ROB 201 - Calculus for the Modern Engineer
HW #7

Prof. Grizzle

Check Canvas for due date and time

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete and turn in.

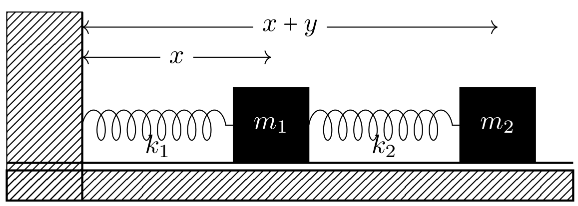
1. Read Chapter 7 of our ROB 201 Textbook, `Calculus for the Modern Engineer`. Based on your reading of the Chapter, summarize in your own words:

   (a) the purpose of Chapter 07;

   (b) two things you found the most DIFFICULT.

   There are no "right" or "wrong" answers, but no answer means no points. The goal is to reflect a bit on what you are learning and why.

2. A kind of Double Slinky: Consider a mechanical system consisting of two springs and two masses sliding on a level frictionless table as depicted below. Note that y is the distance from the center of the first mass to the second mass.



   The system has

   - potential energy, $PE = \frac{1}{2}k_1(x-x_0)^2 + \frac{1}{2}k_2(y-y_0)^2$, where $k_1 > 0$ and $k_2 > 0$ are spring constants having units of N/m (Newtons-per-meter) and where $x_0 > 0$ and $y_0 > 0$ are the rest positions of the springs in units of meters;
   - kinetic energy, $KE = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}+\dot{y})^2$, where $m_1 > 0$ and $m_2 > 0$ are the masses in units of kilograms.

   Derive BY HAND the Equations of Motion (EoM) from Lagrange's equations, assuming there are no external forces. **Provide adequate intermediate steps so that your reasoning can be evaluated**. Optional: You can use the code from JuliaHW06 to check your answer.

3. These are short-answer questions about the Fundamental Theorems of Calculus. For each part, you can just give the final answer followed by a very short comment about why your answer is correct or a one-step solution and a brief comment. There is not really any computational work to show.

   (a) Let $f(x) = 3x^2+2x-1$ and define the function $g(x) := \int_1^x f(t)\,dt$ as an indefinite integral of f. Use the First Fundamental Theorem of Calculus to find the derivative of $g(x)$.

   (b) Suppose $\alpha(x) = \sin(x^3)\cdot\exp\left(\frac{1}{1+x^2}\right)$ and $\alpha'(x) = \frac{x\left(3x(1+x^2)^2\cos(x^3)-2\sin(x^3)\right)}{(1+x^2)^2}\cdot\exp\left(\frac{1}{1+x^2}\right)$. Evaluate $\displaystyle\int_0^{\sqrt[3]{\pi}} \alpha'(x)\,dx$.

   (c) Suppose $h(x)$ is a continuous function satisfying $\int_2^x h(t)\,dt = x^2-4x+4$. Find $h(x)$.

4. By hand, find antiderivatives for the following functions. Show your work and/or explain your reasoning as the problem may require. If you are using a method highlighted in the section of the textbook, **The Art of the Antiderivative: Inverting Differentiation Rules to Find Antiderivatives**, call it out.

   (a) $f(x) = \frac{x^k}{k!}$
   (b) $g(x) = x^2 \exp(x^3)$

5. By hand, determine the following integrals. Show your work and/or explain your reasoning as the problem may require. If you are using a method highlighted in the section of the textbook, **The Art of the Antiderivative: Inverting Differentiation Rules to Find Antiderivatives**, call it out.

   (a) $\int x \cos(x)\, dx$
   (b) $\int \frac{3}{x^2+5x+6}\, dx$

6. By hand, determine the following integrals. Show your work and/or explain your reasoning as the problem may require. If you are using a method highlighted in the section of the textbook, **The Art of the Antiderivative: Inverting Differentiation Rules to Find Antiderivatives**, call it out.

