ROB 201 - Calculus for the Modern Engineer
HW #4

Prof. Grizzle

Check Canvas for due date and time

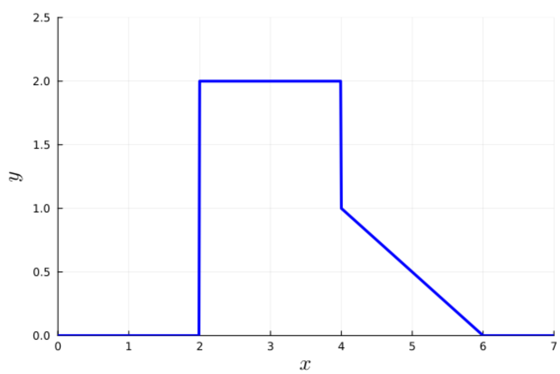
Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete and turn in.

1. (a) Create a "Cheat Sheet" for Chapters 3 and 4 of the textbook. You'll receive the same score for a handwritten solution as a typeset solution. Here is an example from ROB 101.

   (b) Note any material where you found the explanation confusing or difficult to master.

2. Solids of revolution.

   (a) Sketch BY HAND the solid of revolution corresponding to rotating the image below about the $\boldsymbol{y}$**-axis**. Don't worry: we're not looking for art-school-quality sketches.



   (b) Compute the volume of the solid of revolution defined by the two functions

   - $f : [0, 2] \to \mathbb{R}$ by $f(x) = 2x$
   - $g : [0, 2] \to \mathbb{R}$ by $g(x) = x$,

   when rotated about the $\boldsymbol{x}$**-axis**. You are not obliged to make a sketch of any kind, though usually, at least sketching the area out in the (x, y)-plane helps to set up the problem correctly. Show you work.

   In each case, pay attention to the axis of rotation. Parts (a) and (b) are not related. To be exceedingly pedantic, when setting up the problem in part (b), ignore the sketch in part (a); instead, use the given functions.

3. For each of the following sets or functions, compute the maximum value if it exists. If it does not exist, briefly state why, and then compute the supremum. See the hints for example solutions.

   (a) $A := \{x \in \mathbb{R} \mid \frac{x^2-1}{x^2+1} \leq 0.5\}$

   (b) $B := \{x \in \mathbb{R} \mid \frac{1}{x} > 2\}$

   (c) $f : [0, \infty) \to \mathbb{R}$ by $f(x) = 4 - (x-5)^4$

   (d) $g : [0, \infty) \to \mathbb{R}$ by $g(x) = \begin{cases} 0 & x = (2k+1)\frac{\pi}{2}, k \in \mathbb{Z} \\ \tan(x) & \text{otherwise.} \end{cases}$

4. Evaluate the following limits using any method you wish, and then provide a brief (ten words or less) description of your reasoning.

   (a) $\lim_{x \to \pi^-} e^{\sin(x)}$.

   (b) $\lim_{x \to 4^+} \dfrac{x^3 + 4x^2 - 16x - 64}{x - 4}$.

   (c) $\lim_{x \to 0} e^{\frac{1}{x}}$.

5. In each case below, provide ONE example of a function that meets the stated properties. So that your grader has a chance of understanding your thought process, provide a few hints as to why your answer is correct. Those hints could be in the form of text, a plot, or both. Note the domain and codomain are specified for each function. You can choose any range you wish as long as it is contained in the specified codomain.

   (a) $f : [0, 1] \to [2, 4]$ has exactly two points where it is discontinuous.

   (b) $f : (-1, 1) \to \mathbb{R}$ is continuous, $\lim_{x \to -1^+} |f(x)| = \infty$ and $\lim_{x \to 1^-} f(x) = 0$. **Note:** The absolute value allows you to have the limit equal positive or negative infinity; it's your choice.

   (c) $f : \mathbb{R} \to \mathbb{R}$ is bounded and not piecewise continuous.

6. Derive the quotient rule of differentiation, following the method used in the textbook to derive the chain rule and the product rule. Define clearly all notation that you use.

Hints

Prob. 1 Nothing to add.

Prob. 2 For part (b), you have to decide between the **disc-washer method** or the **shell method**. You do not have to make any sketches, though, of course, that may help you.

Prob. 3

- $C := \{x \in \mathbb{R} \mid \frac{2x-1}{x+1} \leq 1.0\}$

  Ans. $x^* := \max(C) = 2.0$

  We first note that $\frac{2x-1}{x+1}\Big|_{x=0} = -1$ and $\frac{2x-1}{x+1}\Big|_{x=5} = \frac{9}{6} = 1.5$, and thus there likely exists a point in between where the function equals 1.0. To find out, we solve

$$\frac{2x-1}{x+1} = 1$$
$$\Downarrow$$
$$2x - 1 = x + 1$$
$$\Downarrow$$
$$x = 2$$

  We note that because the set is defined with a less than or equal to sign, $2 \in C$. Is 2 the largest element?

