Summary:

$$\lim_{x \to x_0^+} f(x) := \lim_{h \to 0^+} f(x_0+h) \quad \text{(right)}$$

$$\lim_{x \to x_0^-} f(x) := \lim_{h \to 0^-} f(x_0+h) \quad \text{(left)}$$

$$\lim_{x \to x_0} f(x) = L \iff \begin{cases} \lim_{x \to x_0^+} f(x) = L \\ \text{AND} \\ \lim_{x \to x_0^-} f(x) = L \end{cases} \quad \text{Double-sided Limit}$$

$$\boxed{\begin{aligned} &f \text{ is cont. at } x_0 \iff \lim_{x \to x_0} f(x) = f(x_0) \\ &\iff \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0) \end{aligned}}$$

Prop. If $\lim_{x \to x_0} g(x) = y_0$ AND $f(x)$ is cont. at y_0. THEN $\lim_{x \to x_0} f(g(x)) = f(y_0)$

that is, $\lim_{x \to x_0} f(g(x)) = f(\lim_{x \to x_0} g(x))$,

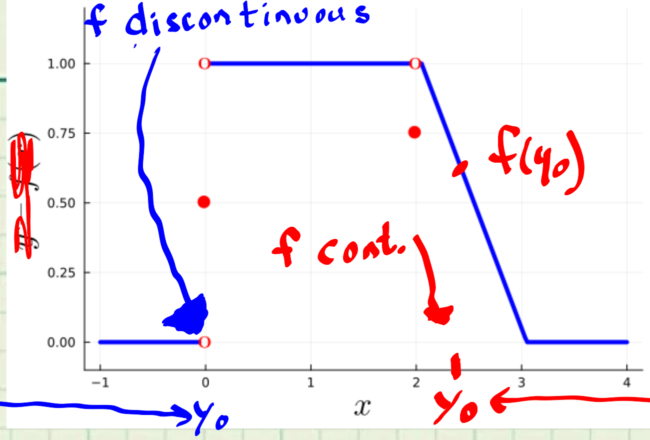
"the limit can be taken inside the function f."

$\lim_{x\to x_0} f(g(x)) = ?$

$y = g(x)$

$y_0 := \lim_{x\to x_0} g(x)$

fails



$\lim_{x\to x_0} f\circ g(x) = ?$

$y = g(x)$

$y_0 := \lim_{x\to x_0} g(x)$

works

Today: Start with the "challenge" problem from last lecture

Given $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n =: e$ Euler's (Bernoulli's) number, evaluate $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ for $x>0$.

fixed

Solution:

Step 1: Change of variable. Define

$$\frac{1}{m} := \frac{x}{n} \iff m = \frac{n}{x} \iff n = mx$$

Observe: $m\to\infty \iff n\to\infty$. Hence

$$\boxed{\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx}\\ &= \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^m\right]^x\end{aligned}}$$

Step 2: Identify f & g

$$g(m) = \left(1 + \frac{1}{m}\right)^m \xrightarrow[m \to \infty]{} e \quad \leftarrow y_0$$

$$f(y) = y^x$$ continuous for all $y > 0$ and hence cont. at e

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = f(y_0) = f(e) = e^x$$

Hard One for Home

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Hint: Do a change of variable

