Summary:

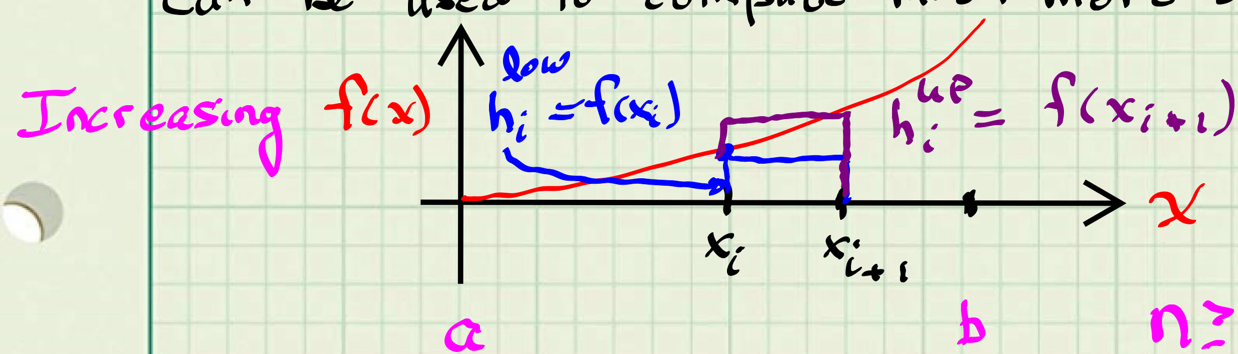
- Proofs differentiate correct from "intuitively obvious"
- Induction: to show $P(n)$ holds for all $n \geq 1$

  Base Case: Show $P(1)$ is true

  Induction Step: Show $P(k) \Rightarrow P(k+1)$ for $k \geq 1$

- Example $P(n) = \underbrace{1^2 + 2^2 + \cdots + n^2}_{L(n)} \overset{?}{=} \underbrace{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}}_{R(n)}$

- Archimedes' Approx. Principle (200 BCE) can be used to compute much more than π.



$n \geq 2$ given

$1 \leq i \leq n+1$ = index

$$\Delta x := \frac{b-a}{n}$$

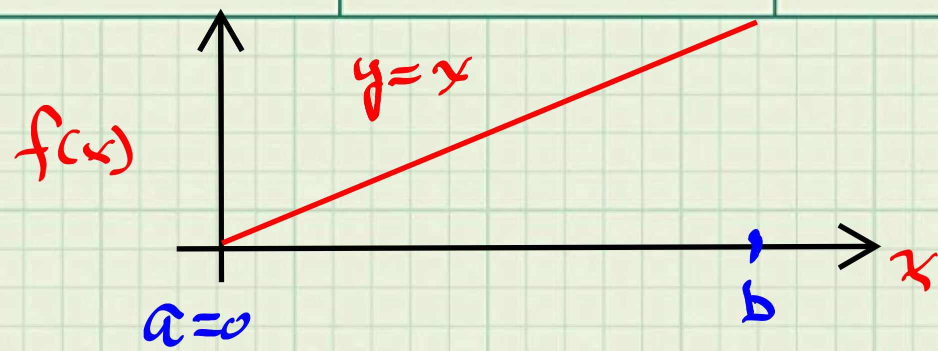
$$x_{i+1} := x_i + \Delta x \quad , \quad x_1 := a$$

$$A_i^{low} := h_i^{low} \cdot \Delta x$$

$$A_i^{up} := h_i^{up} \cdot \Delta x$$

$$\boxed{\text{Area}_n^{low} := \sum_{i=1}^{n} A_i^{low} \leq \text{``}\text{Area}^{true}\text{''} \leq \sum_{i=1}^{n} A_i^{up} =: \text{Area}_n^{up}}$$

If $\text{Area}_n^{up} - \text{Area}_n^{low}$ "$\longrightarrow 0$" for n large then we can define Area^{true}.



