ROB 201 - Calculus for the Modern Engineer
HW #1

Prof. Grizzle

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete.

1. Read Chapters 1 and 2 of our ROB 201 Textbook, `Calculus for the Modern Engineer`. Based on your reading of the chapters, summarize in your own words:

   (a) the purpose of Chapter 01;

   (b) two things you found the most interesting;

   (c) the purpose of Chapter 02;

   (d) and, two things you found the most interesting.

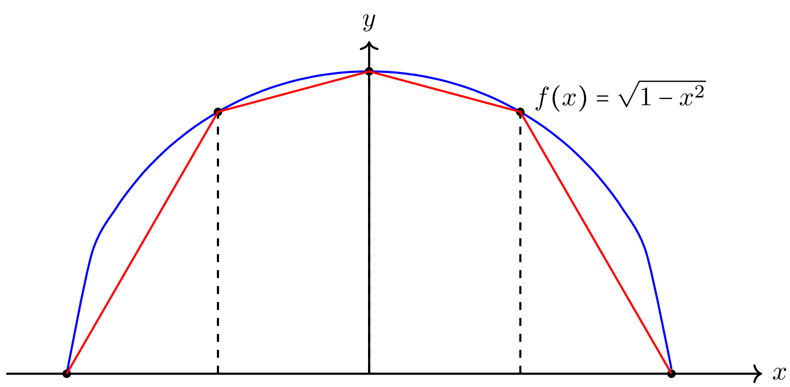
   There are no "right" or "wrong" answers. The goal is to reflect a bit on what you are learning and why.

2. Let $f(x) = \sqrt{1-x^2}$ represent the upper half of the unit circle on the interval $x \in [-1,1]$. Divide the interval into $n = 4$ subintervals of equal width $\Delta x = 0.5$.

   (a) For each subinterval, draw a line segment connecting $f(x_i)$ and $f(x_{i+1})$ and use these to form a piecewise linear approximation of the area under $f(x)$.

   (b) Compute both the overestimate and underestimate of the total area under $f(x)$ from $x = -1$ to $x = 1$ using the areas of trapezoids and rectangles, respectively.

   (c) Compare your result to the known value $\frac{\pi}{2}$ and explain how the Approximation Principle from Chapter 1 is applied.



3. Let $f(\delta) = \ln(1+\delta)$. For a given $\epsilon > 0$, determine the set:

   $$\{\delta > 0 \mid |f(\delta)| < \epsilon\}.$$

   Compare your answer to the analogous set for $e^\delta - 1$ from the textbook problem. How do their behaviors near 0 differ?

4. Let $f(x) = \sqrt{1-x^2}$ defined on $[-1,1]$ and $g(x) = \arcsin(x)$ defined on $[-1,1]$.

   (a) Does $f \circ g$ exist? Why or why not?

(b) Does $g \circ f$ exist? Why or why not?

Support your answer by describing the ranges of the inner functions and the domains of the outer functions.

5. Compute, **the hard way**, $\lim_{x \to \infty} f(x)$ for the following functions. For each answer, give the following information:

- State whether the limit exists or not.
- If the limit exists, give its value, which we'll call $f_\infty := \lim_{x\to\infty} f(x)$.
- If the limit exists **and is finite**, for $\epsilon = 0.1$, find $0 < M < \infty$ such that $|f_\infty - f(x)| \le 0.1$ for all $x \ge M$.
- If the limit exists **and is unbounded** or **does not exist**, you do not need to do anything further.

Remark: We are using $x \ge M$ instead of $x \ge N$ because, below, we use $N(x)$ to denote the numerator of a ratio of two functions.

(a) $f(x) = \dfrac{x^2 + 5x + 4}{x^2 + 3}$

(b) $f(x) = \dfrac{x^3 + 2}{2x^2 + 1}$

(c) $f(x) = \dfrac{1 + x^2}{x^4 + 1}$

Note: We work the example $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \frac{x}{x^2+1}$ so that you are clear on what to do.

- The limit exists.
- $f_\infty = 0.0$.
- $|f_\infty - f(x)| \le 0.1$ for all $x \ge 10$. In other words, $M = 10.0$.