$u := e^h - 1$ so that $h = \ln(u+1)$

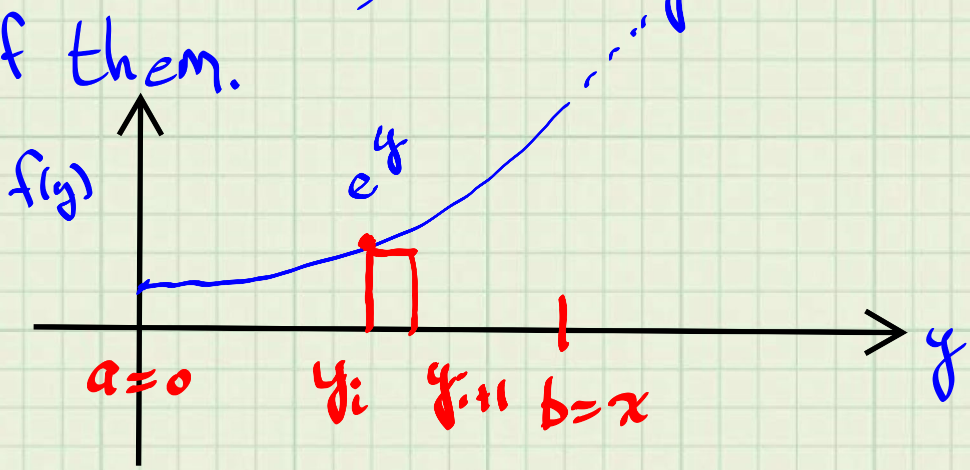
$h \to 0 \iff u \to 0$

Prop. $\displaystyle\int_0^x e^y \, dy = e^x - 1$ for $x > 0$

Pf.

e^y is cont. so the Riemann integral exists on $[0, x]$, x finite.

Both Riemann sums exist and are finite. So, we only compute one of them.



$\Delta y =$ "$\frac{b-a}{n}$" $= \frac{x}{n}$,

$y_i = (i-1)\cdot\Delta y$

$0 = y_1 < y_2 < \cdots < y_n = x$

$$\text{Area}_n^{\text{low}} := \sum_{i=1}^{n} f(y_i)\cdot\Delta y$$

$$= \sum_{i=1}^{n} e^{(i-1)\cdot\Delta y}\cdot\Delta y$$

$e^{\alpha x} = (e^x)^\alpha$

$$= \sum_{i=1}^{n} \underbrace{\left(e^{\Delta y}\right)}_{r}^{(i-1)}\cdot\Delta y$$

$$= \Delta y \sum_{i=1}^{n} \left(e^{\Delta y}\right)^{(i-1)}$$

$$= \frac{x}{n} \sum_{i=1}^{n} r^{i-1} \qquad r = e^{\Delta y}, \ \Delta y = \frac{x}{n}$$

$$= \frac{x}{n} \sum_{k=0}^{n-1} r^{k} \qquad k := i-1 \quad \text{change of index/variable}$$

$$= \frac{x}{n} \frac{1-r^n}{1-r} \qquad (\text{geometric sum})$$

$$r = e^{\Delta y} = e^{x/n}$$

$$r^n = \left(e^{x/n}\right)^n = e^x$$

$$\text{Area}_n^{\text{low}} = \frac{x}{n} \frac{1-e^x}{1-e^{x/n}} = (e^x - 1) \frac{x}{n} \frac{1}{e^{x/n}-1}$$

$$\int_0^x e^y \, dy = \lim_{n\to\infty} (e^x - 1) \frac{x}{n} \frac{1}{e^{x/n}-1}$$

$$= (e^x - 1) \underbrace{\lim_{h \to 0} \frac{h}{e^h - 1}}_{=1} \qquad *$$

$$= e^x - 1$$

$\blacksquare$

Prop. For $x > 0$ finite, $\alpha \in \mathbb{R}$, $\alpha \neq 0$

$$\int_0^x e^{\alpha y} \, dy = \frac{1}{\alpha} \left(e^{\alpha x} - 1\right)$$

Sketch the proof.

We are integrating from $y=0$ to $y=x$

$$\Delta y = \frac{x}{n} \qquad dy := \lim_{n\to\infty} \Delta y$$

Let's do the change of variable

$z := \alpha y$

limits of integration: $\begin{cases} y=0 \Rightarrow z=0 \\ y=x \Rightarrow z=\alpha x \end{cases}$

When integrating with respect to z,

$$\Delta z := \text{“}\frac{b-a}{n}\text{''} = \frac{\alpha x - 0}{n} = \alpha \frac{x}{n} = \alpha \Delta y$$

$$dz = \lim_{n\to\infty} \Delta z = \lim_{n\to\infty} \alpha \Delta y = \alpha \cdot dy$$

Hence, $\displaystyle\int_0^x e^{\alpha y}\,dy = \int_0^{\alpha x} e^z \left(\frac{1}{\alpha}\right) dz$

$$= \frac{1}{\alpha}\left(e^{\alpha x} - 1\right)$$

D ✓

Use Euler's Formula

$$e^{ix} = \cos(x) + i\sin(x)$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$i = \sqrt{-1}$

$\Updownarrow$

$-1 = i^2$

Yields integration formulas for sine & cosine.

Squeeze Theorem or Sandwich Thm

Suppose $f(x) \le g(x) \le h(x)$.