$n \geq 2$ given

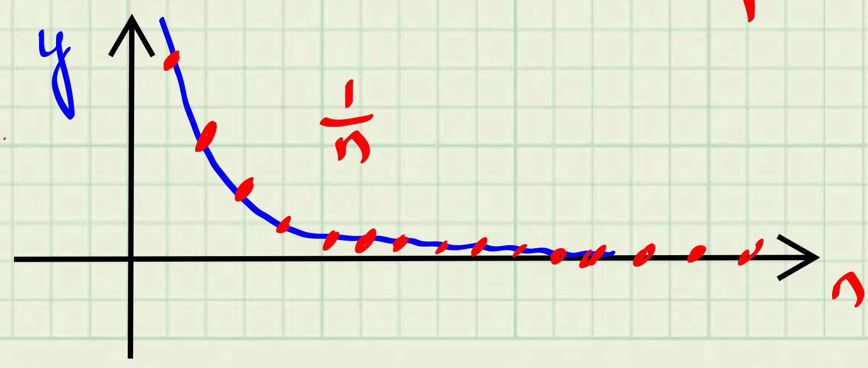
$$\frac{b^2}{2}\left(1-\frac{1}{n}\right) = \overset{\text{low}}{\text{Area}}_n \leq \overset{\text{Triangle}}{\text{Area}} \leq \overset{\text{up}}{\text{Area}}_n = \underbrace{\frac{b^2}{2}\left(1+\frac{1}{n}\right)}_{\text{textbook}}$$

$$\boxed{n := \text{number of rectangles}}$$

Is $\overset{\text{Tri.}}{\text{Area}}$ bracketed by 2 quantities that "approach one another" as n "gets large"?

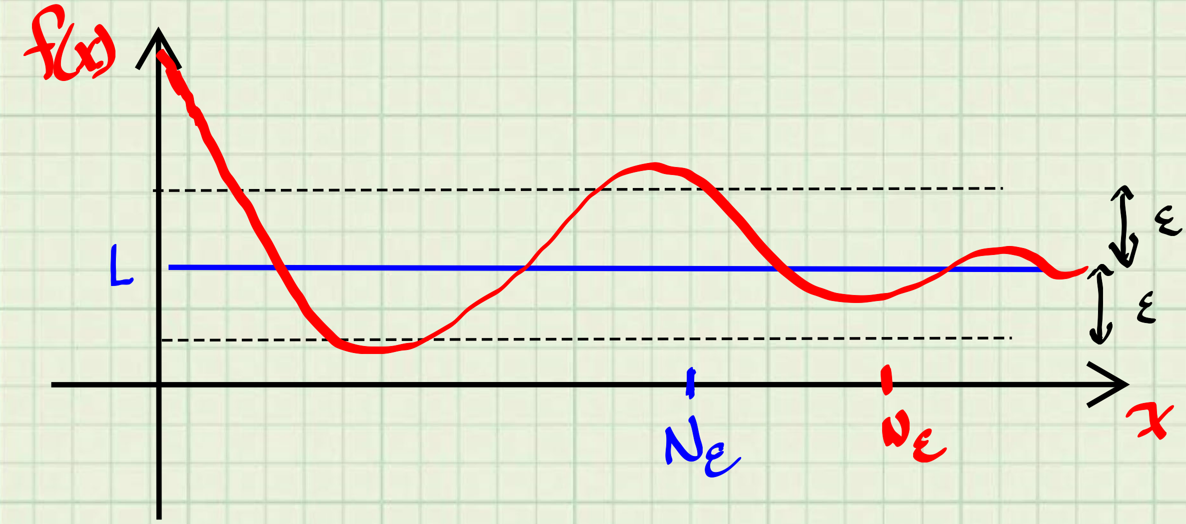
$$\overset{\text{low}}{\text{Area}}_n = \frac{b^2}{2}\left(1-\frac{1}{n}\right) \xrightarrow[n \text{ big}]{?} \frac{b^2}{2} \quad \left\{ \text{derived in lecture} \right.$$

$$\overset{\text{up}}{\text{Area}}_n = \frac{b^2}{2}\left(1+\frac{1}{n}\right) \xrightarrow[n \text{ big}]{?} \frac{b^2}{2} \quad \left\{ \text{see textbook} \right.$$



Today: Limits at ∞ (Infinity)

- Hard way
- Easy way



L is our candidate value for the limit

$\varepsilon > 0$ is our degree of precision

Introduce N as a "knob" for turning $x \geq N$ to achieve the level of precision, $\varepsilon > 0$.

Will see that, in general, N varies with ε, and we denote this by $N(\varepsilon)$ or N_ε.

