Summary: Solution to $\dot{x} = Ax$, $x(t_0) = x_0$

is $e^{A(t-t_0)} x_0$, where $e^{At} = I_n + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots$

- $Av = \lambda v \Rightarrow e^{At} v = e^{\lambda t} v$ [because $A^k v = \lambda^k v$].

- $Av_i = \lambda_i v_i$ $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ distinct $\Rightarrow$ $\{v_1, v_2, \ldots, v_n\}$ linearly independent (basis for $\mathbb{R}^n$)

- Hence, for all $x_0 \in \mathbb{R}^n$, there exist unique $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$x_0 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{\det \neq 0} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

- $e^{At} x_0 = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 + \cdots + \alpha_n e^{\lambda_n t} v_n$

- λ real : $e^{\lambda_j t} \xrightarrow[t \to \infty]{} \begin{cases} 0 & \lambda_j < 0 \\ 1 & \lambda_j = 0 \\ \infty & \lambda_j > 0 \end{cases}$

- $\lambda_j = a_j + i\omega_j$: $e^{\lambda_j t} = e^{a_j t} \cdot e^{i\omega_j t}$

$$= e^{a_j t} \cdot [\cos(\omega_j t) + i \sin(\omega_j t)]$$

$$e^{\lambda_j t} \xrightarrow[t \to \infty]{} \begin{cases} 0 & a_j = \mathrm{Re}\{\lambda_j\} < 0 \\ \text{oscillations} & a_j = 0 \\ \text{exploding oscillations} & a_j > 0 \end{cases}$$

Julia HW #8 explores λ complex.

Remark: $\dot{x} = Ax$, $x_0 = v$, $Av = \lambda v$.

Claim: $\varphi(t) := e^{\lambda t} v$ is the unique solution.

$\varphi(0) = e^{0} \cdot v = v$ I.C. ✓

$\dot{\varphi}(t) = \lambda e^{\lambda t} \cdot v = e^{\lambda t} \cdot \lambda \cdot v = e^{\lambda t} \cdot A \cdot v$

$= A \underbrace{e^{\lambda t} \cdot v}_{\varphi(t)}$

$= A \cdot \varphi(t)$ ■

$$\boxed{\therefore \dot{\varphi}(t) = A\varphi(t) \quad \text{ODE ✓}}$$

TODAY

Chapter 10: Laplace Transforms through the Lens of Feedback Control

Warm-up

a) $\frac{d}{dt} e^{zt} = z e^{zt}$ for $z = a + ib \in \mathbb{C}$

b) $\dot{y} + 3y = e^{-2t}$ $y(0^-) = 0$ $t \geq 0$

$\mathcal{L}\{y(t)\} = Y(s) = \frac{1}{s+3} \cdot \frac{1}{s+2} = \frac{1}{s+2} - \frac{1}{s+3}$ (PFE)

$\therefore y(t) = e^{-2t} - e^{-3t}, \ t \geq 0$

Today: How to differentiate and anti-differentiate e^{zt} where $z = a + \mathrm{i}\,b$. We do it in FULL DETAIL so that we EMBRACE Euler's Formula!

We set everything up so that we can apply our knowledge of derivatives of real-valued functions:

$$
\begin{aligned}
e^{zt} &= e^{(a+\mathrm{i}\,b)t} \\
&= e^{at}\cdot e^{\mathrm{i}\,bt} \quad \text{(exponential rules)} \\
&= e^{at}\cdot[\cos(bt) + \mathrm{i}\sin(bt)] \quad \text{(Euler's Formula)}
\end{aligned}
$$

$-1 = i \cdot i$

$$
\begin{aligned}
\frac{d}{dt}e^{zt} &= \text{(product rule and chain rule)} \\
&= a\cdot e^{at}\cdot \underbrace{[\cos(bt) + \mathrm{i}\sin(bt)]}_{e^{\mathrm{i}\,bt}} + e^{at}\cdot \underbrace{[-\sin(bt)\cdot b + \mathrm{i}\cos(bt)\cdot b]}_{ibe^{ibt}} \quad (\text{Note: } -1 = (\mathrm{i})^2) \\
&= ae^{at}e^{ibt} + e^{at}ibe^{ibt} \\
&= (a + \mathrm{i}\,b)e^{at}\cdot e^{ibt} \\
&= (a + \mathrm{i}\,b)e^{(a+\mathrm{i}\,b)t} \\
&= z\cdot e^{zt}
\end{aligned}
$$

$$
\boxed{\frac{d}{dt}e^{zt} = z\cdot e^{zt} \qquad \int e^{zt}\,dt = \frac{1}{z}e^{zt} + C, \quad z \neq 0}
$$