  We next note that for $x > 2$, $\frac{2x-1}{x+1} > 1$ and hence $x \notin C$. Thus, if $y \in C$, then $y \leq 2$, proving that $x^* = 2$ is the maximum value in the set.

  It's not required, but if you really want to show that $x > 2 \implies \frac{2x-1}{x+1} > 1$, write $x = 2 + \delta$ for $\delta > 0$. Then,

$$\frac{2x-1}{x+1}\Big|_{x=2+\delta} = \frac{4 + 2\delta - 1}{2\delta + 1} = \frac{3 + 2\delta}{3 + \delta} = 1 + \frac{\delta}{3 + \delta} > 1$$

  for $\delta > 0$, and hence $2 + \delta \notin C$.

- $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \frac{4+(x-5)^4}{1+e^{-x}}$

  Ans. The function is unbounded and hence does not have a maximum. $x^* := \sup\limits_{x \in [0,\infty)} f(x) = \infty$

  From Chapter 2, we have a result on the limits of products and ratios, shown in the box below. Applying this result to our problem we have

  - $f(x) = \frac{4+(x-5)^4}{1+e^{-x}} =: \frac{\text{num}(x)}{\text{den}(x)}$
  - $\lim\limits_{x\to\infty} \text{den}(x) = \lim\limits_{x\to\infty} 1 + e^{-x} = 1$
  - $\lim\limits_{x\to\infty} \text{num}(x) = \lim\limits_{x\to\infty} 4 + (x-5)^4 = \infty$

  and hence by the Proposition, the result is established!

Proposition 2.44: Limits of Products and Ratios

Suppose that $g : (0, \infty) \to \mathbb{R}$ has a limit at infinity of one, i.e., $\lim_{x\to\infty} g(x) = 1.0$. Then for any function $f : (0, \infty) \to \mathbb{R}$,

$$\begin{aligned} \lim_{x\to\infty} f(x)\cdot g(x) &= \lim_{x\to\infty} f(x), \text{ and} \\ \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} f(x). \end{aligned} \tag{2.31}$$

In particular,

(a) if $\lim_{x\to\infty} f(x) = L$, for $L \in \mathbb{R}$, then $\lim_{x\to\infty} f(x)\cdot g(x) = L$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$;

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(b) if $\lim_{x\to\infty} f(x) = \pm\infty$, then $\lim_{x\to\infty} f(x)\cdot g(x) = \pm\infty$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \pm\infty$; and

(c) if $\lim_{x\to\infty} f(x)$ does not exist, then $\lim_{x\to\infty} f(x)\cdot g(x)$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ do not exist.

Note: If $\lim_{x\to\infty} g(x) = M \neq 1$, and $M \neq 0$, then $\frac{g(x)}{M}$ has limit one. Hence, using $f(x)\cdot g(x) = (Mf(x))\cdot\frac{g(x)}{M}$,and $\frac{f(x)}{g(x)} = \frac{\frac{1}{M}f(x)}{\frac{1}{M}g(x)}$ reduces the general case of the problem to the given (special) conditions.

If you remember (2.31)**, then the special cases follow immediately.**

Prob. 4 Nothing to add.

Prob. 5 KISS is a good policy.

Prob. 6 Here is your opening move: writing down the definition of the derivative yields,

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\left(\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}\right).$$

From here on, if you read and follow the textbook, it's straightforward. A bit long, but straightforward. You could also watch one of the suggested videos.

Final hints:

- Don't be afraid to place things over a common denominator.
- $\lim_{h\to 0}(g(x)\cdot g'(x)\cdot h) = g(x)\cdot g'(x)\cdot \lim_{h\to 0} h = 0.$

Solutions HW 04

Prob. 1

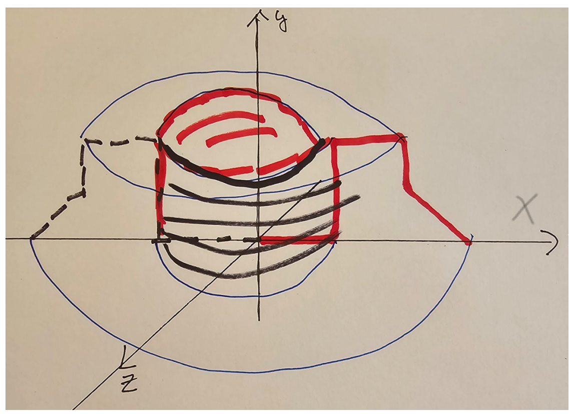
(a) Included at the end of the solution set

