Applications Treated to Date

- Drone position from IMU acceleration (numerical) → +"baby" K.F.
- CoM (of a planar body)
- Moment of Inertia (about z-axis)
- Arc Length (aka, Path Length)
- Roots of $F: \mathbb{R}^n \to \mathbb{R}^n$, $F(x^*) = 0_{n\times 1}$
- Gradient Descent w/o & w/ Equality Constraints

$$x^* = \arg\min_{g_i(x)=0 \;\; 1\le i\le m} f(x) \qquad \begin{cases} \Delta x \bullet \nabla f(x) < 0 \\ \Delta x \bullet \nabla g_i(x) = 0 \quad 1\le i \le m \end{cases}$$

- Lagrangian Dynamics (Planar, point masses)

$1\le i\le N$ $\quad \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} - \dfrac{\partial \mathcal{L}}{\partial q_i} = \Gamma_i$, $\mathcal{L} = KE - PE$, $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \Gamma$

{Robot Equations}

- Path Planning via RBF's
- Numerical "integration" of the Robot Equations

Summary Chapter 7 Antiderivatives and the Fundamental Thms of Calculus

Def $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Today: Video: Proving 1st FTC

Notation $F(x) = \int f(x)\,dx = \int f(y)\,dy$

No limits   No limits

Three types of Integrals

- Definite Integral $\int_a^b \underbrace{f(x)}_{\text{integrand}}\,dx$
- Indefinite Integral $\int_a^x \underbrace{f(y)}_{\text{integrand}}\,dy$
- Antiderivative $\int \underbrace{f(x)}_{\text{integrand}}\,dx$

(a) $f_a(x) = x \cdot \ln(x) - x$ $f_a'(x) = \ln(x) + \cancel{\frac{x}{x}} \cancel{-1} = f_b(x)$

(b) $f_b(x) = \ln(x)$ $f_b'(x) = \frac{1}{x}$

(c) $f_c(x) = \underbrace{\ln(x^2)}_{2\cdot\ln(x)}$ $f_c'(x) = \frac{2}{x}$

} not anti-d's of anything in our list.

Determine if any of f_a, f_b, f_c is an antiderivative of one of f_a, f_b, f_c!

$[f_a(x)]' = f_b(x) \iff f_a(x)$ is an antiderivative of $f_b(x)$

To test $F(x) = \int f(x)\,dx$, simply check $F'(x) = f(x)$. [Apply the def.]

Function	C	$\frac{x^{k+1}}{k+1} + C$	$e^x + C$	$\ln\lvert x\rvert + C$	$\sin(x) + C$	$\cos(x) + C$	$\tan(x) + C$	$\operatorname{atan}(x) + C$	**Antiderivative**
Derivative	0	x^k	e^x	$\frac{1}{x}$	$\cos(x)$	$-\sin(x)$	$1 + \tan^2(x)$	$\frac{1}{1+x^2}$	**Function**

Table 7.9: Common Elementary Functions and their Antiderivatives (Worth committing to memory to showcase your expertise). When doing antiderivatives, it is more convenient to have the monomials presented as given here than how we did the monomial entry in Table 5.14. In addition, we've extended the natural logarithm to negative numbers via an absolute value. Otherwise, the two tables are the same.

Fact: If $F(x)$ is an antiderivative of $f(x)$, then $F(x)+C$ is also an antiderivative for any constant $C \in \mathbb{R}$.

Why: $\frac{d}{dx}\left[F(x)+C\right] = F'(x) + \cancel{C'}^{\,0} = F'(x) = f(x)$.

Corollary: If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then

$F(x) - G(x) = $ a constant.

Why $\left[F(x) - G(x)\right]' = \cancel{F'(x)}^{\,f(x)} - \cancel{G'(x)}^{\,f(x)} = 0$

$[\text{a constant}]' = 0$. $\square$

Fact: Finding a candidate antiderivative is much harder!

In fact, that it is generally Impossible to find one within the class of **Elementary Functions**.