   (a) $\int \frac{1}{\sqrt{9-x^2}}\, dx$, for all $x \in [0, 3)$.
   (b) $\int x^2 \sin(x)\, dx$

Hints

Prob. 1 Write approximately 15 or more words for each part of the question.

Prob. 2 EoM are also called the Robot Equations.

Prob. 3 For part (c), while you may be tempted to answer as was done here, please be a bit more specific! All kidding aside, the problem is a bit different than examples worked in the textbook and may require some thinking. Once you have an answer, it is easy to check it. **We will NOT say anymore in office hours or Piazza.**

Prob. 4 What does it mean to call out a method from **The Art of the Antiderivative: Inverting Differentiation Rules to Find Antiderivatives**? It means to mention "Integrating the Differential", "u-Substitution", etc., if one of them is applicable. Here is part (a) worked out for you. Note the inclusion of the constant of integration.

For $f(x) = \frac{x^k}{k!}$, the antiderivative can be found by recognizing that the derivative of x^{k+1} is $(k+1)x^k$. Therefore, we can reverse this process to find the antiderivative:

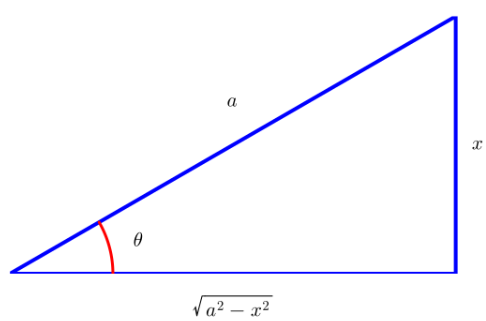
$$F(x) = \int \frac{x^k}{k!}\,dx = \frac{1}{k!}\int x^k\,dx = \frac{x^{k+1}}{(k+1)k!} + C = \frac{x^{k+1}}{(k+1)!} + C$$

To call out the method, you can cite either **the power rule for integration** or **the fundamental rule of recognizing a total differential**.

You still need to turn in a solution to part (a).

Prob. 5 Nothing more to add.

Prob. 6 Part-(a) requires a trig substitution. Recall this diagram from the textbook. The diagram is valid whether the radical is in the numerator or the denominator of the integrand.



Part-(b) requires integration by parts, twice.

Solutions HW 07

Prob. 1 Your answers may vary.

(a) The purpose of Chapter 07 is to bridge the understanding between differential calculus and integral calculus by introducing the concept of an antiderivative. This chapter aims to:

- Illuminate the inverse relationship between differentiation and integration through the Fundamental Theorems of Calculus.
- Demonstrate that while antidifferentiation and definite integration are distinct operations, they are closely related and, often to the peril of young learners, conflated in calculus education.
- Highlight the historical context and the modern computational tools that have made manual computation of antiderivatives less critical, except in specific scenarios.
- Prepare you for further studies in calculus, including indefinite integration, improper integrals, and differential equations, by establishing a solid foundation in understanding antiderivatives.

(b) Two things you may have found DIFFICULT in Chapter 07 are:

- Understanding when and how to apply the various methods for computing antiderivatives, especially distinguishing between situations that call for different techniques. This difficulty stems from the need for significant experience to make these judgments accurately.
- Dealing with the frustration associated with the multi-step process of computing antiderivatives. Errors in calculation, such as sign mistakes or confusion over arithmetic operations, can lead to incorrect results, discouraging learners and learning. This challenge is compounded by the fact that, outside of academic exercises, the practical application of manually computed antiderivatives is limited in modern engineering and mathematical practice.

Prob. 2 The Lagrangian L is defined as the difference between the kinetic and potential energies:

$$L = KE - PE = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}+\dot{y}\right)^2 - \left(\frac{1}{2}k_1(x-x_0)^2 + \frac{1}{2}k_2(y-y_0)^2\right)$$

and Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0_{n\times 1},$$

where q is the vector of generalized coordinates.

From here, two approaches are common. We give our preferred approach first, namely, the one that uses vectors!