(b) These will vary by person, but some of the more challenging topics may have been:

i. Chap 03: Why some functions are Riemann integrable and others not
ii. Chap 03: How to handle integrals when the limits of integration are in a funny order
iii. Chap 03: How to integrate a function that has been shifted or scaled
iv. Chap 03: Center of mass
v. Chap 03: Moment of inertia
vi. Chap 04: Epsilon-delta definition of limits
vii. Chap 04: Formal definition of continuous functions
viii. Chap 04: Supremum and infimum of sets and functions

Prob. 2

(a)



(b) The two functions

- $f : [0,2] \to \mathbb{R}$ by $f(x) = 2x$
- $g : [0,2] \to \mathbb{R}$ by $g(x) = x$,

lie totally in the first quadrant. Hence, we can apply the Proposition on Solids of Revolution (given below).

Ans. $V_{x-axis} = 8\,\pi$

From the Proposition on Solids of Revolution, we know to apply the disc-washer method when rotating a solid about the x-axis. Hence,

$$
\begin{aligned}
V_{\text{x-axis}} &:= \int_a^b \pi \left(f^2(x) - g^2(x)\right)\,dx \\
&= \int_0^2 \pi \left((2x)^2 - x^2\right)\,dx \\
&= \int_0^2 \pi\, 3x^2\,dx \\
&= 3\pi \left.\frac{x^3}{3}\right|_0^2 \\
&= 8\,\pi
\end{aligned}
$$

Proposition 3.35: Volume of a Solid of Revolution

Key Assumptions: The area lies totally in the first-quadrant of the (x,y)-plane, that is, the quadrant where $x \geq 0$ and $y \geq 0$. Moreover, the area is delimited by two functions $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$, and for all $0\leq a\leq x\leq b$, $f(x)\geq g(x)$. The last condition means that f is the upper bound on the area, g is the lower bound, and the two functions do not enclose any "negative area".

Under the above assumptions, the volumes of the two standard solids of revolution are computed as follows:

$$V_{\text{x-axis}} := \int_a^b \pi\left(f^2(x)-g^2(x)\right)\,dx \quad \text{(called the \textbf{disc/washer method} for rotation about the } x\text{-axis)} \tag{3.54}$$

$$V_{\text{y-axis}} := \int_a^b 2\pi\, x\left(f(x)-g(x)\right)\,dx \quad \text{(called the \textbf{shell method} for rotation about the } y\text{-axis).} \tag{3.55}$$

The terms disc/washer and shell are defined in Fig. 3.24 and in the text below.

Prob. 3

(a) $A := \{x\in\mathbb{R} \mid \frac{x^2-1}{x^2+1}\leq 0.5\}$ $\boxed{\textbf{Ans.}\quad x^* := \max(A) = \sqrt{3}}$

Because $x^2+1>0$, we have $\frac{x^2-1}{x^2+1}\leq 0.5 \iff x^2-1\leq 0.5\left(x^2+1\right) \iff 0.5x^2-1.5\leq 0 \iff x^2\leq 3$.

Hence, $A=\{x\in\mathbb{R}\mid x^2\leq 3\}$ from which it follows that $x^* := \max(A)=\sqrt{3}$.

(b) $B := \{x\in\mathbb{R}\mid \frac{1}{x}>2\}$ $\boxed{\textbf{Ans.}\quad x^* := \sup(B) = \frac{1}{2}}$

The set does not have a maximum because

$$B := \{0 < x < \frac{1}{2}\}.$$

In particular, the point $\frac{1}{2}\notin B$, and thus the set does not have a maximum element. Of course, it always has a supremum and it is equal to $\frac{1}{2}$

(c) $f:[0,\infty)\to\mathbb{R}$ by $f(x)=4-(x-5)^4$ $\boxed{\textbf{Ans.}\quad f^* := \max_{x\geq 0}\left(4-(x-5)^4\right)=4}$

Because $-(x-5)^4\leq 0$ for all $x\geq 0$, its smallest value is zero and is achieved when $x=5$. That leaves 4 as the maximum and it is achieved for $x=5$.

(d) $g:[0,\infty)\to\mathbb{R}$ by $g(x)=\begin{cases}0 & x=(2k+1)\frac{\pi}{2}, k\in\mathbb{Z}\\ \tan(x) & \text{otherwise.}\end{cases}$ $\boxed{\textbf{Ans.}\quad g^* := \sup_{x\geq 0} g(x)=\infty}$

Because $\lim x\to\frac{\pi}{2}^{-1}\tan(x)=\infty$, we see that $g(x)$ is unbounded and hence does not have a maximum. It does have a supremum, which is $+\infty$.

