Summary: $\lim\limits_{h\to 0} \frac{e^h - 1}{h} = 1$, uses

$\lim\limits_{x\to x_0}\left[f(g(x)) \right] = f\left(\lim\limits_{x\to x_0} g(x)\right)$ **when**

$f(y)$ is continuous at $y_0 := \lim\limits_{x\to x_0} g(x)$.

Chapter 5 Differentiation

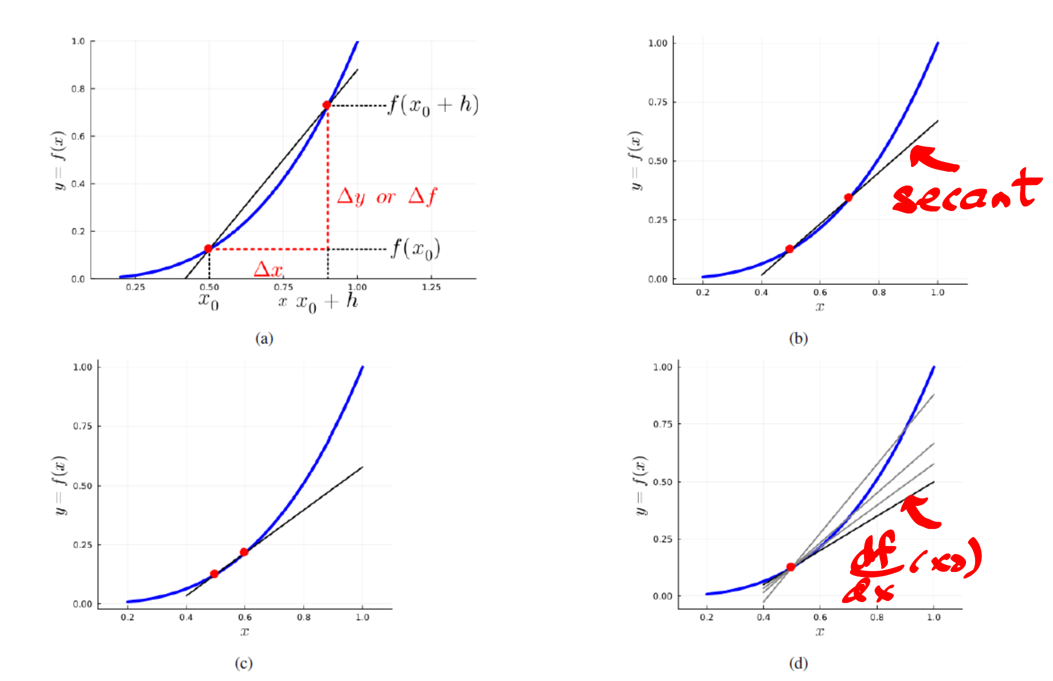
Elementary Functions

Why Calclus is ~~Easy~~ so Easy by the Mathologer

Two valid means to define the derivative of a function

- Local slope = $\frac{\text{rise}}{\text{run}}$
- Local linear approximations

Rise over Run



$$\frac{\Delta f}{\Delta x} \xrightarrow[\Delta x \to 0]{} \frac{df(x_0)}{dx} := \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{(\cancel{x}_0+h) - \cancel{x}_0}$$

$$= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

Let's do a few

a) $f(x) = x^2$, $x_0 \in \mathbb{R}$ arbitrary

$$\frac{df}{dx}(x_0) := \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \qquad \text{calculus}$$

$$= \lim_{h\to 0} \frac{(x_0+h)^2-(x_0)^2}{h} \qquad \text{algebra}$$

$$= \lim_{h\to 0} \frac{\cancel{(x_0)^2}+2x_0h+h^2-\cancel{(x_0)^2}}{h} \qquad \text{algebra}$$

$$\lim_{h\to 0} \frac{2x_0h+h^2}{h} \qquad \text{"}$$

$$\lim_{h\to 0}\left[2x_0 + h\right] = 2x_0$$

$$\boxed{\frac{d}{dx}(x^2) = 2x}$$

b) $f(x) = e^x$, let's leave x as a variable

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h}$$ Apply def.

$$= \lim_{h\to 0} e^x\left[\frac{e^h - 1}{h}\right]$$

$$= e^x \underbrace{\lim_{h\to 0}\frac{e^h - 1}{h}}_{1} = e^x$$

$$\boxed{\frac{d}{dx}e^x = e^x}$$

c) $f(x) = x^k$ $k = 1, 2, 3, \ldots$

$$\frac{d}{dx}(x^k) := \lim_{h\to 0}\frac{(x+h)^k - x^k}{h}$$ def.

