ROB 201 - Calculus for the Modern Engineer
HW #6

Prof. Grizzle

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete.

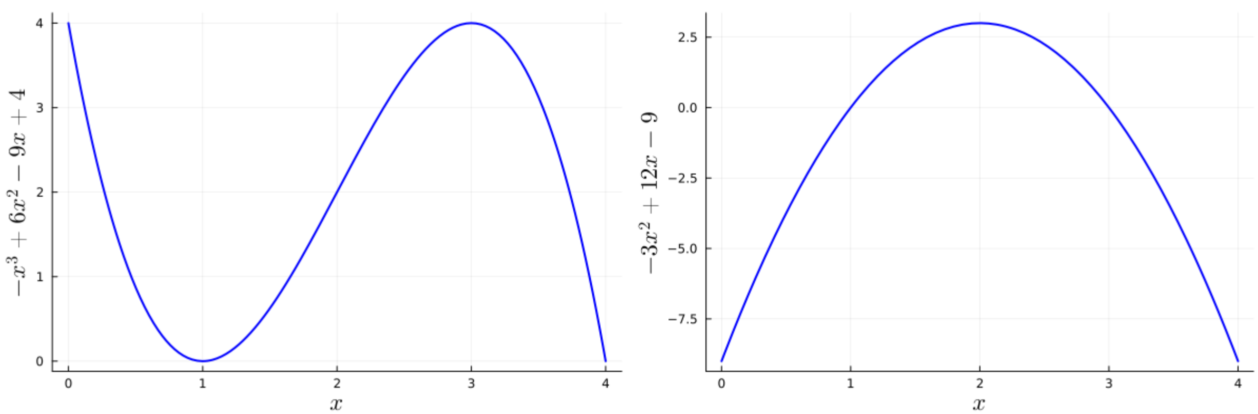
1. (a) Create a "Cheat Sheet" for Chapters 5 and 6 of the textbook. Here is an example from ROB 101.

   (b) Note any material where you found the explanation confusing or difficult to master.

2. Use the two graphs below if you find them helpful. The problem does not require the plots; they can provide a "sanity check" on your answers.

   (a) For the function $p(x) = -x^3 + 6x^2 - 9x + 4$, find all of the critical (aka, extreme or stationary) points and classify each critical point as a local minimum, local maximum, or inflection point.

   (b) Find the absolute minimum and absolute maximum of $p(x)$ on the closed interval $[0, 4]$. Do you have to consider more than just the critical points? Explain.



(a) (b)

3. Mostly thinking, very little computing:

   (a) Let $r(x) = b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$ be a **quartic polynomial**, with $b_4 \neq 0$. A local maximum or minimum occurs at a critical point where the derivative changes sign.

      (i) What is the maximum number of critical points $r(x)$ can have? Justify your answer.

      (ii) Can all of those critical points correspond to local extrema? Why or why not?

      (iii) If two of the critical points are local maxima, what can you conclude about the third (assuming all three are real and distinct)?

   (b) Let $h(x) := k \cdot f(x)$, where $k \in \mathbb{R}$ is a nonzero constant and f is differentiable. Explain why $h(x)$ has the same critical points as $f(x)$. What happens when $k = 0$?

   (c) Suppose $f : (a, b) \to \mathbb{R}$ is differentiable and satisfies $f(x) > 0$ for all $x \in (a, b)$. Compare the critical points of $\log f(x)$ with those of $f(x)$.

4. You are designing a building with a horizontal hallway of unknown width w and a vertical feeder hallway that joins it perpendicularly from above. The feeder hallway is 2 meters wide, and the main hallway runs along the x-axis. You need to determine the *minimum width* w of the main hallway that will allow an 8-meter-long rigid pole to be moved from the feeder hallway into the main hallway. The pole must remain flat on the floor at all times — it cannot tilt or bend.

   (a) Draw or interpret the geometry of the situation.