Video 1 Danger

Video 2 Red Flag

Time Domain (ODE)	Laplace Domain (Algebra)
$e^{-at}, t \geq 0$	$\frac{1}{s+a}$
$y(t)$	$Y(s)$
$\dot{y}(t)$	$sY(s) - y(0^-)$

$$\dot{y} + 3y = e^{-2t} \quad y(0^-) = 0, t \geq 0$$

$$\Downarrow \text{Laplace Transform}$$

$$\begin{aligned} sY(s) + 3Y(s) &= \frac{1}{s+2} \\ (s+3)Y(s) &= \frac{1}{s+2} \\ Y(s) &= \frac{1}{s+3} \cdot \frac{1}{s+2} \\ &= \frac{1}{s+2} - \frac{1}{s+3} \quad \text{PFE} \end{aligned}$$

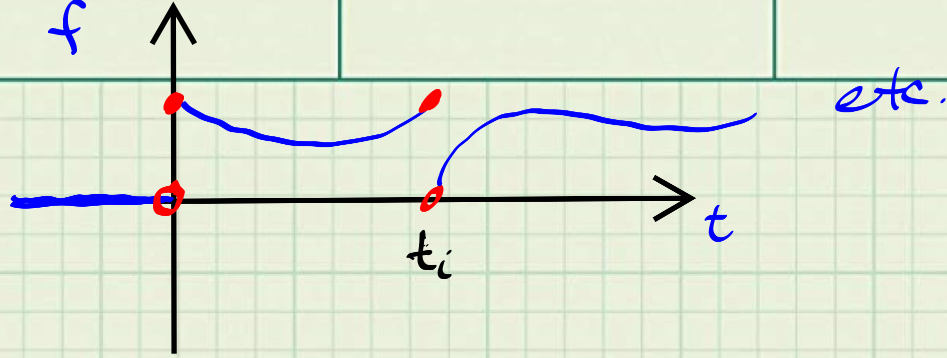
$$\Downarrow \text{inverse Laplace Transform}$$

$$y(t) = e^{-2t} - e^{-3t}, \quad t \geq 0$$

This simple example just scratches the surface of how handy of a tool the Laplace transform can be. We next develop it with roughly the same rigor you would see in EECS 216 Signals and Systems, though we explore fewer of its properties.

Def. A function $f: \mathbb{R} \to \mathbb{R}$ is **right-sided** when $f(t) = 0$ for $t < 0$.

Def. Recall piecewise continuity



For any bounded inter $[a,b]$, $f(t)$ has at most a finite number of discontinuities, t_i,

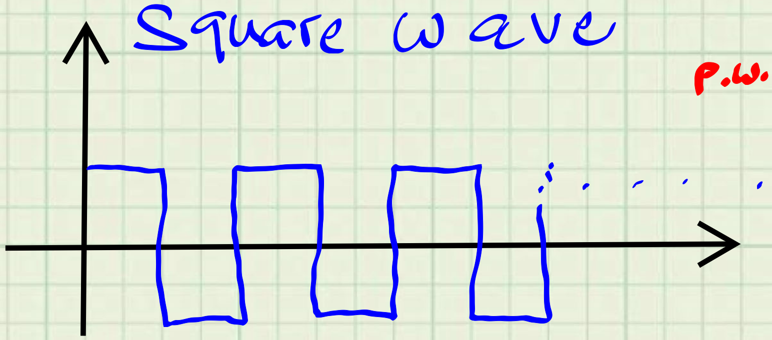
$$\lim_{t\to t_i^-} f(t) \quad \text{and} \quad \lim_{t\to t_i^+} f(t)$$

both exist and are finite.

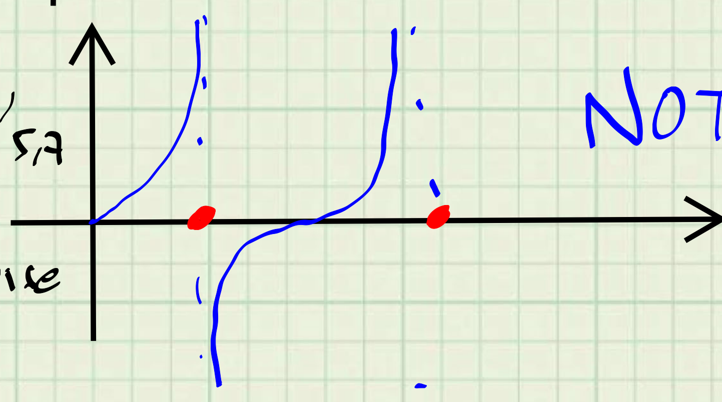
Examples

Square wave

p.w. cont.



$$f(t)=\begin{cases} 0 & t=\frac{\pi}{2}k, \\ & k=\pm 1,3,5,7 \\ \tan(t) & \text{otherwise} \end{cases}$$



NOT p.w. cont.

