ROB 201 29 Jan 2025

Summary: Definite integrals

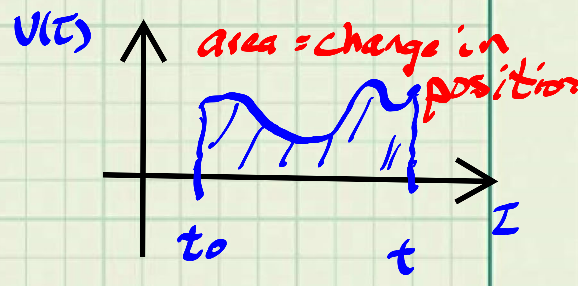
- $f : [a,b] \to \mathbb{R}$ cont. $\Rightarrow \int_a^b f(x)\,dx$ exists
- $a < c < b \Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^b [\alpha_1 f_1(x) + \cdots + \alpha_m f_m(x)]\,dx = \alpha_1 \int_a^b f_1(x)\,dx + \cdots + \alpha_m \int_a^b f_m(x)\,dx$
- Indefinite integral $a \leq x \leq b$

$$g(x) := \int_a^x f(y)\,dy$$

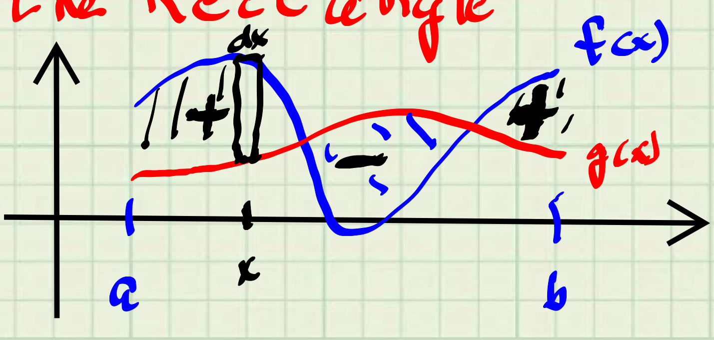
- $\int_a^x y^k\,dy = \left.\frac{y^{k+1}}{k+1}\right|_a^x = \frac{x^{k+1}}{k+1} - \frac{a^{k+1}}{k+1}$

"Evaluated from"

- $p(t) = p(t_0) + \int_{t_0}^{t} v(\tau)\,d\tau$



- Area between two functions: Power of the Rectangle



dA = differential area

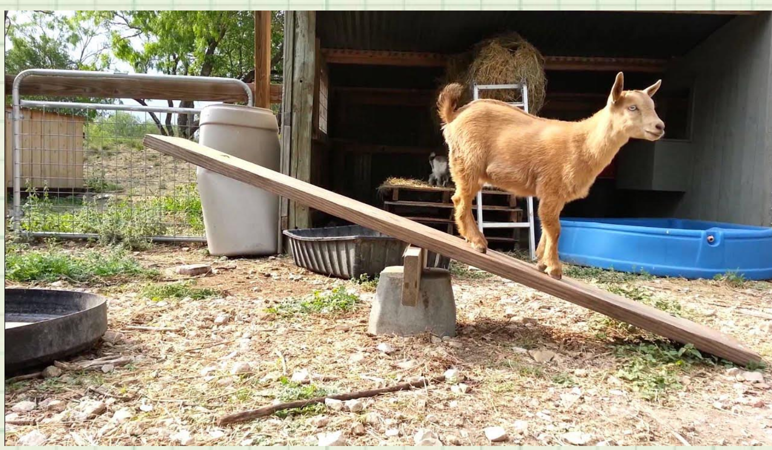
$$= \underbrace{[f(x) - g(x)]}_{\text{height}}\ \underbrace{dx}_{\text{width}}$$

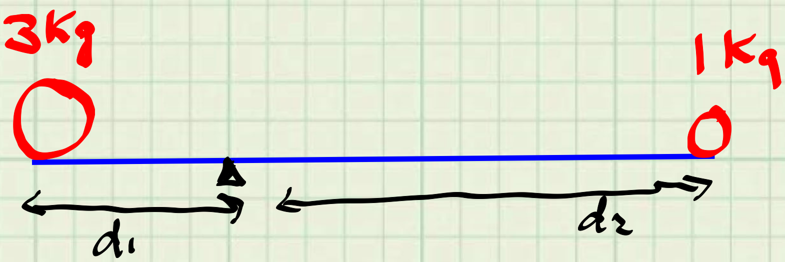
Total Area: $A := \int_a^b dA = \int_a^b [f(x) - g(x)]\, dx$

Three Perspectives on Integrals

- Area under a curve
- Sum of infinitesimal quantities
- Mapping between functions such as velocity to change in position.

