ROB 201 09 April 2025

Summary

$\mathcal{L}\{y(t)\} = Y(s)$ $\mathcal{L}\{u(t)\} = U(s)$

$\mathcal{L}\{\dot{y}(t)\} = sY(s) - y(0^-)$ $\mathcal{L}\{\dot{u}(t)\} = sU(s)$

$\mathcal{L}\{\ddot{y}(t)\} = s\mathcal{L}\{\dot{y}(t)\} - \dot{y}(0^-) = s^2Y(s) - sy(0^-) - \dot{y}(0^-)$

** NEW **

$$\boxed{y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_0 y = b_0 u + b_1\dot{u} + \cdots + b_m u^{(m)}}$$

Called an input-output system because only inputs $u(t)$ and outputs $y(t)$ appear in the model

Let the I.C.'s be $y(0^-), \ldots, y^{(n-1)}(0^-)$, then

$Y(s) = Y_{forced}(s) + Y_{IC}(s)$ called the superposition ("superstition") principle

$Y_{forced}(s)$ = Solution with all I.C. = 0

$Y_{IC}(s)$ = Solution with input = 0

Let $r_1, r_2, \ldots, r_n$ be the roots of the denominator of $Y(s)$, assumed distinct. Then the PFE is

$Y(s) = \dfrac{k_1}{s-r_1} + \dfrac{k_2}{s-r_2} + \cdots + \dfrac{k_n}{s-r_n}$ where

$$k_j = \lim_{s \to r_j} Y(s)(s-r_j)$$ and if $r_{j+1} = r_j^*$ complex conjugate ($r_j = a_j + i\,w_j \Rightarrow r_j^* = a_j - i\,w_j$) then

$$k_{j+1} = \lim_{s \to r_j^*} Y(s)(s-r_j^*) = k_j^*$$

COMPLEX ROOT

$$e^{r_j t} = e^{a_j t} \cdot e^{i w_j t} = e^{a_j t}\left[\cos(w_j t) + i \sin(w_j t)\right]$$

$$\frac{1}{s-r_j} \longleftrightarrow e^{r_j t} u_s(t) \xrightarrow[t \to \infty]{} 0 \Leftrightarrow \operatorname{Re}\{r_j\} < 0$$

(associated with "stability")

Left off: Laplace applied to a state-variable model

$$\dot{x}_1 = -5x_1 + x_2 + 7u \qquad x_1(0^-) = 2$$

$$\dot{x}_2 = -6x_1 + u \qquad x_2(0^-) = 13$$

$$y = x_1 \qquad u(t) = e^{-t} u_s(t)$$

$$sX_1(s) - x_1(0^-) = -5X_1(s) + X_2(s) + 7U(s)$$

$$sX_2(s) - x_2(0^-) = -6X_1(s) + U(s)$$

$$Y(s) = X_1(s)$$

Solve linear equations to obtain $Y(s)$ in terms of $U(s)$, $x_1(0^-)$, $x_2(0^-)$

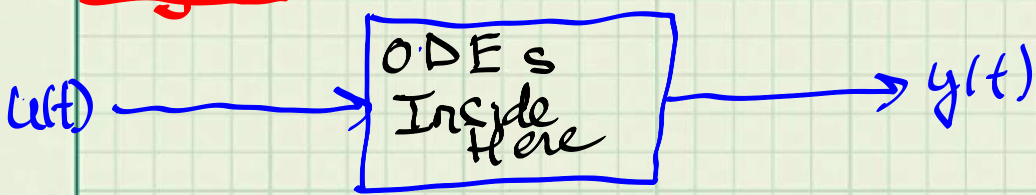
[We'll instead use vectors & matrices]

Today

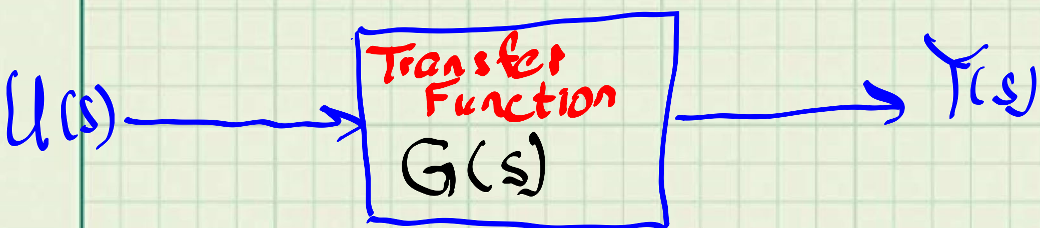
$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned} \quad \longleftrightarrow \quad Y(s) = c(sI-A)^{-1} b\, U(s) + c(sI-A)^{-1} x_0$$

- Transfer Functions
- Poles & Zero
- BIBO Stability

Systems



time domain



- Transfer function maps $U(s)$ [input in the Laplace domain] to $Y(s)$ [output in the Laplace domain]

Def. The transfer function of the input-output model

$$(*) \quad y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_0 u(t) + b_1 \dot{u}(t) + \cdots + b_m u^{(m)}(t)$$

(should be t's everywhere or nowhere)

is the rational function

$$G(s) := \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Obtained by taking the Laplace transform of $(*)$ with all I.C. = 0.