This is true because

$$\begin{aligned} |f_\infty - f(x)| &= \left| 0.0 - \frac{x}{x^2+1} \right| \\ &= \frac{x}{x^2+1} \\ &= \frac{\frac{1}{x}}{1 + \frac{1}{x^2}} \quad \text{(divide through by largest term in the denominator).} \end{aligned}$$

From the Hint: To find an upper bound for a fraction, $\frac{n(x)}{d(x)}$, you replace the numerator with an upper bound and the denominator with a lower bound for the inequality to hold. In symbols, suppose that $0 \le n(x) \le N(x)$. Then,

$$\frac{n(x)}{d(x)} \le \frac{N(x)}{d(x)},$$

because having a larger number in the numerator gives a larger fraction. Next, suppose that $0 < D(x) \le d(x)$. Then,

$$\frac{n(x)}{d(x)} \le \frac{N(x)}{d(x)} \le \frac{N(x)}{D(x)},$$

because having a smaller number in the denominator yields a larger fraction. **While these are "obvious " facts, applying them successfully when you are first learning limits can be hard to master!**

Going back to our problem: For $x \ge 10$, $\frac{1}{x} \le 0.1$ and $1 + \frac{1}{x^2} \ge 1.0$. Hence, for $x \ge 10$,

$$\frac{\frac{1}{x}}{1 + \frac{1}{x^2}} \le \frac{1}{10}.$$

As stated above, if $0 < n(x) \le N(x)$ and $d(x) \ge D(x) > 0$, then $\frac{n(x)}{d(x)} \le \frac{N(x)}{D(x)}$. In our case, $n(x) = \frac{1}{x} \le N(x) := 0.1$ holds for $x \ge 10$, and so does $d(x) = 1 + \frac{1}{x^2} \ge 1.0 =: D(x)$.

This is the end of the drill problems. The second part of the HW set will introduce you to the Julia Programming Language via `jupyter notebooks`. **Please go to** the course GitHub Site and find the assignment titled "Homework 1 (Julia)". This assignment can be completed locally using the Anaconda Distribution or an IDE of your choosing.

Hints

Prob. 1 Write approximately 15 or more words for each part of the question.

Prob. 2 The Approximation Principle in Chapter 1 says we can estimate area under a curve by summing the areas of simpler shapes. For rectangles, we use the function value at one endpoint (here, the left endpoint) to give a constant height over the subinterval—this will underestimate the curve if the function is increasing, because it misses the extra area that rises over the interval.

For trapezoids, we use both endpoint values to draw a straight line between them and measure the area under that line. This usually overestimates a concave-down curve (like a semicircle), because the straight line lies above the curve between the endpoints.

In symbols, recall that for subinterval $[x_i, x_{i+1}]$:

$$\begin{aligned} \text{Rectangle area } &= \Delta x \cdot f(x_i) \\ \text{Trapezoid area } &= \frac{\Delta x}{2}\left(f(x_i) + f(x_{i+1})\right). \end{aligned}$$

By adding up these over each subinterval, you get a total underestimate (rectangle sum) and overestimate (trapezoid sum) for the integral.

Prob. 3 **Notation Reminder:** Many students find set-builder notation confusing at first. That's normal! Our textbook discusses this in Chapter 1 under **"Set Notation and Interpreting Conditions"**.

What does this set mean in the real numbers,

$$\{\delta > 0 \mid |\ln(1+\delta)| < \epsilon\}?$$

First, note that δ is restricted to positive real numbers—so only $\delta > 0$ values are even considered. The vertical bar "|" in the set notation means "such that." Among all $\delta > 0$, we include only those that satisfy the inequality $|\ln(1+\delta)| < \epsilon$.

Remember that

$$|y| < \epsilon \iff -\epsilon < y < \epsilon.$$

So you can rewrite the condition as:

$$-\epsilon < \ln(1+\delta) < \epsilon.$$

The natural logarithm is a **strictly increasing** function on $(0, \infty)$, which means:

$$x_1 < x_2 \implies \ln(1+x_1) < \ln(1+x_2).$$

This property helps you solve the inequality for δ. Compare this process to the textbook example with

$$\left\{\delta > 0 \mid |e^{\delta} - 1| < \epsilon\right\},$$

and notice how the bounding inequalities work in both cases.