   (b) Use trigonometric relationships to express the width w of the main hallway in terms of an angle θ the pole makes with the horizontal hallway.

   (c) Use calculus to determine the minimum width w that will allow the turn to be completed.

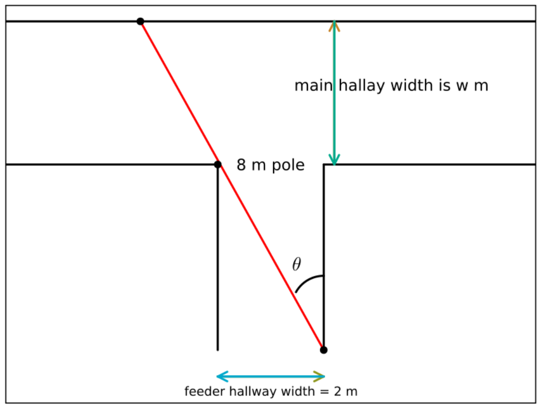


Figure 2: Critical configuration: the rigid 8 m pole touches the far wall of the vertical feeder hallway (width = 2 m) and the far wall of the horizontal main hallway (width = w). The angle θ is formed between the pole and the vertical wall of the feeder hallway. As you vary $0 \le \theta \le \pi/2$, you sweep out the tightest position possible for making the turn without lifting or bending the pole.

5. **Productivity Paradox — The Quest for Optimal Digital Well-being**: You're aiming for a productive day that balances deep work with just enough online time to stay connected. Your daily digital consumption is split between:

   - F: minutes of **focused work**
   - D: minutes of **distracting scrolling**

   Your Satisfaction Score is modeled by

   $$S(F, D) = 100 + 0.4D - 0.1F$$

   because scrolling provides a minor, fleeting boost, while focused work, though rewarding, is draining.

   Your Mental Acuity Level must be maintained at exactly 6.9 for peak performance. It's modeled as:

   $$A(F, D) = 5.0 + \frac{\sqrt{F}}{10} - \frac{D^2}{4000}$$

   Optimization Problem: Maximize

   $$S(F, D) = 100 + 0.4D - 0.1F$$

   subject to the constraint

   $$g(F, D) := 5.0 + \frac{\sqrt{F}}{10} - \frac{D^2}{4000} - 6.9 = 0.$$

   Use the method of Lagrange Multipliers to solve this by hand.

6. **Moment of Inertia of a Triangular Lamina** You are designing a rotating body with the shape of a right triangle. The upper boundary of the triangle is given by the function $f(x) = \frac{x}{4}$, and the lower boundary is $g(x) = 0$, forming a triangular lamina on the interval $x \in [0, L]$. All mass is distributed in the plane of the triangle.

   Assume the material has a uniform mass density of $\rho = 1$ kg/m^2, unless otherwise specified.

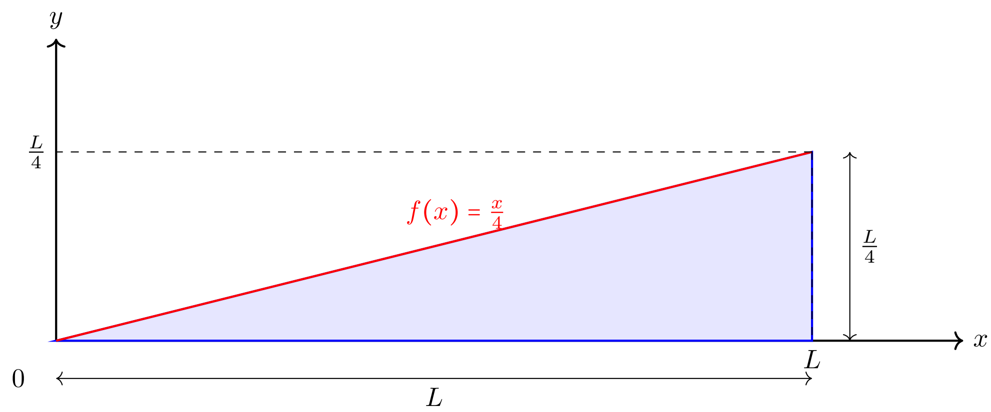


Figure 3: Triangular lamina bounded by $y = \frac{x}{4}$, $y = 0$, and $x \in [0, L]$.

