ROB 201 - Calculus for the Modern Engineer
HW #4

Prof. Grizzle

Check Canvas for due date and time

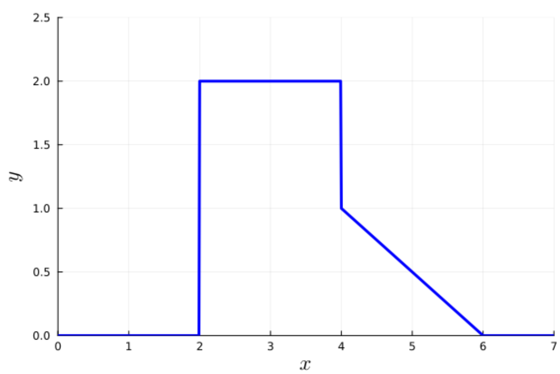
Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete and turn in.

1. (a) Create a "Cheat Sheet" for Chapters 3 and 4 of the textbook. You'll receive the same score for a handwritten solution as a typeset solution. Here is an example from ROB 101.

   (b) Note any material where you found the explanation confusing or difficult to master.

2. Solids of revolution.

   (a) Sketch BY HAND the solid of revolution corresponding to rotating the image below about the $\boldsymbol{y}$**-axis**. Don't worry: we're not looking for art-school-quality sketches.



   (b) Compute the volume of the solid of revolution defined by the two functions

   - $f : [0, 2] \to \mathbb{R}$ by $f(x) = 2x$
   - $g : [0, 2] \to \mathbb{R}$ by $g(x) = x$,

   when rotated about the $\boldsymbol{x}$**-axis**. You are not obliged to make a sketch of any kind, though usually, at least sketching the area out in the (x, y)-plane helps to set up the problem correctly. Show you work.

   In each case, pay attention to the axis of rotation. Parts (a) and (b) are not related. To be exceedingly pedantic, when setting up the problem in part (b), ignore the sketch in part (a); instead, use the given functions.

3. For each of the following sets or functions, compute the maximum value if it exists. If it does not exist, briefly state why, and then compute the supremum. See the hints for example solutions.

   (a) $A := \{x \in \mathbb{R} \mid \frac{x^2-1}{x^2+1} \le 0.5\}$

   (b) $B := \{x \in \mathbb{R} \mid \frac{1}{x} > 2\}$

   (c) $f : [0, \infty) \to \mathbb{R}$ by $f(x) = 4 - (x-5)^4$

   (d) $g : [0, \infty) \to \mathbb{R}$ by $g(x) = \begin{cases} 0 & x = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z} \\ \tan(x) & \text{otherwise.} \end{cases}$

4. Evaluate the following limits using any method you wish, and then provide a brief (ten words or less) description of your reasoning.

   (a) $\lim_{x \to \pi^-} e^{\sin(x)}$.

   (b) $\lim_{x \to 4^+} \dfrac{x^3 + 4x^2 - 16x - 64}{x - 4}$.

   (c) $\lim_{x \to 0} e^{\frac{1}{x}}$.

5. In each case below, provide ONE example of a function that meets the stated properties. So that your grader has a chance of understanding your thought process, provide a few hints as to why your answer is correct. Those hints could be in the form of text, a plot, or both. Note the domain and codomain are specified for each function. You can choose any range you wish as long as it is contained in the specified codomain.

   (a) $f : [0, 1] \to [2, 4]$ has exactly two points where it is discontinuous.

   (b) $f : (-1, 1) \to \mathbb{R}$ is continuous, $\lim_{x \to -1^+} |f(x)| = \infty$ and $\lim_{x \to 1^-} f(x) = 0$. **Note:** The absolute value allows you to have the limit equal positive or negative infinity; it's your choice.

   (c) $f : \mathbb{R} \to \mathbb{R}$ is bounded and not piecewise continuous.

6. Derive the quotient rule of differentiation, following the method used in the textbook to derive the chain rule and the product rule. Define clearly all notation that you use.

Hints

Prob. 1 Nothing to add.

Prob. 2 For part (b), you have to decide between the **disc-washer method** or the **shell method**. You do not have to make any sketches, though, of course, that may help you.

