ROB 201 - Calculus for the Modern Engineer
HW #3

Prof. Grizzle

Check Canvas for due date and time

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete and turn in.

1. Read Chapter 3 of our ROB 201 Textbook, `Calculus for the Modern Engineer`; you will find a copy on our Canvas site, in the `file` folder. Based on your reading of the Chapter, summarize in your own words:

   (a) the purpose of Chapter 03;

   (b) two things you found the most DIFFICULT.

   There are no "right" or "wrong" answers, but no answer means no points. The goal is to reflect a bit on what you are learning and why.

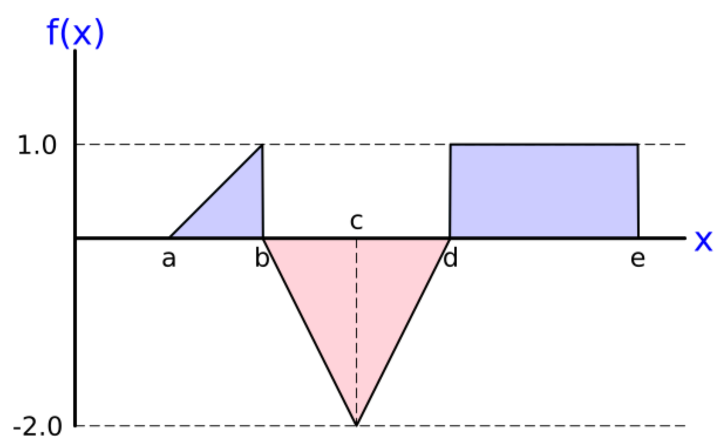
2. Answer any two of the following three **conceptual questions:**

   (a) Explain the concept of the Riemann Integral and its significance in calculus.

   (b) What is the difference between the "signed area under a function" and the "area under the absolute value of a function"?

   (c) Why do we need to impose conditions on functions to define the Riemann Integral?

3. The problem uses the plot below, consisting of three right triangles and a rectangle.



   (a) Compute $\int_a^e f(x)\,dx$. You will obtain a formula in terms of a, b, c, d, e; see the hint.

   (b) Compute $\int_a^c f(x)\,dx$.

   (c) Compute $\int_d^b f(x)\,dx$ (note the order of the limits of integration).

(d) If $\int_a^c f(x)\,dx = -3$ and $\int_c^e f(x)\,dx = 7$, what is the value of $\int_a^e f(x)\,dx$?

4. Suppose that $a < b$ and that $f : [a,b] \to \mathbb{R}$ is monotonically **decreasing**. Following the notation in the textbook, give a formula for the lower Riemann sum, $\text{Area}_n^{\text{Low}}$ of $f(x)$.

5. **Problems 5 and 6 are linked: your overall mission is to derive a cubic version of Simpson's Rule.**

The basic Simpson's Rule uses a quadratic function (polynomial) to estimate the area under a function $f : [a,b] \to \mathbb{R}$. Specifically, given $\Delta x > 0$, x_i, $x_{i+1} = x_i + \Delta x$, and $x_c := \frac{x_i + x_{i+1}}{2} = x_i + \frac{\Delta x}{2}$,

- it computes α, β and γ such that $q(x) := \alpha(x - x_c)^2 + \beta(x - x_c) + \gamma$ interpolates the function, that is, such that $q(x)$ satisfies
$$q(x_i) = f(x_i), \quad q(x_c) = f(x_c), \quad \text{and} \quad q(x_{i+1}) = f(x_{i+1}),$$
- and then it estimates $\int_{x_i}^{x_{i+1}} f(x)\,dx$ by the integral, $\int_{x_i}^{x_{i+1}} q(x)\,dx$, which has a nice closed-form solution, as given in the textbook.

In this problem, we seek to determine the coefficients of a cubic polynomial that interpolates $f : [a,b] \to \mathbb{R}$ **through four equally spaced points**, analogously to the basic Simpson's rule interpolating a quadratic polynomial through three equally spaced points in $[x_i, x_{i+1}]$.