(a) Compute the exact moment of inertia I_z of the lamina about the z-axis (which is perpendicular to the plane and passes through $x = 0$).

(b) Approximate the moment of inertia using the **Trapezoidal Rule** with 4 segments:

$$x_0 = 0, \quad x_1 = \frac{L}{4}, \quad x_2 = \frac{L}{2}, \quad x_3 = \frac{3L}{4}, \quad x_4 = L.$$

(c) Approximate the moment of inertia using a **left Riemann sum** with the same 4 segments.

(d) Repeat part (a), but assume the density varies linearly as:

$$\rho(x) = \frac{x}{L}.$$

(e) Repeat part (a), but assume the density varies linearly as:

$$\rho(x) = \frac{L - x}{L}.$$

Hints

Prob. 1 Nothing more to add.

Prob. 2 Nothing further offered.

Prob. 3 Remember that local extrema occur when the derivative changes sign. A polynomial of degree n has a derivative of degree $n - 1$, and the number of turning points is bounded by the number of real roots of the derivative.

Parts (a)(i)–(ii) are based on material covered in the textbook. However, part (a)(iii) goes beyond what is explicitly addressed. You may need to consult additional sources or challenge yourself with this thought experiment:

> Can you sketch a polynomial that has two local maxima without either a local minimum or an inflection point between them?

Use your knowledge of how derivatives behave, especially their continuity, and remember that local extrema occur where the derivative changes sign.

Prob. 4 To solve this problem, focus on the "critical" configuration where the rigid 8m pipe is just able to make the turn.

- What is the length of the pipe segment inside the vertical feeder hallway? You can form a right triangle using the width of the feeder hallway (2m) and the angle θ that the pipe makes with the vertical wall. Use trigonometry to express this length in terms of θ.
- Similarly, what is the length of the pipe segment inside the horizontal main hallway? Again, form a right triangle using the unknown hallway width w and the angle θ. Express this segment in terms of θ and w.
- The total length of the pipe is 8m. Use your two expressions above to write an equation involving w and θ. Then use calculus to find the minimum value of w.

Prob. 5 (a) Define the Lagrangian $L(F, D, \lambda) = FG(F, D) + \lambda \cdot g(F, D)$.

(b) Compute the three partial derivatives and set them equal to zero.

(c) Solve the first two equations for F and D in terms of λ.

(d) Substitute into the constraint to obtain an equation involving only λ.

(e) Solve this equation, then back-substitute to find F^*, D^*, and FG^*.

Prob. 6 (a) Use the inertia formula derived in the textbook. Do not use the double-integral formulation available in many places online. Stick to the textbook.

(b), (c) Review the appropriate formulas from the textbook!

(d), (e) Repeat the same setup from part (a), but now keep $\rho(x)$ inside the integral as a variable function.

Solutions HW 06

Prob. 1 (a) Included at the end of the solution set

(b) These will vary by person, but some of the more challenging topics may have been:

- Challenging topics from Chapter 5 were discussed in the solutions to HW #5.
- **Optimization Problems Involving Constraints:** Optimization is already a bit of a beast because it involves finding the best possible solution from a set of feasible solutions, which can be vast. When you throw constraints into the mix—like in problems that require using Lagrange multipliers—the complexity can ramp up quickly. The method of Lagrange multipliers, while powerful, introduces additional variables and conditions (the Lagrange multipliers themselves) that must satisfy both the original function and the constraints. It's like trying to balance on a tightrope while juggling; you have to keep everything in balance without losing sight of your goal (the optimal solution). Students often struggle with visualizing these multidimensional spaces and how changes in one variable affect others due to the constraints.
- **Using Gradient Descent for Local Minima:** Gradient descent is a fantastic tool for finding local minima, especially in machine learning and data science. However, it can be a tough concept to grasp because it involves both calculus and a bit of algorithmic thinking. The method uses derivatives to guide the search for minima, but the iterative approach and dependence on parameters like the learning rate can make the process seem opaque. There's also the challenge of understanding its limitations, such as susceptibility to getting stuck in local minima rather than finding the global minimum. This concept often requires a shift from purely analytical solving methods to more experimental, trial-and-error approaches, which can be a significant adjustment for students.
- **Deriving and Applying Lagrange's Equations in Dynamics:** When we move into dynamics and start using Lagrange's equations, we're not just dealing with derivatives anymore; we're applying them to model the motion of systems governed by complex forces. This involves a deep dive into kinetic and potential energy, and how these energies interact within a system. For many students, this is where calculus stops being just about numbers and starts requiring a more conceptual, holistic understanding of physical systems. The abstract nature of potential energy surfaces and the idea of converting these concepts into mathematical equations can be daunting. Moreover, the algebraic complexity of setting up and solving Lagrange's equations can be a hurdle, requiring careful consideration of the physical setup and the corresponding mathematical model.

Prob. 2 (a) The critical points of the function $p(x) = -x^3 + 6x^2 - 9x + 4$ are found by setting the first derivative $p'(x) = -3x^2 + 12x - 9$ to zero. Using the quadratic formula, we get:

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(-3)(-9)}}{2(-3)} = \frac{-12 \pm \sqrt{144 - 108}}{-6} = \frac{-12 \pm \sqrt{36}}{-6} = \frac{-12 \pm 6}{-6}$$

So the critical points are $\boxed{x = 1}$ and $\boxed{x = 3}$.

To classify each critical point, we examine the second derivative $p''(x) = -6x + 12$:

- At $x = 1$, $p''(1) = 6 > 0$, indicating a **local minimum**.
- At $x = 3$, $p''(3) = -6 < 0$, indicating a **local maximum**.

(b) To find the absolute minimum and maximum values of $p(x)$ on $[0, 4]$, we evaluate $p(x)$ at the critical points within this interval and at the endpoints. The critical points are $x = 1$ and $x = 3$, and the endpoints are $x = 0$ and $x = 4$.

Values of Local Min and Max:

- Local minimum at $x = 1$: $p(1) = -1 + 6 - 9 + 4 = 0$.
- Local maximum at $x = 3$: $p(3) = -27 + 54 - 27 + 4 = 4$.

Values at the Endpoints:

- $p(0) = 4$
- $p(4) = -64 + 96 - 36 + 4 = 0$

Global Extrema: Based on the above analysis, and supported by the graph,

- The absolute maximum value of $p(x)$ on $[0, 4]$ is $p(0) = p(3) = 4$.
- The absolute minimum value of $p(x)$ on $[0, 4]$ is $p(1) = p(4) = 0$.

Yes, we must consider more than just the critical points when finding the absolute minimum and maximum values on a closed interval. The endpoints of the interval must also be evaluated because the absolute extrema of a continuous function on a closed interval can occur at critical points within the interval or at the interval's endpoints.

Prob. 3 Let's put our thinking cap on:

(a) **Ans.**
(i) The critical points of $r(x)$ occur where $r'(x) = 0$. Since $r(x)$ is a degree-4 polynomial, $r'(x)$ is a cubic:

$$r'(x) = 4b_4x^3 + 3b_3x^2 + 2b_2x + b_1.$$

A cubic polynomial can have at most three real roots. Therefore, $r(x)$ can have at most three critical points.

(ii) Yes — it is possible for all three critical points to correspond to local extrema. Since each critical point is a point where the derivative is zero, and the sign of the derivative can change across each point, the function can have up to three local extrema: for example, local max–local min–local max, or the reverse.

