ROB 201 - Calculus for the Modern Engineer
HW #3

Prof. Grizzle

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete.

1. Read Chapter 3 of our ROB 201 Textbook, `Calculus for the Modern Engineer`. Based on your reading of the Chapter, summarize in your own words:

   (a) the purpose of Chapter 03;

   (b) two things you found the most DIFFICULT.

   There are no "right" or "wrong" answers. The goal is to reflect a bit on what you are learning and why.

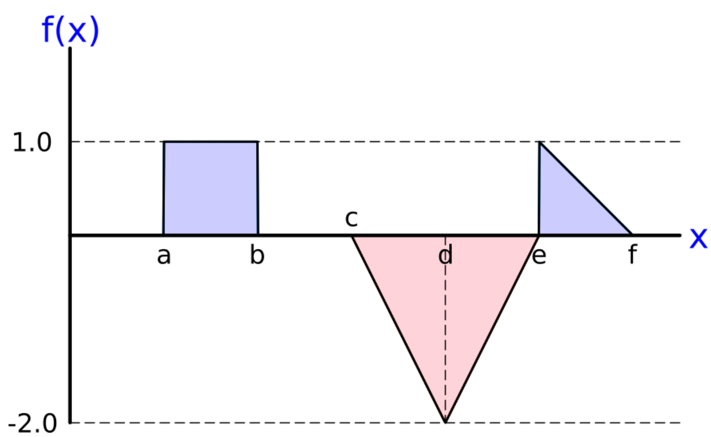
2. Answer any two of the following three conceptual questions:

   (a) Suppose $f(x)$ takes both positive and negative values on $[a, b]$. Explain why the integral $\int_a^b h(x)\, dx$ might be zero even if the function is nonzero.

   (b) In the textbook, we use the formula $p(t) = p(t_0) + \int_{t_0}^{t} v(\tau)\, d\tau$ to compute position from velocity. What is the role of the term $p(t_0)$?

   (c) As an engineer, why would you want to investigate the Riemann integral for discontinuous functions?

3. The problem uses the plot below, consisting of a rectangle and three right triangles.



   (a) Compute $\int_a^c f(x)\, dx$. You will obtain a formula in terms of a, b, c, d, e; see the hint.

   (b) Compute $\int_a^e f(x)\, dx$.

   (c) Compute $\int_d^b f(x)\, dx$ (note the order of the limits of integration).

(d) If $\int_a^c f(x)\,dx = 3$, $\int_c^e f(x)\,dx = -7$, and $\int_c^f f(x)\,dx = -5$, what is the value of $\int_a^f f(x)\,dx$?

4. Suppose that $a < b$ and that $f : [a, b] \to \mathbb{R}$ is monotonically **decreasing**. Following the notation in the textbook, give a formula for the UPPER Riemann sum, $\text{Area}_n^{\text{Up}}$ of $f(x)$.

5. (Conceptual) Suppose you are given a set of basis functions $\varphi_i(t)$, $1 \le i \le N$, each of which can be integrated exactly over any interval $[a, b]$. That is, for each basis function, you know:

$$\text{val}_i := \int_a^b \varphi_i(t)\,dt.$$

In addition, you are given a function $f(x)$ in the form of sampled data: a set of evenly spaced points $\{(x_k, f(x_k))\}_{k=1}^M$ over $[a, b]$. Explain how you can use linear regression to estimate

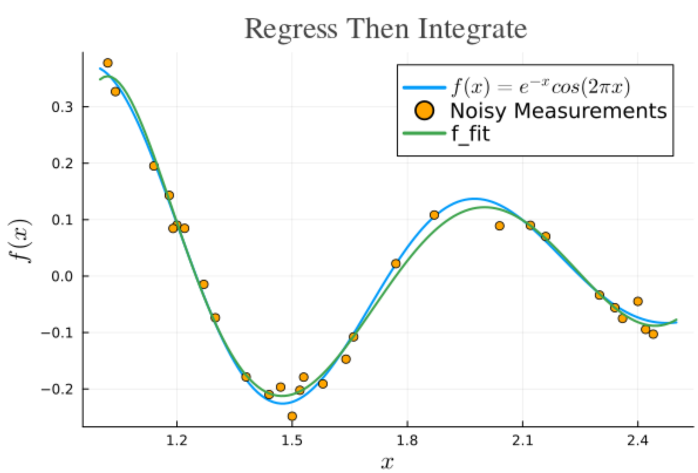
$$\int_a^b f(x)\,dx.$$

If possible, express your method as pseudocode.

6. A complicated function has been regressed against a standard monomial basis, resulting in the fit

$$\hat{f}(x) = -137.91 + 502.81\,x - 731.87\,x^2 + 546.15\,x^3 - 221.16\,x^4 + 46.23\,x^5 - 3.91\,x^6$$

Using the method of Problem 5, estimate $\int_1^{2.5} f(x)\,dx$



Hints

Prob. 1 Write approximately 15 or more words for each part of the question.