Prob. 4 (a) $\lim_{x\to\pi^-} e^{\sin(x)}$. **Ans.** 1.0

We write $f(x) := e^x$ and $g(x) := \sin(x)$, both of which are continuous from our proposition on Common Continuous Functions, and note that $e^{\sin(x)} = f(g(x))$, the composition of two continuous functions. Hence, we can take the limit inside of both of them and compute

$$\lim_{x\to\pi^-}\sin(x)=\sin(\lim_{x\to\pi^-}x)=\sin(\pi)=0$$

$$\lim_{x\to\pi^-}e^{\sin(x)}=\exp\left(\lim_{x\to\pi^-}\sin(x)\right)=e^0=1.$$

(b) $\lim_{x\to 4^+}\dfrac{x^3+4x^2-16x-64}{x-4}$. **Ans.** 64

$x^3+4x^2-16x-64=(x-4)(x+4)^2$. Hence, $\lim_{x\to 4^+}\frac{x^3+4x^2-16x-64}{x-4}=\lim_{x\to 4^+}(x+4)^2=8^2=64$.

(c) $\lim_{x\to 0} e^{\frac{1}{x}}$. **Ans.** Undefined or does not exist (they mean the same thing).

$\lim_{x\to 0^-}\frac{1}{x}=-\infty$ and $\lim_{x\to 0^+}\frac{1}{x}=+\infty$. Hence, $\lim_{x\to 0} e^{\frac{1}{x}}$ does not exist. If you stop here, we'll give you a pass

Continuing with the better solution, we deduce from the above that $\lim_{x\to 0^-} e^{\frac{1}{x}}=0$ and $\lim_{x\to 0^+} e^{\frac{1}{x}}=+\infty$. Hence, $\lim_{x\to 0} e^{\frac{1}{x}}$ does not exist.

Prob. 5

(a) $f:[0,1]\to[2,4]$ has exactly two points where it is discontinuous.

Ans. $f(x):=\begin{cases}2 & 0\le x<\frac{1}{3}\\ 3 & \frac{1}{3}\le x<\frac{2}{3}\\ 4 & \frac{2}{3}\le x\le 1\end{cases}$ is discontinuous at $x=\frac{1}{3}$ and $x=\frac{2}{3}$ and continuous everywhere else.

(b) $f:(-1,1)\to\mathbb{R}$ is continuous, $\lim_{x\to -1^+}|f(x)|=\infty$ and $\lim_{x\to 1^-}f(x)=0$. **Note:** The absolute value allows you to have the limit equal positive or negative infinity; it's your choice.

Ans. $f(x):=\tan\left(\frac{\pi}{4}(x-1)\right)$ satisfies

i. The function $\frac{\pi}{4}(x-1)$ maps the open interval $(-1,1)$ to $(-\frac{\pi}{2},0)$ because $\frac{\pi}{4}(x-1)\Big|_{x=-1}=-\frac{\pi}{2}$ and $\frac{\pi}{4}(x-1)\Big|_{x=1}=0$. Hence, all we have done is map $\tan(y)$ to $\tan\left(\frac{\pi}{4}(x-1)\right)$ via the continuous function $y=\frac{\pi}{4}(x-1)$. Understanding this makes the rest of the calculations more or less obvious?

ii. $\lim_{x\to -1^+}f(x)=\lim_{x\to -1^+}\tan\left(\frac{\pi}{4}(x-1)\right)=\lim_{x\to -2^+}\tan\left(\frac{\pi}{4}x\right)=\lim_{x\to -\frac{\pi}{2}^+}\tan(x)=-\infty$

iii. $\lim_{x\to 1}f(x)=\lim_{x\to 1^-}\tan\left(\frac{\pi}{4}(x-1)\right)=\lim_{x\to 0^-}\tan\left(\frac{\pi}{4}x\right)=\lim_{x\to 0}\tan(x)=0y$

iv. Because $\tan(x)$ is continuous on $(-\frac{\pi}{2},0)$, it follows that $\tan\left(\frac{\pi}{4}(x-1)\right)$ is continuous on $(-1,1)$.

(c) $f:\mathbb{R}\to\mathbb{R}$ is bounded and not piecewise continuous.

Ans. $f(x):=\begin{cases}+1 & x\in\mathbb{Q}\\ -1 & \text{otherwise}\end{cases}$ takes values in $\{-1,1\}$ and is everywhere discontinuous.