<u>Note:</u> $Y(s) = G(s) \cdot U(s)$

multiplication of "rational" functions

<u>Example</u> Compute the transfer function for $\ddot{y} + 7\dot{y} - 6y = 3u - 4\dot{u}$

$$G(s) = \frac{3-4s}{s^2+7s-6} = \frac{-4s+3}{s^2+7s-6}$$

$G(s) = \frac{Y(s)}{U(s)}$ or $\frac{U(s)}{Y(s)}$?

$$(s^2+7s-6)Y(s) = (3-4s)U(s)$$

$$Y(s) = \boxed{\frac{(-4s+3)}{s^2+7s-6}}\, U(s)$$

$G(s)$

$$\therefore \frac{Y(s)}{U(s)} = \frac{-4s+3}{s^2+7s-6} =: G(s)$$

Def. The transfer function of

$$\dot{x} = Ax + bu \quad , \; u \in \mathbb{R} \quad , \; x \in \mathbb{R}^n$$
$$y = cx \quad , \; y \in \mathbb{R}$$

is the Laplace transform, $Y(s)$, divided by the Laplace transform, $U(s)$, when all I.C. are zero $\left(x(0^-) = 0_{n\times 1} \right)$.

$$Y(s) = \underbrace{c(sI-A)^{-1}b}_{G(s)}\, U(s)$$

$$\boxed{\therefore \frac{Y(s)}{G(s)} = c(sI-A)^{-1}b}$$

Cannot compute by inspection. Use software.

- For now, invite you to read about
  1) Segway model
  2) Linear Approximations of NL ODEs

Poles-Zeros-BIBO Stability

Def. For a rational transfer function $G(s) = \frac{N(s)}{D(s)}$ with all common factors REMOVED, the