Today: Video 1 | Video 2 | Video 3





$$3 * d_1 = 1 \cdot d_2$$

$$\sum_{i=1}^{n} (x_i - x_c) \cdot m_i = 0$$

unknown

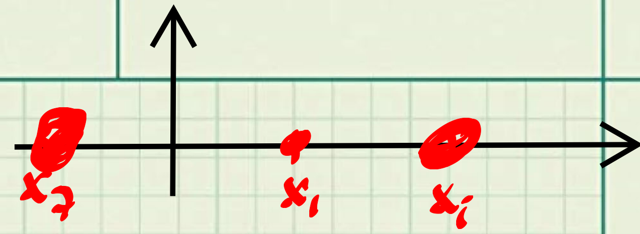
position of m_i

Sum of the moments $= 0$

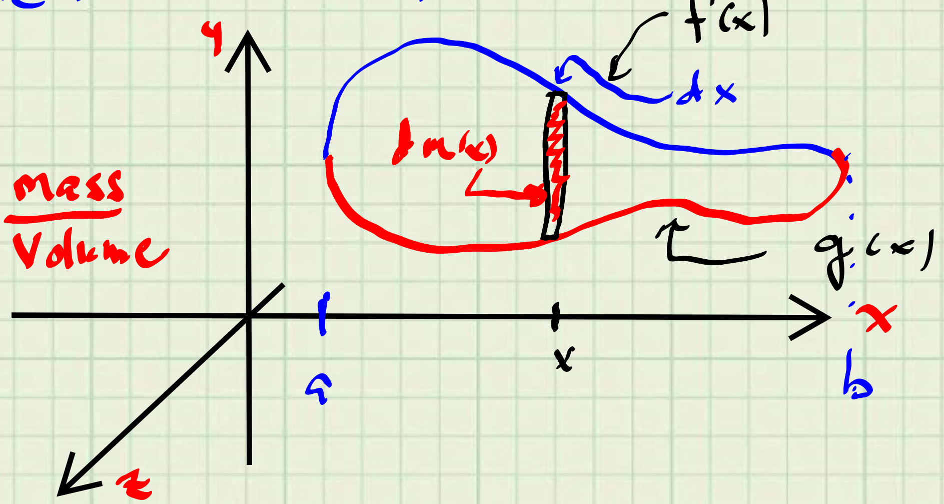
x-center of mass

com

$$x_c = \frac{\sum_{i=1}^{n} x_i \cdot m_i}{\underbrace{\sum_{i=1}^{n} m_i}_{\text{Total Mass}}}$$



x_c = Balance point



density = $\frac{\text{mass}}{\text{Volume}}$

Assumptions About the Link

- Constant thickness in the z-direction
- The density ρ is constant throughout the link.

From Calculus, $M_T := \int_a^b dm(x)$

$$dm(x) = \rho \cdot dV = = \rho \cdot h \cdot dA = \rho \cdot h \cdot [f(x) - g(x)] \cdot dx$$

$$M_T = \int_a^b dm(x) = \int_a^b \rho \cdot h\,[f(x) - g(x)] \cdot dx$$

CoM for our "distributed" mass.