Prob. 2 Read the Chapter. Google if necessary. Think!

Prob. 3 The area of a triangle is one-half base times height. Applying this to the the function $f(x)$ we have

$$\text{Area} = \frac{(b-a)}{2} - 2\,\frac{(d-b)}{2} + (e-d).$$

Because the Riemann integral and the area are the same thing for triangles and rectangles, we have

$$\int_a^e f(x)\,dx = \frac{(b-a)}{2} - 2\,\frac{(d-b)}{2} + (e-d) = -\frac{a}{2} + \frac{3b}{2} - 2d + e$$

It was not necessary to simplify the answer, meaning, if you had left the answer as $\frac{(b-a)}{2} - 2\,\frac{(d-b)}{2} + (e-d)$, you'd still earn full credit.

Prob. 4 No hints provided.

Prob. 5 Think it through, calmly. The problems combines ROB 101, Linear Regression, with ROB 201.

Prob. 6 Remember that integrating the polynomial fit term by term is straightforward.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

Find the antiderivative of each term in $\hat{f}(x)$ and evaluate it from 1 to 2.5.

Solutions HW 03

Prob. 1 Based on your reading of the Chapter, summarize in your own words:

(a) **The purpose of Chapter 03:** is to deepen the understanding of the fundamental concepts of calculus by merging programming and numerical calculations with traditional mathematical analysis. It aims to provide a solid mathematical foundation by introducing critical concepts such as one-sided limits and continuity in detail, thereby enhancing comprehension of how functions behave near specific points. The chapter further explores the derivation of closed-form solutions for integrals of exponential functions, which paves the way for integrating trigonometric functions like sine and cosine. It addresses the concepts of bounded and unbounded functions, going beyond the simple identification of maximum and minimum values to embrace a broader understanding of function behavior. Through an exploration of piecewise continuous functions and conditions for Riemann integrability, the chapter extends the learner's insight into a wider class of functions. Overall, the chapter is designed to equip learners with both intuitive and formal understandings of these critical concepts, enabling them to apply this knowledge to complex calculus problems effectively.

(b) **Two things you may have found the most DIFFICULT**, among many other possibilities:

i. **The Epsilon-Delta Definition of Continuity at a Point**: This formal definition is a cornerstone of mathematical analysis, providing a rigorous way to understand continuity. It requires a precise handling of limits and an understanding of how function values can be controlled within arbitrary proximity to a point. The abstract nature of the epsilon-delta criterion, which involves conceptualizing distances in the input and output of functions to ensure they remain within specified bounds, can be difficult for students to grasp initially.

ii. **Acquiring Intuition for a Broader Class of Functions that are Riemann Integrable**: Understanding the conditions under which a function is Riemann integrable extends beyond the simple cases often introduced early in calculus courses. Students must come to terms with the idea that even functions with certain types of discontinuities or peculiar behaviors can still be integrated in the Riemann sense. This requires a deeper understanding of the concept of integration, the significance of partitions, and how the sum of areas under a curve can converge to a finite value, despite the presence of irregularities in the function.

Both topics demand a solid conceptual foundation and the ability to engage with abstract mathematical ideas, which can pose significant challenges to young learners.

Prob. 2 (a) If $f(x)$ takes both positive and negative values on $[a,b]$, then the integral $\int_a^b f(x)\,dx$ computes the **signed area**. Positive regions add to the total, and negative regions subtract. If these contributions cancel exactly, the integral can be zero even though $f(x) \neq 0$ on parts of the interval.

(b) The term $p(t_0)$ represents the **initial position** of the object at time t_0. The integral $\int_{t_0}^t v(\tau)\,d\tau$ computes the net change in position, or displacement, over time. So the full formula,

$$p(t) = p(t_0) + \int_{t_0}^{t} v(\tau)\,d\tau,$$

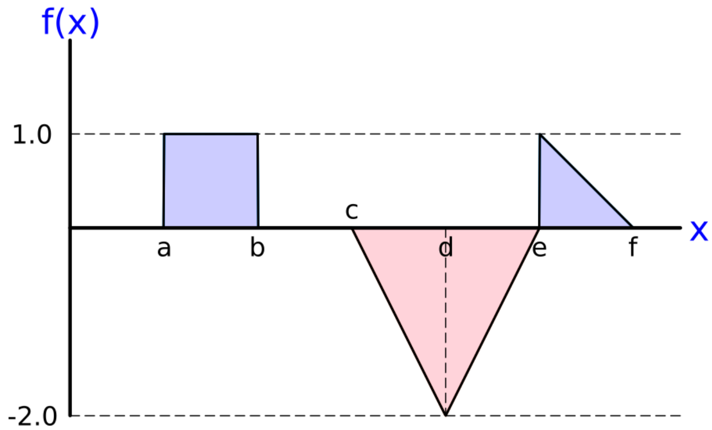
gives the position at time t by adding the initial position to the displacement.