- Let the generalized coordinates be $q = \begin{bmatrix} x \\ y \end{bmatrix}$, where x and y are the positions of the masses.
- Compute the partial derivatives of L with respect to q:

$$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \end{bmatrix} = \begin{bmatrix} -k_1(x-x_0) \\ -k_2(y-y_0) \end{bmatrix}$$

  This vector represents the gradient of the Lagrangian L with respect to the generalized coordinates q.

- Compute the partial derivatives of L with respect to $\dot{q}$ (the generalized velocities)

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{x}} \\ \frac{\partial L}{\partial \dot{y}} \end{bmatrix} = \begin{bmatrix} m_1\dot{x} + m_2(\dot{x}+\dot{y}) \\ m_2(\dot{x}+\dot{y}) \end{bmatrix} = \begin{bmatrix} (m_1+m_2)\,\dot{x} + m_2\dot{y} \\ m_2(\dot{x}+\dot{y}) \end{bmatrix}$$

- Compute $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{bmatrix} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) \end{bmatrix} = \begin{bmatrix} m_1\ddot{x} + m_2(\ddot{x}+\ddot{y}) \\ m_2(\ddot{x}+\ddot{y}) \end{bmatrix} = \begin{bmatrix} (m_1+m_2)\,\ddot{x} + m_2\ddot{y} \\ m_2(\ddot{x}+\ddot{y}) \end{bmatrix}$$

- Assemble the pieces according to Lagrange's equation,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0_{2\times 1} \iff \begin{bmatrix} m_1\ddot{x} + m_2(\ddot{x}+\ddot{y}) + k_1(x-x_0) \\ m_2(\ddot{x}+\ddot{y}) + k_2(y-y_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} k_1(x-x_0) \\ k_2(y-y_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

This is a vector form of the Equations of Motion for the system, derived from Lagrange's equations when there are no external forces.

An alternative derivation. This scalar approach is worth equal credit.

The Lagrangian L is defined as the difference between the kinetic and potential energies:

$$L = KE - PE = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}+\dot{y}\right)^2 - \left(\frac{1}{2}k_1(x-x_0)^2 + \frac{1}{2}k_2(y-y_0)^2\right)$$

To derive the Equations of Motion (EoM) using Lagrange's equations, we apply:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

To derive the Equations of Motion (EoM) for both x and y, we perform the following steps:

- **Compute $\frac{\partial L}{\partial \dot{x}}$:**

$$\frac{\partial L}{\partial \dot{x}} = m_1\dot{x} + m_2(\dot{x}+\dot{y})$$

- **Compute $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$:**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_1\ddot{x} + m_2(\ddot{x}+\ddot{y})$$

- **Compute $\frac{\partial L}{\partial x}$:**

$$\frac{\partial L}{\partial x} = -k_1(x-x_0)$$

- **Assemble the equation for x:**

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m_1\ddot{x} + m_2(\ddot{x}+\ddot{y}) - k_1(x-x_0) = 0}$$

- **Compute $\frac{\partial L}{\partial \dot{y}}$:**

$$\frac{\partial L}{\partial \dot{y}} = m_2(\dot{x}+\dot{y})$$

- **Compute $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)$:**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m_2(\ddot{x}+\ddot{y})$$

- **Compute $\frac{\partial L}{\partial y}$:**

$$\frac{\partial L}{\partial y} = -k_2(y-y_0)$$

- **Assemble the equation for y:**

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m_2(\ddot{x}+\ddot{y}) - k_2(y-y_0) = 0}$$

Prob. 3 Short answers.

(a) Let $f(x) = 3x^2 + 2x - 1$ and define the function $g(x) := \int_1^x f(t)\,dt$ as an indefinite integral of f. Use the First Fundamental Theorem of Calculus to find the derivative of $g(x)$.

Ans. By the First Fundamental Theorem of Calculus, $g'(x) = f(x) = 3x^2 + 2x - 1$.

