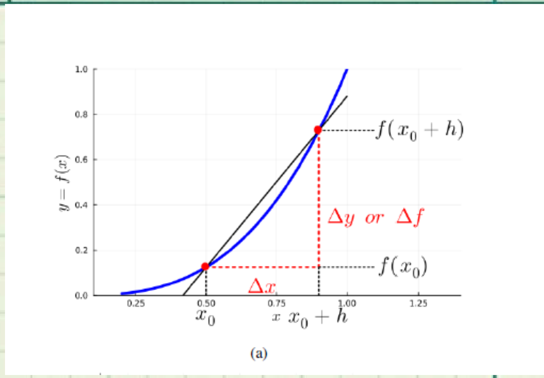
ROB 201 12 Feb 2025

Review



(a)

$$f'(x) := \frac{df}{dx}(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

if the limit exists.

f differentiable at $x_0 \Rightarrow$ continuous at x_0

Function	c	x^n	e^x	$\ln(x)$	$\sin(x)$	$\cos(x)$	$\tan(x)$	$\operatorname{atan}(x)$
Derivative	0	nx^{n-1}	e^x	$\frac{1}{x}$	$\cos(x)$	$-\sin(x)$	$1+\tan^2(x)$	$\frac{1}{1+x^2}$

Sum Rule $\left(a_1 f_1(x) + a_2 f_2(x)\right)' = a_1 f_1'(x) + a_2 f_2'(x)$

Product Rule $\left(f(x) * g(x)\right)' = f'(x) * g(x) + f(x) * g'(x)$

Quotient Rule $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}$

Chain Rule: $\left(f(g(x))\right)' = f'(g(x)) * g'(x)$

$$\frac{d}{dx}\left[f(g(x))\right] = \left.\frac{df(y)}{dy}\right|_{y=g(x)} * \frac{dg}{dx}(x)$$

Power Rule $\left(x^\alpha\right)' = \alpha x^{\alpha-1}$ $\alpha \in \mathbb{R}$

Symbolic Differentiation

Julia & Wolfram

Today: Euler's Formula $\left(\begin{array}{l} i = \sqrt{-1} \\ (i)^2 = -1 \end{array} \right)$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$\frac{d}{dx}(e^{ix}) = \boxed{\frac{d}{dx}[\cos(x)]} + i \boxed{\frac{d}{dx}[\sin(x)]}$$

$$\begin{array}{c} \shortparallel \\ i e^{ix} \end{array} = \boxed{-\sin(x)} + i \boxed{\cos(x)}$$

__Practice:__ $f(x) = \cos(x^2+1)$

$$f'(x) = -\sin(x^2+1) \cdot [x^2+1]'$$

$$= -\sin(x^2+1) \cdot (2x)$$

Use Cases of Single-variable Derivatives

From Physics: $v(t) := \frac{d}{dt} p(t)$

$$a(t) := \frac{d}{dt} v(t)$$

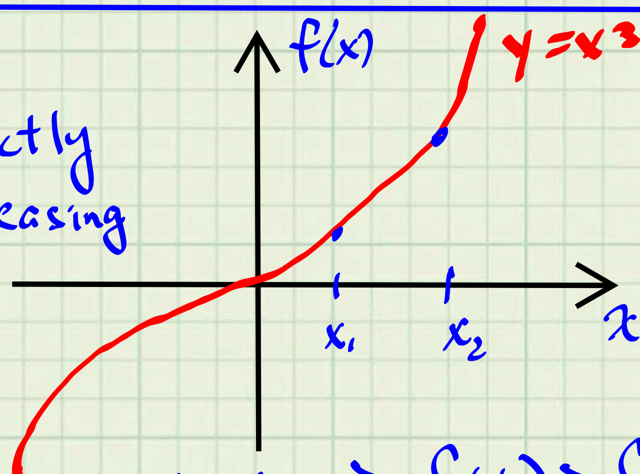
__From Project 1.__

$$p(t) = p(t_0) + \int_{t_0}^{t} v(\tau)\, d\tau$$

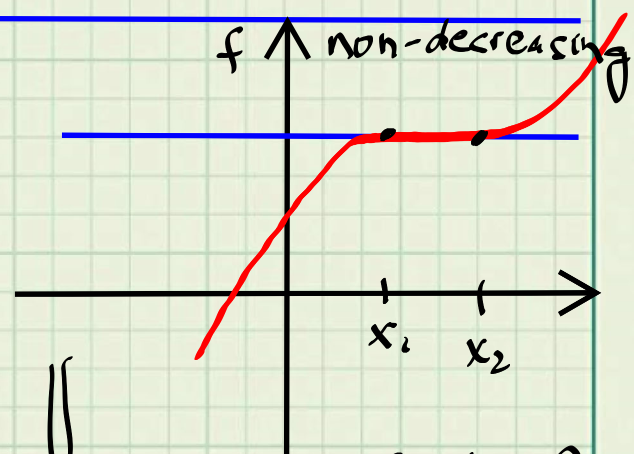
$$v(t) = v(t_0) + \int_{t_0}^{t} a(\tau)\, d\tau$$