Whether this actually happens depends on the specific shape of the polynomial (i.e., the signs and relative sizes of the coefficients), but it is certainly allowed by the theory. So a quartic can have three local extrema, all of which are genuine maxima or minima.

(iii) Suppose all three critical points are real and distinct, and two of them are local maxima. Because $r'(x)$ is continuous, the Intermediate Value Theorem implies that between the two local maxima, the derivative must become positive again to form the second peak. This would require the derivative to increase between the two maxima, which can only happen if it passes through a local minimum in between.

Therefore, the third critical point must be a local minimum. Local extrema must alternate in type — e.g., max–min–max or min–max–min.

(b) **Ans.** The derivative of $h(x) = k \cdot f(x)$ is $h'(x) = k \cdot f'(x)$. If $k \neq 0$, then $h'(x) = 0$ exactly when $f'(x) = 0$. Thus, $h(x)$ and $f(x)$ have the same critical points.

If $k = 0$, then $h(x)$ is constant and its derivative is zero everywhere. In that case, every point is a critical point — but this is a degenerate situation and doesn't reflect any variation in the function.

(c) **Ans.** Since $f(x) > 0$, the function $\log f(x)$ is well-defined and differentiable. By the chain rule,

$$(\log f(x))' = \frac{f'(x)}{f(x)}.$$

This derivative is zero exactly when $f'(x) = 0$. Hence, the critical points of $\log f(x)$ are the same as those of $f(x)$.
Note: Since the logarithm is strictly increasing on $(0, \infty)$, the shape of the graph (i.e., whether a critical point is a local max or min) is preserved between $f(x)$ and $\log f(x)$.

Prob. 4 We are asked to find the minimum hallway width w that allows a rigid 8-meter pipe to turn the corner between two perpendicular hallways. In the "critical" configuration, the pipe is just barely able to clear the corner while remaining in contact with

- the far wall of the vertical feeder hallway (width 2 m), and - the far wall of the main hallway (width w).

Let θ be the angle between the pipe and the vertical wall of the feeder hallway (as shown in the diagram).

- The segment of the pipe in the **feeder hallway** spans a horizontal distance of 2 m. With angle θ, its length is

$$\frac{2}{\cos\theta}.$$

- The segment in the **main hallway** spans a vertical distance of w. With the same angle θ, its length is

$$\frac{w}{\sin\theta}.$$

- Since the total pipe length is 8 m, we have

$$\frac{2}{\cos\theta} + \frac{w}{\sin\theta} = 8.$$

- Solving for w:

$$\frac{w}{\sin\theta} = 8 - \frac{2}{\cos\theta} \implies w(\theta) = \Big(8 - 2\sec\theta\Big)\sin\theta = 8\sin\theta - 2\tan\theta.$$

Our goal is to minimize

$$w(\theta) = 8\sin\theta - 2\tan\theta, \quad 0 < \theta < \frac{\pi}{2}.$$

Differentiate:

$$w'(\theta) = 8\cos\theta - 2\sec^2\theta.$$

Set $w'(\theta) = 0$:

$$8\cos\theta = 2\sec^2\theta \implies 8\cos^3\theta = 2 \implies \cos\theta = \left(\tfrac{1}{4}\right)^{1/3}.$$

Hence

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\tfrac{1}{4}\right)^{2/3}}, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Numerically,

$$\cos\theta \approx 0.630, \quad \sin\theta \approx 0.777, \quad \tan\theta \approx 1.233,$$

so

$$w_{\min} = 8\sin\theta - 2\tan\theta \approx 8(0.777) - 2(1.233) \approx 3.75 \text{ m}.$$

$$\boxed{w_{\min} \approx 3.75 \text{ meters.}}$$

Prob. 5 We define the Lagrangian function $\mathcal{L}$:

$$\mathcal{L}(F, D, \lambda) = 100 + 0.4D - 0.1F - \lambda\left(\frac{\sqrt{F}}{10} - \frac{D^2}{4000} - 1.9\right)$$