If $\lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} h(x)$

both exist, are finite, and equal to one another, then

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} f(x)$$

[same as $\lim\limits_{x \to x_0} h(x)$] $\square$

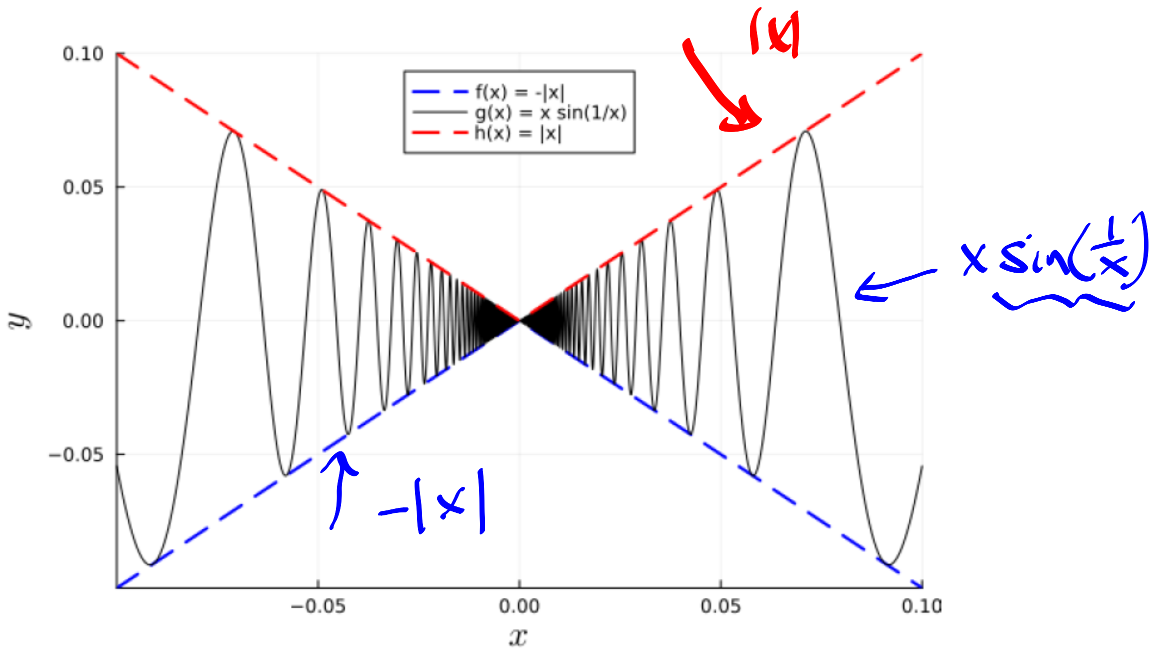
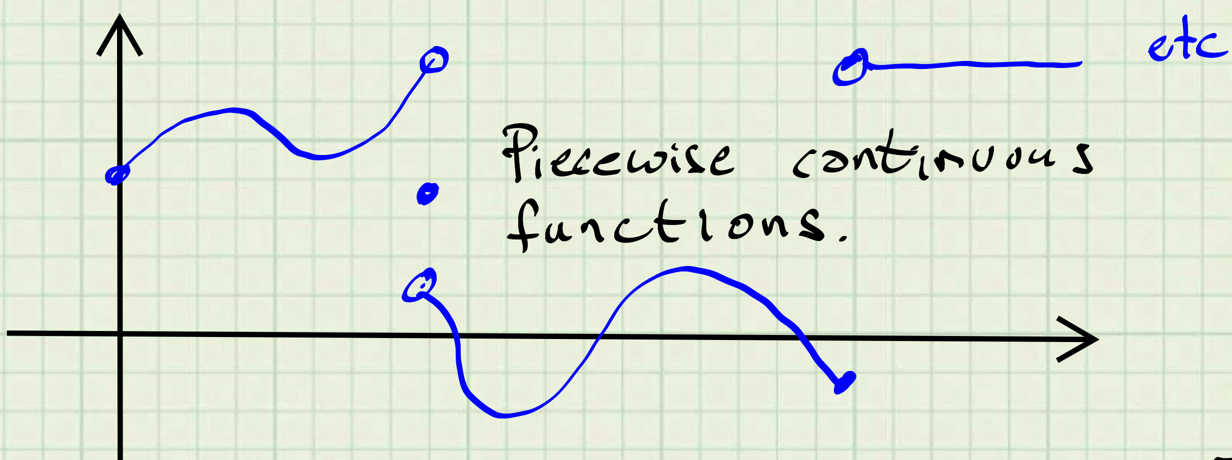


Figure 4.6: **(Squeeze or Sandwich Theorem:)** If a function is sandwiched between two functions that have a common limit, then the function itself also has the same limit.