To make the derivation "clean", we suggest the following notation and definitions:

- $h := \Delta x$
- $\bar{x}_a := x_i$
- $\bar{x}_b := x_i + h/3$
- $\bar{x}_c := x_i + 2h/3$
- $\bar{x}_d := x_i + h = x_{i+1}$
- $p(x) := \alpha_3(x - x_i)^3 + \alpha_2(x - x_i)^2 + \alpha_1(x - x_i) + \alpha_0$

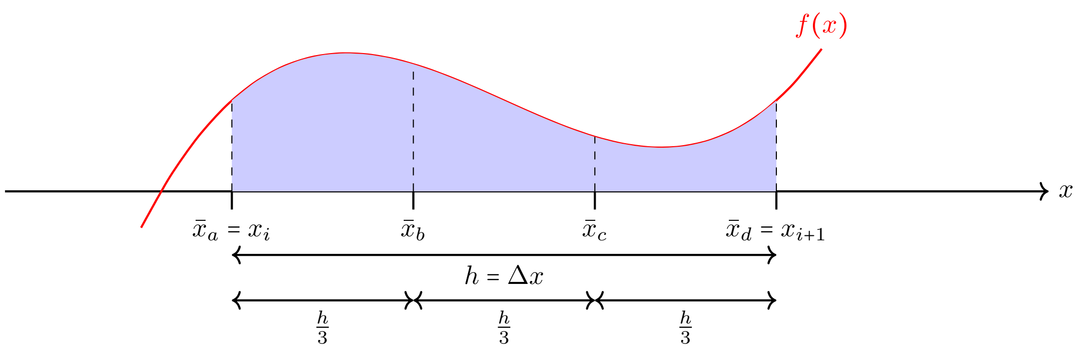


Figure 1: Simpson's 3/8 rule illustrated. Interpolating a function f with a cubic polynomial over four points: $\bar{x}_a = x_i$, $\bar{x}_b = x_i + \frac{h}{3}$, $\bar{x}_c = x_i + \frac{2h}{3}$, and $\bar{x}_d = x_{i+1} = x_i + h$.

To do: Set up four simultaneous equations for the unknown coefficients $\alpha_3, \ldots, \alpha_0$ such that the polynomial p interpolates f, that is, such that $p(x_0) = f(x_0), \ldots, p(x_3) = f(x_3)$. For your final answer, write the equations in the form

$$\Phi_{4\times 4} \cdot \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix},$$

where $\Phi_{4\times 4}$ is a 4×4 real matrix, with entries depending on h. Draw a box around $\Phi_{4\times 4}$.

Note: This problem does not involve Calculus. It only requires you to set up a system of equations. The matrix Φ should depend explicitly on h and its powers. **You are NOT asked to solve the equations.**

6. Using the cubic polynomial $p(x)$ from Prob. 5, the goal of this problem is to evaluate the integral $\int_{x_i}^{x_{i+1}} p(x)\, dx$ term-by-term using either the Shifting Property or a change of variable. To make your work easy to grade, please integrate the following four terms

   (a) $\int_{x_i}^{x_{i+1}} (x - x_i)^3\, dx$
   (b) $\int_{x_i}^{x_{i+1}} (x - x_i)^2\, dx$
   (c) $\int_{x_i}^{x_{i+1}} (x - x_i)\, dx$
   (d) $\int_{x_i}^{x_{i+1}} 1\, dx$;

   you do not need to include the coefficients, $\alpha_3, \ldots, \alpha_0$. **In each case, your answer should depend explicitly on $h := \Delta x = x_{i+1} - x_i$, and nothing else.**

Hints

Prob. 1 Write approximately 15 or more words for each part of the question.

Prob. 2 Read the Chapter. Google if necessary. Think!