link with

Physics

$$\int_a^b (x - x_c)\,dm(x) = 0$$

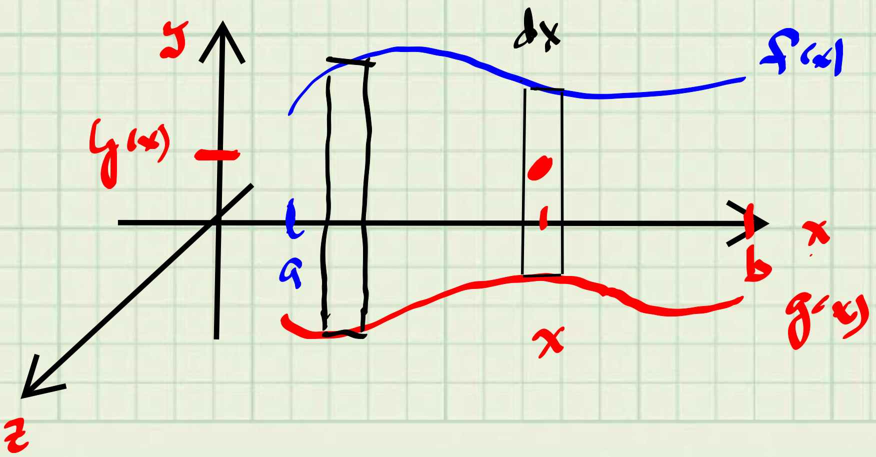
unknown

x_c = CoM in the x-direction

$$\underbrace{\int_a^b x\,dm(x)}_{M_y} = x_c \underbrace{\int_a^b dm(x)}_{m_T}$$

$$x_c = \frac{\int_a^b x\,\rho \cdot h\,[f(x)-g(x)]\,dx}{\int_a^b \rho \cdot h\,[f(x)-g(x)]\,dx} = \frac{M_y}{m_T}$$

Finding x_c, the y-position of the center of mass is more "interesting"



$$y(x) = \frac{f(x)+g(x)}{2}$$

$$M_x = \int_a^b y(x) \cdot dm(x)$$

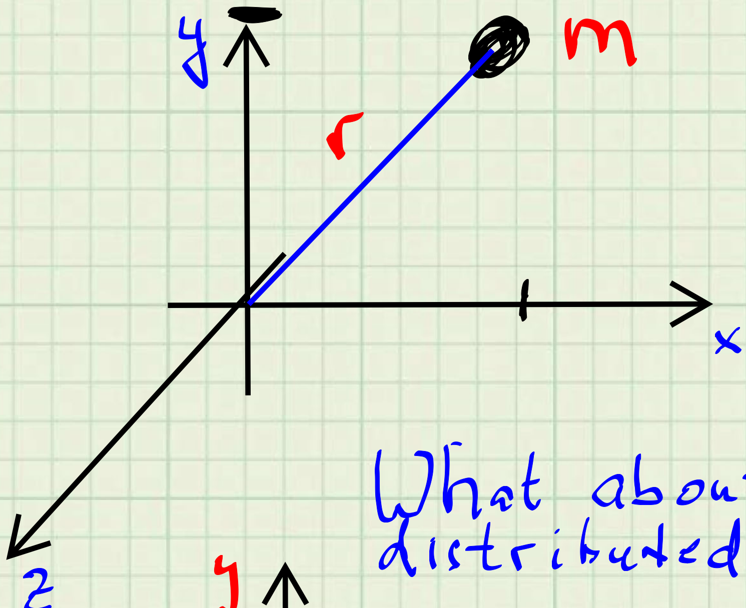
$$= \int_a^b \frac{[f(x)+g(x)]}{2} \rho \cdot h \cdot [f(x) - g(x)] dx$$

$$= \int_a^b \frac{1}{2} \cdot \rho \cdot h \left[f^2(x) - g^2(x) \right] dx$$

quadGK to the rescue!!!