Certainly looks like differentiation and integration are "inverse" operations! In Chapt. 7 we explore this more deeply.

strictly increasing



$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$



$x_2 \geq x_1 \Rightarrow f(x_2) \geq f(x_1)$

- $\dfrac{df(x)}{dx} > 0$ for all $x \in (a, b) \Rightarrow f$ is **strictly** increasing

- $\dfrac{df(x)}{dx} \geq 0$ for all $x \in (a, b) \Rightarrow f$ is **non-decreasing**

See textbook: **If $\dfrac{df(x)}{dx} > 0$ EXCEPT** for a finite number of points in (a, b), then **$f(x)$ is strictly increasing.**

L'Hôpital's Rule: its a means of finding limits for the indeterminate forms $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$

Suppose $f(x_0) = g(x_0) = 0$. Then

$$f(x) \approx \cancel{f(x_0)}^{0} + f'(x_0)(x-x_0)$$
$$g(x) \approx \cancel{g(x_0)}^{0} + g'(x_0)(x-x_0)$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x_0)\cancel{(x-x_0)}}{g'(x_0)\cancel{(x-x_0)}} = \frac{f'(x_0)}{g'(x_0)}$$

L'Hôpital's Rule Extends this to cases where $f'(x), g'(x)$ do not necessarily exist at x_0.

L'Hôpital's Rule Let $I = (a, b)$, $x_0 \in I$, $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ that are differentiable on I **except possibly at x_0.** If f and g satisfy

i) For all $x \in I$, $x \neq x_0$, $g'(x) \neq 0$

ii) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$ **OR**

$$\left| \lim_{x \to x_0} f(x) \right| = \left| \lim_{x \to x_0} g(x) \right| = \infty$$

iii) $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ **exists** (can be finite or unbounded)

THEN

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

$\square$

Examples

1) $\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$

all conditions for L'Hôpital are met.

ii) $\lim_{x \to \infty} x e^{-x}$ not $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

$$\Vert$$

$$\lim_{x\to\infty} \frac{x}{e^x}$$ now is $\frac{\infty}{\infty}$

$$\Vert$$

$$\lim_{x\to\infty} \frac{1}{e^x} = 0$$

iii) $\lim_{x\to\infty} x \ln\left(\frac{1}{x}\right) = \infty \times (-\infty) \to -\infty$

not an allowed indeterminate form for L'Hôpital

See textbook: trying to apply L'Hôpital $\Rightarrow$ answer is $+\infty$

iv) Prof. Nima Fazeli posed to the grad students and they struggled:

$$\lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right) = ?$$

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}x\cdot\left(\sqrt{1+\frac{1}{x}}-1\right)\quad\text{(now looks like }\infty\cdot 0\text{)}$$

$$\text{(define }\eta:=\frac{1}{x}\text{)}$$

$$=\lim_{\eta\to 0^+}\left(\frac{1}{\eta}\right)\cdot\left(\sqrt{1+\eta}-1\right)\quad\text{(still looks like }\infty\cdot 0\text{)????}$$

$$=\lim_{\eta\to 0^+}\frac{\sqrt{1+\eta}-1}{\eta}\quad\text{(is now the indeterminate form }\frac{0}{0}\text{)}$$

$$=\lim_{\eta\to 0^+}\frac{\frac{1}{2}\cdot(1+\eta)^{-\frac{1}{2}}}{1}\quad\text{(apply L'Hôpital)}$$

$$=\frac{1}{2}$$

Put your pens down & let's think through

Taylor Polynomials & Series

We often write monomials as

$$x^n \Rightarrow \frac{d}{dx}(x^n) = n\,x^{n-1}$$

However, if we normalize them properly

$$\frac{x^n}{n!} \Rightarrow \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \frac{n}{n!}\,x^{n-1} = \frac{x^{n-1}}{(n-1)!}$$

because $\frac{n}{n!} = \frac{\cancel{n}}{1\cdot 2\cdot 3\cdots(n-1)\cdot\cancel{n}} = \frac{1}{(n-1)!}$

Consider $p(x) = a_3\frac{x^3}{3!} + a_2\frac{x^2}{2!} + a_1\frac{x^1}{1!} + a_0$, then

$p(0) = a_0$ zeroth-order coeff.

