ROB 201 - Calculus for the Modern Engineer
HW #6

Prof. Grizzle

Check Canvas for due date and time

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete and turn in. **Red Flag:** Problems 2, 3, and 4 are quite short, while Problems 5 and 6 are much longer. We highlight this so that when you have completed Problems 1 through 4, you do not think you are almost done with the HW set.

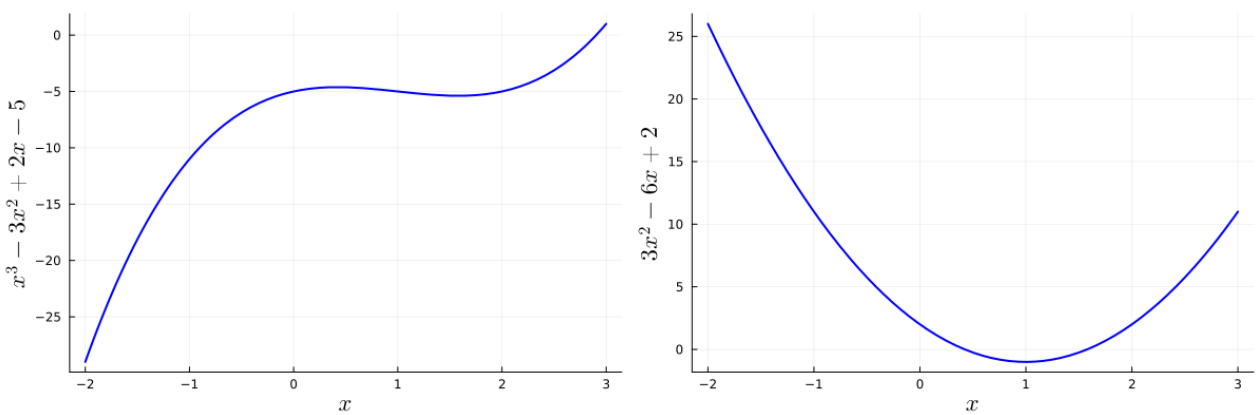
1. (a) Create a "Cheat Sheet" for Chapters 5 and 6 of the textbook. You'll receive the same score for a handwritten solution as a typeset solution. Here is an example from ROB 101.

   (b) Note any material where you found the explanation confusing or difficult to master.

2. Use the two graphs below if you find them helpful. The problem does not require the plots; they can provide a "sanity check" on your answers.

   (a) For the function $q(x) = x^3 - 3x^2 + 2x - 5$, find all of the critical (aka, extreme or stationary) points and classify each critical point as a local minimum, local maximum, or inflection point.

   (b) Find the absolute minimum and absolute maximum of $q(x)$ on the closed interval $[-2, 3]$. Do you have to consider more than just the critical points? Explain.



(a) (b)

3. Mostly thinking, very little computing:

   (a) Let $q : \mathbb{R} \to \mathbb{R}$ by $q(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ be a **cubic polynomial**, with $a_3 \neq 0$. Is it possible for there to be three distinct points where one is a local minimum, one a local maximum, and yet another a **stationary point of inflection**[1]: Yes or No? Explain briefly your answer.

   (b) For $C \in \mathbb{R}$ a constant and $f : (a, b) \to \mathbb{R}$ a differentiable function, explain why $g(x) := f(x) + C$ has the same critical points as $f(x)$.

[1] Check the textbook for the definition.

(c) Suppose that $f : (a,b) \to \mathbb{R}$ is a differentiable function and $f(x) \neq 0$ for all $x \in (a,b)$. Relate the critical points of $\frac{1}{f(x)}$ to those of $f(x)$.

4. A company manufactures two types of products, A and B, in separate facilities. The profit functions (mega-dollars earned per unit of product produced) for products A and B are given by $P_A(x) = -x^2 + 10x - 15$ and $P_B(x) = -2x^2 + 16x - 20$, respectively, where x is the number of units produced. Their shapes can be explained by, if you produce nothing, you still have to pay rent, your workers, material storage, etc., and if you produce too much, a flooded market suppresses prices.

   (a) For each product, find the number of units that should be produced to maximize profit. They do not have to be whole numbers.

   (b) What is the maximum total profit?

5. **Lagrange Multipliers and The Quest for Balanced Joy in STEM:** You have been studying STEM for a long time, totally winging the allocation of study time and socialization so as to "have a life" and yet not disappoint yourself or those close to you. In your pursuit of happiness and academic excellence, you've decided to formulate a constrained optimization problem whose solution will give you an irreproachable basis[2] to allocate your time between studying and socializing. Here we go!

   Data:

   - Your week has disposable time which you can divide between studying (S hours) and socializing (Z hours).
   - Your Joy (J) is quantified as $J = 60 - S + 2Z$, meaning you are an atypical engineer who likes people (lol).
   - However, your GPA, G, still needs to stay at a respectable level. After much soul searching and talking to friends, you decide that your GPA model is $G = 3.0 + \frac{\sqrt{S}}{10} - \frac{Z^2}{800}$ and you are aiming for a GPA of exactly 3.6 so that you look totally competent to employers without setting the bar too high.

   Optimization Problem: Solve:

$$\begin{aligned} \text{Maximize} \quad & J(S,Z) := 60 - S + 2Z \\ \text{subject to} \quad & g(S,Z) = 0.0, \end{aligned}$$

   where $g(S,Z) := G(S,Z) - 3.6 = 3.0 + \frac{\sqrt{S}}{10} - \frac{Z^2}{800} - 3.6$

   Solve the problem using Lagrange Multipliers and hand computations. Document your thought process. It is OK if you prefer to set up a minimization problem. You would then minimize negative joy; makes sense, right? It's a glass-half-empty vs half-full view of life.

6. **While this problem looks super long, the calculations at each stage are very similar, and you can take advantage of that. This will provide you with a refresher on Riemann Integration being a fancy way to compute sums, while helping you to understand the concept of moment of inertia**.