Prob. 4 In each of the problems, you are given the **domains** of the functions (i.e., the functions' allowed "inputs"). To solve the problems, you must determine the **range** of each function (i.e., the "outputs" of the function). **Remember, in general, the co-domain is not the same as the range.**

Prob. 5 **Intuition:** To find an upper bound for a fraction, $\frac{n(x)}{d(x)}$, you can replace the numerator with an upper bound and/or the denominator with a lower bound for the inequality to hold. Indeed, going back to the standard fractions we learned in junior high, if $\frac{n}{d} := \frac{3}{5}$, then both $\frac{N}{d} =: \frac{4}{5}$ and $\frac{n}{D} =: \frac{3}{4}$ are upper bounds. Returning to $\frac{n(x)}{d(x)}$, suppose that $0 \le n(x) \le N(x)$. Then,

$$\frac{n(x)}{d(x)} \le \frac{N(x)}{d(x)},$$

because having a larger function in the numerator gives a larger fraction. Next, suppose that $0 < D(x) \le d(x)$. Then,

$$\frac{n(x)}{d(x)} \le \frac{N(x)}{d(x)} \le \frac{N(x)}{D(x)},$$

because having a smaller function in the denominator yields a larger fraction. **While these are "obvious" facts, applying them successfully when you are first learning limits can be hard to master.**

Solutions HW 01

Prob. 1 (a) **The purpose of Chapter 01:** Chapter 01 reviews in detail how mathematics is written using good notation. It introduces what is likely a new concept for you, the Approximation Principle, which shows how to estimate quantities with firm lower and upper bounds. We'll come back to this over and over in the early part of learning Calculus. The remainder of Chapter 01 reviews basic facts about algebra, functions, trigonometry, powers, radicals, exponentials, and logarithms. Only a few of you will have seen Euler's formula during your HS career. The Binomial Theorem should be familiar to you, though you may be rusty at using it.

(b) **Two things I found the most interesting:** The history behind some of the mathematical inventions is cool. How one goes about defining x^y when y is an irrational number is pretty neat. If I were to choose a third idea, it would be Euler's formula.

(c) **The purpose of Chapter 02:** Chapter 02 begins by underlining that mathematics is hard to create. Cantor discovered that the concept of infinity comes in more than one size, an idea so shocking that he was ostracized by the mathematical community of his day. Yet, we were able to go through a very simple proof that showed Cantor was correct. Proofs are how mathematicians try to avoid letting errors slip into their body of work. In ROB 201, you will see some proofs in lectures, some you will read in the textbook, and yet others are too challenging for your level of mathematical training. Chapter 02 introduced (or maybe reviewed for some of you) how to construct a proof by induction. It then brought us to our first contact with the notion of a limit.

(d) **Two things I found the most interesting:** The first time I saw it, I was super surprised that rational numbers can be put into one-to-one correspondence with the counting numbers. Cantor's Diagonal Argument is super elegant, and hence, I found that interesting. Limits are the workhorse of Calculus. I did not really understand them until I took the equivalent of Michigan's Math 451 Advanced Calculus I. My hope is that our textbook does a better job than the one I had for Calculus I and II back in the day.

Prob. 2 (a) We divide the interval $[-1, 1]$ into $n = 4$ equal subintervals of width

$$\Delta x = \frac{1 - (-1)}{4} = 0.5.$$

The nodes are:

$$x_0 = -1.0, \quad x_1 = -0.5, \quad x_2 = 0.0, \quad x_3 = 0.5, \quad x_4 = 1.0.$$

Function evaluations:

$$\begin{aligned} f(x) &= \sqrt{1 - x^2}. \\ f(-1) &= 0 \\ f(-0.5) &= \sqrt{0.75} \approx 0.8660 \\ f(0) &= 1 \\ f(0.5) &= 0.8660 \\ f(1) &= 0. \end{aligned}$$

(b) **Approximating the Area**

(i) **Trapezoidal Rule (Overestimate):**

The area over each subinterval is

$$A_i = \frac{\Delta x}{2}\big[f(x_i) + f(x_{i+1})\big].$$

$$\begin{aligned} A_1 &= 0.25 \times (0 + 0.8660) = 0.2165 \\ A_2 &= 0.25 \times (0.8660 + 1) = 0.4665 \\ A_3 &= 0.25 \times (1 + 0.8660) = 0.4665 \\ A_4 &= 0.25 \times (0.8660 + 0) = 0.2165. \end{aligned}$$

Total trapezoidal approximation:

$$0.2165 + 0.4665 + 0.4665 + 0.2165 = 1.366.$$

(ii) **Left-Endpoint Rule (Underestimate):**

The area over each subinterval is

$$R_i = \Delta x \cdot f(x_i).$$

$$\begin{aligned} R_1 &= 0.5 \times 0 = 0 \\ R_2 &= 0.5 \times 0.8660 = 0.4330 \\ R_3 &= 0.5 \times 1 = 0.5 \\ R_4 &= 0.5 \times 0.8660 = 0.4330. \end{aligned}$$

Total left-endpoint approximation:

$$0 + 0.4330 + 0.5 + 0.4330 = 1.366.$$

(c) **Comparison to Exact Value**
The exact area under $f(x)$ over $[-1, 1]$ is half the area of the unit circle:

$$A_{\text{exact}} = \frac{\pi}{2} \approx 1.5708.$$

Error in approximation:

$$|1.5708 - 1.366| \approx 0.2048.$$

Note: Both rules give the same numerical approximation here due to the symmetry and choice of partition points. Still, both slightly underestimate the true semicircular area.