Imagine you program a speed profile into your favorite robot, such as an MBot. In one case, you start the robot from your driveway. In another, you start it from your favorite restaurant. If the robot follows the same speed profile in both cases, will it end up in the same place? Definitely not — the initial position $p(t_0)$ matters. Yeah, it's quite important.

(c) Many engineering applications rely on Riemann integration to connect physical quantities — such as turning acceleration into velocity, and velocity into position. The Riemann integral provides a mathematically rigorous way to do this. But if it only worked for perfectly smooth, continuous functions, we'd be in a world of trouble.

In the real world, systems often experience jumps, shocks, or switching — for example, bang-bang control signals, relay circuits, or instantaneous braking. These introduce discontinuities into our models. As an engineer, we still want to compute things like total energy, net displacement, or accumulated input — even when the signal isn't "super nice".

Here's another angle: if the conditions for the Riemann integral to exist were too restrictive — applying only to very special or idealized functions — then many real-world problems would fall outside its scope. In such cases, we'd be forced to find alternative mathematical tools or avoid modeling specific systems altogether.



Prob. 3

(a) **Ans.**

$$\int_a^c f(x)\,dx = \underbrace{\int_a^b f(x)\,dx}_{\text{rectangle}} + \underbrace{\int_b^c f(x)\,dx}_{\text{zero}} = (b-a)\cdot 1 + 0 = b-a.$$

(b) **Ans.**

$$\int_a^e f(x)\,dx = \underbrace{\int_a^b f(x)\,dx}_{\text{rectangle}} + \underbrace{\int_b^c f(x)\,dx}_{\text{zero}} + \underbrace{\int_c^c ef(x)\,dx}_{\text{two similar triangles}} + \underbrace{\int_e^f f(x)\,dx}_{\text{triangle}} = (b-a)\cdot 1 + 0 - 2\cdot(e-c) + \frac{1}{2}(f-e) = b-a+2c-\frac{5}{2}e+\frac{1}{2}f$$

because

$$(b-a)\cdot 1 + 0 - 2\cdot(e-c) + \frac{1}{2}(f-e) = b-a-2e+2c+\frac{1}{2}f-\frac{1}{2}e = \boxed{b-a+2c-\frac{5}{2}e+\frac{1}{2}f}$$

(c) **Ans.**

$$\int_d^b f(x)\,dx = -\int_b^d f(x)\,dx = -\left(-2\cdot\frac{d-c}{2}\right) = d-c.$$

(d) **Ans.**

We are given:

$$\int_a^c f(x)\,dx = 3$$
$$\int_c^e f(x)\,dx = -7$$
$$\int_c^f f(x)\,dx = -5$$

Then:

$$\int_e^f f(x)\,dx = \int_c^f f(x)\,dx - \int_c^e f(x)\,dx = (-5) - (-7) = 2$$

Now substitute everything into the original expression:

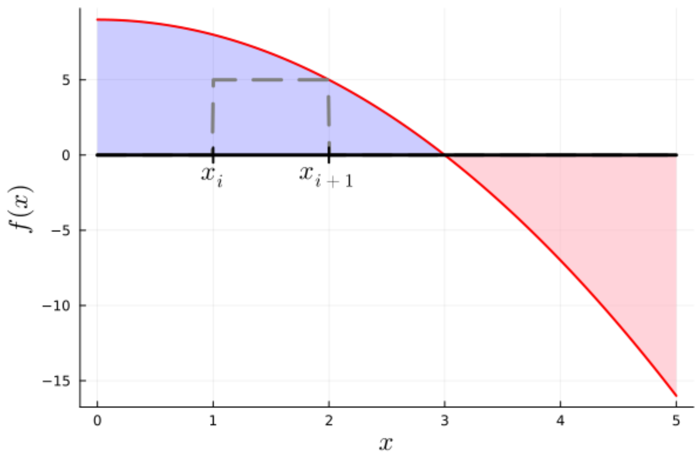
$$\int_a^f f(x)\,dx = 3 + (-7) + 2 = \boxed{-2}$$

Prob. 4 We divide the interval $[a,b]$ into $n > 1$ **evenly spaced subintervals**, $[x_i, x_{i+1}]$, where

$$\Delta x := \frac{b-a}{n}$$
$$x_i := a + (i-1)\cdot\Delta x,$$

so that

$$a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b \tag{1}$$
$$x_{i+1} - x_i = \Delta x,\ 1 \le i \le n. \tag{2}$$