Poles of $G(s)$ = roots of $D(s)$

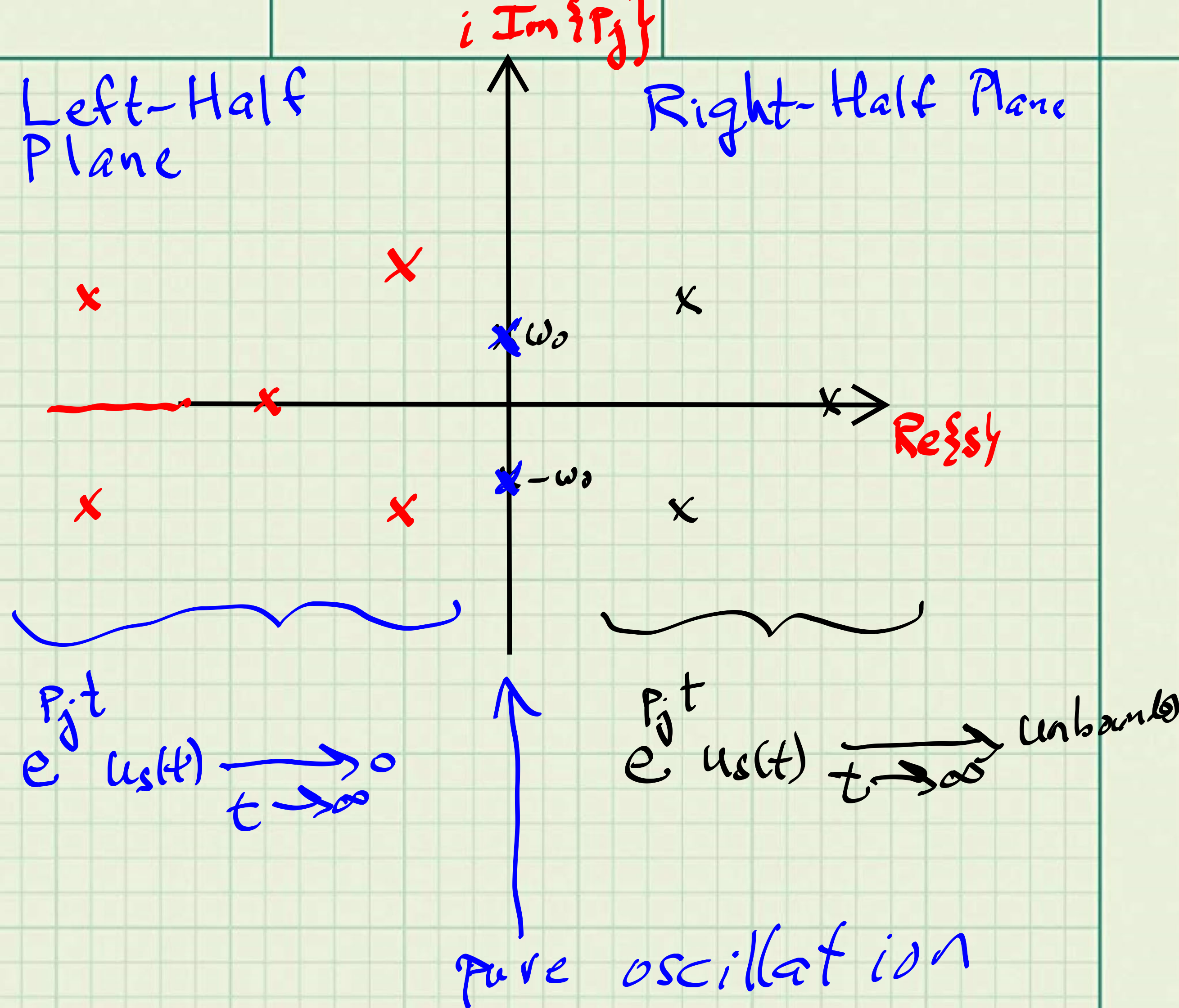
Zeros of $G(s)$ = roots of $N(s)$

$$Y(s) = G(s) \cdot U(s) \quad , \quad U(s) = \frac{1}{s} \longleftrightarrow u(t) = u_s(t)$$

$$Y(s) = \frac{k_0}{s} + \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n} \qquad \text{PFE}$$

as long as the poles are distinct. Note that $p_1, p_2, \ldots, p_n$ are the poles of $G(s)$

$$y(t) = \underbrace{k_0 \, u_s(t)}_{\text{Bounded}} + \underbrace{\sum_{j=1}^{n} k_j \, e^{p_j t} \, u_s(t)}_{\xrightarrow{t \to \infty} 0 \quad \mathrm{Re}\{p_j\} < 0}$$



[redacted]

Def. A system $\xrightarrow{u(t)} \boxed{\text{ODE}} \longrightarrow y(t)$ is BIBO STABLE (Bounded-Input Bounded-Output) if bounded inputs always produce bounded outputs.

Prop. Suppose the system is defined by a transfer function $G(s) = \frac{N(s)}{D(s)}$ where $\deg(N(s)) \leq \deg(D(s))$. Then, it is

BIBO STABLE $\Longleftrightarrow$ all poles of $G(s)$ have negative real parts.

Examples

$G_a(s) = \frac{s+2}{s^2-1} = \frac{s+2}{(s+1)(s-1)}$

Zeros -2

Poles ± 1

BIBO NO, because of a pole with a positive real part, $+1$

$G_b(s) = \frac{s+1}{s^2-1} = \frac{\cancel{s+1}}{\cancel{(s+1)}(s-1)}$

Zeros None

Poles $+1$

BIBO No, due to pole at $+1$

$G_c(s) = \frac{2s-4}{s^2+2s+2} = \frac{2(s-2)}{(s+1+i)(s+1-i)}$

Zeros $+2$

Poles $-1 \pm i$

BIBO YES

$\mathrm{Re}\{-1 \pm i\} = -1 < 0$

$$G(s) = \frac{1}{s^2+1}$$

$$\begin{cases} \text{Zeros} & \text{None} \\ \text{Poles} & \pm i \\ \text{BIBO} & \text{No? Do we believe it?} \end{cases}$$

$$u(t) = \underbrace{\sin(t)\, u_s(t)}_{\text{Bounded}} \longleftrightarrow \frac{s}{s^2+1} = U(s)$$

$$Y(s) = G(s) \cdot U(s) = \frac{s}{(s^2+1)^2} \underset{\text{Laplace Transforms}}{\overset{\text{Table}}{\longleftrightarrow}}$$