Prob. 3

- $C := \{x \in \mathbb{R} \mid \frac{2x-1}{x+1} \leq 1.0\}$

  Ans. $x^* := \max(C) = 2.0$

  We first note that $\frac{2x-1}{x+1}\Big|_{x=0} = -1$ and $\frac{2x-1}{x+1}\Big|_{x=5} = \frac{9}{6} = 1.5$, and thus there likely exists a point in between where the function equals 1.0. To find out, we solve

$$\begin{gathered}\frac{2x-1}{x+1} = 1 \\ \Downarrow \\ 2x - 1 = x + 1 \\ \Downarrow \\ x = 2\end{gathered}$$

  We note that because the set is defined with a less than or equal to sign, $2 \in C$. Is 2 the largest element?

  We next note that for $x > 2$, $\frac{2x-1}{x+1} > 1$ and hence $x \notin C$. Thus, if $y \in C$, then $y \leq 2$, proving that $x^* = 2$ is the maximum value in the set.

  It's not required, but if you really want to show that $x > 2 \implies \frac{2x-1}{x+1} > 1$, write $x = 2 + \delta$ for $\delta > 0$. Then,

$$\frac{2x-1}{x+1}\Big|_{x=2+\delta} = \frac{4+2\delta-1}{2\delta+1} = \frac{3+2\delta}{3+\delta} = 1 + \frac{\delta}{3+\delta} > 1$$

  for $\delta > 0$, and hence $2 + \delta \notin C$.

- $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \frac{4+(x-5)^4}{1+e^{-x}}$

  Ans. The function is unbounded and hence does not have a maximum. $x^* := \sup_{x \in [0,\infty)} f(x) = \infty$

  From Chapter 2, we have a result on the limits of products and ratios, shown in the box below. Applying this result to our problem we have

  - $f(x) = \frac{4+(x-5)^4}{1+e^{-x}} =: \frac{\text{num}(x)}{\text{den}(x)}$
  - $\lim_{x\to\infty} \text{den}(x) = \lim_{x\to\infty} 1 + e^{-x} = 1$
  - $\lim_{x\to\infty} \text{num}(x) = \lim_{x\to\infty} 4 + (x-5)^4 = \infty$

  and hence by the Proposition, the result is established!

Proposition 2.44: Limits of Products and Ratios

Suppose that $g:(0,\infty)\to\mathbb{R}$ has a limit at infinity of one, i.e., $\lim\limits_{x\to\infty} g(x) = 1.0$. Then for any function $f:(0,\infty)\to\mathbb{R}$,

$$\begin{aligned}\lim_{x\to\infty} f(x)\cdot g(x) &= \lim_{x\to\infty} f(x), \text{ and}\\ \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} f(x).\end{aligned} \tag{2.31}$$

In particular,

(a) if $\lim\limits_{x\to\infty} f(x) = L$, for $L\in\mathbb{R}$, then $\lim\limits_{x\to\infty} f(x)\cdot g(x) = L$ and $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$;

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(b) if $\lim\limits_{x\to\infty} f(x) = \pm\infty$, then $\lim\limits_{x\to\infty} f(x)\cdot g(x) = \pm\infty$ and $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \pm\infty$; and

(c) if $\lim\limits_{x\to\infty} f(x)$ does not exist, then $\lim\limits_{x\to\infty} f(x)\cdot g(x)$ and $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ do not exist.

Note: If $\lim\limits_{x\to\infty} g(x) = M \neq 1$, and $M \neq 0$, then $\frac{g(x)}{M}$ has limit one. Hence, using $f(x)\cdot g(x) = (Mf(x))\cdot\frac{g(x)}{M}$,and $\frac{f(x)}{g(x)} = \frac{\frac{1}{M}f(x)}{\frac{1}{M}g(x)}$ reduces the general case of the problem to the given (special) conditions.

If you remember (2.31)**, then the special cases follow immediately.**

Prob. 4 Nothing to add.

Prob. 5 KISS is a good policy.

Prob. 6 Here is your opening move: writing down the definition of the derivative yields,

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\left(\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}\right).$$

From here on, if you read and follow the textbook, it's straightforward. A bit long, but straightforward. You could also watch one of the suggested videos.

Final hints:

- Don't be afraid to place things over a common denominator.
- $\lim\limits_{h\to 0}\left(g(x)\cdot g'(x)\cdot h\right) = g(x)\cdot g'(x)\cdot \lim\limits_{h\to 0} h = 0.$