$x \geq N_\varepsilon \quad |f(x) - L| \leq \varepsilon.$

Def. (Finite limit at ∞) Suppose $f:(0,\infty)\to\mathbb{R}$ is a function and $L\in\mathbb{R}$ is a (finite) real number. Then we say

$$\lim_{x\to\infty} f(x) = L$$

if, for all $\varepsilon>0$ (no matter how small), there exists $N_\varepsilon<\infty$ such that $x\geq N_\varepsilon \Rightarrow |f(x)-L|\leq\varepsilon$. $\square$

Examples

(a) For $f(x)=\frac{1}{x}$, we guess that $L=0$. For $\varepsilon=0.1=\frac{1}{10}$. What is a value for N_ε?

$$|f(x)-L| \leq \varepsilon = \frac{1}{10}$$

$$\overset{\shortparallel}{\left|\tfrac{1}{x}-0\right|} = \left|\tfrac{1}{x}\right| \leq \tfrac{1}{10} \quad \text{for } x\geq N_{\frac{1}{10}}$$

Clearly $N_{\frac{1}{10}}=10$

For general $\varepsilon > 0$, $\left|\frac{1}{x}\right| \leq \varepsilon$ for $x \geq \frac{1}{\varepsilon}$

$\therefore N_\varepsilon := \frac{1}{\varepsilon}$

(b) $f(x) = \frac{1}{x^2}$, $L = 0$. $\varepsilon = \frac{1}{4}$

$$|f(x) - L| = \left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2}$$

We seek $N_{\frac{1}{4}}$ such that $x \geq N_{\frac{1}{4}} \Rightarrow \frac{1}{x^2} \leq \frac{1}{4}$

$$x^2 \geq 4 \Rightarrow x \geq 2 \Rightarrow N_{\frac{1}{4}} = 2$$

For general $\varepsilon > 0$, $\frac{1}{x^2} \leq \varepsilon \Leftrightarrow x^2 \geq \frac{1}{\varepsilon}$

$\Rightarrow x \geq \sqrt{\frac{1}{\varepsilon}} = \frac{1}{\sqrt{\varepsilon}}$. $\therefore N_\varepsilon = \frac{1}{\sqrt{\varepsilon}}$

(c) $f(x) = \frac{x}{1+2x}$. What do you think L is?

$L = \frac{1}{2}$?

$$|f(x) - L| = \left|\frac{x}{1+2x} - \frac{1}{2}\right|$$

$$= \left|\frac{\cancel{2x} - 1(1+\cancel{2x})}{2(1+2x)}\right| \qquad \text{common den.}$$

$$= \left|\frac{-1}{2(1+2x)}\right|$$

$$= \frac{1}{2(1+2x)} \quad \text{for } x > 0$$

Let $\varepsilon > 0$. Find N_ε.