   Kinetic Energy of Bodies with Distributed Mass (means, does not consist of a finite number of point masses): Figure 2 defines two sets of coordinates on a uniform bar in the plane, modeled as a rectangle. Its total length is L, and its total mass is M. From this, we compute a uniform (linear) density of $\rho := \frac{M}{L}$ and that each segment has mass $\frac{M}{n}$, where n is the number of segments in the bar. As in the textbook, the kinetic energy of the bar will be a quadratic function of $\dot{\theta}$, namely

$$KE = \frac{1}{2} I_z \left(\dot{\theta}\right)^2,$$

   where I_z is called the "moment of inertia about the z-axis " (i.e., the axis coming out of the plane and passing through the origin). In this problem, we'll see how I_z changes depending on how we model the bar. You do not need to know anything about moments of inertia in order to work the problem.

   (a) Compute the center of mass of the bar, (x_c, y_c), in the world frame coordinates (always assuming uniform density).

[2] The problem and all numbers in it are figments of the imagination of your instructor. They bear no resemblance to any person living or dead.

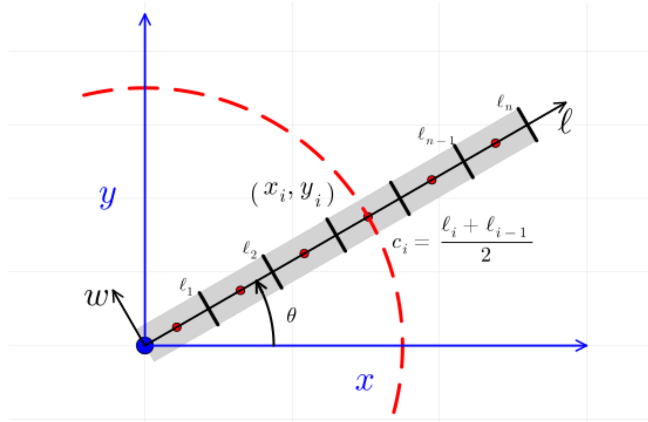


Figure 2: **(For Prob. 6)** A uniform bar in the plane, of length L and total mass M. (x, y) are standard Cartesian coordinates called **world coordinates** in mechanics, while (ℓ, w) are **body coordinates**, that is, coordinates measured in a coordinate frame attached to the bar itself (meaning, if the bar rotates, the coordinates rotate with the bar). We will ignore the width coordinate, w, and focus on the length coordinate, ℓ, to tell us where we are along the length of the bar. The figure shows the bar partitioned into n **segments** of equal length, $\Delta\ell := \frac{L}{n}$. It follows that $\ell_i := i\Delta\ell = \frac{i}{n}L$ and $\ell_0 = 0$. Each red dot is in the center of the corresponding segment. The red dashed line shows that the path traced out by each segment's center is identical to that of a bob (aka, a point mass on the end of a pendulum); see Section "*Kinetic and Potential Energy of a Point Mass in the Plane*" of the textbook.

(b) In the body frame coordinates, compute a simple formula for the position of the center of each segment of the bar, denoted c_i in the Figure. Because the w-coordinate is zero, you can ignore it and only use the length coordinate, ℓ. Your formula should contain L, n, and i. As a check, if you plug in $L = 12$ and $n = 24$, you should obtain $c_6 = 3 - \frac{1}{4} = 2.75$ (yes, $i = 6$). **Suggestion:** Put everything over a common denominator of $2n$.

(c) Compute the kinetic energy KE of the bar, assuming all of the mass is lumped at the center of mass. This means you replace the distributed mass of the bar with a single point mass, having mass equal to the total mass of the bar. Equivalently, you view the bar as having $n = 1$ segments and use your formula from the previous sub-problem. You can find a simple formula for KE of a point pass attached to a pivot in Section "*Kinetic and Potential Energy of a Point Mass in the Plane*" of the textbook.

(d) Compute the kinetic energy KE of the bar when it is divided into $n = 3$ segments of equal length. For each segment, lump its mass of $\frac{M}{3}$ at c_i. **As part of your solution, give the formula for** KE_i**,** $1 \le i \le 3$. KE will be the sum of the three individual kinetic energies, KE_i. Simplify it as much as you can. The algebra just involves fractions. **Suggestion:** Put the c_i over a common denominator before squaring them.

(e) Compute the kinetic energy KE of the bar when it is divided into $n > 1$ segments of equal length. For each segment, lump its mass of $\frac{M}{n}$ at c_i. **As part of your solution, give the formula for** KE_i. KE will be the sum of the n individual kinetic energies, KE_i. You can use the hint to obtain a nice formula for KE.

(f) Compute the kinetic energy of the bar in the limit as the number of segments $n \to \infty$. Obviously, you can take the limit of the previous problem. Alternatively, you can use your knowledge of Riemann integration to pass directly to an integral instead of taking the limit in the previous problem. It's your choice, and each approach will obtain the same credit. We'll provide both solutions.

(g) Compare the formula for KE when lumping the mass at the center of the bar against the "real" KE when considering a distributed mass. In fact, compute the ratio,

$$\frac{KE_{\text{lumped}}}{KE_{\text{distributed}}}.$$

One used to say, "10% is close enough for engineering work!" Is the relative error within this folkloric margin?

Fact: When using 3-segments,

$$\frac{KE_{\text{lumped}}}{KE_{\text{distributed}}} = \frac{35}{36},$$

which means the relative error is one part in 36, or less than 3%. That's pretty incredible!

We repeat: While this problem looks super long, the calculations at each stage are very similar, and you can take advantage of that. This will provide you with a refresher on Riemann Integration being a fancy way to compute sums.

Hints

Prob. 1 Nothing more to add.

Prob. 2 Nothing further offered.

Prob. 3 Nothing further offered.

Prob. 4 Nothing further offered.

Prob. 5 While the problem is a bit of a grind, it is a straightforward application of the Method of Lagrange Multipliers.

(a) Write down the stationary conditions as in the textbook. The first two equations, which are partial derivatives with respect to the decision variables, are sufficiently nice that you can solve for S and Z in terms of λ.

(b) Substituting these values into the constraint gives an equation in only λ. Your equation should contain a constant, a constant times λ, and a constant times $\frac{1}{\lambda^2}$.

(c) Upon multiplying through by λ^2, you will obtain a cubic equation in λ, which has one real solution you can obtain with the Bisection Algorithm or ask your favorite LLM.

(d) If you use the Bisection Algorithm, you need good bracketing points, which you can obtain from a plot.

(e) If you use an LLM, remember to verify the number it gives you is actually a solution.

(f) Now that you know λ^*, look at the first hint again!