(Binomial Thm) $$= \lim_{h\to 0}\frac{\cancel{x^k} + kx^{k-1}h + \sum_{i=2}^{k}\binom{k}{i}x^{k-i}h^i - \cancel{x^k}}{h}$$

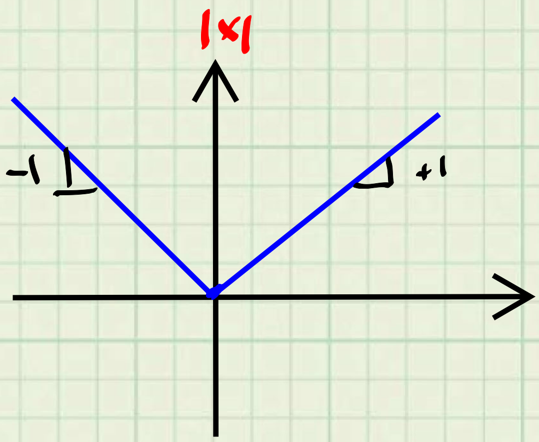
$$= \lim_{h\to 0} \frac{k x^{k-1} h + \sum_{i=2}^{k} \binom{k}{i} x^{k-i} h^i}{h}$$

$$= k x^{k-1} + \underbrace{\lim_{h\to 0} \sum_{i=2}^{k} \binom{k}{i} x^{k-i} h^{i-1}}_{0}$$

$$\boxed{\frac{d}{dx}(x^k) = k x^{k-1}}$$

A Few Non-examples

i) Is $f(x) = |x|$ differentiable at the origin?



Not looking good

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h - 0}{h} = 1$$

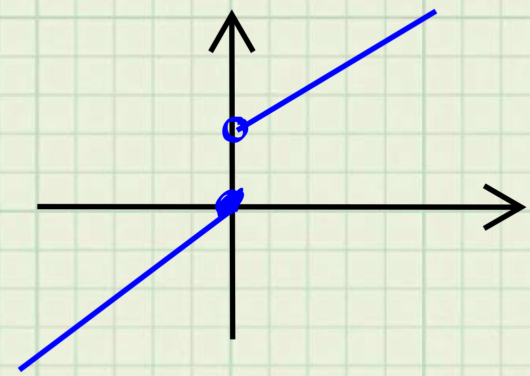
$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-h - 0}{h} = -1$$

Left & right limits both exist,

are finite, but they do not agree.

The function is differentiable for all $x_0 \neq 0$.

ii) $f(x) = \begin{cases} 1+x & x>0 \\ x & x \leq 0 \end{cases}$ $\qquad x_0 = 0$



$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(1+h)-0}{h}$$

$$= \lim_{h\to 0^+} \left[1+\frac{1}{h}\right] = +\infty$$

Not diff. at the origin.

General Fact

a) If f is discontinuous at x_0, then f is not differentiable at x_0.

b) If f is differentiable at x_0, then f is continuous at x_0.

$$A \Rightarrow B \quad \Leftrightarrow \quad \sim B \Rightarrow \sim A$$

not or logical negation

Notation for derivatives

① $$\frac{df}{dx}(x_0) := \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$ Leibniz

② $$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ Lagrange

Second derivatives

$$\frac{d^2 f(x)}{dx^2} := \frac{d}{dx}\left(\frac{df(x)}{dx}\right)$$

$$f''(x) := \left(f'(x)\right)'$$

Derivatives of derivatives

N-th Derivatives

$$\frac{d^n f(x)}{dx^n} := \frac{d}{dx}\left(\frac{d^{n-1} f(x)}{dx^{n-1}}\right)$$

$$f^{(n)}(x) := \left(f^{(n-1)}(x)\right)'$$

Worth memorizing, just as you

Function	c	x^n	e^x	$\ln(x)$	$\sin(x)$	$\cos(x)$	$\tan(x)$	$\mathrm{atan}(x)$
Derivative	0	nx^{n-1}	e^x	$\frac{1}{x}$	$\cos(x)$	$-\sin(x)$	$1+\tan^2(x)$	$\frac{1}{1+x^2}$

would your multiplication table.

