**Summary:** First FTC If $f:[a,b]\to\mathbb{R}$ is continuous, then $F(x) := \int_a^x f(y)dy$ satisfies $F'(x) = f(x)$ [Every cont. function has an antiderivative]

Alternative notation: $\frac{d}{dx}\int_a^x f(y)dy = f(x)$.

Second FTC If $F:[a,b]\to\mathbb{R}$ is an antiderivative of $f$, then $\int_a^b f(x)dx = F(x)\Big|_a^b := F(b)-F(a)$

[The $\overbrace{\text{integral}}^{\text{definite}}$ of every cont. function can be evaluated via antiderivatives] Alternative forms ① $\int_a^b F'(x)dx = F(x)\Big|_a^b$ ② $\int_a^b \frac{dF}{dx}(x)\,dx = F(x)\Big|_a^b$ ③ $\int_a^b dF(x) = F(x)\Big|_a^b$

# The Art of Finding Antiderivatives

A. Power Rule

$$\int x^k dx = \frac{x^{k+1}}{k+1} + C, \quad x\in\mathbb{R},\ k=0,1,2,\dots$$

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad x>0,\ \alpha\in\mathbb{R},\ \alpha\neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C, \quad x\neq 0$$

B. Fundamental Rule (Second FTC) $\int f'(x)dx = f(x) + C$

C. u-substitution (Inverting Chain Rule)

$$\int f'(g(x))\cdot g'(x)\,dx = f(g(x)) + C$$

$u = g(x) \Rightarrow du = g'(x)\,dx$

$$\int f'(g(x))g'(x)dx = \int f'(u)\,du = f(u) + C = f(g(x)) + C$$

**Today** Compute $\int_0^{2\pi} 2x\cos(x^2+1)\,dx$

Let

$$u(x) = x^2 + 1 \qquad \begin{aligned} x = 0 &\implies u = 1 \\ x = \pi &\implies u = \pi^2 + 1 \end{aligned}$$

$$du = (2x)\,dx$$

$$\begin{aligned} \int_{x=0}^{x=\pi} 2x\cos(x^2+1)\,dx &= \int_{u=1}^{u=\pi^2+1} \cos(u)\,du \\ &= \sin(u)\Big|_{u=1}^{u=\pi^2+1} \\ &= \sin(\pi^2+1) - \sin(1) \end{aligned}$$

**Alternative**

$$\begin{aligned} \int_{x=0}^{x=\pi} \underbrace{2x\cos(x^2+1)}_{f(x)=F'(x)}\,dx &= \underbrace{\sin(x^2+1)}_{F(x)}\Big|_{x=0}^{x=\pi} \\ &= \sin(\pi^2+1) - \sin(1) \end{aligned}$$

D. Product Rule: Inverting it

gives integration by parts.

$$\frac{d}{dx}(u\cdot v) = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx}$$

multiply both sides by $dx$

$$d(u\cdot v) = u\cdot dv + v\cdot du$$

Integrate both sides after re-arranging a bit.

$$u\cdot dv = d(u\cdot v) - v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Looks like we are re-arranging the deck chairs on th Titanic!

## Examples $\int x \cdot \sin(x)\,dx$

$$\int u\,dv = uv - \int v\,du$$

$u = x \implies du = dx$

$dv = \sin(x)dx \implies v(x) = -\cos(x)$

$$\int \underset{u}{x}\,\underbrace{\sin(x)\,dx}_{dv} = (x)[-\cos(x)] - \int [-\cos(x)]\,dx$$

$$= -x\cos(x) + \sin(x) + C$$

At home: try $u = \sin(x)$, $dv = x\,dx$ } yields an even harder integral

Example: $\int e^t \cdot \cos(e^t)\,dt = \sin(e^t) + C$

u-substitution: $u = e^t$

$du = e^t\,dt$

---

$\int t\,e^t\,dt =$

want to make it go away via

differentiation.