Problem: In practice, it's hard to find good bounding functions, f(x) and h(x).

Numerical exploration in Julia HW4



etc.

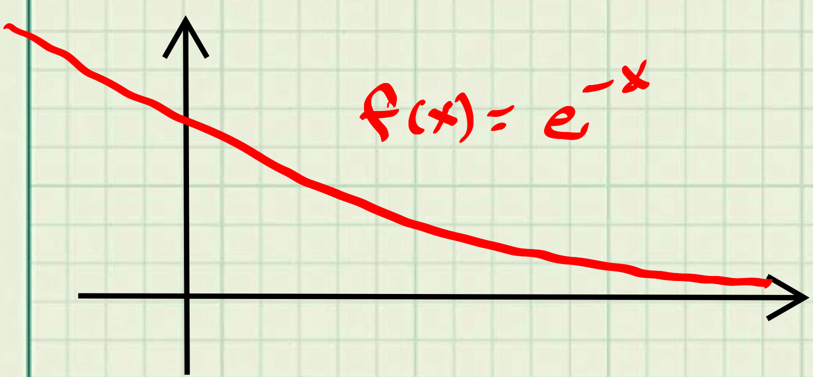
Piecewise continuous functions.

i) $[a, b] = [a, a_1) \cup [a_1, a_2) \cup \cdots \cup [a_{n-1}, b]$

ii) Function is cont. on (a_i, a_{i+1}), limits at endpoints exist and FINITE

"Largest" & "Smallest" Values

max = Not exist
min = 0.0



max = Not exist
min = Not exist

Same for sets $A \subset \mathbb{R}$

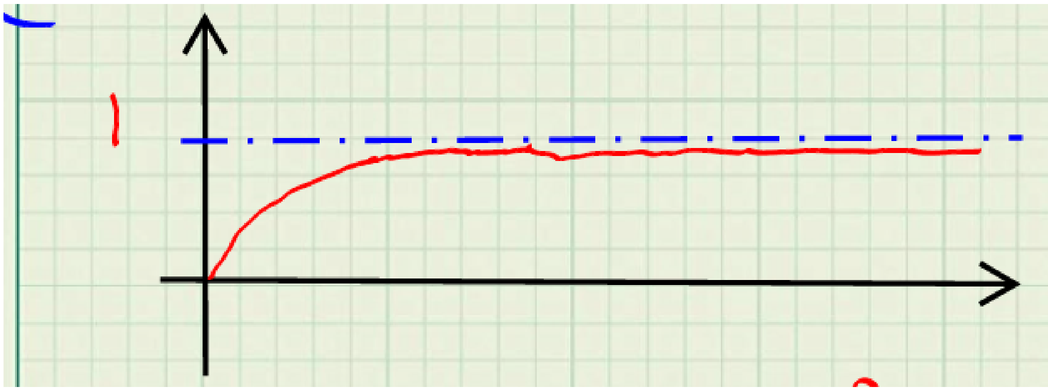
$$A = (1,2) := \{ x \in \mathbb{R} \mid 1 < x < 2 \}$$

$\max\{A\}$ = Not exist
$\min\{A\}$ = Not exist

Notions of greatest lower bound and least upper bound: infimum and supremum.

Maximum and Minimum vs Supremum and Infimum

Motivating Example: $f : [0, \infty) \to \mathbb{R}$ by $f(x) = 1 - e^{-x}$. We could also look at



- The function has no maximum value. The functions gets closer and closer to 1.0, but never attains it.
- 1.0 is the smallest number $y \in \mathbb{R}$ such that $f(x) \leq y$ for all x in the domain of the function.

Create a function where our notion of the Riemann integral breaks, but by using sup and inf, we could still compute tight lower and upper bounds via Archimedes' Approximation Principle.

Or just mention the connection.