$$\frac{1}{2(1+2x)} \le \varepsilon$$

$$2(1+2x) \ge \frac{1}{\varepsilon}$$

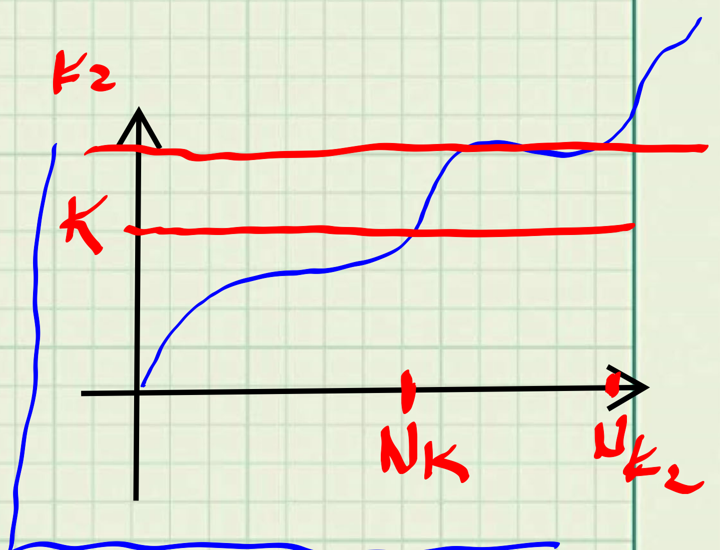
$$1+2x \ge \frac{1}{2\varepsilon}$$

$$2x \ge \frac{1}{2\varepsilon} - 1$$

$$x \ge \frac{1}{4\varepsilon} - \frac{1}{2}$$

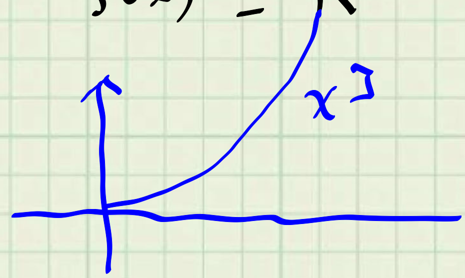
$$\therefore N_\varepsilon := \frac{1}{4\varepsilon} - \frac{1}{2}$$

We can do the grungy algebra for TOY PROBLEMS and very little else!



<u>Def</u> $\lim_{x\to\infty} f(x) = \infty$

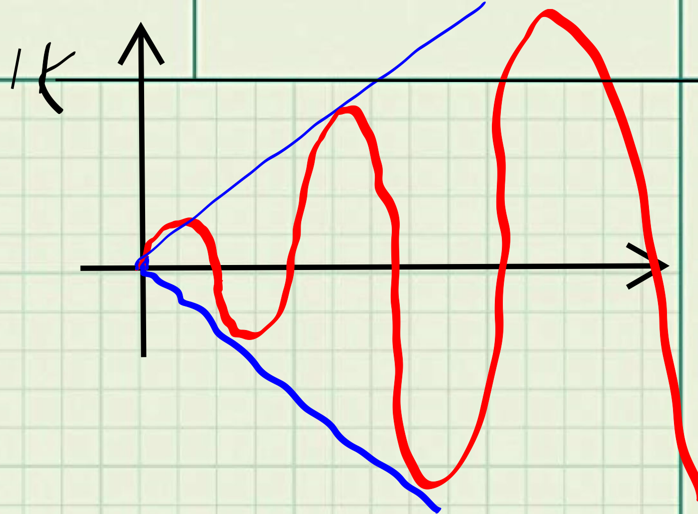
if, for all $K > 0$ (no matter how large), there exists $N_K < \infty$ such that $x \ge N_K \Rightarrow f(x) \ge K$

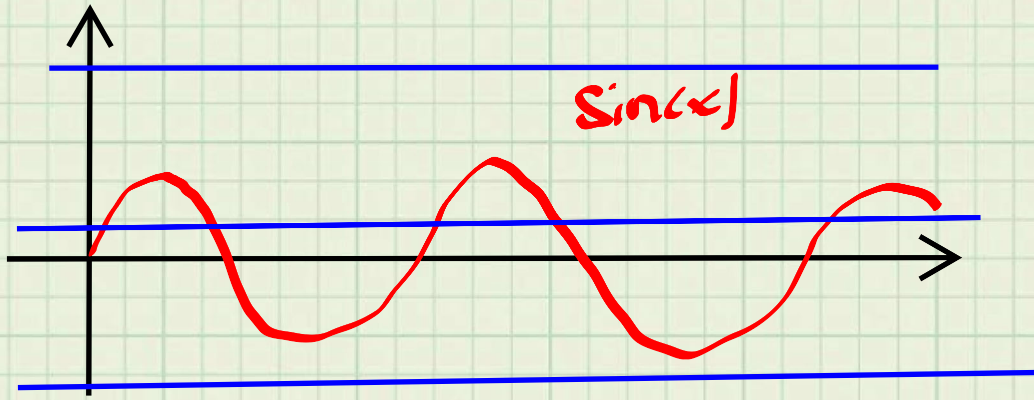


- Examples
  - $f(x) = x^3$

    $\lim_{x\to\infty} f(x) = \infty$

$$\underbrace{f(x) = x \cdot \sin(x)}_{\text{has no limit as } x \to \infty.}$$





Does $f(x) \to L$ as $x \to \infty$?

$\lim_{x \to \infty} \sin(x)$ is undefined (same as does not exist).

Limits at Infinity the Easy Way!