Prob. 6 As in the textbook, using the prime notation for derivatives, we write the linear approximations for f and g about a point x as

$$f(x+h)\approx f(x)+f'(x)\cdot h \text{ for } h \text{ small} \iff \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)\quad(A)$$

$$g(x+h)\approx g(x)+g'(x)\cdot h \text{ for } h \text{ small} \iff \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=g'(x)\quad(B).$$

We assume that $g(x) \neq 0$. Then,

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\left(\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}\right) \text{ (opening move from the hint)}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right) \text{ (factor out } h\text{)}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)+f'(x)\cdot h}{g(x)+g'(x)\cdot h} - \frac{f(x)}{g(x)}\right) \text{ (apply (A) and (B))}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)+f'(x)\cdot h}{g(x)+g'(x)\cdot h} - \frac{f(x)}{g(x)}\right) \text{ (put over a common denominator)}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)\cdot g(x)+f'(x)\cdot h\cdot g(x) - f(x)\cdot g(x) - f(x)g'(x)\cdot h}{g(x)\cdot g(x)+g(x)\cdot g'(x)\cdot h}\right) \text{ (algebra)}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{f'(x)\cdot h\cdot g(x) - f(x)g'(x)\cdot h}{g(x)\cdot g(x)+g(x)\cdot g'(x)\cdot h}\right) \text{ (more algebra)}$$

$$= \lim_{h\to 0}\left(\frac{f'(x)\cdot g(x) - f(x)g'(x)}{g(x)\cdot g(x)+g(x)\cdot g'(x)\cdot h}\right) \text{ (cancel } h\text{)}$$

$$= \left(\frac{f'(x)\cdot g(x) - f(x)g'(x)}{g(x)\cdot g(x)}\right) \text{ (take the limit)}$$

ROB 201 CHAPTER 3 STUDY GUIDE

Riemann Integral

Definite Integral to functions defined over a closed interval $[a,b]$.

- Partition of Interval: $\Delta x = \dfrac{b-a}{n}$ ($b-a$: interval, n: subintervals) $\quad x_i = a+(i-1)\Delta x \quad$ for $i = 0,1,2,\dots,n$

$\boxed{\text{Ex: } f(x) = x^2 \text{ over interval } [1,3] \text{ using } n = 4 \text{ subintervals}}$

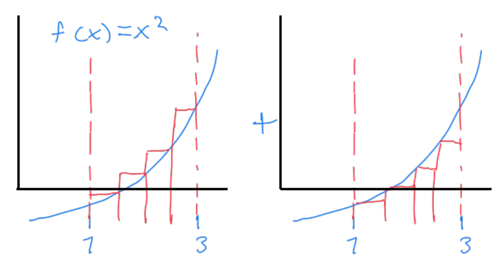
① Interval and subintervals
- interval: $[1,3]$ ($a = 1$, $b = 3$)
- subintervals 4 ($n = 4$)

② Calculate Δx:
$\Delta x = \frac{3-1}{4} = 0.5$

③ Define Partition Points x_i:
- $x_i := 1+(i-1)\cdot 0.5$ $\Rightarrow$
- $x_1 = 1+(1-1)\cdot 0.5 = 1$
- $x_2 = 1+(2-1)\cdot 0.5 = 1.5$
- $x_3 = 1+(3-1)\cdot 0.5 = 2$
- $x_4 = 1+(4-1)\cdot 0.5 = 2.5$

④ Compute $f(x_i)$ at each x_i:
- $f(x) = x^2$
- $f(x_1) = (1)^2 = 1$
- $f(x_2) = (1.5)^2 = 2.25$
- $f(x_3) = (2)^2 = 4$
- $f(x_4) = (2.5)^2 = 6.25$

⑤ Calculate Riemann Sum S_n:
$S_n = \sum_{i=1}^{4} f(x_i)\cdot \Delta x$
$S_4 = f(x_1)\cdot 0.5 + f(x_2)\cdot 0.5 + f(x_3)\cdot 0.5 + f(x_4)\cdot 0.5$
$S_4 = 1\cdot 0.5 + 2.25\cdot 0.5 + 4\cdot 0.5 + 6.25\cdot 0.5$
$\boxed{S_4 \approx 6.75}$



On the graph of $f(x) = x^2$, in bounds $[1,3]$ the Riemann Sum S_n combines the lower and upper sums

Non Riemann Integral Functions:
① Function w/ infinite discontinuity
② unbounded on interval
③ oscillate infinitely
④ Interval extends to infinity

Integral Notation

$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ ($+C$ For indefinite) $\quad$ Ex: $\int x^2 dx = \frac{x^3}{3} + C$

Integral Notation of a function: $\int_a^b f(x)\,dx := \begin{cases} \lim_{n\to\infty} \text{Area}_n^{\text{low}} \\ \lim_{n\to\infty} \text{Area}_n^{\text{up}} \end{cases}$