Consider the set $A := [0, 1]$. Then zero is the smallest element in the set and we denote it by $\min\{A\} = 0.0$. Similarly, one is the largest element of the set and we denote it by $\max\{A\} = 1.0$.

Key Points: the maximum and minimum of a set must belong to the set.

Potential useful video Real Analysis 6 | Supremum and Infimum . Do not play to the end.

Def. or Vocabulary

- Sup = Supremum = Least Upper Bound. It's maybe easier to remember it is the TIGHTEST upper bound.
- Inf = Infimum = Greatest Lower Bound It's maybe easier to remember it is the TIGHTEST lower bound.

Why do we need them? Consider $A := (0, 1)$. It has neither a minimum (smallest) element nor a maximum (largest) element.

Proof

Indeed, let $0 < x < 1$ be arbitrary.

- Then, $x > \frac{x}{2} \in A$ and hence x is not the smallest element in A
- Moreover, x is not the largest element in the set. We define

$$y := \frac{x+1}{2}.$$

and show that $x < y < 1$.

Using $0 < x < 1$, we can

- check that $x < y$:

$$x = \frac{x+x}{2} < \frac{x+1}{2} =: y.$$

- check that $y < 1$:

$$y = \frac{x+1}{2} < \frac{1+1}{2} = 1.$$

Conclusion: Using $y = \frac{x+1}{2}$, we have $0 < x < y < 1$, showing that x is not the maximum in the interval $(0,1)$.

"Clearly," 1 is the least (aka, smallest, tightest) upper bound for the set A and similarly, 0 is the greatest (aka, largest, tightest) lower bound for the set A. We denote them by

- $\sup\{A\} = 1.0$
- $\inf\{A\} = 0.0$.

$A = (0, 1)$

Def. Suppose that $A \subset \mathbb{R}$. Then x^* is the **maximum** of A, denoted $x^* = \max\{A\}$, if

(a) $x^* \in A$ ✗

(b) $x^* \geq a$ for all $a \in A$

In words, it is the largest element **in** the set.

Def. Suppose that $A \subset \mathbb{R}$. Then x^* is the **supremum** of A, denoted $x^* = \sup\{A\}$, if

(a) $x^* \geq a$ for all $a \in A$, and,

(b) if $y \in \mathbb{R}$ satisfies $y \geq a$ for all $a \in A$, then $x^* \leq y$ (x^* must be the LEAST upper bound).

Note: By convention, $\sup\{[0,\infty)\} := \infty$ and $\sup\{\mathbb{N}\} := \infty$. More generally, the supremum of sets that are unbounded from the right is defined to be infinity.

Relationships between max and sup for a set $A \subset \mathbb{R}$:

- The supremum always exists, while a max may or may not exist.
- $x^* := \max\{A\}$ if, and only if, $x^* := \sup\{A\}$ **and** $x^* \in A$.

Similar statements apply to min and inf.

Supremum and Infimum Applied to Functions

Why do we need them? Because when we defined the Riemann integral, we used **max and min** of a function over an interval $[x_i, x_{i+1}]$ to define the lower sum and the upper sum. We've just understood that max and min do not always exist, while sup and inf do. That's why!

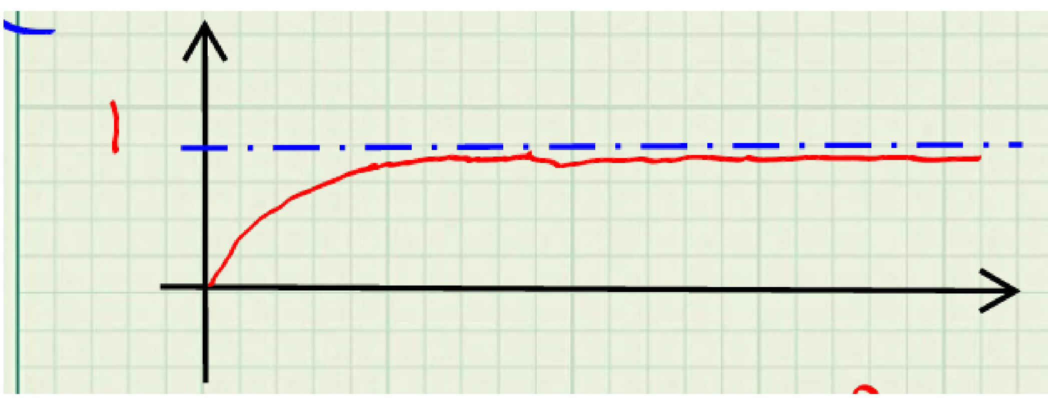
Def. Consider $f : A \to B$ with A and B subsets of the real numbers. Recall that A is the domain, B is the co-domain, and $C := f(A) := \{y = f(a) \mid a \in A\}$ is called the range.