Def. (Laplace Transform) For a right-sided piecewise continuous function, the Laplace transform is

$$\mathcal{L}\{f(t)\} := \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

where $s = \sigma + i\omega \in \mathbb{C}$ is called the Laplace variable.

Warning = Heads-Up: the 0^- only shows up for initial conditions of ODEs.

Let's compute our first Laplace transform: $f(t) = u_s(t)$ unit step

$$u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$ right-sided & p.w. cont.

$$\mathcal{L}\{u_s(t)\} := \int_{0^-}^{\infty} \underbrace{u_s(t)}_{1}\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_{0^-}^{\infty}$$

$$= -\frac{1}{s}\Big[\underbrace{\lim_{t\to\infty} e^{-st}}_{?} - \underbrace{\lim_{t\to 0^-} e^{-st}}_{e^{-st} \text{ cont} \Rightarrow e^{-st}\big|_{t=0^-} = 1} \Big]$$

$$\lim_{t\to\infty} e^{-st} = \lim_{t\to\infty} e^{-(\sigma+i\omega)t}$$

$$= \lim_{t\to\infty} e^{-\sigma t} \cdot e^{-i\omega t}$$

$$= \lim_{t\to\infty} e^{-\sigma t}\left[\cos(-\omega t) + i \sin(-\omega t)\right]$$

$$= \begin{cases} 0 & \sigma > 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Hence

$$\mathcal{L}\{u_s(t)\} = \begin{cases} -\frac{1}{s}[0 - 1] & \sigma = \mathrm{Re}\{s\} > 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{s} & \mathrm{Re}\{s\} > 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

and we write this

$$\mathcal{L}\{u_s(t)\} = \frac{1}{s} \quad \text{with ROC} = \{s \in \mathbb{C} \mid \mathrm{Re}\{s\} > 0\}$$

$$\boxed{\text{ROC} = \text{Region of Convergence}}$$

Function $f(t), t \geq 0$	**Laplace Transform** $F(s)$	**ROC** (can be ignored)
~~$\delta(t)$~~	~~1~~	~~all s~~
$u_{\text{stp}}(t)$	$\frac{1}{s}$	real(s) > 0
$t^n\, u_{\text{stp}}(t)$	$\frac{n!}{s^{n+1}}$	real(s) > 0
$e^{at}\, u_{\text{stp}}(t)$	$\frac{1}{s-a}$	real(s – a) > 0
$t^n e^{at}\, u_{\text{stp}}(t)$	$\frac{n!}{(s-a)^{n+1}}$	real(s – a) > 0
$\sin(\omega t)\, u_{\text{stp}}(t)$	$\frac{\omega}{s^2+\omega^2}$	real(s) > 0
$\cos(\omega t)\, u_{\text{stp}}(t)$	$\frac{s}{s^2+\omega^2}$	real(s) > 0
$e^{at}\sin(\omega t)\, u_{\text{stp}}(t)$	$\frac{\omega}{(s-a)^2+\omega^2}$	real(s – a) > 0
$e^{at}\cos(\omega t)\, u_{\text{stp}}(t)$	$\frac{s-a}{(s-a)^2+\omega^2}$	real(s – a) > 0
$t\sin(\omega t)\, u_{\text{stp}}(t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	real(s) > 0
$t\cos(\omega t)\, u_{\text{stp}}(t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	real(s) > 0
$f(t-b)\cdot u_{\text{stp}}(t-b), b \geq 0$	$e^{-bs}F(s)$	ROC of $F(s)$

Let's do a few more by hand.

$f(t) = e^{at} u_s(t)$ right-sided p.w. cont.

$$\mathcal{L}\{e^{at} u_s(t)\} = \int_{0^-}^{\infty} e^{at} e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} e^{-(s-a)t}\, dt$$

$$= \frac{-1}{s-a} e^{-(s-a)t} \Big|_{0^-}^{\infty}$$

$$= \begin{cases} \frac{1}{s-a} & \mathrm{Re}\{s-a\} > 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$= \frac{1}{s-a} \quad \text{for ROC} = \mathrm{Re}\{s-a\} > 0$$

Note, we used the same limit analysis as we did for e^{-st} as $t \to \infty$.