(b) Suppose $\alpha(x) = \sin(x^3) \cdot \exp\left(\frac{1}{1+x^2}\right)$ and $\alpha'(x) = \frac{x\left(3x(1+x^2)^2\cos(x^3) - 2\sin(x^3)\right)}{(1+x^2)^2} \cdot \exp\left(\frac{1}{1+x^2}\right)$. Evaluate $\displaystyle\int_0^{\sqrt[3]{\pi}} \alpha'(x)\,dx$.

Ans. By the Second Fundamental Theorem of Calculus, $\displaystyle\int_0^{\sqrt[3]{\pi}} \alpha'(x)\,dx = \alpha(\sqrt[3]{\pi}) - \alpha(0) = 0.0 - 0.0 = 0.0$, because $\sin(\pi) = \sin(0) = 0.0$.

(c) Suppose $h(x)$ is a continuous function satisfying $\int_2^x h(t)\,dt = x^2 - 4x + 4$. Find $h(x)$.

Ans. By the First Fundamental Theorem of Calculus,

$$h(x) = \frac{d}{dx}\underbrace{\left(\int_2^x h(t)\,dt\right)}_{x^2-4x+4} = \frac{d}{dx}\left(x^2 - 4x + 4\right) = 2x - 4.$$

Prob. 4 To find antiderivatives for the given functions, we use methods highlighted in the section of the textbook, **The Art of the Antiderivative: Inverting Differentiation Rules to Find Antiderivatives**.

(a) **Ans.** $\frac{x^{k+1}}{(k+1)!} + C$

For $f(x) = \frac{x^k}{k!}$, the antiderivative can be found by recognizing that the derivative of x^{k+1} is $(k+1)x^k$. Therefore, we can reverse this process to find the antiderivative:

$$F(x) = \int \frac{x^k}{k!}\,dx = \frac{1}{k!}\int x^k\,dx = \frac{x^{k+1}}{(k+1)k!} + C = \frac{x^{k+1}}{(k+1)!} + C$$

To call out the method, you can cite either **the power rule for integration** or **the fundamental rule of recognizing a total differential**.

(b) **Ans.** $\frac{1}{3}\exp(x^3) + C$

To find the antiderivative of $g(x) = x^2\exp(x^3)$, we use the **method of u-substitution**, because x^3 appears in the exponential function. Let $u = x^3$, then $du = 3x^2dx$. Therefore,

$$G(x) = \int x^2\exp(x^3)\,dx = \frac{1}{3}\int 3x^2\exp(x^3)\,dx = \frac{1}{3}\int \exp(u)\,du = \frac{1}{3}\exp(u) + C = \frac{1}{3}\exp(x^3) + C$$

Prob. 5 Determine the following integrals, using methods highlighted in the section of the textbook, **The Art of the Antiderivative: Inverting Differentiation Rules to Find Antiderivatives**, where applicable.

(a) **Ans.** $\int x\,\cos(x)\,dx = x\sin(x) + \cos(x) + C$

For $\int x\,\cos(x)\,dx$, we use **Integration by Parts**, which is based on the formula $\int u\,dv = uv - \int v\,du$.

Let $u = x$ and $dv = \cos(x)\,dx$, then $du = dx$ and $v = \sin(x)$. Applying Integration by Parts:

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C$$

(b) **Ans.** $\int \frac{3}{x^2+5x+6}\,dx = 3\ln|x+2| - 3\ln|x+3| + C$. Note the absolute value signs. You were not required to provide the domain for the antiderivative, but it is all $x \in \mathbb{R}$ except $x = -2$ and $x = -3$.