Prob. 3 **Claim:** For $\epsilon > 0$, determine

$$\{\delta > 0 \mid |\ln(1 + \delta)| < \epsilon\}.$$

Step 1: Simplify the Inequality
For $\delta > 0$, note that $\ln(1 + \delta) > 0$. Hence

$$|\ln(1 + \delta)| = \ln(1 + \delta).$$

The inequality becomes:

$$\ln(1 + \delta) < \epsilon.$$

Step 2: Solve for δ
Exponentiating both sides gives:

$$1 + \delta < e^{\epsilon},$$

so

$$\delta < e^{\epsilon} - 1.$$

Since $\delta > 0$, the set is:

$$\boxed{\{0 < \delta < e^{\epsilon} - 1\}}.$$

Step 3: Comparison to the Textbook Example
The textbook gave:

$$\{\delta > 0 \mid |e^{\delta} - 1| < \epsilon\} = \{0 < \delta < \ln(1 + \epsilon)\}.$$

Step 4: Behavior Near 0
For small ϵ, we use approximations:

$$e^{\epsilon} - 1 \approx \epsilon,$$

$$\ln(1 + \epsilon) \approx \epsilon.$$

Both sets approach $\{0 < \delta < \epsilon\}$ as $\epsilon \to 0$.
Interpretation: This shows they behave similarly near 0. But further out:

$\ln(1 + \delta)$ grows slowly for large δ.

- $e^{\delta} - 1$ grows rapidly for large δ.

Conclusion
The set of δ satisfying $|\ln(1+\delta)| < \epsilon$ is:

$$\boxed{\{0 < \delta < e^{\epsilon} - 1\}}.$$

It behaves similarly near 0 to the textbook example, but the bounding functions grow at different rates for larger δ.

Prob. 4 (a) **Ans.** $\boldsymbol{f \circ g}$ **fails to exist (N)**

Reason: We have

- $f : [-1,1] \to [0,1]$ given by $f(x) = \sqrt{1-x^2}$
- $\text{dom}(f) = [-1,1]$
- $\text{range}(f) = [0,1]$

and

- $g : [-1,1] \to [-\pi/2, \pi/2]$ given by $g(x) = \arcsin(x)$
- $\text{dom}(g) = [-1,1]$
- $\text{range}(g) = [-\pi/2, \pi/2]$

$f \circ g$ fails to exist because $\text{range}(g) \notin \text{dom}(f)$

(b) **Ans.** $\boldsymbol{g \circ f}$ **exists (Y)**

Reason: $g \circ f$ exists because $\text{range}(f) \subset \text{dom}(g)$

Prob. 5 (a) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{x^2 + 5x + 4}{x^2 + 3}$.

- The limit exists.
- $f_\infty = 1.0$.
- $|f_\infty - f(x)| \leq 0.1$ for all $x \geq 50$. In other words, $M = 50$.

This is true because dividing numerator and denominator by x^2 shows

$$f(x) = \frac{1 + \frac{5}{x} + \frac{4}{x^2}}{1 + \frac{3}{x^2}} \to 1 \text{ as } x \to \infty.$$

For $\epsilon = 0.1$, choosing $x \geq 50$ makes $\frac{5}{x} \leq 0.1$, ensuring the overall difference is within ϵ.

(b) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{x^3 + 2}{2x^2 + 1}$.

- The limit does not exist (unbounded).
- $f_\infty = \infty$.
- No other work is required.

For completeness, note that

$$f(x) = \frac{x + \frac{2}{x^2}}{2 + \frac{1}{x^2}} \approx \frac{x}{2} \to \infty.$$

(c) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{1 + x^2}{x^4 + 1}$.

- The limit exists.
- $f_\infty = 0.0$.
- $|f_\infty - f(x)| \leq 0.1$ for all $x \geq 4$. In other words, $M = 4$.

This is true because

$$f(x) = \frac{1 + x^2}{x^4 + 1} \leq \frac{x^2 + 1}{x^4} = \frac{1}{x^2} + \frac{1}{x^4}.$$

For $x \geq 4$, both terms sum to at most 0.1.