ROB 201 | 24 March 2025

Summary:

nonlinear (NL) term

$$m\dot{v} = -mg + \kappa v^2, \quad v(t_0) = v_0$$

Solution cannot be given with elementary functions

- Same is true for most NL ODEs
  - Numerical solutions are valuable, BUT
  - Also need qualitative insight for DESIGN

- System (or vector) of 1st order ODEs

$$\boxed{\dot{x} = f(x), \quad x(t_0) = x_0 \in \mathbb{R}^3}$$

- $\dddot{\theta} + a_2\ddot{\theta} + a_1\dot{\theta} + a_0\theta = b_0 V$

$$\left.\begin{aligned} x_1 &:= \theta \\ x_2 &:= \dot{\theta} \\ x_3 &:= \ddot{\theta} \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -a_0 x_1 - a_1 x_2 - a_2 x_3 + b_0 V \end{aligned}$$

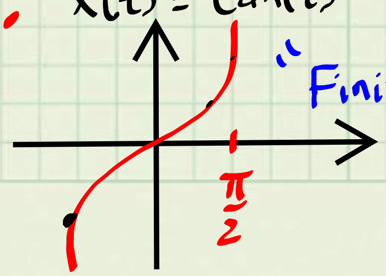
- $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \Gamma \qquad q \in \mathbb{R}^n$

$$\begin{aligned} x_1 &:= q & \dot{x}_1 &= x_2 \\ x_2 &:= \dot{q} & \dot{x}_2 &= D(x_1)\backslash\left[-C(x_1,x_2)\cdot x_2 - G(x_1) + \Gamma\right] \end{aligned}$$

$\boxed{\text{Rarely do this by hand!}}$ **TY!**

$\dot{x} = 1 + x^2, \quad x(0) = 0$

$\therefore \quad x(t) = \tan(t)$



"Finite Escape Time"

$\dot{x} = x^{2/3}, \quad x(0) = 0$

For all $c > 0$

$$\varphi_c(t) := \begin{cases} 0 & 0 \le t < c \\ \dfrac{(t-c)^3}{27} & t \ge c \end{cases}$$

"∞ of solutions"

Euler's Method: $\dot{x} = f(t, x)$, $x(t_0) = x_0$

$$\dot{x}(t) \approx \frac{x(t+\Delta t) - x(t)}{\Delta t} \qquad x(t+\Delta t) = x(t) + f(t, x(t)) \cdot \Delta t$$

Today

More on solutions of ODEs.

Exist? Unique?

__Example__ Verify that $x(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$ satisfies the ODE

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} =: x_0$$

Solution:

a) Check the I.C.

$$x(0) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_0 \checkmark$$

b) Check the ODE

LHS: $\frac{d}{dt}\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix}$

RHS: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix}$

LHS = RHS $\checkmark$

Could there be other solutions?

Verify that $\varphi : [0, \infty) \to \mathbb{R}^2$ by

$$\varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{(t+1)^2} \\ \frac{-2}{(t+1)^3} \end{bmatrix} \quad \text{is a sol.}$$

to $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 6(x_1)^2 \end{bmatrix}$, $\quad x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = x_0.$

Question: Could there be other solutions?

9.4.2 Solutions of First-order Vector ODEs

What is a Solution?

Definition 9.16. *A function,* $\varphi : [t_0, T) \to \mathbb{R}^n$, $T > t_0$, *is a* **solution** *to the Ordinary Differential Equation (ODE)*

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0, \tag{9.31}$$

if

(a) $\varphi : [t_0, T) \to \mathbb{R}^n$ *is continuous (meaning, each of its components is continuous);*

(b) $\varphi(t_0) = x_0$ *(satisfies the initial condition); and*

(c) for all $t \in (t_0, T)$, *the derivative* $\dot{\varphi}(t) := \frac{d}{dt}\varphi(t)$ *exists and* $\dot{\varphi}(t) - f(\varphi(t)) = 0_{n\times 1}$ *(satisfies the differential equation).*

Note: It is very common to drop the time variable and write the ODE as $\dot{x} = f(x)$, $x(t_0) = x_0$. The "dot" notation for the derivative indicates the dependence on time.

Drop the "t" for compactness of notation.

Prop (Existence & Uniqueness)

Consider a vector ODE $\dot{x} = f(x)$, $x(t_0) = x_0 \in \mathbb{R}^n$.

(a) [Local Existence & Uniqueness] If for some $r > 0$, the Jacobian $J_f(x) := \frac{\partial f}{\partial x}(x)$ exists and is continuous for all

$\{x \mid \|x - x_0\| < r\}$ (x's near the I.C.), then there exists $\delta > 0$ such that the ODE has a unique solution on $[t_0, t_0 + \delta)$.

"local" small interval of time beyond t_0

Recall: $\dot{x} = 1 + x^2$, $x(0) = x_0 = 0 \in \mathbb{R} \Rightarrow x(t) = \tan(t)$ "blows up" at $t = \frac{\pi}{2}$. [solution ceases to exist at $\frac{\pi}{2}$].

$J_f(x) = \frac{\partial}{\partial x}[1 + x^2] = 2x$ is continuous everywhere, not just our initial condition of $x_0 = 0$.

$\therefore$ There exists $\delta > 0$ and $\varphi : [t_0, t_0 + \delta) \to \mathbb{R}$ that satisfies the ODE. Moreover, it is unique.

Thm does not provide a value for $\delta > 0$. It just assures existence.