Next, we find the partial derivatives and set them to zero:

$$\frac{\partial\mathcal{L}}{\partial F} = -0.1 - \lambda\left(\frac{1}{20\sqrt{F}}\right) = 0 \quad \Rightarrow \quad \lambda = -2\sqrt{F}$$

$$\frac{\partial\mathcal{L}}{\partial D} = 0.4 - \lambda\left(-\frac{2D}{4000}\right) = 0.4 + \lambda\left(\frac{D}{2000}\right) = 0 \quad \Rightarrow \quad \lambda = -\frac{800}{D}$$

By equating the two expressions for λ, we establish a relationship between F and D:

$$-2\sqrt{F} = -\frac{800}{D} \quad \Rightarrow \quad \sqrt{F} = \frac{400}{D}$$

Now, we substitute this relationship into the constraint equation:

$$\frac{1}{10}\left(\frac{400}{D}\right) - \frac{D^2}{4000} - 1.9 = 0$$

$$\frac{40}{D} - \frac{D^2}{4000} - 1.9 = 0$$

To solve for D, we multiply the entire equation by $4000D$ to eliminate the denominators:

$$4000D\left(\frac{40}{D}\right) - 4000D\left(\frac{D^2}{4000}\right) - 4000D(1.9) = 0$$

$$160000 - D^3 - 7600D = 0$$

Rearranging gives us a cubic equation in D:

$$D^3 + 7600D - 160000 = 0$$

By inspection or numerical methods, we find that $D = 20$ is the only positive real solution.

Now we back-substitute to find the optimal value for F:

$$\sqrt{F^*} = \frac{400}{20} = 20 \quad \Rightarrow \quad F^* = 400$$

Finally, we compute the maximum Satisfaction Score:

$$S^* = 100 + 0.4(20) - 0.1(400) = 100 + 8 - 40 = \boxed{68}$$

Final Answer and Interpretation:

$$F^* = 400 \text{ minutes} \quad D^* = 20 \text{ minutes} \quad S^* = 68$$

The optimal strategy for this day is 400 minutes of focused work (6 hours and 40 minutes) and 20 minutes of distracting scrolling. This yields a focus-to-distraction ratio of $\boxed{F/D = 20/1}$. This seems incredibly unrealistic!

Prob. 6 (a) With $f(x) = \frac{x}{4}$, the integrand becomes:

$$h(x) = x^2 \cdot \frac{x}{4} + \frac{1}{3} \cdot \left(\frac{x}{4}\right)^3 = \frac{x^3}{4} + \frac{x^3}{192} = \frac{49x^3}{192}.$$

Thus,

$$I_z = \int_0^L \frac{49x^3}{192} dx = \frac{49}{192} \cdot \left[\frac{x^4}{4}\right]_0^L = \frac{49L^4}{768} \approx \boxed{0.06380\, L^4}$$

(b) Let $\Delta x = \frac{L}{4}$. The nodes are:

$$x_0 = 0, \quad x_1 = \frac{L}{4}, \quad x_2 = \frac{L}{2}, \quad x_3 = \frac{3L}{4}, \quad x_4 = L$$

The function values are:

$$\begin{aligned}
h(x_0) &= 0 \\
h(x_1) &= \frac{49}{192} \cdot \left(\frac{L}{4}\right)^3 = \frac{49L^3}{12288} \approx 0.00399L^3 \\
h(x_2) &= \frac{49}{192} \cdot \left(\frac{L}{2}\right)^3 = \frac{49L^3}{1536} \approx 0.03190L^3 \\
h(x_3) &= \frac{49}{192} \cdot \left(\frac{3L}{4}\right)^3 = \frac{1323L^3}{12288} \approx 0.10770L^3 \\
h(x_4) &= \frac{49}{192} \cdot L^3 = \frac{49L^3}{192} \approx 0.25521L^3
\end{aligned}$$