Prob. 3 The area of a triangle is one-half base times height. Applying this to the the function $f(x)$ we have

$$\text{Area} = \frac{(b-a)}{2} - 2\,\frac{(d-b)}{2} + (e-d).$$

Because the Riemann integral and the area are the same thing for triangles and rectangles, we have

$$\int_a^e f(x)\,dx = \frac{(b-a)}{2} - 2\,\frac{(d-b)}{2} + (e-d) = -\frac{a}{2} + \frac{3b}{2} - 2d + e$$

It was not necessary to simplify the answer, meaning, if you had left the answer as $\frac{(b-a)}{2} - 2\,\frac{(d-b)}{2} + (e-d)$, you'd still earn full credit.

Prob. 4 No hints provided.

Prob. 5 Hints were given as part of the problem statement.

Prob. 6 So that you can check your work, $\int_{x_i}^{x_i+h}(x-x_i)^3\,dx = \frac{h^4}{4}$.

Solutions HW 03

Prob. 1 Based on your reading of the Chapter, summarize in your own words:

(a) **The purpose of Chapter 03:** is to deepen the understanding of the fundamental concepts of calculus by merging programming and numerical calculations with traditional mathematical analysis. It aims to provide a solid mathematical foundation by introducing critical concepts such as one-sided limits and continuity in detail, thereby enhancing comprehension of how functions behave near specific points. The chapter further explores the derivation of closed-form solutions for integrals of exponential functions, which paves the way for integrating trigonometric functions like sine and cosine. It addresses the concepts of bounded and unbounded functions, going beyond the simple identification of maximum and minimum values to embrace a broader understanding of function behavior. Through an exploration of piecewise continuous functions and conditions for Riemann integrability, the chapter extends the learner's insight into a wider class of functions. Overall, the chapter is designed to equip learners with both intuitive and formal understandings of these critical concepts, enabling them to apply this knowledge to complex calculus problems effectively.

(b) **Two things you may have found the most DIFFICULT**, among many other possibilities:

i. **The Epsilon-Delta Definition of Continuity at a Point**: This formal definition is a cornerstone of mathematical analysis, providing a rigorous way to understand continuity. It requires a precise handling of limits and an understanding of how function values can be controlled within arbitrary proximity to a point. The abstract nature of the epsilon-delta criterion, which involves conceptualizing distances in the input and output of functions to ensure they remain within specified bounds, can be difficult for students to grasp initially.

ii. **Acquiring Intuition for a Broader Class of Functions that are Riemann Integrable**: Understanding the conditions under which a function is Riemann integrable extends beyond the simple cases often introduced early in calculus courses. Students must come to terms with the idea that even functions with certain types of discontinuities or peculiar behaviors can still be integrated in the Riemann sense. This requires a deeper understanding of the concept of integration, the significance of partitions, and how the sum of areas under a curve can converge to a finite value, despite the presence of irregularities in the function.

Both topics demand a solid conceptual foundation and the ability to engage with abstract mathematical ideas, which can pose significant challenges to young learners.

Prob. 2 (a) The Riemann Integral is a fundamental concept in calculus that provides a systematic way to define and compute the area under a curve. Defined for a function f on an interval $[a, b]$, it represents the total accumulation of quantities, such as displacement or volume, across that interval. The essence of the Riemann Integral lies in partitioning the interval into smaller subintervals, measuring the function's value at specific points within these subintervals, and then summing up the product of these values and the widths of the subintervals. As we refine the partition by increasing the number of subintervals and decreasing their width, if the sum approaches a limit, it is defined to be the integral of f from a to b.

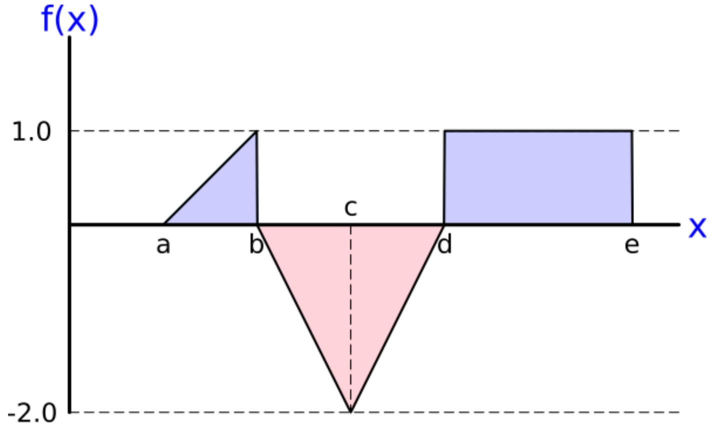
The beauty of the Riemann Integral lies in its ability to precisely define and calculate areas, even for complex functions, by breaking down the problem into simpler, more manageable parts. This concept is not only pivotal for theoretical mathematics but also has practical applications in physics, engineering, and economics, where calculating such quantities is essential.