$u = t \Rightarrow du = dt$
$dv = e^t dt \qquad v = e^t$

$$\int u\,dv = uv - \int v\,du$$

$$\int t e^t dt = (t e^t) - \int e^t \cdot dt$$

$$= te^t - e^t + C$$

__Example__ $\int \underline{\ln(x)}\,\underline{dx}$

Thought process: some term we get to differentiate and some term we get to integrate

$u = \ln(x) \Rightarrow du = \frac{1}{x}dx$
$dv = dx \Rightarrow v = x$

$$\int u\,dv = u \cdot v - \int v\,du$$

$$\int \ln(x)dx = [\ln(x)][x] - \int x \cdot \frac{1}{x}dx$$

$$= x\ln(x) - x + C$$

## E. Antiderivatives for Rational Functions (with real roots):

Partial Fraction Expansion

**Fact:** For $r$ a real number,

$$\int \frac{1}{x \pm r}\,dx = \ln|x \pm r| + C$$

Hence, $\int \left( \frac{1}{x+3} + \frac{1}{x-1} \right) dx = \ln|x+3| + \ln|x-1| + C$

However, if I give you give you the function like this

$$\frac{1}{x+3} + \frac{1}{x-1} = \frac{x-1}{(x+3)(x-1)} + \frac{x+3}{(x+3)(x-1)}$$

$$= \frac{2x+2}{x^2+2x-3}$$

$$\int \frac{2x+2}{x^2+2x-3}\,dx = ?$$

What do you do? PFE to the rescue!!!

## How to perform it?

We deliberately use a new example

$$\frac{x+3}{(x+2)(x-4)}$$

## Step 1: Find the roots

Already done for us: $r_1 = -2$, $r_2 = 4$

## Step 2: Expand the rational function into 1st order terms

$$\frac{x+3}{(x+2)(x-4)} = \frac{k_1}{\underbrace{x+2}_{1^{st}\text{-order}}} + \frac{k_2}{\underbrace{x-4}_{1^{st}\text{-order}}}$$

## Step 3: Brute force solve for $k_1$ and $k_2$ by placing over a common denominator

$$\frac{x+3}{(x+2)(x-4)} = \frac{k_1(x-4) + k_2(x+2)}{(x+2)(x-4)}$$

$$= \frac{(k_1+k_2)x + (-4k_1 + 2k_2)}{(x+2)(x-4)}$$

Equate numerators because of the common denominator

$$x+3 = (k_1+k_2)x + (-4k_1+2k_2)$$

$$\left.\begin{aligned} 1 &= k_1+k_2 \\ 3 &= -4k_1+2k_2 \end{aligned}\right\} \begin{aligned} k_1 &= -\frac{1}{6} \\ k_2 &= \frac{7}{6} \end{aligned}$$

Step 4: Use the expansion

$$\int \frac{x+3}{(x+2)(x-4)}\,dx = \int\left(-\frac{1}{6}\right)\frac{1}{x+2}\,dx + \int\left(\frac{7}{6}\right)\frac{1}{x-4}\,dx$$

$$= -\frac{1}{6}\ln|x+2| + \frac{7}{6}\ln|x-4| + C$$

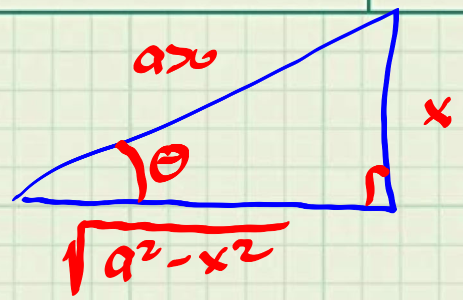
---

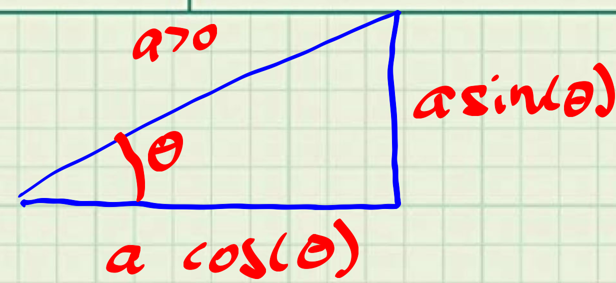
F. Trigonometric Substitution

Hard because it uses Pythagorean Thm and Trigonometry

Special: u-substitution

Example: $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ for $a>0$, $0\le x<a$




$$\sqrt{a^2-x^2} = a\cos(\theta)$$

$$x = a\sin(\theta) \Rightarrow dx = a\cos(\theta)\,d\theta$$

$$x = a\sin(\theta) \Rightarrow \theta = \arcsin\left(\frac{x}{a}\right)$$

$$\underbrace{\int \frac{1}{\sqrt{a^2-x^2}}\,dx}_{\left[\arcsin\left(\frac{x}{a}\right)\right]'} = \int \underbrace{\frac{1}{\cancel{a\cos(\theta)}}}_{\frac{1}{\sqrt{a^2-x^2}}} \cdot \underbrace{\cancel{a\cos(\theta)}\,d\theta}_{dx}$$

$$= \int d\theta = \theta + C$$

$$= \arcsin\left(\frac{x}{a}\right) + C.$$

**Secrets of the Arcane 7.39: Elementary Functions and Antiderivatives**

In mathematics, when we talk about **elementary functions**, we are referring to functions that are constructed using basic arithmetic operations (addition, subtraction, multiplication, and division) and compositions of,