Example Problem. You have many others in the textbook. Compute the area of the largest rectangle that can be inscribed inside a circle of radius $r > 0$.

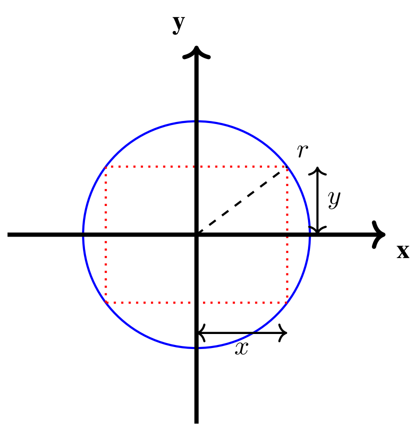


Figure 3: Rectangle inscribed in a circle with labeled dimensions. Note that the width of the rectangle is $2x$ and the height is $2y$.

- Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the area of the rectangle. Then, from the figure,

$$f(x, y) = 4xy.$$

  This is what we want to maximize.

- Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the constraint. We require that the rectangle touches the circle, so

$$g(x, y) = x^2 + y^2 - r^2 = 0.$$

- Using the method of Lagrange multipliers, define

$$L(x,y,\lambda) := f(x,y) + \lambda\, g(x,y) = 4xy + \lambda\,(x^2 + y^2 - r^2).$$

To find the stationary points, we compute the partial derivatives:

$$\frac{\partial L}{\partial x} = 4y + 2\lambda x = 0,$$
$$\frac{\partial L}{\partial y} = 4x + 2\lambda y = 0,$$
$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - r^2 = 0;$$

our maximizing values will be one of the solutions to the above equations.

- Solve the first two equations for λ. From

$$4y + 2\lambda x = 0 \quad \Rightarrow \quad \lambda = -\frac{2y}{x},$$

and

$$4x + 2\lambda y = 0 \quad \Rightarrow \quad \lambda = -\frac{2x}{y}.$$

Equate these two expressions:

$$-\frac{2y}{x} = -\frac{2x}{y} \quad \Rightarrow \quad \frac{y}{x} = \frac{x}{y} \quad \Rightarrow \quad y^2 = x^2.$$

Since we are considering the first quadrant ($x, y > 0$), we have

$$y = x.$$

- Substitute $y = x$ into the constraint:

$$x^2 + x^2 = r^2 \quad \Rightarrow \quad 2x^2 = r^2 \quad \Rightarrow \quad x^2 = \frac{r^2}{2} \quad \Rightarrow \quad x^* = \frac{r}{\sqrt{2}}.$$

Thus,

$$y^* = \frac{r}{\sqrt{2}}.$$

- The maximum area is then computed as:

$$A_{\text{max}} = 4x^* y^* = 4\left(\frac{r}{\sqrt{2}}\right)\left(\frac{r}{\sqrt{2}}\right) = 4 \cdot \frac{r^2}{2} = 2r^2.$$

■

Prob. 6 The sum of the first n ODD integers is equal to $\frac{n(4n^2-1)}{3}$. Indeed, consider,

$$\sum_{i=1}^{n}(2i-1)^2 = \sum_{i=1}^{n}(4i^2 - 4i + 1)$$

$$= 4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$= 4\left(\frac{n(n+1)(2n+1)}{6}\right) - 4\left(\frac{n(n+1)}{2}\right) + n$$

$$= \left(\frac{4n^3}{3} + 2n^2 + \frac{2n}{3}\right) - \left(2n^2 + 2n\right) + n$$

$$= \frac{n(4n^2-1)}{3}$$

Solutions HW 06

Prob. 1 (a) Included at the end of the solution set

(b) These will vary by person, but some of the more challenging topics may have been:

- Challenging topics from Chapter 5 were discussed in the solutions to HW #5.
- **Optimization Problems Involving Constraints:** Optimization is already a bit of a beast because it involves finding the best possible solution from a set of feasible solutions, which can be vast. When you throw constraints into the mix—like in problems that require using Lagrange multipliers—the complexity can ramp up quickly. The method of Lagrange multipliers, while powerful, introduces additional variables and conditions (the Lagrange multipliers themselves) that must satisfy both the original function and the constraints. It's like trying to balance on a tightrope while juggling; you have to keep everything in balance without losing sight of your goal (the optimal solution). Students often struggle with visualizing these multidimensional spaces and how changes in one variable affect others due to the constraints.
- **Using Gradient Descent for Local Minima:** Gradient descent is a fantastic tool for finding local minima, especially in machine learning and data science. However, it can be a tough concept to grasp because it involves both calculus and a bit of algorithmic thinking. The method uses derivatives to guide the search for minima, but the iterative approach and dependence on parameters like the learning rate can make the process seem opaque. There's also the challenge of understanding its limitations, such as susceptibility to getting stuck in local minima rather than finding the global minimum. This concept often requires a shift from purely analytical solving methods to more experimental, trial-and-error approaches, which can be a significant adjustment for students.
- **Deriving and Applying Lagrange's Equations in Dynamics:** When we move into dynamics and start using Lagrange's equations, we're not just dealing with derivatives anymore; we're applying them to model the motion of systems governed by complex forces. This involves a deep dive into kinetic and potential energy, and how these energies interact within a system. For many students, this is where calculus stops being just about numbers and starts requiring a more conceptual, holistic understanding of physical systems. The abstract nature of potential energy surfaces and the idea of converting these concepts into mathematical equations can be daunting. Moreover, the algebraic complexity of setting up and solving Lagrange's equations can be a hurdle, requiring careful consideration of the physical setup and the corresponding mathematical model.

Prob. 2 (a) The critical points of the function $q(x) = x^3 - 3x^2 + 2x - 5$ are found by setting the first derivative $q'(x) = 3x^2 - 6x + 2$ to zero. From the quadratic formula, the critical points are $\boxed{x = 1 - \dfrac{\sqrt{3}}{3}}$ and $\boxed{x = 1 + \dfrac{\sqrt{3}}{3}}$.

> **To classify each critical point**, we examine the second derivative $q''(x) = 6x - 6$ at these points:
>
> - At $x = 1 - \frac{\sqrt{3}}{3}$, $q''(x) = -2\sqrt{3} < 0$, indicating a local maximum.
> - At $x = 1 + \frac{\sqrt{3}}{3}$, $q''(x) = 2\sqrt{3} > 0$, indicating a local minimum.
>
> If you forget how the signs work for the second derivative, always consider quadratic functions, such as $y = x^2$ and $y = -x^2$.

