ROB 201 - Calculus for the Modern Engineer
HW #3

Prof. Grizzle

Check Canvas for due date and time

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete and turn in.

1. Read Chapter 3 of our ROB 201 Textbook, `Calculus for the Modern Engineer`; you will find a copy on our Canvas site, in the `file` folder. Based on your reading of the Chapter, summarize in your own words:

   (a) the purpose of Chapter 03;

   (b) two things you found the most DIFFICULT.

   There are no "right" or "wrong" answers, but no answer means no points. The goal is to reflect a bit on what you are learning and why.

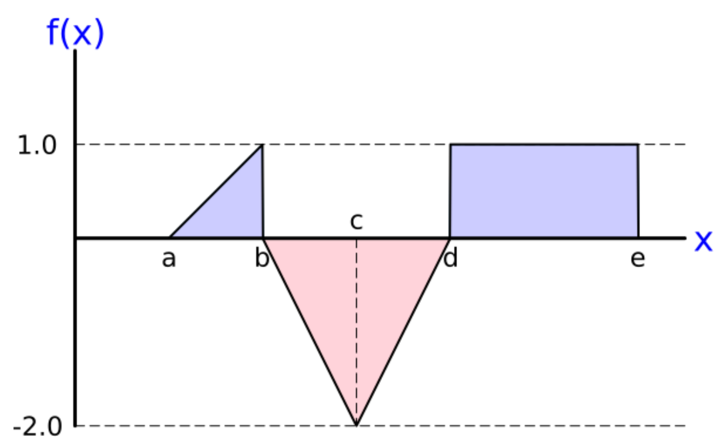
2. Answer any two of the following three **conceptual questions:**

   (a) Explain the concept of the Riemann Integral and its significance in calculus.

   (b) What is the difference between the "signed area under a function" and the "area under the absolute value of a function"?

   (c) Why do we need to impose conditions on functions to define the Riemann Integral?

3. The problem uses the plot below, consisting of three right triangles and a rectangle.



   (a) Compute $\int_a^e f(x)\,dx$. You will obtain a formula in terms of a, b, c, d, e; see the hint.

   (b) Compute $\int_a^c f(x)\,dx$.

   (c) Compute $\int_d^b f(x)\,dx$ (note the order of the limits of integration).

(d) If $\int_a^c f(x)\,dx = -3$ and $\int_c^e f(x)\,dx = 7$, what is the value of $\int_a^e f(x)\,dx$?

4. Suppose that $a < b$ and that $f : [a, b] \to \mathbb{R}$ is monotonically **decreasing**. Following the notation in the textbook, give a formula for the lower Riemann sum, $\mathrm{Area}_n^{\mathrm{Low}}$ of $f(x)$.

5. **Problems 5 and 6 are linked: your overall mission is to derive a cubic version of Simpson's Rule.**

The basic Simpson's Rule uses a quadratic function (polynomial) to estimate the area under a function $f : [a, b] \to \mathbb{R}$. Specifically, given $\Delta x > 0$, x_i, $x_{i+1} = x_i + \Delta x$, and $x_c := \frac{x_i + x_{i+1}}{2} = x_i + \frac{\Delta x}{2}$,

- it computes α, β and γ such that $q(x) := \alpha(x - x_c)^2 + \beta(x - x_c) + \gamma$ interpolates the function, that is, such that $q(x)$ satisfies
$$q(x_i) = f(x_i), \quad q(x_c) = f(x_c), \quad \text{and} \quad q(x_{i+1}) = f(x_{i+1}),$$
- and then it estimates $\int_{x_i}^{x_{i+1}} f(x)\,dx$ by the integral, $\int_{x_i}^{x_{i+1}} q(x)\,dx$, which has a nice closed-form solution, as given in the textbook.

In this problem, we seek to determine the coefficients of a cubic polynomial that interpolates $f : [a, b] \to \mathbb{R}$ through four equally spaced points, analogously to the basic Simpson's rule interpolating a quadratic polynomial through three equally spaced points in $[x_i, x_{i+1}]$.