Assume $b_m \neq 0$, $a_n \neq 0$ and consider

$$\frac{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0}{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0} = \boxed{\frac{x^m}{x^n}} \cdot \underbrace{\frac{b_m + b_{m-1}\frac{1}{x} + \cdots + \frac{b_0}{x^m}}{a_n + a_{n-1}\frac{1}{x} + \cdots + \frac{a_0}{x^n}}}_{\lim_{x \to \infty} \frac{b_m}{a_n}}$$

$$\lim_{x\to\infty} \frac{b_m x^m + \cdots + b_0}{a_n x^n + \cdots + a_0} = \begin{cases} 0 & m < n \\ \frac{b_m}{a_n} & m = n \\ \operatorname{sign}\left(\frac{b_m}{a_n}\right)\cdot\infty & m > n \end{cases}$$

Examples

a) $f_a(x) = \dfrac{20x^2 + 11x + 14}{1 + 4x^2} \xrightarrow[x\to\infty]{} \frac{20}{4} = 5$

b) $f_b(x) = \dfrac{-4x^3 + 11x + 14}{1 + 4x^2} \xrightarrow[x\to\infty]{} -\infty$

c) $f_c(x) = \dfrac{-4x^4 + 11x + \sqrt{2}}{2x^6 + 1} \xrightarrow[x\to\infty]{} 0$

How does the story change for something like

$$\lim_{n\to\infty} \frac{n^3 + 6n^2}{4n^2 + 11} \; ?$$

No substantial difference

$$\lim_{n\to\infty} \frac{n^3}{n^2} \frac{1 + 6\frac{1}{n}}{4 + \frac{11}{n^2}} \to +\infty$$

See examples 2.35 and 2.36 in

the textbook.

$$\lim_{N \to \infty} \underbrace{\frac{\left(x+\frac{1}{N}\right)^2 - x^2}{\frac{1}{N}}}_{\text{Simplify via Algebra}} = ?$$

Not a ratio of polynomials

$$\frac{\cancel{x^2} + \frac{2}{N}x + \frac{1}{N^2} - \cancel{x^2}}{\frac{1}{N}} \frac{N}{N} = \frac{2x + \frac{1}{N}}{}$$

$1 - 0.9 = 0.1$

$1 - 0.99 = 0.01$

$1 - 0.999 = 0.001$

etc.

$$0.99999\ldots = 1$$

Limits Are Everywhere!

Real numbers

$x \in [0,1]$ — decimal expansion

$x = 0.d_1d_2d_3d_4\ldots$ — $d_i \in \{0,1,2,\ldots,9\}$

$$\boxed{= \lim_{N\to\infty} \sum_{i=1}^{N} d_i 10^{-i}}$$

(Math 350 & 450)

Powers : x^y, $x \geq 0$

$r = \frac{p}{q}$ rational number, $p \in \mathbb{Z}$, $q \in \mathbb{N}$

Integers

$$x^r = x^{p/q} := \sqrt[q]{x^p} = \left(\sqrt[q]{x}\right)^p$$

How about x^y for $y \in \mathbb{R}$?

Suppose $|y - r_i| = \left|y - \frac{p_i}{q_i}\right| < 10^{-i}$ $i \geq 1$

$$x^y := \lim_{i\to\infty} x^{\frac{p_i}{q_i}} = \lim_{i\to\infty} \sqrt[q_i]{x^{p_i}}$$