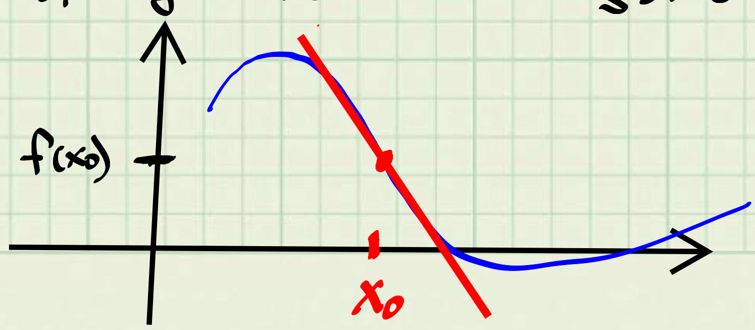
We did a Julia Demo from Chapter 5.

A Second Way to View Derivatives:
Local linear approximations of functions

Let $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$, and suppose we wish to approx. f "near x_0" by

$$f(x)\approx y_0+m(x-x_0)$$

$|x-x_0|<\delta$ "near" some $\delta>0$.



$$f(x_0) \approx y_0 + \cancel{m(x_0 - x_0)}^{\to 0} \Rightarrow y_0 = f(x_0)$$

$$f(x) \approx f(x_0) + m(x - x_0)$$

$$m \approx \frac{f(x) - f(x_0)}{x - x_0}$$ Let $x = x_0 + h, |h| < \delta$

$$m = \frac{f(x_0+h) - f(x_0)}{h} \xrightarrow[h \to 0]{} f'(x_0)$$

$\boxed{\begin{array}{l}\textbf{Def.} \\ \quad y(x) = f(x_0) + f'(x_0)(x - x_0) \\ \text{is the linear approximation of } f \\ \text{about } x_0.\end{array}}$

Proposition 5.11: Differentiation Rules

Here are the rules of differentiation:

(a) **Sum/Difference Rule**: If $f(x) = g(x) + h(x)$ or $f(x) = g(x) - h(x)$, then

$$f'(x) = g'(x) + h'(x) \text{ or } f'(x) = g'(x) - h'(x).$$

(b) **Product Rule**: If $f(x) = g(x) \cdot h(x)$, then

$$f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x).$$

In particular, if $g(x) = c$, a constant, then

$$f'(x) = c \cdot h'(x).$$

(c) **Quotient Rule or Ratio Rule**: If $f(x) = \frac{g(x)}{h(x)}$ and $h(x) \neq 0$, then

$$f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{(h(x))^2}.$$

In particular, if $g(x) = c$, a constant, then

$$f'(x) = -c \cdot \frac{h'(x)}{(h(x))^2}.$$

(d) **Chain Rule**: If $f(x) = g(h(x))$, then

$$f'(x) = g'(h(x)) \cdot h'(x).$$

(e) **Chain Rule** (Another way to state it): Mentally, decompose $f(x) = g(h(x))$ as $g(y)$ evaluated at $y = h(x)$. Then,

$$\frac{d}{dx}(g(h(x))) = \left.\frac{dg(y)}{dy}\right|_{y=h(x)} \cdot \frac{dh(x)}{dx},$$

- Product Rule
- Quotient Rule
- Chain Rule ← (bane of young learners)
- Power Rule

All engineers expected to know

where $\Big|_{y=h(x)}$ means to evaluate y at $h(x)$, or equivalently, substitute in $h(x)$ for y.

Find one of these forms of the Chain Rule that you can master and stick with it! The Chain Rule shows u everywhere in Calculus.

(f) **Exponential Rule**: If $f(x) = a^x$, where a is a positive constant, then

$$f'(x) = \ln(a) \cdot a^x.$$

In particular, if $a = e$, Euler's constant, then

$$f'(x) = e^x.$$

(g) **Logarithm Rule**: If $f(x) = \log_a(x)$, where a is a positive constant, then

$$f'(x) = \frac{1}{x\ln(a)}.$$

In particular, if $a = e$, Euler's constant, then

$$f'(x) = \frac{1}{x}.$$

(h) **Power Rule**: If $f(x) = x^n$, where $n \in \mathbb{N}$, the counting numbers, then

$$f'(x) = nx^{n-1}$$

and there is no restriction on x.