(b) The "signed area under a function" accounts for the position of the function relative to the x-axis, where regions of the function below the x-axis are considered negative, and those above are positive. This approach reflects net changes and can result in positive, negative, or zero area, depending on the function's behavior. In contrast, the "area under the absolute value of a function" measures the total magnitude of the function's deviation from the x-axis, treating all parts of the function as if they lie above the x-axis. This calculation ignores the direction of the deviation, computing total area without regard to direction or sign.

While domain knowledge is required to decide which integral to compute, in this course, we almost always compute the signed area.

(c) **The short answer would be: because we need two limits to exist and to be equal in value, the limits arising from the Riemann lower and upper sums, and limits are never automatic!** A longer answer is: To define the Riemann Integral, conditions are imposed on functions to guarantee that the process of summing over increasingly finer partitions of an interval $[a, b]$ leads to a unique and existent limit. A function must be bounded within the interval, meaning it cannot stretch to infinity or drop to negative infinity, and it should not have discontinuities that are "excessively severe or numerous". These criteria ensure the Riemann lower and upper sums converge to a definitive and common value, allowing for the definition of the integral.

Prob. 3 (a) **Ans.** Given for free in the hints.



(b) **Ans.** $\int_a^c f(x)\,dx == \frac{(b-a)}{2} - 2\,\frac{(c-b)}{2} = \frac{(b-a)}{2} - (c-b) = -\frac{a}{2} + \frac{3b}{2} - c.$

(c) **Ans.** $\int_d^b f(x)\,dx = d - b$

Because the lower limit is less than the upper limit, we have

$$\begin{aligned}\int_d^b f(x)\,dx &= -\int_b^d f(x)\,dx \\ &= -\left(-2\frac{d-b}{2}\right) \\ &= d-b\end{aligned}$$

(d) **Ans.** $\int_a^e f(x)\,dx = 4.$

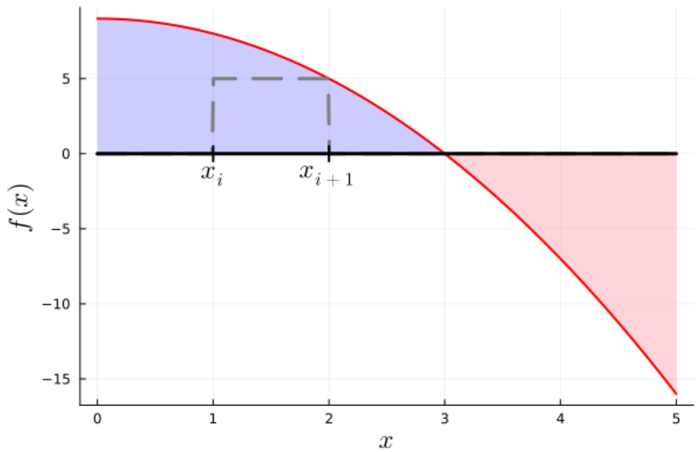
$$\int_a^e f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{-3} + \underbrace{\int_c^e f(x)\,dx}_{7} = -3 + 7 = 4$$

Prob. 4 We divide the interval $[a, b]$ into $n > 1$ **evenly spaced subintervals**, $[x_i, x_{i+1}]$, where

$$\begin{aligned}\Delta x :=& \frac{b-a}{n} \\ x_i :=& a + (i-1)\cdot\Delta x,\end{aligned}$$

so that

$$a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b \tag{1}$$
$$x_{i+1} - x_i = \Delta x,\ 1 \le i \le n. \tag{2}$$



