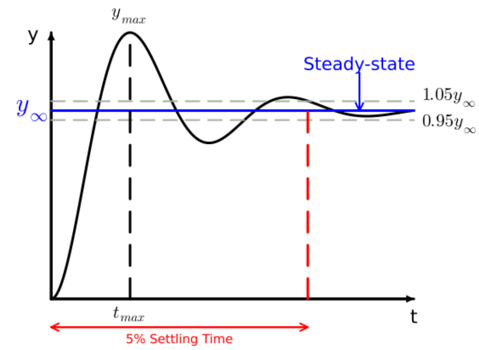


Figure 10.8: Typical step response of an underdamped system. The 5% settling time is the time it takes for the system to enter and then remain within the interval $[0.95y_\infty, 1.05y_\infty]$.

Figure 3: Underdamped System Response with Overshoot and Settling Time

Key Qualitative Features

- % overshoot := $\frac{y_{\max} - y_\infty}{y_\infty} \times 100\%$
- 5% Settling time := $\min_{T>0} \{|y(t) - y_\infty| \le 0.05 y_\infty \ \text{ for all } \ t \ge T\}$

Standard Second-order System:

$$\boxed{G(s) := \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \tag{1}$$

- ω_n is called the **undamped natural frequency**
- ζ is called the **damping ratio**
- 5% $T_s \approx \frac{3}{\zeta\omega_n}$

How to choose parameters:

- Choose ζ to set overshoot
- Then solve for ω_n to meet the settling time

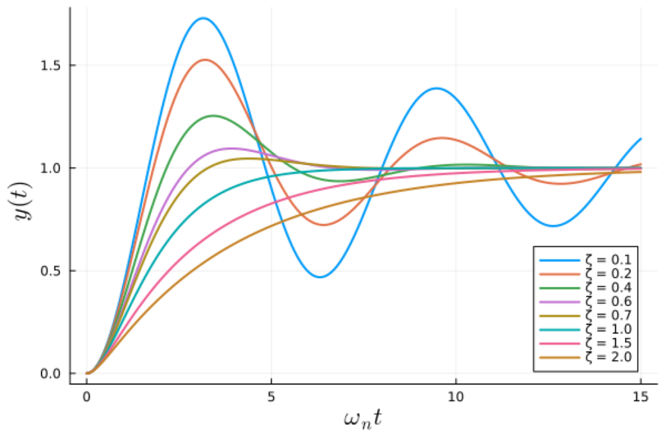


Figure 10.10: Step response of a standard second-order system, $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$. There is overshoot for $0 < \zeta < 1$, while there is no overshoot for $\zeta \geq 1$. There is a trade-off between fast response and overshoot. **Note that the x-axis is scaled and equal to $\omega_n t$.**

$$\boxed{\begin{array}{ccccc} \zeta & \leftrightarrow & \text{percent overshoot} & = & 100 e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\ \% \\ \omega_n & \leftrightarrow & 5\% \text{ settling time} & \approx & \frac{3}{\zeta\omega_n}. \end{array}}$$

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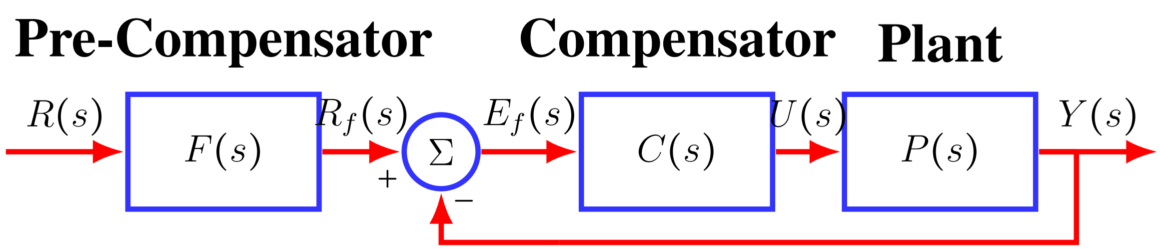
ζ	$\approx$ Overshoot	True Overshoot
1.0	0 %	0.00 %
0.9	0 %	0.15 %
0.8	2 %	1.5 %
0.7	5 %	4.6 %
0.6	10 %	9.5 %
0.5	15 %	16.3 %



A unity feedback loop with a pre-compensator (aka, pre-filter) $F(s)$ added to the overall system. The pre-filter can be used to ameliorate the effects of left-half plane zeros and/or to "fix" steady-state error.