Apply the Trapezoidal Rule:

$$\begin{aligned}
I_z^{\text{Trap}} &= \frac{\Delta x}{2} \left[h(x_0) + 2h(x_1) + 2h(x_2) + 2h(x_3) + h(x_4)\right] \\
&= \frac{L}{8} \cdot \left[0 + 2(0.00399) + 2(0.03190) + 2(0.10770) + 0.25521\right] L^3 \\
&= \frac{L^4}{8} \cdot (0.00798 + 0.06380 + 0.21540 + 0.25521) = \frac{L^4}{8} \cdot 0.54239 \\
&= \boxed{0.06780\, L^4}
\end{aligned}$$

(c) Left Riemann sum (use x_0 to x_3):

$$\begin{aligned} I_z^{\text{Left}} &= \Delta x \cdot [h(x_0) + h(x_1) + h(x_2) + h(x_3)] \\ &= \frac{L}{4} \cdot (0 + 0.00399 + 0.03190 + 0.10770)\, L^3 \\ &= \frac{L^4}{4} \cdot 0.14359 = \boxed{0.03590\, L^4} \end{aligned}$$

(d) Now use $\rho(x) = \frac{x}{L}$:

$$I_z = \int_0^L \frac{x}{L} \cdot \frac{49x^3}{192} dx = \frac{49}{192L} \int_0^L x^4 dx = \frac{49}{192L} \cdot \frac{L^5}{5} = \frac{49L^4}{960} \approx \boxed{0.05104\, L^4}$$

(e) Use $\rho(x) = \frac{L-x}{L}$:

$$\begin{aligned} I_z &= \int_0^L \frac{L-x}{L} \cdot \frac{49x^3}{192} dx = \frac{49}{192L} \int_0^L (Lx^3 - x^4) dx \\ &= \frac{49}{192L} \left(\frac{L^5}{4} - \frac{L^5}{5} \right) = \frac{49L^4}{192} \cdot \left(\frac{1}{4} - \frac{1}{5} \right) = \boxed{\frac{49L^4}{3840}} \approx \boxed{0.01276\, L^4} \end{aligned}$$

Remark (Right Riemann Sum and Comparison):

We now compute the moment of inertia using the **right Riemann sum** with the same 4 subintervals.

$$x_1 = \frac{L}{4}, \quad x_2 = \frac{L}{2}, \quad x_3 = \frac{3L}{4}, \quad x_4 = L, \quad \Delta x = \frac{L}{4}$$

Using $h(x) = x^2 \cdot f(x) + \frac{1}{3} f(x)^3 = \frac{49x^3}{192}$, we evaluate:

$$\begin{aligned} h(x_1) &\approx 0.00399L^3 \\ h(x_2) &\approx 0.03190L^3 \\ h(x_3) &\approx 0.10770L^3 \\ h(x_4) &\approx 0.25521L^3 \end{aligned}$$

Then the right Riemann sum becomes:

$$\begin{aligned} I_z^{\text{Right}} &= \Delta x \cdot [h(x_1) + h(x_2) + h(x_3) + h(x_4)] \\ &= \frac{L^4}{4} \cdot (0.39880) = \boxed{0.09970\, L^4} \end{aligned}$$

From part (c), we had the left Riemann sum:

$$I_z^{\text{Left}} = \boxed{0.03590\, L^4}$$

Taking their average:

$$I_z^{\text{Avg}} = \frac{I_z^{\text{Left}} + I_z^{\text{Right}}}{2} = \frac{0.03590 + 0.09970}{2} = \boxed{0.06780\, L^4}$$

This agrees exactly with the Trapezoidal Rule result from part (b):

$$I_z^{\text{Trap}} = \boxed{0.06780\, L^4}$$

as expected, since the Trapezoidal Rule is algebraically equivalent to the average of the left and right Riemann sums when using equally spaced nodes.