$\left.\frac{dp(x)}{dx}\right|_{x=0} = \left.\left(a_3\frac{x^2}{2!} + a_2\frac{x}{1!} + a_1\right)\right|_{x=0} = a_1$ 1st-order coeff

$\left.\frac{d^2p(x)}{dx^2}\right|_{x=0} = \left.\left(a_3 x + a_2\right)\right|_{x=0} = a_2$ 2nd-order coeff

$$\left.\frac{d^3 p(x)}{dx^3}\right|_{x=0} = a_3$$

3rd-order coeff

$$\boxed{\therefore\ p(x) = p'''(0)\frac{x^3}{3!} + p''(0)\frac{x^2}{2!} + p'(0)\frac{x}{1!} + p(0)}$$

In a similar manner, Taylor & Maclaurin discovered that one can form higher-order polynomial approximations (beyond linear = first-order approx) about the origin ($x_0 = 0$) by

$$f(x) \approx f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + \dots + f^{(n)}(0)\frac{x^n}{n!}$$

Goes by the name of Maclaurin expansion

OR, more generally about $x_0 \in \mathbb{R}$

$$f(x) \approx f(x_0) + f'(x_0)\frac{(x-x_0)}{1!} + f''(x_0)\frac{(x-x_0)^2}{2!} + \dots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$$

Goes by the name of Taylor expansion

Justification of L'Hôpital's Rule via Taylor

Suppose $f(x_0) = g(x_0) = 0$, so that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \quad \text{yields} \quad \frac{0}{0} \quad \text{(indeterminate form)}$$

By Taylor

$$\left.\begin{aligned} f(x) &\approx \cancel{f(x_0)}^{\,0} + f'(x_0)(x - x_0) \\ g(x) &\approx \cancel{g(x_0)}^{\,0} + g'(x_0)(x - x_0) \end{aligned}\right\} \text{Exact in the limit}$$

$$\therefore \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x_0)(x - x_0)}{g'(x_0)(x - x_0)} = \frac{f'(x_0)}{g'(x_0)}$$

Suppose $f'(x_0) = g'(x_0) = 0$. Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f''(x_0) \frac{(x - x_0)^2}{2!}}{g''(x_0) \frac{(x - x_0)^2}{2!}} = \frac{f''(x_0)}{g''(x_0)}$$

Etc. Etc. Etc.

L'Hôpital's Rule is a tiny bit more sophisticated, as it says

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad \text{(got it)}$$

98% of the uses of "Taylor Expansions" in STEM are "first-order"

$$f(x) = f(x_0) + f'(x_0)\cdot(x-x_0) = f(x_0) + \frac{df}{dx}(x_0)\cdot(x-x_0),$$

our beloved linear approximation!

Potential Exceptions: When the function is "infinitely differentiable"

$$\boxed{\frac{d^n f(x_0)}{dx^n} \text{ exists for all } n \geq 1.}$$

$$\frac{d}{dx}(e^x) = e^x \implies \frac{d^n}{dx^n}(e^x) = e^x \quad \text{all } n \geq 1$$

$$\frac{d}{dx}(\sin(x)) = \cos(x) \quad \& \quad \frac{d}{dx}(\cos(x)) = -\sin(x)$$

In fact, the k-th derivative of $\sin(x)$ is

Convention: zero-th derivative = the function

$f^{(0)}(x) = f(x)$

$$\frac{d^k}{dx^k}\sin(x) = \begin{cases} \sin(x) & \text{if } k \equiv 0 \pmod 4 \\ \cos(x) & \text{if } k \equiv 1 \pmod 4 \\ -\sin(x) & \text{if } k \equiv 2 \pmod 4 \\ -\cos(x) & \text{if } k \equiv 3 \pmod 4 \end{cases}$$

and the k-th derivative of $\cos(x)$ is

$$\frac{d^0}{dx^0} f(x) = f(x)$$

$$\frac{d^k}{dx^k}\cos(x) = \begin{cases} \cos(x) & \text{if } k \equiv 0 \pmod 4 \\ -\sin(x) & \text{if } k \equiv 1 \pmod 4 \\ -\cos(x) & \text{if } k \equiv 2 \pmod 4 \\ \sin(x) & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

Fact 5.31: (Infinite) Taylor Series Expansion

The Taylor series expansion of an infinitely differentiable function $f(x)$ around a point x_0 is given by,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \tag{5.21}$$

where $f^{(k)}(x_0)$ denotes the k-th derivative of f evaluated at the point x_0.