(b) To find the absolute minimum and maximum values of $q(x)$ on $[-2, 3]$, we evaluate $q(x)$ at the critical points within this interval and at the endpoints. The critical points within this interval are the same as found in (a), and the endpoints are $x = -2$ and $x = 3$.

Values of Local Min and Max:

- Local minimum at $x = 1 + \frac{\sqrt{3}}{3}$: $q\left(1 + \frac{\sqrt{3}}{3}\right) = -3\left(\frac{\sqrt{3}}{3} + 1\right)^2 - 3 + \frac{2\sqrt{3}}{3} + \left(\frac{\sqrt{3}}{3} + 1\right)^3 \approx -5.38.$
- Local maximum at $x = 1 - \frac{\sqrt{3}}{3}$: $q\left(1 - \frac{\sqrt{3}}{3}\right) = -3 - \frac{2\sqrt{3}}{3} - 3\left(1 - \frac{\sqrt{3}}{3}\right)^2 + \left(1 - \frac{\sqrt{3}}{3}\right)^3 \approx -4.62.$

Values at the Endpoints:

- The value of $q(x)$ at -2 is $q(-2) = -29$.
- The value of $q(x)$ at 3 is $q(3) = 1$.

Global Extrema: Based on the above analysis, and supported by the graph,

- The absolute minimum value of $q(x)$ on $[-2,3]$ is $q(-2) = -29$.
- The absolute maximum value of $q(x)$ on $[-2,3]$ is $q(3) = 1$.

Yes, we must consider more than just the critical points when finding the absolute minimum and maximum values on a closed interval. The endpoints of the interval must also be evaluated because the absolute extrema of a continuous function on a closed interval can occur at critical points within the interval or at the interval's endpoints.

Prob. 3 Let's put our thinking cap on:

(a) **Ans.** No. **Why:** Critical points are zeros of the first derivative, $q'(x)$. When $q(x)$ is a cubic, we see that $q'(x)$ is a quadratic. A quadratic polynomial can have at most two roots, and thus, there can be at most two critical points.

(b) **Ans.** Critical points occur where the derivative of a function vanishes. $g(x) = f(x)+C$ and $f(x)$ have the same critical points because the derivative of a constant is zero. In other symbols, $g'(x) = f'(x) + 0$, and thus their critical points are equal.

(c) **Ans.** The critical points of $\frac{1}{f(x)}$ coincide with those of $f(x)$, as long as $f(x) \neq 0$, because the derivative of the reciprocal function is zero where the derivative of the original function is zero. Indeed, by the ratio rule,

$$\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f^2(x)},$$

and thus

$$\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f^2(x)} = 0 \iff f'(x) = 0.$$

Note: Where $f(x)$ is big, $\frac{1}{f(x)}$ is small, and vice-versa. Hence, even though their critical points are the same, the roles of local max and min may be swapped.

Prob. 4 A company manufactures two types of products, A and B, in separate facilities. The profit functions (mega-dollars earned per unit of product produced) for products A and B are given by $P_A(x) = -x^2 + 10x - 15$ and $P_B(x) = -2x^2 + 16x - 20$, respectively, where x is the number of units produced.

(a) Given the profit functions for products A and B as $P_A(x) = -x^2 + 10x - 15$ and $P_B(x) = -2x^2 + 16x - 20$, respectively, where x is the number of units produced, we need to determine the critical points and then check for local min and max.

For Product A: The first derivative of $P_A(x)$ with respect to x is given by $P_A'(x) = -2x + 10$. Setting $P_A'(x) = 0$ gives the critical point $x = 5$.

To determine if this critical point corresponds to a local maximum or minimum (or neither), we take the second derivative: $P_A''(x) = -2$. Since $P_A''(x) < 0$, the critical point $x = 5$ corresponds to a local maximum.

For Product B: Similarly, the first derivative of $P_B(x)$ with respect to x is $P_B'(x) = -4x + 16$. Setting $P_B'(x) = 0$ gives the critical point $x = 4$.

The second derivative of $P_B(x)$ is $P_B''(x) = -4$. Since $P_B''(x) < 0$, the critical point $x = 4$ also corresponds to a local maximum.

(b) **Calculating the Maximum Total Profit:** The maximum profit for Product A when producing 5 units is $P_A(5) = -5^2 + 10(5) - 15 = 10$ mega-dollars. The maximum profit for Product B when producing 4 units is $P_B(4) = -2(4)^2 + 16(4) - 20 = 12$ mega-dollars.

Therefore, the maximum total profit is $10 + 12 = 22$ mega-dollars!

Prob. 5 We use the method of Lagrange Multipliers and therefore define

$$L(S,Z,\lambda) := J(S,Z) + \lambda \cdot g(S,Z) = 60 - S + 2Z + \lambda\left(-0.6 + \frac{\sqrt{S}}{10} - \frac{Z^2}{800}\right).$$

Next, we solve for the stationary conditions:

$$0 = \frac{\partial L}{\partial S} = -1 + \lambda\left(\frac{1}{20}\frac{1}{\sqrt{S}}\right)$$

$$0 = \frac{\partial L}{\partial Z} = 2 + \lambda\left(\frac{-Z}{400}\right)$$

$$0 = \frac{\partial L}{\partial \lambda} = -0.6 + \frac{\sqrt{S}}{10} - \frac{Z^2}{800}.$$

We are lucky because the first two equations are easy to solve,

$$0 = -1 + \lambda\left(\frac{1}{20}\frac{1}{\sqrt{S}}\right) \quad \Longrightarrow \sqrt{S} = \frac{\lambda}{20}$$

$$0 = 2 + \lambda\left(\frac{-Z}{400}\right) \quad \Longrightarrow Z = \frac{800}{\lambda}.$$

Substituting these into the last equation gives

$$0 = -0.6 + \frac{\sqrt{S}}{10} - \frac{Z^2}{800} \quad \Longrightarrow -0.6 + \frac{\lambda}{200} - \frac{1}{800}\left(\frac{800}{\lambda}\right)^2 = 0.$$