Objective: Derive the Chain Rule

(i) **Generalized Power Rule**: If $f(x) = x^\alpha$, where $\alpha \in \mathbb{R}$ and $x > 0$, then

$$f'(x) = \alpha x^{\alpha-1}.$$

$\frac{d}{dx}\left(f(g(x))\right) = ??$

Here, when the exponent (power) is an arbitrary real number, we need $x > 0$. When the exponent is a counting numbe $f'(x) = nx^{n-1}$, and there is no restriction on x.

It's pretty incredible what the `Symbolics` and `ForwardDiff` packages are doing for us! The are many online sources f the proofs of the differentiation rules. Here are a few.

Method: Use linear approx and the prime notation

- Paul's Online Notes
- openstax.org Differentiation Rules
- Mathologer: Why is calculus so ... EASY ? (Recall that Leibniz's notation for the derivative is $\frac{df}{dx}$, an increment f over an increment in x. The prime notation is due to Newton. **Mathologer shows how the $\frac{df}{dx}$ notation eases t derivation of the rules of differentiation**.)

As we show below, it's really all about linear approximations of functions about a point.

Where do these rules come from.

$$\frac{d}{dx}\left(f(g(x))\right) = ?$$

Lin. Approx.

$$\begin{cases} g(x+h) \approx g(x) + g'(x)\cdot h & \text{for } |h| \text{ small} \\ f(y+\delta) \approx f(y) + f'(y)\cdot\delta & \text{for } |\delta| \text{ small} \end{cases}$$

y and δ are to be found

$$\left(f(g(x))\right)' := \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(\overbrace{g(x)}^{y} + \overbrace{g'(x)h}^{\delta}) - f(g(x))}{h}$$

$y = g(x)$

$\delta = g'(x)h$

$h \to 0 \Rightarrow \delta \to 0$

$$= \lim_{h\to 0} \frac{\cancel{f(g(x))} + f'(g(x))\cdot \overbrace{g'(x)h}^{\delta} \cancel{- f(g(x))}}{h}$$

$$= \lim_{h\to 0} \frac{f'(g(x))\cdot g'(x)\cancel{h}}{\cancel{h}}$$

$$\boxed{= f'(g(x))\cdot g'(x)}$$

$$\frac{d}{dx}\left[f(g(x))\right] = \left.\frac{df(x)}{dx}\right|_{x=g(x)} \cdot \frac{dg(x)}{dx}$$

Examples

i) $f(x) = 3x^2 + 2x + 4$

$f'(x) = 6x + 2$

ii) $f(x) = \frac{1}{x}$

Quotient $f'(x) = \frac{\overset{0}{\cancel{(1)'}} \cdot x - 1 \cdot \overset{1}{\cancel{(x)'}}}{(x)^2} = \frac{0-1}{x^2} = \frac{-1}{x^2}$

Power: $f(x) = x^{-1}$

$f'(x) = (-1)\, x^{(-1-1)} = -x^{-2} = \frac{-1}{x^2}$

iii) $f(x) = \cos(e^{x^2+1})$

Chain Rule
+ Power Rule

$(\cos(y))' = -\sin(y)$

$(x^2+1)' = 2x$

$\left(e^{x^2+1}\right)' = \left(e^{x^2+1}\right) \cdot (x^2+1)' = e^{x^2+1} \cdot 2x$

iv) $f(x) = \ln(2 + \sin^3(x) + x^\pi)$

$$f'(x) = \frac{1}{2+\sin^3(x)+x^\pi} \cdot [2 + \sin^3(x) + x^\pi]'$$

$$= \frac{1}{2+\sin^3(x)+x^\pi}\left[0 + 3\sin^2(x)\cdot\cos(x) + \pi x^{\pi-1}\right]$$

v) $f(x) = x^x$

Key: $f(x)^{g(x)} = e^{\ln[f(x)^{g(x)}]} = e^{g(x)\ln[f(x)]}$

$$f(x) = e^{\ln(x^x)} = e^{x\ln(x)}$$

$$f'(x) = \underbrace{e^{x\ln(x)}}_{x^x} \cdot [x\ln(x)]'$$

$$[x\ln(x)]' = \ln(x) + \frac{x}{x} = \ln(x) + 1$$

$$\therefore f'(x) = x^x \cdot [\ln(x) + 1]$$

QED