Whether or not the limit in an infinite Taylor series converges is a highly technical question that we will not explore. Nevertheless, there are three CONVERGENT infinite Maclaurin series that show up frequently in Engineering and STEM. We highlight them below:

Expansion for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}, \tag{5.22}$$

memorize

Expansion for $\sin(x)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \tag{5.23}$$

odd terms

Expansion for $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}. \tag{5.24}$$

even terms

If we were to add a fourth one, it would be the expansion for $\ln(1 + x)$.

"Everyone" recalls $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

"Most" mess up $\sin(x)$ & $\cos(x)$

Fun Fact: $e^{ix} = \cos(x) + i\sin(x)$ follows from

$$\boxed{\begin{array}{l} \text{"}i = \sqrt{-1}\text{"} \\ i^2 := -1 \end{array}}$$

$$i^k = \begin{cases} 1 & k = 0 \mod 4 \\ i & k = 1 \mod 4 \\ -1 & k = 2 \mod 4 \\ -i & k = 3 \mod 4 \end{cases}$$

Looking ahead

$A = n \times n$ matrix

$A^2 = A \times A = n \times n$ matrix

$A^k = A(A^{k-1}) = n \times n$ matrix

$A^0 := I_n$ indentity matrix

$$e^A := I_n + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^k}{k!} + \cdots$$

Turns out to be crucial in the solution of systems of linear differential equations

e^{at} $\quad \dot{x} = ax \quad , x(0) = 1$

e^{At} $\quad \dot{x} = Ax \quad x(0) = I_n$

$\underline{\underline{x^{4}}}$ $\quad e^x$

Partial Derivatives

Consider $f: \mathbb{R}^2 \to \mathbb{R}$ such as

$$f(x_1, x_2) = x_1 + x_1 x_2 + (x_2)^2$$

The **partial derivative of f** with respect to x_1 is

$$\frac{\partial f}{\partial x_1}(x_1, x_2) := \lim_{h \to 0} \frac{f(x_1+h, x_2) - f(x_1, x_2)}{h}$$

in other words, we hold x_2 constant and differentiate f with respect to x_1.

The symbol "∂" is called "partial"

[in latex \partial]

As you might guess

$$\frac{\partial f(x_1, x_2)}{\partial x_2} := \lim_{h \to 0} \frac{f(x_1, x_2+h) - f(x_1, x_2)}{h}$$

Hold x_1 constant, compute "slope" w.r.t.

x_2.

Back to $f(x_1,x_2) = x_1 + \underline{x_1 x_2} + (x_2)^2$

$$\frac{\partial f}{\partial x_1}(x_1,x_2) = 1 + x_2 + 0 = 1 + x_2$$

because

$$\frac{\partial}{\partial x_1}(x_1) = 1$$

$$\frac{\partial (x_2)}{\partial x_1} = 0$$ x_2 treated as a constant

$$\frac{\partial ((x_2)^2)}{\partial x_1} = 0$$

$f(x_1,x_2) = x_1 + x_1 x_2 + (x_2)^2$

$$\frac{\partial f}{\partial x_2}(x_1,x_2) = 0 + x_1 + 2x_2 = x_1 + 2x_2$$

Because $\frac{\partial}{\partial x_2}(x_1) = 0$

$$\frac{\partial}{\partial x_2}(x_2) = 1$$

$$\frac{\partial}{\partial x_2}\left[(x_2)^2\right] = 2x_2$$

$$h(x_1,x_2) = \cos[x_1 (x_2)^2]$$

$$\frac{\partial h}{\partial x_1}(x_1,x_2) = -\sin[x_1 (x_2)^2] \cdot \frac{\partial}{\partial x_1}[x_1 (x_2)^2]$$

$$= -\sin[x_1(x_2)^2] \cdot (x_2)^2$$

$$\frac{\partial h}{\partial x_2}(x_1, x_2) = -\sin\left[x_1 (x_2)^2\right] \cdot \frac{\partial}{\partial x_2}\left[x_1 (x_2)^2\right]$$

$$= -\sin\left[x_1 (x_2)^2\right] * \underbrace{\left[x_1\, 2x_2\right]}_{2x_1x_2}$$