To make the derivation "clean", we suggest the following notation and definitions:

- $h := \Delta x$
- $\bar{x}_a := x_i$
- $\bar{x}_b := x_i + h/3$
- $\bar{x}_c := x_i + 2h/3$
- $\bar{x}_d := x_i + h = x_{i+1}$
- $p(x) := \alpha_3(x - x_i)^3 + \alpha_2(x - x_i)^2 + \alpha_1(x - x_i) + \alpha_0$

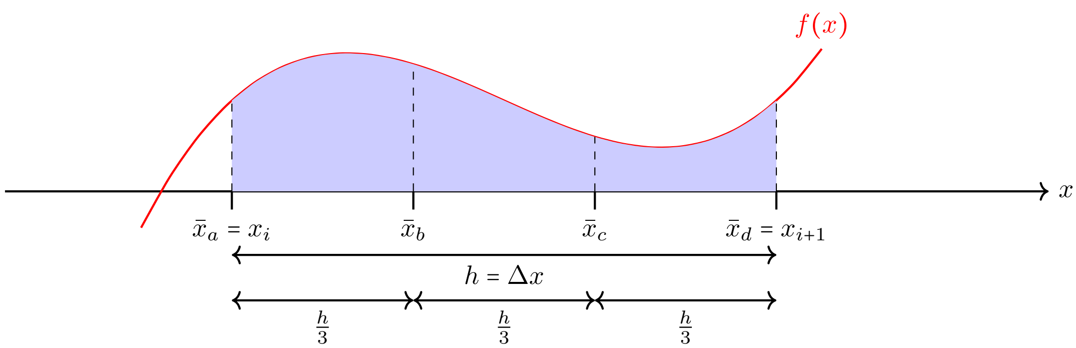


Figure 1: Simpson's 3/8 rule illustrated. Interpolating a function f with a cubic polynomial over four points: $\bar{x}_a = x_i$, $\bar{x}_b = x_i + \frac{h}{3}$, $\bar{x}_c = x_i + \frac{2h}{3}$, and $\bar{x}_d = x_{i+1} = x_i + h$.

To do: Set up four simultaneous equations for the unknown coefficients $\alpha_3, \ldots, \alpha_0$ such that the polynomial p interpolates f, that is, such that $p(x_0) = f(x_0), \ldots, p(x_3) = f(x_3)$. For your final answer, write the equations in the form

$$\Phi_{4\times 4} \cdot \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix},$$

where $\Phi_{4\times 4}$ is a 4×4 real matrix, with entries depending on h. Draw a box around $\Phi_{4\times 4}$.

Note: This problem does not involve Calculus. It only requires you to set up a system of equations. The matrix Φ should depend explicitly on h and its powers. **You are NOT asked to solve the equations.**

6. Using the cubic polynomial $p(x)$ from Prob. 5, the goal of this problem is to evaluate the integral $\int_{x_i}^{x_{i+1}} p(x)\,dx$ term-by-term using either the Shifting Property or a change of variable. To make your work easy to grade, please integrate the following four terms

   (a) $\int_{x_i}^{x_{i+1}} (x - x_i)^3\,dx$
   (b) $\int_{x_i}^{x_{i+1}} (x - x_i)^2\,dx$
   (c) $\int_{x_i}^{x_{i+1}} (x - x_i)\,dx$
   (d) $\int_{x_i}^{x_{i+1}} 1\,dx$;

   you do not need to include the coefficients, $\alpha_3, \ldots, \alpha_0$. **In each case, your answer should depend explicitly on $h := \Delta x = x_{i+1} - x_i$, and nothing else.**

Hints

Prob. 1 Write approximately 15 or more words for each part of the question.

Prob. 2 Read the Chapter. Google if necessary. Think!

Prob. 3 The area of a triangle is one-half base times height. Applying this to the the function $f(x)$ we have

$$\text{Area} = \frac{(b-a)}{2} - 2\,\frac{(d-b)}{2} + (e-d).$$

Because the Riemann integral and the area are the same thing for triangles and rectangles, we have

$$\int_a^e f(x)\,dx = \frac{(b-a)}{2} - 2\,\frac{(d-b)}{2} + (e-d) = -\frac{a}{2} + \frac{3b}{2} - 2d + e$$

It was not necessary to simplify the answer, meaning, if you had left the answer as $\frac{(b-a)}{2} - 2\,\frac{(d-b)}{2} + (e-d)$, you'd still earn full credit.

Prob. 4 No hints provided.

Prob. 5 Hints were given as part of the problem statement.

Prob. 6 So that you can check your work, $\int_{x_i}^{x_i+h} (x-x_i)^3\,dx = \frac{h^4}{4}$.