1. **Polynomials**: Functions like $f(x) = x^2$, $g(x) = 3x^3 - 2x + 1$.
2. **Rational Functions**: Ratios of two polynomials, e.g., $R(x) = \frac{x^2+1}{x-2}$.
3. **Root Functions**: Functions involving square roots, cube roots, etc., like $h(x) = \sqrt{x}$.
4. **Trigonometric Functions**: The standard sine, cosine, tangent, etc., and their inverses.
5. **Exponential and Logarithmic Functions**: Functions like $e^x$, $\ln(x)$.

These functions are termed "elementary" because they are among the first functions introduced in mathematics education and are foundational in calculus and analysis. They are well-understood and have known properties, such as **the derivative of an elementary function is also elementary**. When you combine elementary functions using operations like addition, multiplication, or composition, you get a wide variety of functions that can model many different phenomena, but they still retain the "elementary" label because they're built from these basic components.

**Most elementary functions do not have antiderivatives that are also elementary functions.**[a] Even the very common function, $e^{-x^2}$, used in Radial Basis Functions and Gaussian Probability Distributions, does not have an antiderivative that can be expressed in terms of anything we would recognize. The same is true for the antiderivative of $\cos(x^2)$ in Example 7.4-(d). In these cases, the antiderivative is literally defined through the First Fundamental Theorem of Calculus, that is,

$$\text{the antiderivative of } f(x) \text{ is } F(x) = \int_a^x f(t)\,dt + C,$$

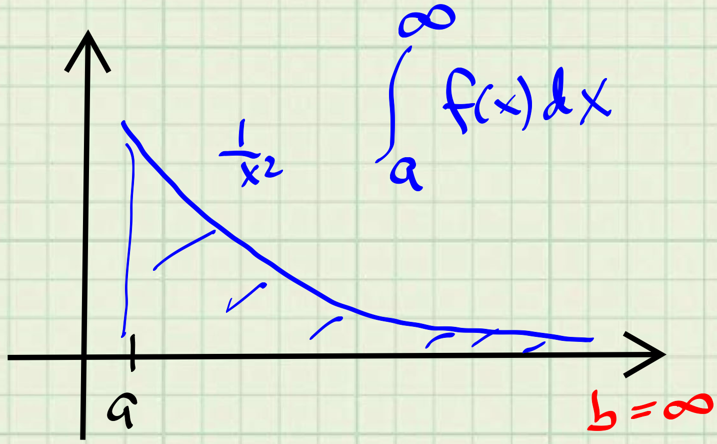
which is the integral you were trying to compute in the first place. Fortunately, for functions that arise sufficiently often, numerical analysts seek fast and accurate means to compute the new antiderivative. If you are lucky, your antiderivative may be one of the non-elementary antiderivatives that have been tabulated. Are you ready to roll the dice?

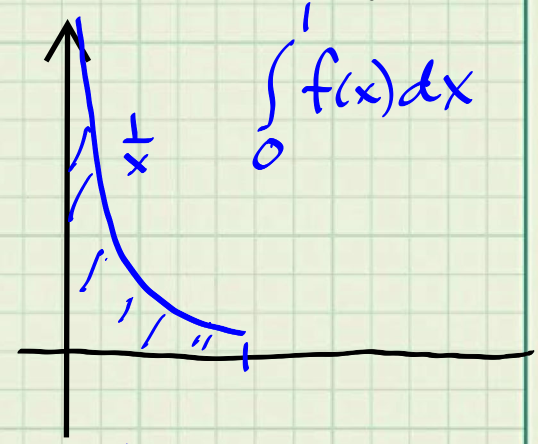
**The coup de grâce in all of this:** the hand methods for determining antiderivatives allow you to find antiderivatives that are already available in what used to be called **Integral Tables**. You can simply look up your antiderivative. Back in the day, engineering students purchased books with thousands of antiderivatives in them. Today, no one needs to do that. We have ChatGPT+Wolfram, which provides access to more integrals than all of the books we students of yore used to lug around.