Ex: $\int_1^3 x^2 dx \Leftarrow$ Definite Integral

Ex: $\int_1^x x^2 dx \Leftarrow$ Indefinite Integral

- $\int_a^b f(x)dx$ when $a > b \Rightarrow \int_a^b f(x)dx := -\int_b^a f(x)dx$

Ex: $\int_5^2 x^2 dx = -\int_2^5 x^2 dx$

Shift Property

Evaluate definite Integral
$\int_0^1 (x-2)^2 dx = \int_{0-2}^{1-2} x^2 dx$
$= \int_{-2}^{-1} x^2 dx = \frac{x^3}{3}\Big|_{-2}^{-1}$
$= \frac{(-1)^3}{3} - \frac{(-2)^3}{3} = -\frac{1}{3} + \frac{8}{3} = \boxed{2\tfrac{1}{3}}$

Scaling Property

$\int_0^1 (2x)^2 dx = 4\int_0^1 x^2 dx = 4\frac{x^3}{3}\Big|_0^1 = 4\left(\frac{1}{3} - 0\right) = \frac{4}{3}$
$\int_0^1 (2x)^2 dx = \frac{1}{2}\int_0^2 x^2 dx = \frac{1}{2}\frac{x^3}{3}\Big|_0^2$
$= \frac{1}{2}\left(\frac{8}{3} - 0\right) = \boxed{\tfrac{4}{3}}$

One Sided Limits

- $f(x)$ is continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$

Ex: $f(x) = \begin{cases} x^2 & x < 1 \\ 2x-1 & x \geq 1 \end{cases}$ Is $f(x)$ continuous at $x = 1$?

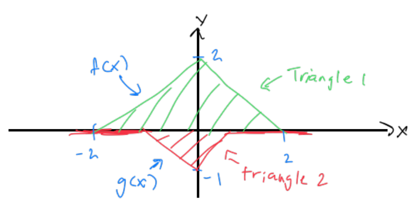
- left hand limit: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$
- right hand limit: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x-1) = 2\cdot 1 - 1 = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$ ✓

continuous at $x = 1$ ✓

Area between two functions

Ex: $f(x) = \begin{cases} 2-|x| & |x| \leq 2 \\ 0 & \text{o.w.} \end{cases} \quad g(x) = \begin{cases} |x|-1 & |x| \leq 1 \\ 0 & \text{o.w.} \end{cases}$



- Area $= \underbrace{\frac{1}{2}\cdot 4\cdot 2}_{1} + \underbrace{\frac{1}{2}\cdot 2\cdot 1}_{2} = 5.0$

Simpson's Rule

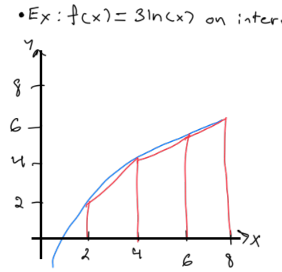
Uses Parabolas to approximate area

$\alpha \cdot \dfrac{(\Delta x)^3}{12} + \gamma \cdot \Delta x$

Trapezoid Rule

$\lim_{n\to\infty} \sum_{i=1}^{n} 0.5 \cdot \dfrac{f(x_i) + f(x_{i+1})}{x_{i+1} - x_i}$

- Ex: $f(x) = 3\ln(x)$ on interval $[2,8]$



use trapezoids for closer approximation

□ Balistic Motion

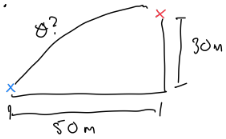
$$p(t) \xrightarrow{\text{Derivative}} v(t) \xrightarrow{\text{Derivative}} a(t)$$

$$p(t) \xleftarrow{\text{Integral}} v(t) \xleftarrow{\text{Integral}} a(t)$$

position   velocity   acceleration

□ $v(t) = v_0 + \int_{t_0}^{t} a(\tau)\, d\tau$

Ex:



- $m = 140g$
- $v_0 = 15 m/s$
- $g = 9.81 m/s^2$

$$\underbrace{\begin{bmatrix} v_x(0) \\ v_y(0) \end{bmatrix}}_{\text{start velocity}} = \begin{bmatrix} 15\cos(\theta) \\ 15\sin(\theta) \end{bmatrix}$$

$$p(T) = \begin{bmatrix} 15\cos(\theta)\cdot T \\ 15\sin(\theta)\cdot T - \frac{1}{2}(0.14)(9.81)T^2 \end{bmatrix} = \begin{bmatrix} 50 \\ 30 \end{bmatrix}$$

$$T = \frac{50}{15\cos(\theta)} \Rightarrow 30 = 50\tan(\theta) - 0.6867\left(\frac{10}{3\cos\theta}\right)^2$$