$$f^* := \sup_{a \in A} f(a) = \sup\{\text{ range of } f\}$$

Similarly,

$$f_* := \inf_{a \in A} f(a) = \inf\{\text{ range of } f\}$$

Example: $f : [0, \infty) \to \mathbb{R}$ by $f(x) = 1 - e^{-x}$



$$\sup_{x \geq 0} f(x) = 1.0 \text{ while } \max_{x \geq 0} f(x) \quad \text{does not exist}$$

Example: $A := \{\underbrace{x \in \mathbb{R}}_{\text{"domain"}} \mid \underbrace{\frac{3x^2-1}{x^2+1} < 2}_{\text{"condition"}}\}$ **Note:** the bar symbol "$|$" means "such that". A is the set of all real numbers such that $\frac{3x^2-1}{x^2+1} < 2$.

Because $x^2 + 1 > 0$, we have that

$$\begin{aligned} \frac{3x^2 - 1}{x^2 + 1} < 2 &\iff (3x^2 - 1) \leq 2(x^2 + 1) \\ &\iff x^2 < 3 \\ &\iff |x| < \cancel{3}\ \sqrt{3} \end{aligned}$$

Hence, $A = \{x \in \mathbb{R} \mid -3 < x < 3\}$. Therefore,

- $\max\{A\}$ and $\min\{A\}$ do not exist.
- $\sup\{A\} = 3$ and $\inf\{A\} = -3$.

In Fall 2024, I included asymptotes and the notion of bounded functions. These can be covered in Chapter 07: Improper Integrals.

Next 2 pages have a Bonus Example

Bonus

Compute $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h}$

<u>Solution:</u> Two changes of variables

a) $u := e^h - 1 \iff u + 1 = e^h \iff h = \ln(1+u)$

Important: $h \to 0 \iff u \to 0$

$$\therefore \lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{u\to 0} \frac{u}{\ln(1+u)}$$

$$= \lim_{u\to 0} \frac{1}{\frac{1}{u}\ln(1+u)}$$

$$= \lim_{u\to 0} \frac{1}{\ln(1+u)^{\frac{1}{u}}}$$

because $y \cdot \ln(x) = \ln(x^y)$ (log power rule)

Do we recognize $(1+u)^{1/u}$? Kind of, sort of!

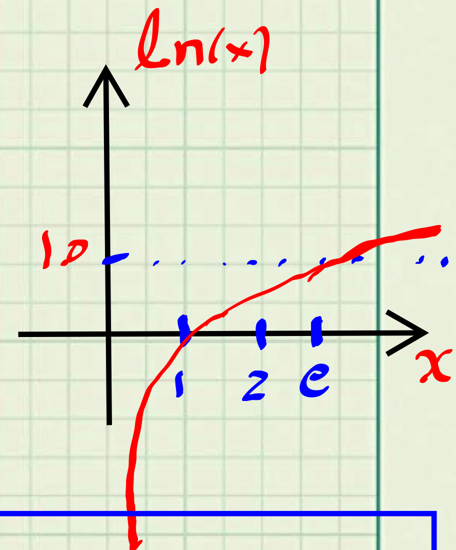
Let $\frac{1}{n} = u \iff n = \frac{1}{u}$

$u \to 0^+$ $\qquad n \to \infty$

$$\therefore \lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{u\to 0} \frac{1}{\ln(1+u)^u} = \lim_{n\to\infty} \frac{1}{\ln\left[\left(1+\frac{1}{n}\right)^n\right]}$$

$$\lim_{n\to\infty} \underbrace{\left(1+\frac{1}{n}\right)^n}_{g(n)} = e$$

$\underbrace{\ln(x)}_{f(x)}$ is cont. at $x_0 = e$



$$\boxed{\therefore \lim_{h\to 0}\frac{e^h-1}{h} = \frac{1}{\ln\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]} = \frac{1}{1} = 1.0}$$