Upon multiplying through by λ^2, we obtain

$$(-0.6)\ \lambda^2 + \frac{\lambda^3}{200} - 800 = 0.0,$$

which has one real solution,

$$\lambda^* = 129.5355$$

and two complex solutions, $-4.7678 \pm \mathbf{i}\,34.8203$, Substituting the real solution into the earlier equations and doing a bit of algebra, we obtain

$$S^* = \left(\frac{\lambda^*}{20}\right)^2 = 41.95$$

$$Z^* = \frac{800}{\lambda^*} = 6.18$$

You were not asked to compute it, but your optimal Joy is

$$J^* = 30.40\ \text{(units unknown!)}.$$

We share also a solution via JuMP:

```julia
using Plots, JuMP, Ipopt

# Create a new model with Ipopt as the solver
model = Model(Ipopt.Optimizer)

# Define variables
```

```
@variable(model, S >= 0, start = 40.0)  # Ensure S is non-negative
@variable(model, Z >= 0, start = 4.0)   # Ensure Z is non-negative and adjust start if necessary

# Define Joy using the nonlinear objective macro
@NLobjective(model, Max, 60.0 - S + 2*Z)

# Define the constraint using the nonlinear constraint macro
@NLconstraint(model, g, -0.6 + sqrt(S)/10.0 - Z^2/800 == 0)

# Suppress most of the output
set_optimizer_attribute(model, "print_level", 4)

# Tweak solver options
set_optimizer_attribute(model, "max_iter", 1000)  # Increase max iterations
set_optimizer_attribute(model, "tol", 1e-9)        # Set a tighter convergence tolerance

# Solve the optimization problem
optimize!(model)

# Display results
println("Optimal parameters (S, Z): (", value(S), ", ", value(Z), ")")
println("Maximum objective value: ", objective_value(model))

# Access and display the Lagrange multiplier for the constraint
lagrange_multiplier = dual(g)
println("Lagrange multiplier for the constraint: ", lagrange_multiplier)
```

Output

```
EXIT: Optimal Solution Found.

Optimal parameters (S, Z): (41.94860077255686, 6.175914169920733)

Maximum objective value: 30.40322756728461

Lagrange multiplier for the constraint: 129.53547895827631
```

Prob. 6 (a) Compute the center of mass of the bar, (x_c, y_c) (always assuming uniform density).

Ans. Because the bar has uniform density, in the body frame, the center of mass is in the middle, $\boxed{\frac{L}{2}}$. In the world frame (that is, the Cartesian coordinates), we have

$$\boxed{\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \frac{L}{2} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.}$$

Either one of the answers is accepted.

(b) Compute a simple formula for the position of the center of each segment of the bar, denoted c_i in the Figure. Your formula should contain L, n, and i. As a check, if you plug in $L = 12$ and $n = 24$, you should obtain $c_6 = 3 - \frac{1}{4} = 2.75$ (yes, $i = 6$).

Ans. $\boxed{C_i = \frac{(2i-1)}{2n} L}$

Why: By definition, $\ell_i := i\frac{L}{n}$. Hence,

$$c_i = \frac{\ell_i + \ell_{i-1}}{2} = \frac{\frac{i\,L}{n} + \frac{(i-1)\,L}{n}}{2} = \frac{i\,L}{n} - \frac{L}{2n} = \frac{(2i-1)}{2n}L.$$

(c) Compute the kinetic energy KE of the bar, assuming all of the mass is lumped at the center of mass. This means you replace the distributed mass of the bar with a single point mass, equal to the total mass of the bar. Equivalently, you view the bar as having $n = 1$ segments, and we are assuming that the mass of each segment is lumped at its center.

Ans. From the textbook, the kinetic energy of a point mass $\bar{M}$ arranged as a pendulum is

$$KE = \frac{1}{2}\,\bar{M}\,\bar{L}^2\,\dot{\theta}^2,$$

where $\bar{L}$ is the distance of the mass from the pivot point. For us, $\bar{M} = M$ and $\bar{L} = \frac{L}{2}$. Hence

$$\boxed{KE = \frac{1}{2}M\,\frac{L^2}{4}\,\dot{\theta}^2}$$

(d) Compute the kinetic energy KE of the bar when it is divided into $n = 3$ segments of equal length. For each segment, lump its mass of $\frac{M}{3}$ at c_i. **As part of your solution, give the formula for** KE_i. KE will be the sum of the three individual kinetic energies, KE_i. Simplify it as much as you can. The algebra just involves fractions.

Ans. Kinetic energy of the i-th segment is $KE_i = \frac{1}{2}\frac{M}{3}c_i^2\dot{\theta}^2$ because $\bar{M} = \frac{M}{3}$ and $\bar{L} = c_i$. In our case, we obtain,

$$c_1 = \frac{L}{3} - \frac{L}{6} = \frac{L}{6}$$

$$c_2 = 2\frac{L}{3} - \frac{L}{6} = \frac{L}{2}$$

$$c_3 = 3\frac{L}{3} - \frac{L}{6} = \frac{5L}{6}.$$

Plugging in, we obtain,

$$KE_1 = \frac{1}{2}\,\frac{M}{3}\,\frac{L^2}{36}\dot{\theta}^2$$

$$KE_1 = \frac{1}{2}\,\frac{M}{3}\,\frac{L^2}{4}\dot{\theta}^2$$

$$KE_1 = \frac{1}{2}\,\frac{M}{3}\,\frac{25L^2}{36}\dot{\theta}^2$$

Therefore,

$$KE = KE_1 + KE_2 + KE_3 = \frac{1}{2}\,\frac{M}{3}\,\frac{35L^2}{36}\dot{\theta}^2$$

because $\frac{1}{36} + \frac{9}{36} + \frac{25}{36} = \frac{35}{36}$.

(e) Compute the kinetic energy KE of the bar when it is divided into $n > 1$ segments of equal length. For each segment, lump its mass of $\frac{M}{n}$ at c_i. **As part of your solution, give the formula for** KE_i. KE will be the sum of the n individual kinetic energies, KE_i. Do your best to obtain a nice formula for KE.