As in the textbook, we define

$$\text{Area}_n^{\text{Low}} := \sum_{i=1}^{n} h_i^{\text{Low}} \cdot b_i,$$

where $b_i = \Delta x$ is the base of the rectangle and h_i^{Low} is the smallest value of the function over the interval $[x_i, x_{i+1}]$. From the figure, we have to select

$$h_i^{\text{Low}} = f(x_{i+1}).$$

Therefore,

$$\boxed{\text{Area}_n^{\text{Low}} := \sum_{i=1}^{n} h_i^{\text{Low}} \cdot b_i = \sum_{i=1}^{n} f(x_{i+1}) \cdot \Delta x.} \tag{3}$$

Prob. 5 We are given

- $h := \Delta x$
- $\bar{x}_a := x_i$
- $\bar{x}_b := x_i + h/3$
- $\bar{x}_c := x_i + 2h/3$
- $\bar{x}_d := x_i + h = x_{i+1}$
- $p(x) := \alpha_3(x - x_i)^3 + \alpha_2(x - x_i)^2 + \alpha_1(x - x_i) + \alpha_0$

and we seek conditions on the coefficients such that $f(\bar{x}_a) = p(\bar{x}_a), \ldots, f(\bar{x}_d) = p(\bar{x}_d)$.

Hence, we have

$$\begin{aligned} \alpha_3(\bar{x}_a - x_i)^3 + \alpha_2(\bar{x}_a - x_i)^2 + \alpha_1(\bar{x}_a - x_i) + \alpha_0 &= f(\bar{x}_a) \\ \vdots \quad &\vdots \\ \alpha_3(\bar{x}_d - x_i)^3 + \alpha_2(\bar{x}_d - x_i)^2 + \alpha_1(\bar{x}_d - x_i) + \alpha_0 &= f(\bar{x}_d). \end{aligned}$$

Substituting in for $\bar{x}_a - x_i = 0$, $\bar{x}_b - x_i = \frac{h}{3}$, $\bar{x}_c - x_i = \frac{2h}{3}$, and $\bar{x}_d - x_i = h$ yields

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ \left(\frac{h}{3}\right)^3 & \left(\frac{h}{3}\right)^2 & \frac{h}{3} & 1 \\ \left(\frac{2h}{3}\right)^3 & \left(\frac{2h}{3}\right)^2 & \frac{2h}{3} & 1 \\ h^3 & h^2 & h & 1 \end{bmatrix}}_{\Phi} \cdot \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} f(\bar{x}_a) \\ f(\bar{x}_b) \\ f(\bar{x}_c) \\ f(\bar{x}_d) \end{bmatrix}$$

or, if you prefer,

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{h^3}{27} & \frac{h^2}{9} & \frac{h}{3} & 1 \\ \frac{8h^3}{27} & \frac{4h^2}{9} & \frac{2h}{3} & 1 \\ h^3 & h^2 & h & 1 \end{bmatrix}$$

You were not asked to compute the solutions, but it is straightforward in code:

$$\begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{-9f(x_i)}{2h^3} + \frac{9f(x_{i+1})}{2h^3} + \frac{27f(x_i+h/3)}{2h^3} + \frac{-27f(x_i+2h/3)}{2h^3} \\ \frac{9f(x_i)}{h^2} + \frac{18f(x_i+2h/3)}{h^2} + \frac{-9f(x_{i+1})}{2h^2} + \frac{-45f(x_i+h/3)}{2h^2} \\ \frac{f(x_{i+1})}{h} + \frac{-\frac{9}{2}f(x_i+2h/3)}{h} + \frac{-\frac{11}{2}f(x_i)}{h} + \frac{9f(x_i+h/3)}{h} \\ f(x_i) \end{bmatrix}$$

Prob. 6 By the Shifting Property of Integration,

$$\int_{x_i}^{x_i+h} g(x-x_i)\,dx = \int_{x_i-x_i}^{x_i+h-x_i} g(x)\,dx = \int_0^h g(x)\,dx.$$