Fundamental Theorems of Calculus

First FTC (Says every continuous function has an antiderivative; **does not say you can compute it in closed form**). Suppose $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$. Then, $F:[a,b]\to\mathbb{R}$ defined by

$$F(x) := \int_a^x f(y)\,dy \quad , \quad x\in[a,b]$$

is continuous on $[a,b]$, differentiable on (a,b), and for all $a<x<b$,

$$F'(x) = f(x).$$

You will also see this written as

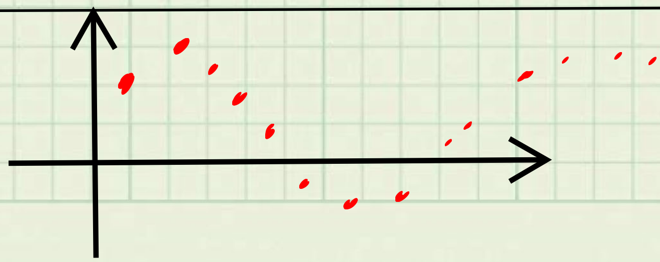
$$\frac{d}{dx}\underbrace{\int_a^x f(y)\,dy}_{F(x)} = f(x)$$

to underline that differentiation is the inverse of integration.

Relation to Project 1:

$$p(t_{k+1}) := p(t_k) + \int_{t_k}^{t_{k+1}} v(\tau)\, d\tau$$

produces a numerical evaluation of the antiderivative of $v(t)$ for all $t \in \{t_1, t_2, \dots, t_N\}$



Second FTC (Says every definite integral can be evaluated through antiderivatives, assuming you can find one) Suppose $f:[a,b]\to\mathbb{R}$ has antiderivative $F(x)$, then $f(x)$ is Riemann integrable. Moreover,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b := F(b) - F(a)$$

You will also see this written as

$$\int_a^b F'(x)\,dx = F(x)\Big|_a^b$$

or

$$\int_a^b \frac{dF}{dx}(x)\,dx = F(x)\Big|_a^b$$

or

$$\int_a^b dF(x) = F(x)\Big|_a^b$$

$\square$

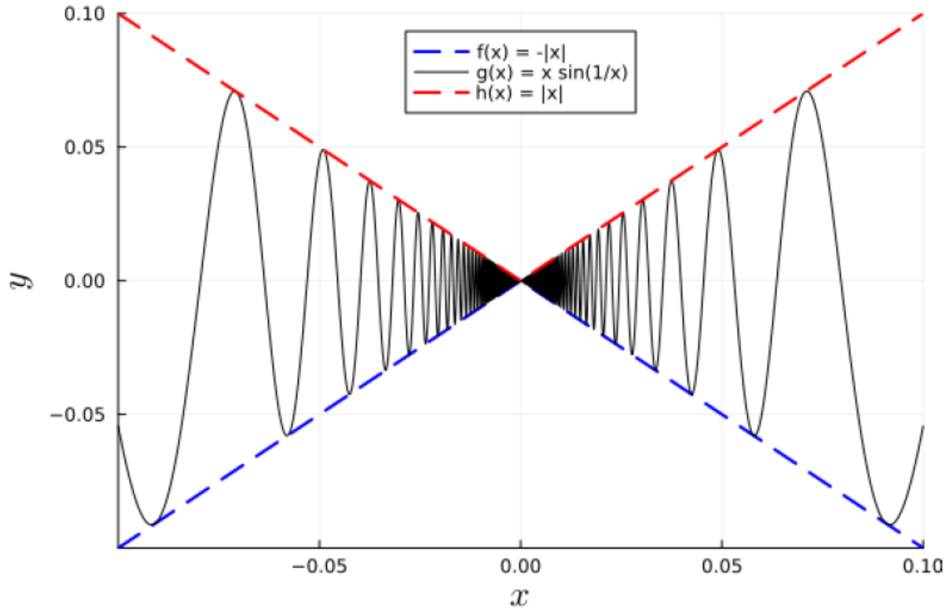


Figure 4.6: **(Squeeze or Sandwich Theorem:)** If a function is sandwiched between two functions that have a common limit, then the function itself also has the same limit.