Ans. kinetic energy of the i-th segment is $KE_i = \frac{1}{2}\frac{M}{n}c_i^2\dot{\theta}^2$ because $\bar{M} = \frac{M}{n}$ and $\bar{L} = c_i$. Hence,

$$\boxed{KE_i = \frac{1}{2}\frac{M}{n}c_i^2\dot{\theta}^2 = \frac{1}{2}\frac{M}{n}\left(\frac{(2i-1)}{2n}L\right)^2\dot{\theta}^2 = \frac{1}{2}\frac{M}{n}\frac{(2i-1)^2}{4n^2}L^2\dot{\theta}^2}$$

Summing this up gives,

$$\begin{aligned}
KE &= \sum_{i=1}^{n} KE_i \\
&= \sum_{i=1}^{n}\left(\frac{1}{2}\frac{M}{n}\frac{(2i-1)^2}{4n^2}L^2\dot{\theta}^2\right) \\
&= \left(\frac{1}{2}\frac{M}{n}\frac{1}{4n^2}L^2\dot{\theta}^2\right)\cdot\sum_{i=1}^{n}(2i-1)^2 \\
&= \frac{1}{2}ML^2\dot{\theta}^2\cdot\frac{1}{4n^3}\cdot\underbrace{\sum_{i=1}^{n}(2i-1)^2}_{\text{see the hint}} \\
&= \frac{1}{2}ML^2\dot{\theta}^2\cdot\frac{1}{4n^3}\cdot\underbrace{\frac{n(4n^2-1)}{3}}_{\text{from the hint}} \\
&= \frac{1}{2}ML^2\dot{\theta}^2\cdot\frac{(1-\frac{1}{4n^2})}{3}
\end{aligned}$$

Hence,

$$\boxed{KE = \frac{1}{2}ML^2\dot{\theta}^2\cdot\frac{(1-\frac{1}{4n^2})}{3}}$$

(f) Compute the kinetic energy of the bar in the limit as the number of segments $n \to \infty$.

Ans. $\boxed{KE = \frac{1}{2}ML^2\dot{\theta}^2\cdot\frac{1}{3} = \frac{1}{2}\underbrace{\frac{ML^2}{3}}_{I_z}\dot{\theta}^2}$. Hence, because $KE =: \frac{1}{2}I_z\dot{\theta}^2$, we can identify $I_z = \frac{ML^2}{3}$ for a uniformly distributed bar.

The limit is so "obvious" that you do not need to show any details. However, for the record, $\lim_{n\to\infty}(1-\frac{1}{4n^2}) = 1$.

Alternative solution: directly building the integral:

$$\begin{bmatrix} x(\ell,\theta) \\ y(\ell,\theta) \end{bmatrix} = \ell\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$

Because the mass is NOT moving in the body frame, $\dot{\ell} = 0$. Hence, we only differentiate θ with respect to time, giving us,

$$\begin{bmatrix} \dot{x}(\ell,\dot{\theta}) \\ \dot{y}(\ell,\dot{\theta}) \end{bmatrix} = \ell\begin{bmatrix} -\dot{\theta}\,\sin(\theta) \\ \dot{\theta}\,\cos(\theta) \end{bmatrix}.$$

And then,

$$\left(\dot{x}(\ell,\dot{\theta})\right)^2 + \left(\dot{y}(l,\dot{\theta})\right)^2 = \left(-\ell\dot{\theta}\,\sin(\theta)\right)^2 + \left(\ell\dot{\theta}\,\cos(\theta)\right)^2 = \ell^2(\dot{\theta})^2$$

We have $\Delta m := \rho \, \Delta \ell$, where $\rho := \frac{M}{L}$ is the density KE per unit length and $\delta \ell$ is a differential length. Position ℓ along the bar corresponds to

$$\begin{bmatrix} x(\ell) \\ y(\ell) \end{bmatrix} = \ell \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$

Hence, the infinitesimal KE at a point ℓ along the bar is $\Delta KE(\ell) = y(\ell) \, \Delta m \, g = \ell \, g \, \sin(\theta) \rho \, \Delta \ell$. Adding these up in the limit as $\Delta \ell \to d\ell$ gives

$$KE = \int_0^L dKE(\ell) = \int_0^L \frac{1}{2} \rho \left(\ell^2 \, \dot{\theta} \right)^2 \, d\ell = \frac{1}{2} \rho \left(\dot{\theta} \right)^2 \int_0^L \ell^2 \, d\ell = \frac{1}{2} \rho \left(\dot{\theta} \right)^2 \frac{\ell^3}{3} \Big|_0^L = \frac{1}{2} \rho \left(\dot{\theta} \right)^2 \frac{L^3}{3}.$$

This looks different than what we were expecting, namely $KE = \frac{1}{2} \frac{M \, L^2}{3} \dot{\theta}^2$. However, once we note that $\rho \, L = M$, the above becomes

$$\boxed{KE = \frac{1}{2} L \, \rho \left(\dot{\theta} \right)^2 \frac{L^2}{3} = \frac{1}{2} M \, \frac{L^2}{3} \, \left(\dot{\theta} \right)^2 .}$$

Hence, $I_z = \frac{1}{3} \, M \, L^2$.

Note: When computing KE, you can leave in ρL or replace it with M; we'll accept either answer.

(g) The ratio, $\frac{KE_{\text{lumped}}}{KE_{\text{distributed}}} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$ and thus the relative error is 25%. Thus, NO, it does not satisfy the folkloric adage, "10% is close enough for engineering work!"

However: When using 3-segments,

$$\frac{KE_{\text{lumped}}}{KE_{\text{distributed}}} = \frac{35}{36},$$

which means the error is one part in 36 or less than 3%. That's pretty incredible!

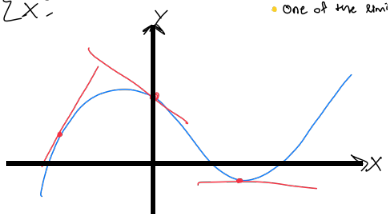
ROB 201 CHAPTER 5 STUDY GUIDE

Differentiation

☐ The Derivative of a function f at a point x is the slope of the tangent line to the graph f at that point.