Applying this to the cubic through constant terms yields,

- $\int_{x_i}^{x_i+h}(x-x_i)^3\,dx = \int_0^h x^3\,dx = \frac{h^4}{4}\Big|_0^h = \frac{h^4}{4}$.
- $\int_{x_i}^{x_i+h}(x-x_i)^2\,dx = \int_0^h x^2\,dx = \frac{h^3}{3}\Big|_0^h = \frac{h^3}{3}$.
- $\int_{x_i}^{x_i+h}(x-x_i)\,dx = \int_0^h x\,dx = \frac{h^2}{2}\Big|_0^h = \frac{h^2}{2}$.
- $\int_{x_i}^{x_i+h} 1\,dx = \int_0^h 1\,dx = x\Big|_0^h = h$.

You were not asked to compute this: Combining the answers from Problems 5 and 6, we obtain

$$\int_{x_i}^{x_i+h} p(x)\,dx = \frac{h}{8}\cdot\left(f(x_i) + 3f(x_i+h/3) + 3f(x_i+2h/3) + f(x_{i+1})\right),$$

which goes by Simpson's 3/8's Rule. Once again, Simpson's Rule becomes a weighted sum of the function evaluated at the two endpoints, with weights $\frac{h}{8}$, and the two midpoints, with $\frac{3h}{8}$; moreover, the sum of the weights is $h := \Delta x$.

Alternative Solution Using a Change of Variable as Illustrated in Lecture

Let $y = x - x_i$, then $x = y + x_i$. The bounds of integration change accordingly:

- When $x = x_i$, we have $y = x_i - x_i = 0$,
- When $x = x_i + h$, we have $y = x_i + h - x_i = h$.

Thus, the integrals over $[x_i, x_i+h]$ become integrals over $[0,h]$ in terms of y.

Now applying the change of variables to the integrals:

1. **Cubic term** $(x-x_i)^3$**:**

$$\int_{x_i}^{x_i+h}(x-x_i)^3\,dx = \int_0^h y^3\,dy.$$

Solving the integral:

$$\int_0^h y^3\,dy = \frac{y^4}{4}\Big|_0^h = \frac{h^4}{4}.$$

2. **Quadratic term** $(x - x_i)^2$**:**

$$\int_{x_i}^{x_i+h} (x - x_i)^2 \, dx = \int_0^h y^2 \, dy.$$

Solving the integral:

$$\int_0^h y^2 \, dy = \frac{y^3}{3}\Big|_0^h = \frac{h^3}{3}.$$

3.**Linear term** $(x - x_i)$**:**

$$\int_{x_i}^{x_i+h} (x - x_i) \, dx = \int_0^h y \, dy.$$

Solving the integral:

$$\int_0^h y \, dy = \frac{y^2}{2}\Big|_0^h = \frac{h^2}{2}.$$

4. **Constant term (1):**

$$\int_{x_i}^{x_i+h} 1 \, dx = \int_0^h 1 \, dy.$$

Solving the integral:

$$\int_0^h 1 \, dy = y\Big|_0^h = h.$$

Explanation of the Change of Variable: For each integral:

- The integrand $(x - x_i)$ is replaced with y, since $y = x - x_i$.
- The bounds change accordingly from $[x_i, x_i + h]$ to $[0, h]$, where:
- The lower bound is $x = x_i$, which gives $y = 0$,
- The upper bound is $x = x_i + h$, which gives $y = h$.

Thus, the integration becomes simpler as the integrals are now in terms of y, and we can directly compute standard polynomial integrals.

Final Result: After performing the integrals, we again arrive at the same expression for the integral of the cubic polynomial $p(x)$ over $[x_i, x_i + h]$:

$$\int_{x_i}^{x_i+h} p(x) \, dx = \frac{h}{8} \cdot \left(f(x_i) + 3f(x_i + h/3) + 3f(x_i + 2h/3) + f(x_{i+1}) \right),$$

which is Simpson's 3/8's Rule. The rule once again becomes a weighted sum of the function evaluated at the endpoints and the midpoints, with weights $\frac{h}{8}$ for the endpoints and $\frac{3h}{8}$ for the midpoints.