**Further Reading:** **Non-elementary Integrals**; **Proving the Non-existence of Elementary Anti-derivatives for Certain Elementary Functions**; **Impossibility Theorems for Elementary Integration**; How WolframAlpha defines these nonelementary integrals; and **Closure of the Set of Elementary Functions**.

[a] In plain words, seeking an antiderivative that you can express in understandable terms can be a fool's errand.

# Chapter 8: Improper Integrals




$a = 0^+$ $b = 1$

**Type-I**
**unbounded limit**

**Type-II**
**Unbounded Function**

## Why you need to be careful:

First limit Type-I $= \int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

Second Case Type-II $\int_{a^+}^b f(x)\,dx = \lim_{\alpha\to a^+} \int_\alpha^b f(x)\,dx$

## Second Reason

$$\lim_{b\to\infty} \int_a^b f(x)\,dx = \lim_{b\to\infty} \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\,\Delta x$$

$$\Delta x = \frac{b-a}{N}$$

Limits of limits are "tricky"

Really a subject of Math 451.

## Easier Warmup Problems

Consider $x>0$

$$\int_1^x \frac{1}{y}\,dy = \ln(y)\Big|_1^x = \ln(x) - \cancel{\ln(1)}^{\,0}$$

$$= \ln(x) \xrightarrow[x\to\infty]{} \infty$$

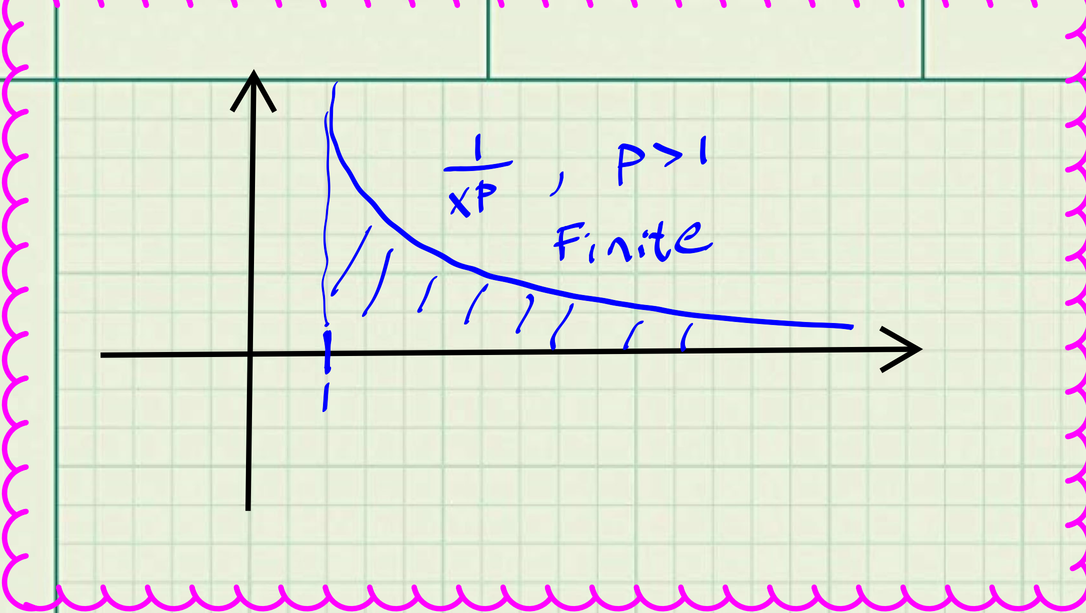
- $x > 0, \quad p \neq 1$

$$\int_1^x \frac{1}{y^p}\,dy = \int_1^x y^{-p}\,dy = \frac{y^{-p+1}}{-p+1}\Bigg|_1^x$$

$$= \frac{x^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

$$= \frac{x^{1-p}}{1-p} - \frac{1}{1-p}$$

$$\lim_{x\to\infty} \frac{x^{1-p}}{1-p} - \frac{1}{1-p} = \begin{cases} -\frac{1}{1-p} & p>1 \\ \infty & p<1 \end{cases}$$




$$\boxed{\int_1^x \frac{1}{y^2}\,dy = \frac{1}{2-1} = 1}$$

Next time, we'll use this class of functions to bound more interesting functions.