$$y_c = \frac{M_x}{m_T}$$

Moment of Inertia About the z-Axis

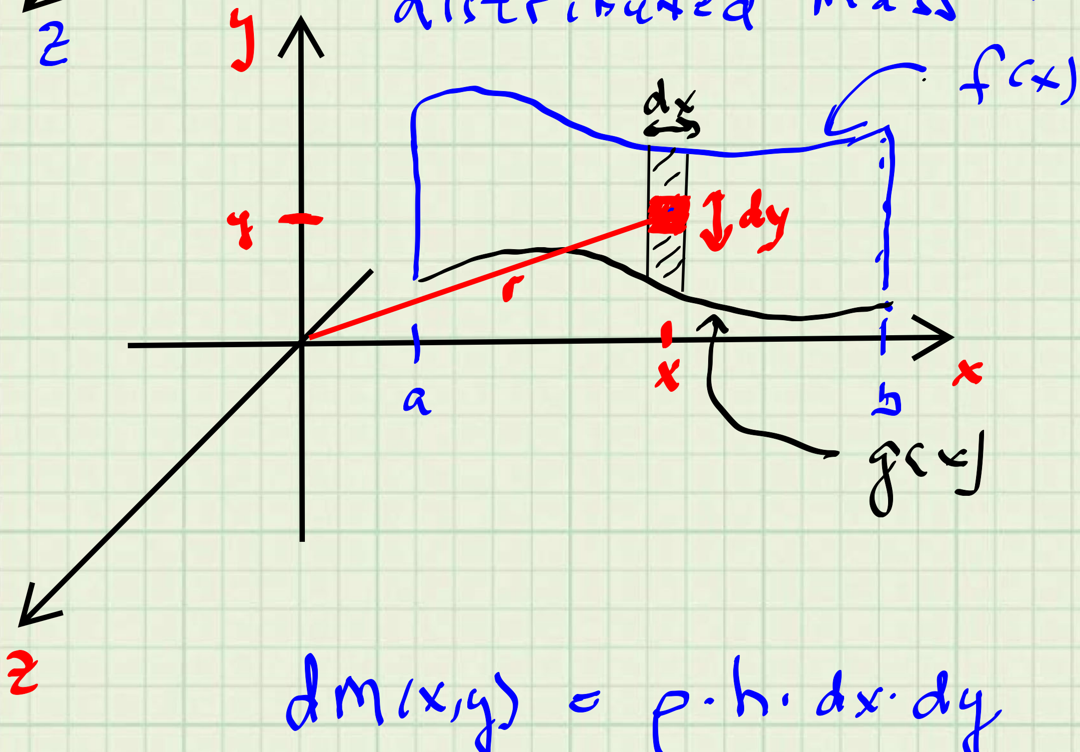


$I_z := r^2 \cdot m$ (Physics)

$r = \sqrt{x^2+y^2}$

$r^2 = x^2+y^2$

What about a body with distributed mass



$$dm(x,y) = \rho \cdot h \cdot dx \cdot dy$$

$$r^2 = x^2+y^2$$

(rectangle) $dI_z := \int_{y=g(x)}^{y=f(x)} r^2(x,y)\, dm(x,y)$

Integrate along - y

with x being constant.

$$= \int_{y=g(x)}^{y=f(x)} (x^2+y^2)\, \rho \cdot h \cdot dx\, dy$$

Now, evaluate this integral using the table below

$\therefore$
$$dI_z = \rho h dx \int_{g(x)}^{f(x)} (x^2+y^2)\, dy$$

$$= \rho \cdot h \cdot dx \left[x^2 \cdot y + \frac{y^3}{3} \right]_{g(x)}^{f(x)}$$

$$= \rho \cdot h \cdot dx \left[x^2 (f(x) - g(x)) + \frac{(f^3(x) - g^3(x))}{3} \right]$$

$$I_z = \int_a^b dI_z = \int_a^b \rho \cdot h \left[x^2 (f(x) - g(x)) + \frac{(f^3(x) - g^3(x))}{3} \right] dx$$

Need numerical methods to compute this.