$\hookrightarrow$ rad $\simeq 0.5404$ $\theta \simeq 41$

Robot Links

Key Concepts

1. **Newton's Second Law:**
   - Formula: $F = ma$
   - Describes the motion of a single point mass.
2. **Robot Link as a Body:**
   - A link is composed of infinitely many point masses.
   - Using calculus, we treat the link as a continuous collection of points to understand its properties.
3. **Total Mass:**
   - The mass of a link is an integral over its volume.
   - Formula: $M_T = \rho \cdot V = \rho \cdot h \cdot A$
   - For a function $f(x)$ representing the upper boundary and $g(x)$ the lower boundary, the total mass M_T is given by:
     $M_T = \rho \cdot h \cdot \int_{x_{\min}}^{x_{\max}} (f(x) - g(x))\, dx$
4. **Center of Mass:**
   - The center of mass is the point where the total mass of the system is balanced.
   - For discrete point masses, the x-coordinate of the center of mass x_c is:
     $x_c = \frac{\sum_{i=1}^{N} m_i \cdot x_i}{\sum_{i=1}^{N} m_i}$
   - This formula represents a weighted average of the positions of the point masses, where each position is weighted by its corresponding mass.

Center of Mass for Point Masses

To find the center of mass for a system of discrete point masses:

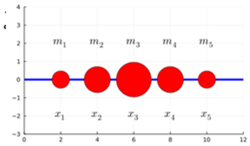
1. **Total Mass:**
   $M_T = \sum_{i=1}^{N} m_i$
2. **Total x-Moment about the Origin:**
   $\text{Total x-Moment} = \sum_{i=1}^{N} m_i \cdot x_i$
3. **x-coordinate of the Center of Mass:**
   $x_c = \frac{\text{Total x-Moment}}{M_T} = \frac{\sum_{i=1}^{N} m_i \cdot x_i}{\sum_{i=1}^{N} m_i}$

Notation:

Notation

- A: Total area of a link.
- h: Thickness of a link.
- V: Total volume of a link ($V = A \cdot h$).
- ρ: Material density of a link.
- M_T: Total mass of a link ($M_T = \rho \cdot V = \rho \cdot h \cdot A$).

Ex:



$(m_1, m_2, m_3, m_4, m_5) = (2, 3, 4, 3, 2)$

$(x_1, x_2, x_3, x_4, x_5) = (2, 4, 6, 8, 10)$

$M_T = 14$ x-moment $= m_1 \cdot x_1 + m_2 \cdot x_2 + m_3 \cdot x_3 + m_4 \cdot x_4 + m_5 \cdot x_5 = 84$

$$\boxed{x_c = \frac{84}{14} = 6 = x_3}$$

Simpson's Rule

□ uses Parabolas to approximate area

$$\alpha \cdot \frac{(\Delta x)^3}{12} + \gamma \cdot \Delta x$$

ROB 201 CHAPTER 4 STUDY GUIDE

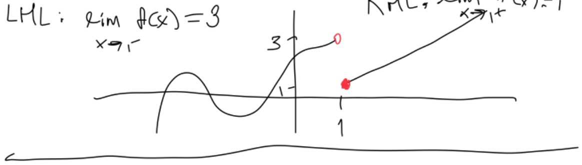
Left and Right Limits

- Left hand limit : $\lim_{x \to x_0^-} f(x)$
- Right hand limit : $\lim_{x \to x_0^+} f(x)$

Ex:

LHL : $\lim_{x \to 1^-} f(x) = 3$

RHL : $\lim_{x \to 1^+} f(x) = 1$



Ex: $f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$ $\lim_{x \to 1^-} f(x) = 2(1)+1 = 3$

$\lim_{x \to 1^+} f(x) = 1^2 = 1$

Continuity

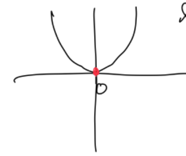
$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$

$\Downarrow$

$\lim_{x \to x_0} f(x)$

- A function is continuous at $x = x_0$ if $\Rightarrow$
  1. $\lim_{x \to x_0} f(x)$ exists
  2. $f(x_0)$ is defined
  3. $\lim_{x \to x_0} f(x) = f(x_0)$

Ex: $f(x) = x^2$



$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \rightarrow$ function continuous

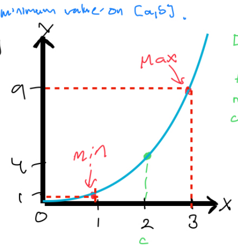
Function continuous on both the range and everywhere

Key Properties

- If f is continuous on $[a,b]$ and k is between $f(a)$ and $f(b)$, then there exists c in (a,b) such that $f(c) = k$ (Intermediate Value Theorem)
- Extreme Value Theorem: if f is continuous on a closed interval $[a,b]$, then f attains a maximum and a minimum value on $[a,b]$.

Ex: For $f(x) = x^2$ on $[1,3]$

- $f(x)$ is continuous



- Since the function ranges from $f(x) = 1$ to $f(x) = 3$ on $[1,3]$, there must be a point when $c = 2$ b/c $a \leq c \leq b$.