Function	C	$\frac{x^{k+1}}{k+1}+C$	e^x+C	$\ln\lvert x\rvert+C$	$\sin(x)+C$	$\cos(x)+C$	$\tan(x)+C$	$\mathrm{atan}(x)+C$	**Antiderivative**
Derivative	0	x^k	e^x	$\frac{1}{x}$	$\cos(x)$	$-\sin(x)$	$1+\tan^2(x)$	$\frac{1}{1+x^2}$	**Function**

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Most engineers carry this table in their head and not much more.

What is $\frac{d}{dx}\,\mathrm{atan}(x)$?

Idea: Apply the Chain Rule

(*) $x = \tan(\mathrm{atan}(x)) =: f(g(x))$

$f(y) = \tan(y)$

$$g(x) = \text{atan}(x)$$

Differentiate both sides of (#)

$$1 = [1 + \tan^2(y)]\Big|_{y=\text{atan}(x)} * \underbrace{[\text{atan}(x)]'}_{\text{we seek this}}$$

$$1 = [1 + x^2] * [\text{atan}(x)]'$$

$$[\text{atan}(x)]' = \frac{1}{1+x^2}$$

HW [if you want]

- $\frac{d}{dx} \arcsin(x) = ?$
- Show $\frac{d}{dx} \ln|x| = \frac{1}{x}$ Hint $|x| = \begin{cases} x & x>0 \\ -x & x<0 \end{cases}$

$$\frac{d}{dx}(\ln(\underbrace{-x}_{|x|})) = ? \quad \text{for } x<0$$

Let's practice integrals via the 2nd FTC

Recall: $F(x)\Big|_a^b := F(b) - F(a)$

(a) $\int_0^5 (2x^3 + e^x)\,dx = \left(2\frac{x^4}{4} + e^x\right)\Big|_0^5$

$= \left(\frac{1}{2}625 + e^5\right) - (0+1)$

= Arithmetic

(b) $\int_1^e \left(\frac{1}{x} + \frac{1}{1+x^2}\right)dx = \left(\ln|x| + \operatorname{atan}(x)\right)\Big|_1^e$

= Arithmetic

(c) $\int_0^1 \frac{x}{1+x^2}\,dx$ = No antiderivative in our table :(

Hence, we need methods for finding antiderivatives

The Art of the Antiderivative by Inverting Differentiation Rules

A.) Power Rule $\frac{d}{dx}\left(x^{k+1}\right) = (k+1)x^k$

$x \in \mathbb{R}$, $k = 0, 1, 2, \ldots$

By the Second FTC $\int x^k \, dx = \frac{x^{k+1}}{k+1} + C$

Similarly,

$$\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad , x > 0, \alpha \in \mathbb{R}, \alpha \neq -1$$

Find an antiderivative for $3\sqrt{x}$

$$\int 3\sqrt{x} \, dx = \int 3x^{\frac{1}{2}} dx = 3\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{\cancel{3}\, x^{3/2}}{\cancel{3}/2} + C$$

$$= 2x^{3/2} + C$$

B) Fundamental Rule $\int f'(x) \, dx = f(x) + C$

If we recognize the integrand as the derivative of a known function, then it's game over!

$$\int \frac{1}{1+x^2}\,dx = \operatorname{atan}(x) + C$$

C. Inverting the Chain Rule by u-substitution

why u? Because mathematicians commonly use $u(x)$ as a change of variable

Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$

Issue $\int f'(g(x))\,g'(x)\,dx$ can be hard to recognize

Change of variable

$$u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$$

$$\Rightarrow \frac{du}{dx}\cdot dx = g'(x)\,dx$$

$$\Rightarrow du = g'(x)\,dx$$

Hence

$$\int \underbrace{f'(g(x))}_{f'(u)}\,\underbrace{g'(x)\,dx}_{du} = \int f'(u)\,du = f(u) + C$$

$$= f(g(x)) + C$$

Example

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln(1+x^2) + C$$

$u(x) = 1+x^2$

$du = 2x\,dx$

because $1+x^2 > 0$