$y = \pi = 3.14159265\ldots$

$r_1 = 3.1$

$r_2 = 3.14$

$r_3 = 3.145$

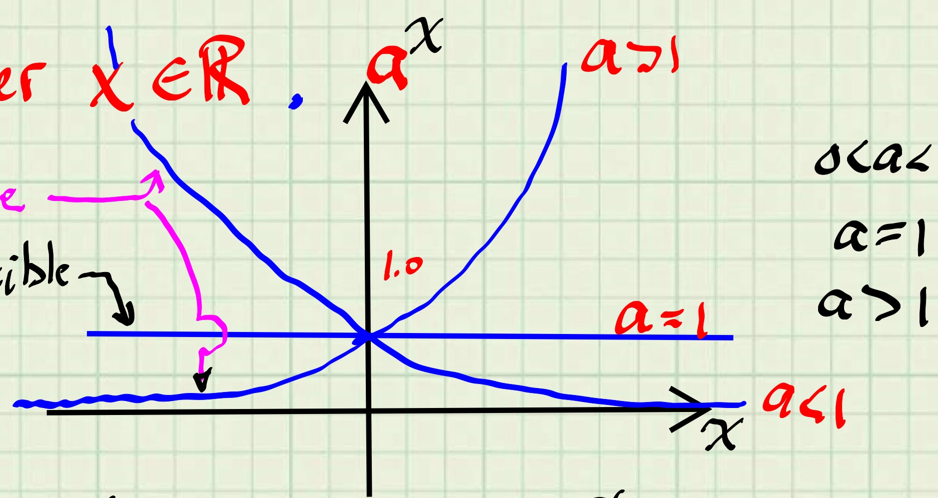
$\vdots$

$$3^\pi = \lim_{i\to\infty} 3^{r_i}$$

Roots : Inverse functions of Powers

$$z := \sqrt[y]{x} \iff z^y = x \quad , \quad x \geq 0$$

Exponentials : a^x are "powers" where we fix the base $a > 0$ and vary the power $x \in \mathbb{R}$.



Def. $$a^{-x} := \frac{1}{a^x} = \left(\frac{1}{a}\right)^x$$

Logarithms: Just like roots, they are inverse functions to exponentials.

Def. For $x > 0$, $y := \log_a x \iff a^y = x$

logarithm base a of x $[a > 0, a \neq 1]$

Three Key Properties $a>0, a\neq 1$

1) For $x>0, y>0$, $\log_a(x\cdot y)=\log_a(x)+\log_a(y)$

2) $\alpha\in\mathbb{R}$, $x>0$: $\log_a(x^\alpha)=\alpha\cdot\log_a(x)$

3) $x>0$, $f:(0,\infty)\to\mathbb{R}$

$$\log_a\left(x^{f(x)}\right)=f(x)\cdot\log_a(x)$$

Natural Exponential & Logarithm:

base is $e :=$ Euler's number

$$:=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$\left\{\begin{matrix}\text{Chapter}\\ 1.3.3\end{matrix}\right\}$

$$\approx 2.71828183\ldots$$

$$y=\ln(x)\iff e^y=x$$

$\{\ln(x):=\log_e(x)$

In Engineering, you see bases 2, e, and 10.

$\square$

Exponentials a^x, $a>1$, grow faster than any monomial x^m, $m = 1, 2, \ldots$

For all $a>1$, $m = 1, 2, \ldots$

- $\lim\limits_{x\to\infty} \frac{a^x}{x^m} = \infty$
- $\lim\limits_{x\to\infty} \frac{x^m}{a^x} = 0$

Examples

$0.5 = \frac{1}{2}$ $\quad \left(\frac{1}{2}\right)^x = \frac{1}{2^x}$

a) $f_a(x) = (0.5)^x \cdot x^{37} \xrightarrow[x\to\infty]{} 0$ $\quad \frac{x^{37}}{2^x}$

b) $f_b(x) = \frac{3^x}{x^{200}} \xrightarrow[x\to\infty]{} \infty$

c) $f_c(x) = x e^x \xrightarrow[x\to\infty]{} \infty$

d) $f_d(x) = \dfrac{e^{3x} + e^{4x} + x^{100} e^{x}}{x^{201} + 3x^{400} + 5x^{10^{100}}} \to \infty$ as $x \to \infty$

Algebra of Limits

<u>Intuition</u>

- $\alpha \in \mathbb{R}$, $\alpha + \infty = \infty$ $\left(\lim_{x\to\infty} (\alpha + x) = \infty\right)$
- Similarly $\alpha - \infty = -\infty$ $\left(\lim_{x\to\infty} (\alpha - x) = -\infty\right)$
- Can extend to $\infty + \infty = \infty$

$\left(\lim_{x\to\infty} x^3 + 3x = \infty\right)$

- However: $\infty - \infty$ in undefined.

It is called an <u>indeterminate</u> form

<u>Why</u>

$\lim_{x\to\infty} (x - x^2) = -\infty$

$\lim_{x\to\infty} \left(x - \frac{x^2}{x+1}\right) = \lim_{x\to\infty} \left(\frac{\cancel{x^2} + x - \cancel{x^2}}{x+1}\right) = 1$

$$\lim_{x \to \infty} (x^2 - x) = +\infty$$