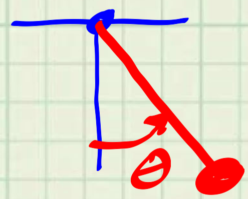
b) [Global] If there exists $0 \le L < \infty$ such that, for all $x \in \mathbb{R}^n$,

$$\boxed{\max_{i,j} \left| \frac{\partial f_i}{\partial x_j}(x) \right| \le L}$$

then the ODE has a unique solution on $[t_0, \infty)$, for all $x_0 \in \mathbb{R}^n$.

<u>Example:</u>

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ -\sin(x_1) \end{bmatrix}}_{f(x)}$$



$x_1 = \theta$,
$x_2 = \dot{\theta}$

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2}{\partial x_2}(x) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & 0 \end{bmatrix}$$

$$\left| \frac{\partial f_i}{\partial x_j}(x) \right| \le \underset{\substack{\uparrow \\ L}}{1} \qquad 1 \le i, j \le 2$$

$\therefore$ Global Existence & Uniqueness for all $x_0 \in \mathbb{R}^2$.

Cool: Know a lot about the solution even though we cannot write it down in terms of elementary functions.

Example $\dot{x} = \underbrace{x^{2/3}}_{f(x)}$ $\quad x(0) = 0$

$$J_f(x) = \frac{\partial}{\partial x}\left[x^{2/3}\right] = \frac{2}{3} x^{-1/3} = \frac{2}{3} \frac{1}{\sqrt[3]{x}}$$

$\lim_{x \to 0^+} J_f(x) = \infty \Rightarrow$ Prop. does not apply.

Proof is based on Contraction Mapping Thm, see ROB 501, and also in EECS 562, NL Control.

Example (Linear Vector ODEs)

$$\dot{x} = \underbrace{Ax}_{f(x)} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

Claim $\dfrac{\partial f_i}{\partial x_j}(x) = a_{ij}$

Proof: $f_i(x) = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n$

$$\boxed{\frac{\partial f_i(x)}{\partial x_j} = a_{ij}}$$

$$\max_{i,j} \left| \frac{\partial f_i(x)}{\partial x_j} \right| = \max_{i,j} |a_{ij}| < \infty$$

$\therefore$ Solutions globally exist and are unique. $\square$

What is the solution to $\dot{x} = Ax$,

$x(t_0) = x_0$???

Case 1: $A = a = 1\times 1$

$$\dot{x} = ax \quad , \quad x(t_0) = x_0 \in \mathbb{R}$$

Claim: $\varphi(t) = e^{a(t-t_0)} x_0$ is the unique solution

Pf. $\varphi(t_0) = e^{a(t_0-t_0)} x_0 = e^{0} \cdot x_0 = x_0$ ✓ I.C.

$$\frac{d}{dt}\varphi(t) = \frac{d}{dt}\left[e^{a(t-t_0)} x_0 \right] = a \underbrace{e^{a(t-t_0)} x_0}_{\varphi(t)} = a\varphi(t)$$

✓ ODE

$\square$

Case 2: $\dot{x} = Ax \quad , \quad x(t_0) = x_0 \in \mathbb{R}^n$

Wouldn't it be wild if the answere were $\varphi(t) = e^{A(t-t_0)} \cdot x_0$?

Taylor/Maclaurin Series for e^x and e^{at}

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{where} \quad x^0 := 1, \quad 0! := 1$$

$$e^{at} = 1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \cdots + \frac{(at)^k}{k!} + \cdots$$

$$= \sum_{k=0}^{\infty} a^k \frac{t^k}{k!}$$

$$e^{a\underline{(t-t_0)}} = \sum_{k=0}^{\infty} a^k \frac{(t-t_0)^k}{k!}$$

Taylor Series

$\dot{x} = ax$

Notation: $A = n \times n$ constant matrix,

$A^0 : I_n = n \times n$ identity matrix

Def. For A an $n \times n$ matrix,

$$e^{At} := I_n + A \cdot \frac{t}{1!} + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \cdots + A^k \frac{t^k}{k!} + \cdots$$

$$= \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$$

Makes sense because for A square

$$A^k = \underbrace{A \cdots A}_{k\text{-times}} \qquad n \times n$$

We'll see how to compute this in Julia, shortly.

Side Computation

$$e^{At} = I_n + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$$

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\left[I_n\right] + A\frac{d}{dt}\left[t\right]\frac{1}{1!} + A^2\frac{d}{dt}\left[t^2\right]\frac{1}{2!}$$

$$+ A^3\frac{d}{dt}\left[t^3\right]\frac{1}{3!} + \cdots A^k\frac{d}{dt}\left[t^k\right]\frac{1}{k!} +$$

$$+ A^{k+1}\frac{d}{dt}\left[t^{k+1}\right]\frac{1}{(k+1)!} + \cdots$$

$$= O_n + A\frac{1}{1!} + A^2\frac{t}{1!} + A^3 t^2\frac{1}{2!} + \cdots$$

$$+ A^k t^{k-1}\frac{1}{(k-1)!} + A^{k+1} t^k \frac{1}{k!} + \cdots$$

$$= A\left[I_n + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots\right.$$

$$\left. + A^{k-1}\frac{t^{k-1}}{(k-1)!} + A^k\frac{t^k}{k!} + \cdots\right]$$

$$= A \cdot e^{At}$$

We really want to understand how to compute the matrix

exponential and understand
its qualitative properties!!!