$$\int_c^d 1\,dy = y\Big|_c^d = d-c$$

$$\int_c^d y\,dy = \frac{y^2}{2}\Big|_c^d = \frac{d^2-c^2}{2}$$

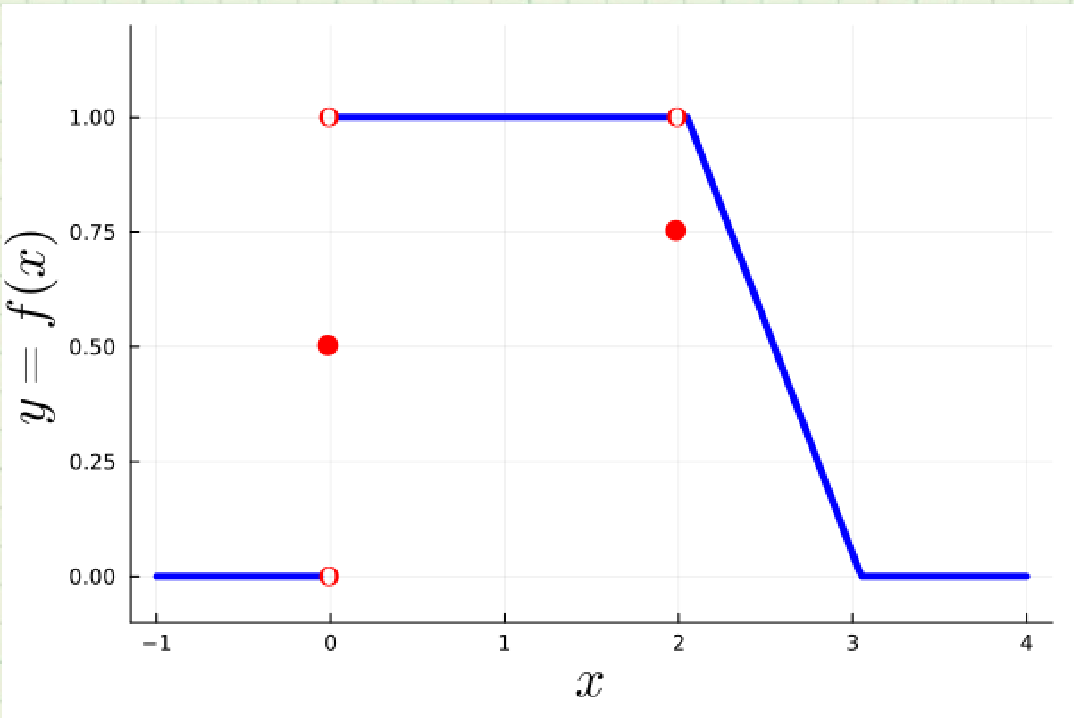
$$\int_c^d y^2\,dy = \frac{y^3}{3}\Big|_c^d = \frac{d^3-c^3}{3}$$

Chapter 4: Properties of Functions

Toughest Chapter in the book

Fun factor is low

Payoff is high



$\lim_{x \to 0^-} f(x) = 0$

approach from left of zero

$\lim_{x \to 0^+} f(x) = 1$

approach from the right of zero

Observations

- For $n > 0$, $\frac{1}{n} > 0$ and $\lim_{n \to \infty} \frac{1}{n} = 0$

Because $\frac{1}{n} > 0$ we are approaching zero from the right. The right side of zero is denoted 0^+

Similarly, for $n<0$, $\frac{1}{n}<0$ and $\lim_{n\to-\infty} \frac{1}{n}=0$, but this time we are approaching zero from the LEFT.

The left side of zero is denoted 0^-

Preliminary Def.

$$\lim_{x\to x_0^+} f(x) := \lim_{n\to\infty} f(x_0+\tfrac{1}{n}) \quad \left[\begin{array}{c}\text{limit}\\ \text{from}\\ \text{right}\end{array}\right]$$

$$\lim_{x\to x_0^-} f(x) := \lim_{n\to-\infty} f(x_0+\tfrac{1}{n}) \quad \left[\begin{array}{c}\text{limit from}\\ \text{the}\\ \text{left}\end{array}\right]$$

Examples

a) $f(y)=y^2$

$$\lim_{y\to 0^+} f(y) = 0$$

$$\lim_{y\to 0^-} f(y) = 0$$

b) $f(\theta) = \begin{cases} \text{undefined} & \theta = \frac{\pi}{2} \pm k\pi, \quad k = 0, 1, 2, \ldots \\ \tan(\theta) & \text{otherwise} \end{cases}$

$$\lim_{\theta \to \frac{\pi}{2}^+} f(\theta) = -\infty$$

$$\lim_{\theta \to \frac{\pi}{2}^-} f(\theta) = +\infty$$