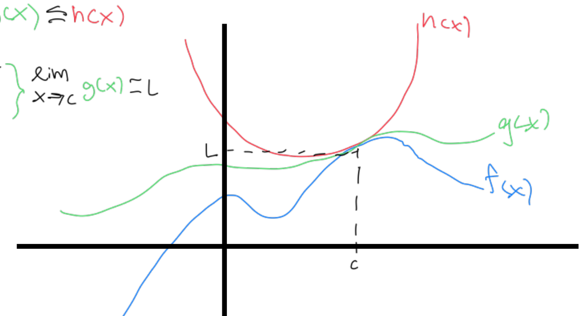
Squeeze Theorem

- $f(x) \leq g(x) \leq h(x)$

Ex:

- $\lim_{x \to c} f(x) = L$
- $\lim_{x \to c} h(x) = L$

$\left.\right\}$ $\lim_{x \to c} g(x) = L$

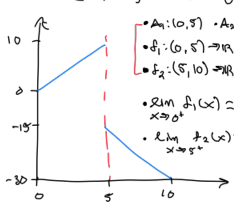


Piecewise Continuity

- A function is piecewise continuous if it is continuous on each piece of its domain, with only a finite number of discontinuities

Ex:

$f : [0, 10] \rightarrow \mathbb{R}$ by $f(x) = \begin{cases} 2x & 0 \leq x < 5 \\ -3x & 5 \leq x \leq 10 \end{cases}$



- $A_1 : [0, 5)$ • $A_2 (5, 10)$
- $f_1 : (0,5) \rightarrow \mathbb{R}$ $f_1(x) = 2x$
- $f_2 : (5, 10) \rightarrow \mathbb{R}$ $f_2(x) = -3x$

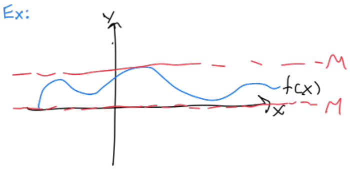
$\}$ Continuous

- $\lim_{x \to 0^+} f_1(x) = 0 = f(0)$
- $\lim_{x \to 5^-} f_1(x) = 10$
- $\lim_{x \to 5^+} f_2(x) = -15$
- $\lim_{x \to 10^-} f_2(x) = -30 = f(10)$

Unbounded VS Bounded

- f is bounded from below if there exists $M > -\infty$ such that $f(x) \geq M$ for all $x \in A$.

Ex:

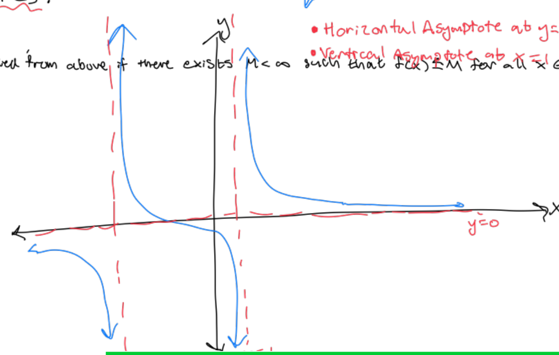


Unbounded $\Rightarrow$ If

- Asymptotes:

- f is bounded from above if there exists $M < \infty$ such that $f(x) \leq M$ for all $x \in A$.

- Horizontal Asymptote at $y = 0$
- Vertical Asymptote at $x = 1$, $x = -3$



Max/Min Element +Sup/Inf Values

Ex: $A := [3, 4]$ $B := (3, 4)$ (Max/Min Element)

- $\text{Min}\{A\} = 3$
- $\text{Max}\{A\} = 4$
- $\text{Min}\{B\}$ DNE
- $\text{Max}\{B\} = 4$

- Infimum and Supremum (greatest / least upper Bound)

$A := [3, 4]$ $B := (3, 4)$

- $\inf\{A\} = 3$
- $\sup\{A\} = 4$
- $\inf\{B\} = 3$
- $\sup\{B\} = 4$

Proof of Prop. 4.18 (Common Continuous Functions

We'll prove a few of the dozen or more functions listed in Prop. 4.18: the following functions are continuous for all $x \in \mathbb{R}$ unless indicated otherwise:

(a) Monomials: $f(x) = x^k$

(b) Polynomials: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

(c) Sine: $f(x) = \sin(x)$

(d) Natural Exponential: $f(x) = e^x$

(e) Natural Logarithm (continuous for $x > 0$): $f(x) = \ln(x)$

(f) Power (continuous for $x > 0$ and $y \in \mathbb{R}$): $f(x) = x^y$

(g) Square Root (continuous for $x \geq 0$): $f(x) = \sqrt{x}$