$$\longrightarrow y(t) = \underbrace{t \sin(t)\, u_s(t)}_{\text{unbounded.}}$$

"Resonance" yields unbounded behavior

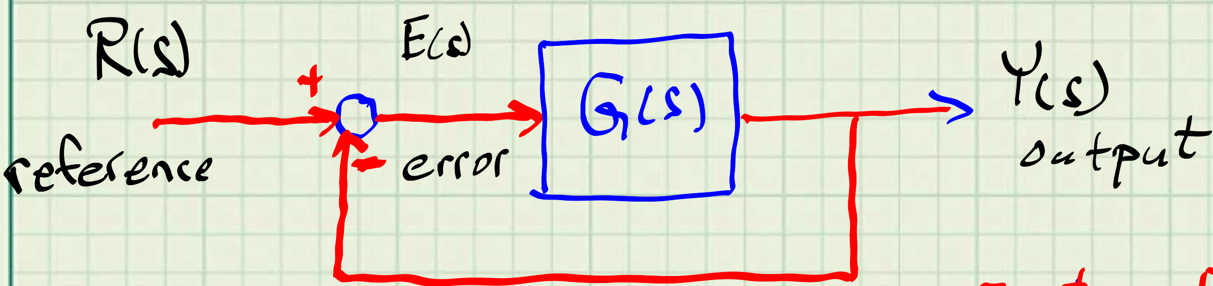
Poles in the Left Half Plane

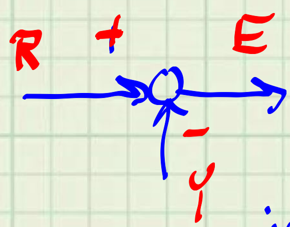
$$\Updownarrow$$

$$\mathrm{Re}\{p_j\} < 0 \quad 1 \le j \le n$$

Start Feedback Control

Def.: Block Diagram of a Unity Feedback System





Summing junction

$$E(s) = R(s) - Y(s)$$

$$Y(s) = G(s)\,E(s)$$

$$Y(s) = G(s)\left[R(s) - Y(s)\right]$$

$$Y(s) = G(s)R(s) - G(s)Y(s)$$

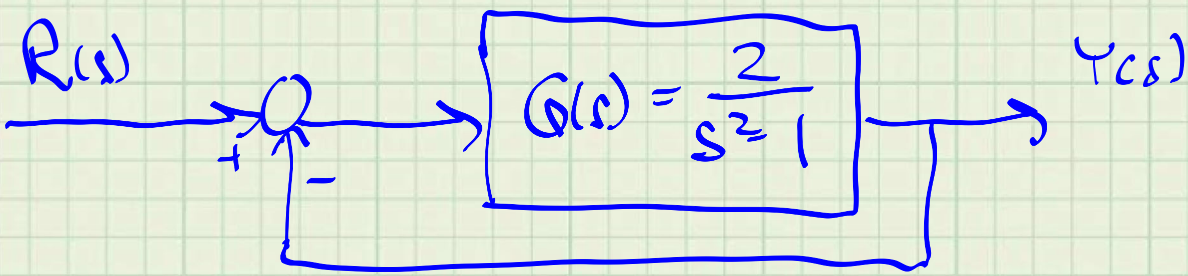
$$Y(s) + G(s)Y(s) = G(s)R(s)$$

$$(1 + G(s))Y(s) = G(s)R(s)$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}}$$

closed-loop transfer function

Aside: $$\frac{\frac{3}{4}}{1+\frac{3}{4}}\cdot\frac{4}{4} = \frac{3}{4+3} = \frac{3}{7}$$



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{2}{s^2-1}}{1+\frac{2}{s^2-1}} = \frac{2}{s^2-1+2}$$

$$= \frac{2}{s^2+1}$$

$$G_2(s) = \frac{s+2}{s^2-1}$$ in the unity fdbk loop

$$\frac{Y(s)}{R(s)} = \frac{\frac{s+2}{s^2-1}}{1+\frac{s+2}{s^2-1}}\cdot\frac{s^2-1}{s^2-1} = \frac{s+2}{s^2-1+s+2} = \frac{s+2}{s^2+s+1}$$

$$G_2(s) = \frac{s+2}{s^2-1}\quad\begin{cases}\text{poles: } \pm 1\\ \text{BIBO: NO}\end{cases}$$

open-loop TF unstable

$$\frac{Y(s)}{R(s)} = \frac{s+2}{s^2+s+1} \quad \begin{cases} \underline{\text{Poles:}} & -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \\ \text{BIBO} & \text{Yes} \end{cases}$$

Closed-loop TF
BIBO Stable

Fact 10.33: Simplified Stability Test

For real numbers a_1 and a_0, the roots of the quadratic equation

$$s^2 + a_1 s + a_0 = 0 \tag{10.49}$$

have negative real parts if, and only if, $a_1 > 0$ and $a_0 > 0$.

Note: To apply the above result to $\bar{a}_2 s^2 + \bar{a}_1 s + \bar{a}_0 = 0$, with $\bar{a}_2 \neq 0$, you simply divide through by $\bar{a}_2$, yielding $a_1 = \frac{\bar{a}_1}{\bar{a}_2}$ and $a_0 = \frac{\bar{a}_0}{\bar{a}_2}$.

For a cubic equation, the result is, for real numbers a_2, a_1, and a_0, the roots of the equation

$$s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

$$(s+r_1)(s+r_2) = s^2 + (r_1+r_2)s + r_1 r_2$$

$$p_1 = -r_1, \quad p_2 = -r_2$$

$$\underbrace{(s+r)^2+\omega^2}_{-r \pm i\omega} = s^2 + 2rs + r^2 + \omega^2$$

See textbook for cubic case