Define: $E(s) := R(s) - Y(s)$ true error, not the filtered error $E_f(s)$

After a bit of algebra, and assuming closed-loop stability, one computes for a unit-step input,

$$e_{SS} = \frac{1 + (1 - F(0))\, C(0)\, P(0)}{1 + C(0)\, P(0)}. \tag{2}$$

To achieve $e_{SS} = 0$, we solve the above for $F(0)$ (aka, the DC-gain for the pre-compensator) yielding

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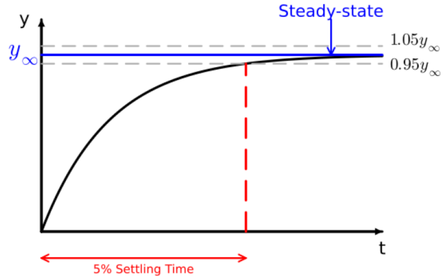
Handout on Tuning PD Controllers for Second-order Systems



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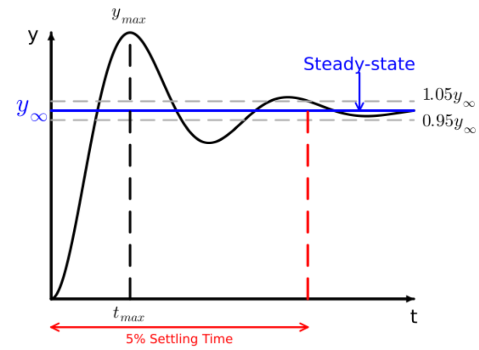

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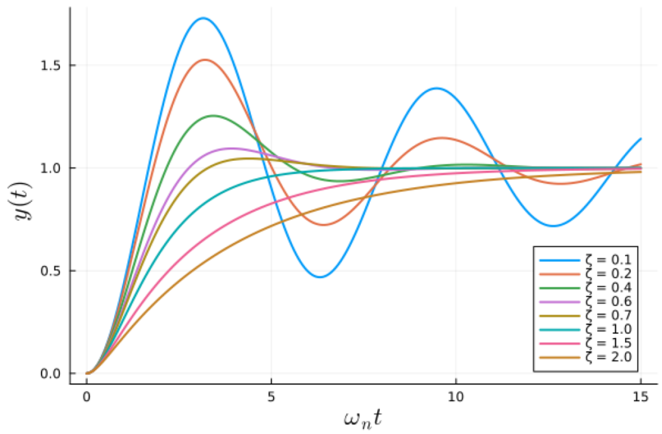

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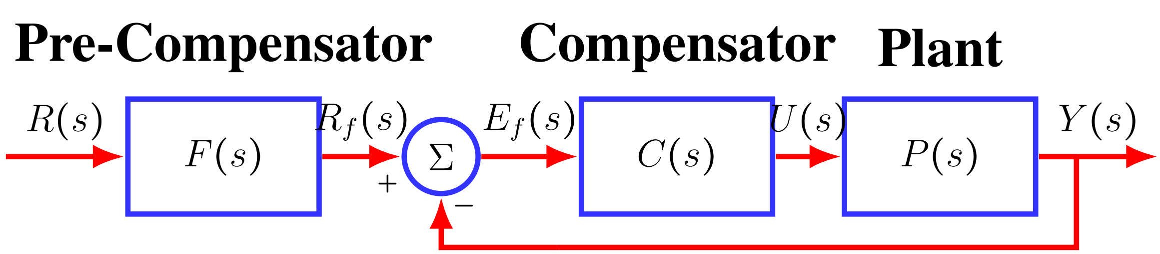

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