As in the textbook, we define

$$\text{Area}_n^{\text{Up}} := \sum_{i=1}^{n} h_i^{\text{Up}} \cdot b_i,$$

where $b_i = \Delta x$ is the base of the rectangle and h_i^{Up} is the largest value of the function over the interval $[x_i, x_{i+1}]$. From the figure, we have to select

$$h_i^{\text{Up}} = f(x_i).$$

Therefore,

$$\boxed{\text{Area}_n^{\text{Up}} := \sum_{i=1}^{n} h_i^{\text{Up}} \cdot b_i = \sum_{i=1}^{n} f(x_i) \cdot \Delta x.} \tag{3}$$

Prob. 5 We approximate $f(x)$ by a linear combination of known basis functions:

$$f(x) \approx \sum_{i=1}^{N} \alpha_i \varphi_i(x),$$

where the coefficients α_i are to be determined. We construct a regression problem by evaluating each basis function at each sample point x_k, giving rise to a regressor matrix $\Phi \in \mathbb{R}^{M \times N}$, where

$$\Phi_{k,i} = \varphi_i(x_k).$$

Let $Y \in \mathbb{R}^M$ be the vector of observed data values, $Y_k = f(x_k)$. The least-squares regression problem is:

$$\Phi\alpha = Y,$$

and we solve for α by minimizing $\|\Phi\alpha - Y\|^2$. If Φ has full column rank, the solution is given by:

$$\alpha = (\Phi^\top \Phi)^{-1} \Phi^\top Y.$$

Once the coefficients α_i are computed, we estimate the integral of $f(x)$ over $[a, b]$ by integrating the basis approximation:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{N} \alpha_i \cdot \int_a^b \varphi_i(x)\,dx = \sum_{i=1}^{N} \alpha_i \cdot \text{val}_i.$$

Pseudocode:

- **Input:**
  - Basis functions $\varphi_1(t), \ldots, \varphi_N(t)$
  - Exact integrals $\text{val}_1, \ldots, \text{val}_N$ over $[a, b]$
  - Sampled data $(x_1, f(x_1)), \ldots, (x_M, f(x_M))$
- **Step 1: Construct the regressor matrix Φ of size $M \times N$**
  For $k = 1$ to M:
  For $i = 1$ to N:
  $\Phi[k, i] := \varphi_i(x_k)$
- **Step 2: Construct the data vector Y**
  Let $Y := [f(x_1), f(x_2), \ldots, f(x_M)]^\top$
- **Step 3: Solve the least-squares regression problem**
  Compute $\alpha := (\Phi^\top \Phi)^{-1} \Phi^\top Y$
- **Step 4: Estimate the integral**
  Compute $I_{\text{estimate}} := \sum_{i=1}^{N} \alpha_i \cdot \text{val}_i$
- **Output:** $\int_a^b f(x)\, dx \approx I_{\text{estimate}}$

Prob. 6 We assume the regression fit $\hat{f}(x)$ is a good approximation of $f(x)$, so we estimate the integral using

$$\int_1^{2.5} f(x)\, dx \approx \int_1^{2.5} \hat{f}(x)\, dx.$$

Each monomial has a known definite integral, so we compute:

$$\begin{aligned}\int_1^{2.5} \hat{f}(x)\, dx = -137.91 \int_1^{2.5} 1\, dx + 502.81 \int_1^{2.5} x\, dx - 731.87 \int_1^{2.5} x^2\, dx \\ + 546.15 \int_1^{2.5} x^3\, dx - 221.16 \int_1^{2.5} x^4\, dx + 46.23 \int_1^{2.5} x^5\, dx - 3.91 \int_1^{2.5} x^6\, dx.\end{aligned}$$

Evaluating each term:

$$\int_1^{2.5} x^n\, dx = \left. \frac{x^{n+1}}{n+1} \right|_1^{2.5}, \quad \text{for } n = 0, \ldots, 6.$$

Hence, compute and sum:

$$\int_1^{2.5} f(x)\, dx \approx \boxed{0.0076}.$$

Optional Code to Confirm the Answer:

```
# Problem 6 (continued)

using QuadGK

f(x) = cos(2pi * x) * exp(-x)

result, err = quadgk(f, 1, 2.5)

result2, err2 = quadgk(f_hat, 1, 2.5)

@show [result result2]
@show [err err2]
```

Output

```
[result result2] = [0.011116156866440963 0.0075940626487636925]
[err err2] = [2.2923270920349736e-11 9.488243524202744e-14]
1×2 Matrix{Float64}:
 2.29233e-11  9.48824e-14
```

```
# Problem 6 (continued)

mySum = 0.0
for k = 1:length(alpha)
    mySum = mySum + alpha[k]*(2.5^k - 1^k)/k
end
 mySum
```

Output

```
.00759406264842255
```