For $\int \frac{3}{x^2+5x+6}\,dx$, we use **Partial Fraction Expansion**, which is also called **Partial Fraction Decomposition**. First, factor the denominator: $x^2 + 5x + 6 = (x+2)(x+3)$. Then, decompose the fraction,

$$\frac{3}{x^2+5x+6} = \frac{k_1}{x+2} + \frac{k_2}{x+3}$$

Solving for k_1 and k_2, we find $k_1 = 3$ and $k_2 = -3$. Thus, the integral becomes:

$$\int \frac{3}{x^2+5x+6}\,dx = 3\int \frac{1}{x+2}\,dx - 3\int \frac{1}{x+3}\,dx = 3\ln|x+2| - 3\ln|x+3| + C$$

Calculating Constants: (if you failed to show the details, that's OK)

For k_i (where $i = 1$ and $r_1 = -2$):

$$k_1 = \lim_{x\to -2} \frac{(x+2)\cdot 3}{(x+2)(x+3)} = \frac{3}{(-2+3)} = 3$$

For k_i (where $i = 2$ and $r_2 = -3$):

$$k_2 = \lim_{x\to -3} \frac{(x+3)\cdot 3}{(x+2)(x+3)} = \frac{3}{(-3+2)} = -3$$

Prob. 6 More challenging integration!

(a) **Ans.** For all $x \in [0,3)$, $\int \frac{1}{\sqrt{9-x^2}}\,dx = \arcsin\left(\frac{x}{3}\right) + C$

Inspired by the textbook or the diagram in the hint, we make the **Trigonometric Substitution** $x = 3\sin(\theta)$, for which we have $dx = 3\cos(\theta)d\theta$. Substituting these into the integral gives:

$$\int \frac{3\cos(\theta)}{\sqrt{9-9\sin^2(\theta)}}\,d\theta$$

Simplifying the integrand using a standard trig identity, we have

$$\int \frac{3\cos(\theta)}{|3\cos(\theta)|}\,d\theta$$

The absolute value sign looks like a real problem. However, we are given that $x = 3\ \sin(\theta) \in [0,3)$. We observe that for $0 \le 3\ \sin(\theta) < 3$ we have $0 \le \theta < \frac{\pi}{2}$. Finally, for $0 \le \theta < \frac{\pi}{2}$, $1 \ge \cos(\theta) \ge 0$ and therefore we can remove the absolute value sign. This gives us

$$\int \frac{3\cos(\theta)}{|3\cos(\theta)|}\,d\theta = \int 1\,d\theta = \theta + C$$

But from $x = 3\sin(\theta)$ we deduce that $\frac{x}{3} = \sin(\theta)$, and hence,

$$\theta = \arcsin\left(\frac{x}{3}\right).$$

Thus, the solution to the integral is:

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \arcsin\left(\frac{x}{3}\right) + C$$

and is valid for $0 \le x < 3$. In fact, even though it was not required to note this, because $\arcsin(1) = \frac{\pi}{2}$, we can even compute the solution at $x = 3$.

(b) **Ans.** $\int x^2 \sin(x)\,dx = -x^2\cos(x) + 2(x\sin(x) + \cos(x)) + C = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$

Standard Integration by Parts: For $\int x^2 \sin(x)\,dx$, let $u = x^2$ and $dv = \sin(x)\,dx$, then $du = 2x\,dx$ and $v = -\cos(x)$. Applying Integration by Parts:

$$\int x^2 \sin(x)\,dx = -x^2\cos(x) + 2\int x\cos(x)\,dx$$

We need to apply Integration by Parts a second time to $\int x\cos(x)\,dx$, which, fortunately, we worked it in the previous problem. Putting everything together gives

$$-x^2\cos(x) + 2(x\sin(x) + \cos(x)) + C$$

Alternative Solution via the DI Method: The DI method involves creating a table with differentiation on one side and integration on the other, alternating between decreasing the power of x and integrating $\sin(x)$ or $\cos(x)$.

Sign	D	I
$+$	x^2	$\sin(x)$
$-$	$2x$	$-\cos(x)$
$+$	2	$-\sin(x)$
$-$	0	$\cos(x)$

To extract the integral from the table, follow these steps:

- Multiply D and I entries diagonally, starting from the top left.
- Alternate signs starting with a positive for the first product.

$$\int x^2 \sin(x)\,dx = (x^2)(-\cos(x)) - (2x)(-\sin(x)) + (2)(\cos(x)) + C = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$