For $a \in \mathbb{R}$, $\mathrm{Re}\{s-a\} = \mathrm{Re}\{s\} - \mathrm{Re}\{a\} = \mathrm{Re}\{s\} - a$

$$\mathrm{Re}\{s-a\} > 0 \iff \mathrm{Re}\{s\} > a$$

For $a = i\omega_0$, $\text{Re}\{s-a\} = \text{Re}\{s\} - \cancel{\text{Re}\{a\}}^{\to 0} = \text{Re}\{s\}$

$\text{Re}\{s-a\} > 0 \iff \text{Re}\{s\} > 0$

Why do we care?

$$\cos(\omega_0 t)\, u_s(t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \quad (\text{Euler})$$

$$\Downarrow$$

$$\mathcal{L}\{\cos(\omega_0 t)\, u_s(t)\} = \frac{1}{2}\mathcal{L}\{e^{i\omega_0 t} u_s(t)\} + \frac{1}{2}\mathcal{L}\{e^{-i\omega_0 t} u_s(t)\}$$

$$= \frac{1}{2}\frac{1}{s - i\omega_0} + \frac{1}{2}\frac{1}{s + i\omega_0}$$

$\text{Re}\{s\} > 0$ $\qquad$ $\text{Re}\{s\} > 0$

$$= \frac{1}{2}\frac{(s + \cancel{i\omega_0}) + (s - \cancel{i\omega_0})}{(s - i\omega_0)(s + i\omega_0)}$$

$$= \frac{1}{\cancel{2}}\frac{\cancel{2}s}{s^2 + \omega_0^2}$$

$$= \frac{s}{s^2 + \omega^2}$$

ROC $\text{Re}\{s\} > 0$

Now that we know how to compute Laplace transforms of funtions how do we it to solve ODEs???

<u>Prop.</u> Suppose that $f(t)$ is differentiable, and that both $f(t)$ and $f'(t)$ have Laplace transforms. Then,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-) .$$

Remark: $F(s) = \mathcal{L}\{f(t)\}$.

$$\boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0^-)}$$

True because of integration by parts

$$\int_{0^-}^{\infty} \underbrace{f'(t)}_{dv} \overbrace{e^{-st}}^{u} dt = uv\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v\, du$$

$dv = f'(t)\,dt \Rightarrow v = f(t)$

$u = e^{-st} \Rightarrow du = (-s)e^{-st}\,dt$

$$= e^{-st}f(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)(-s)e^{-st}\,dt$$

$\underbrace{\qquad}_{\text{delicate}}$ $\underbrace{\qquad\qquad}$

$= -f(0^-)$ $\qquad s\mathcal{L}\{f(t)\}$

$= sF(s) - f(0^-)$

$s \longleftrightarrow \frac{d}{dt}$ $\quad$ (Yes $\frac{1}{s} \longleftrightarrow \int (\ )\,dt$)

Time Domain	**Laplace Domain**
$f'(t)$	$sF(s) - f(0^-)$
$f''(t)$	$s^2F(s) - sf(0^-) - f'(0^-)$
$f'''(t)$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
$\vdots$	$\vdots$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
$f^{(n)}(t)$	$s^nF(s)$ when all initial conditions are zero.

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{(f'(t))'\}$$
$$= s\mathcal{L}\{f'(t)\} - f'(0^-)$$
$$= s(sF(s) - f(0^-))$$
$$= s^2 - sf(0^-) - f'(0^-)$$

Superpower of Laplace: ODEs

become Algebraic Equations!!

Solve our first ODE using Laplace

$$\dot{y}(t) + 3y(t) = e^{-3t}u_s(t), \quad y(0^-) = 4$$

$\mathcal{L}\{y(t)\} =: Y(s)$

$\mathcal{L}\{\dot{y}(t)\} = sY(s) - y(0^-) = sY(s) - 4$

$\mathcal{L}\{e^{-3t}u_s(t)\} = \frac{1}{s+3}$ from our Table

$$(sY(s) - 4) + 3Y(s) = \frac{1}{s+3}$$

$$(s+3)Y(s) - 4 = \frac{1}{s+3}$$

$$Y(s) = \frac{1}{s+3}\left(\frac{1}{s+3} + 4\right)$$

$$= \frac{1}{(s+3)^2} + \frac{4}{s+3}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= te^{-3t}u_s(t) + 4e^{-3t}u_s(t)$$

From our Table.