Def: $f'(x) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$

Ex:



*A function is NOT differentiable if:
- Both two sided limits exist and are finite, but have diff values
- one of the limits is infinite
- One of the limits DNE

Ex: $f(x) = 5x^4 + 3x - 8$

$f'(x) = 20x^3 + 3$

Software Tools for Differentiation

Symbolic

Adv:
- Compute exact derivatives
- Analysis
- Closed-form derivatives

Dis-Adv:
- Expression swell

Numerical (Finite Differences)

Adv:
- Simplest Implementation
- Any Differentiable function

Dis-Adv:
- inaccurate w/ large step-size
- expensive to compute
- poor iteration

Automatic

Adv:
- Exact derivatives to machine precision
- Wide range of functions

Dis-Adv:
- Overkill
- Specialized Software or libraries

Key Properties

Proposition 5.9: Differentiation Rules

Here are the rules of differentiation:

(a) **Sum/Difference Rule**: If $f(x) = g(x) + h(x)$ or $f(x) = g(x) - h(x)$, then

$$f'(x) = g'(x) + h'(x) \text{ or } f'(x) = g'(x) - h'(x).$$

(b) **Product Rule**: If $f(x) = g(x) \cdot h(x)$, then

$$f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x).$$

In particular, if $g(x) = c$, a constant, then

$$f'(x) = c \cdot h'(x).$$

(c) **Quotient Rule or Ratio Rule**: If $f(x) = \frac{g(x)}{h(x)}$ and $h(x) \neq 0$, then

$$f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{(h(x))^2}.$$

In particular, if $g(x) = c$, a constant, then

$$f'(x) = -c \cdot \frac{h'(x)}{(h(x))^2}.$$

(d) **Chain Rule**: If $f(x) = g(h(x))$, then

$$f'(x) = g'(h(x)) \cdot h'(x).$$

(e) **Chain Rule** (Another way to state it): If $y = f(u)$ and $u = g(v)$, then

$$\frac{dy}{dv} = \left.\frac{df(u)}{du}\right|_{u=g(v)} \cdot \frac{dg(v)}{dv}.$$

The result is a function of v because all of the u's in $\frac{df(u)}{du}$ have been substituted by $u = g(v)$. Indeed, the meaning of the notation $\left.\frac{df(u)}{du}\right|_{u=g(v)}$ is "the derivative of f with respect to u, **evaluated at** $u = g(v)$." Using the prime notation for the derivative, we can also write it as $f'(g(v)) \cdot g'(u)$. **Find one of these forms of the Chain Rule that you can master and stick with it! The Chain Rule shows up everywhere in Calculus.**

(f) **Exponential Rule**: If $f(x) = a^x$, where a is a positive constant, then

$$f'(x) = \ln(a) \cdot a^x.$$

In particular, if $a = e$, Euler's constant, then

$$f'(x) = e^x.$$

(g) **Logarithm Rule**: If $f(x) = \log_a(x)$, where a is a positive constant, then

$$f'(x) = \frac{1}{x \ln(a)}.$$

In particular, if $a = e$, Euler's constant, then

$$f'(x) = \frac{1}{x}.$$

(h) **Power Rule**: If $f(x) = x^n$, where $n \in \mathbb{N}$, the counting numbers, then

$$f'(x) = nx^{n-1}$$

and there is no restriction on x.

Examples

(a) $f(x) = 3x^2 + 5x + 4 \quad f'(x) = 6x + 5$

(b) $f(x) = (3x^2)(2x-1) \quad f'(x) = (6x)(2x-1) + (3x^2)(2)$

$12x^2 - 6x + 6x^2 \Rightarrow 18x^2 + 6x$

(c) $f(x) = \frac{6x^3+1}{9x} \quad f'(x) = \frac{(18x^2)(9x) - (6x^3+1)(9)}{(9x)^2} = \frac{162x^3 - 54x^3 + 9}{81x^2}$

(d) $f(x) = \sin(3x^2+x) \quad f'(x) = \underbrace{\cos}_{\text{derivative of outside}}\underbrace{(3x^2+x)}_{\text{leave inside}} \cdot \underbrace{(6x+1)}_{\text{derivative of inside}}$

(f) $f(x) = e^x \quad f'(x) = e^x$

(g) $f(x) = \ln(x+5) \quad u = x+5 \quad u' = 1 \quad f'(x) = \frac{1}{x+5}$

(h) $f(x) = 9x^3 \quad f'(x) = 27x^2$

L'Hôpital's Rule

- For Limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Ex:
$\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{0}{0} \to \lim_{x \to 0} \frac{\cos(x)}{1} = 1$

Partial Derivatives

- Derivative of a multivariable function with respect to one variable

Ex: $f(x,y) = x^2 + y^2$ $\frac{\partial f}{\partial x} = 2x$ $\frac{\partial f}{\partial y} = 2y$

Jacobians

- Matrix w/ all first-order partial derivatives of a vector-valued function

Ex: $F(x,y) = \begin{pmatrix} x^2+y \\ x-y \end{pmatrix}$

$J = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 1 \\ 1 & -2y \end{pmatrix}$

Gradients

- Vector of partial derivatives of a scalar function

Ex: $f(x,y) = x^2 + y^2$ $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

Monotonicity

Function ($f(x)$)	Derivative ($f'(x)$)	Monotonicity
x^3	$3x^2$	Strictly Increasing
$-x^3$	$-3x^2$	Strictly Decreasing
$\ln(x)$	$\frac{1}{x}$	Strictly Increasing for $x > 0$
e^x	e^x	Strictly Increasing
e^{-x}	$-e^{-x}$	Strictly Decreasing
$\tan(x)$	$1 + \tan^2(x)$	Strictly Increasing (in its domain)
$\frac{1}{1+x^2}$	$-\frac{2x}{(1+x^2)^2}$	Strictly Decreasing for $x \geq 0$
$\frac{1}{1+x^2}$	$-\frac{2x}{(1+x^2)^2}$	Strictly Increasing for $x \leq 0$
$\frac{x^2}{x^2+1}$	$\frac{2x}{(x^2+1)^2}$	Strictly Increasing for $x \geq 0$

Taylor and Maclaurin Polynomials

Taylor Polynomial:

A Taylor polynomial of a function $f(x)$ centered at $x = a$ approximates $f(x)$ using the function's derivatives at a. The formula for the n-th degree Taylor polynomial is:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin Polynomial:

A special case of the Taylor polynomial centered at $a = 0$. The formula for the n-th degree Maclaurin polynomial is:

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

ROB 201 CHAPTER 6 STUDY GUIDE

Path Length

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Ex: $y = f(x) = x^2$ from $x=0$ to $x=1$

$$S = \int_0^1 \sqrt{1 + (2x)^2}\, dx$$

≈ 1.479

For a function

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex: $x(t) = \sin(t)$ $y(t) = \cos(t)$ from $t=0$ to $t=\pi$

$$S = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$\frac{dx}{dt} = \cos(t)$ $\quad dy(t) = -\sin(t)$

$S = \sqrt{(\cos(t))^2 + (-\sin(t))^2} = \sqrt{1} = 1$

Arc Length for Parametric Curve

Root Finding

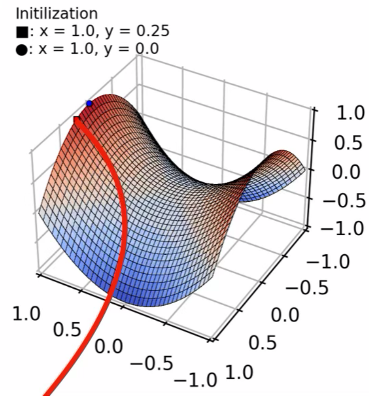
- Aims to locate solutions x for $f(x) = 0$

Linear Approximation Formula:
- $f(x) \approx f(x_0) + \frac{\partial f(x_0)}{\partial x} \cdot (x - x_0)$ Root Finding
- $f(x) \approx f(x_0) + \nabla f(x_0) \bullet (x - x_0)$ Minimization

Gradient Descent

- Optimization algorithm used to minimize a function by iteratively moving in the direction of steepest descent.

① Start w/ initial Guess ② calculate gradient
③ update parameters ④ iterate



- $f''(x) > 0 \rightarrow$ local min
- $f''(x) < 0 \rightarrow$ local max
- $f''(x) = 0 \Rightarrow$ inconclusive, possible inflection point

Ex:
$f(x) = x^2 - x^3 + x^4$
$f'(x) = 2x - 3x^2 + 4x^3$
$f''(x) = 2 - 6x + 12x^2$
$f''(0) = 2 - 6(0) + 12(0)^2 = 2$
$2 > 0$ LOCAL MIN

Lagrange Multipliers

- technique to find local maxima or minima subject to one or more constraints

Ex w/ single equality constraint:

Minimize xy to the constraint $x^2 + y^2 = 1$

① Lagrangian function $L(x, \lambda) := f(x) + \lambda g(x)$

$\hookrightarrow L(x, y, \lambda) := (x + y) + \lambda (x^2 + y^2 - 1)$

② Solve for stationary points of L

$$\frac{\partial L(x,y,\lambda)}{\partial x} = \boxed{1 + 2x\lambda} \qquad \frac{\partial L(x,y,\lambda)}{\partial y} = \boxed{1 + 2y\lambda}$$

$$\frac{\partial L(x,y,\lambda)}{\partial \lambda} = \boxed{x^2 + y^2 - 1}$$

$1 + 2x\lambda = 0 \rightarrow 2x = -\frac{1}{\lambda} \rightarrow x = -\frac{1}{2\lambda}$

$1 + 2y\lambda = 0 \rightarrow y = -\frac{1}{2\lambda}$

③ Plug into Constraint

$0 = -1 + x^2 + y^2 = -1 + \left(\frac{1}{-2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2$

$= -1 + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = -1 + \frac{1}{2} \cdot \frac{1}{\lambda^2}$

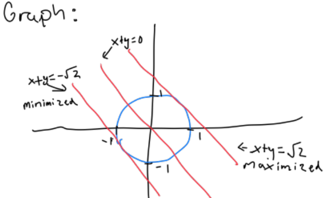
$\hookrightarrow \lambda = \pm \frac{\sqrt{2}}{2}$

$$\begin{bmatrix} x^* \\ y^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} \text{ and } \begin{bmatrix} x^* \\ y^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

Cost Function $(x, y) = x + y$

$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2}$ $\quad f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

Graph:



KE and PE 3-Link

- PE of a point mass: $V := m \cdot g \cdot y$
- KE of a point mass: $K := \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T D(q) \dot{q}$ ← KE of robot composed of rigid links

Ex: find mass inertia matrices for a point mass in a plane moving under the influence of gravity

$$K = \frac{1}{2} m \cdot (\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}^T \cdot \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$D(x, y) = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

Vocab for Constrained Optimization

- Cost Function: function wanted to minimize or maximize
- constraints:
  - Equality constraints $g(x) = 0$
  - Inequality constraints
- Feasible Set: all x where $g(x) = 0$
- Constraint Qualifications: set of conditions ensuring well defined solution
- Lagrange Multipliers: λ associated w/ each constraint to measure sensitivity of objective function's optimal value.
- KKT Conditions: conditions for solution to be optimal
- Slack Variables: introduced to turn inequality constraints to equality constraints

Lagrange's Equations

- Lagrangian summarizes dynamics of a system

$$\mathcal{L}(q, \dot{q}) := \underbrace{K(q, \dot{q})}_{\text{Kinetic Energy}} - \underbrace{V(q)}_{\text{Potential Energy}}$$

- Equations of Motion describes how a system evolves over time using a Lagrangian

$$\frac{d}{dt} \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}} - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q} = \Gamma$$

Ex: Apply Lagrange's EoM to an unconstrained point mass

$V = (x, y) = m \cdot g \cdot y$

$K = (x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \Rightarrow \mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - m \cdot g \cdot y$

- Compute Partials

$$\frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_1} = \frac{\partial}{\partial x} \left(\frac{1}{2} m \cdot (\dot{x}^2 + \dot{y}^2) - m \cdot g \cdot y \right) = 0$$

$$\frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_2} = \frac{\partial}{\partial y} \rightarrow = -mg \qquad \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_1} = \dot{x}_m \qquad \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_2} = \dot{y}$$

now... $\frac{d}{dt} \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_1} = \frac{d}{dt}(\dot{x}) = \ddot{x}$

$\frac{d}{dt} = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_2} = \frac{d}{dt}(\dot{y}) = \ddot{y}$

$$\hookrightarrow \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}}_{\frac{d}{dt} \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q}} - \underbrace{\begin{bmatrix} 0 \\ -mg \end{bmatrix}}_{\frac{\partial \mathcal{L}(q, \dot{q})}{\partial q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\Gamma}$$