

monotonic function

$$\Delta x := \frac{b-a}{n} \quad a = x_1 < x_2 < \dots < x_n < x_{n+1} = b$$

$n+1$ points to define n -rectangles

$$A_i^{\text{low}} := h_i^{\text{low}} \cdot \Delta x$$

$$A_i^{\text{up}} := h_i^{\text{up}} \cdot \Delta x$$

$$\text{Area}_n^{\text{low}} := \sum_{i=1}^n A_i^{\text{low}} \leq A \leq \sum_{i=1}^n A_i^{\text{up}} = \text{Area}_n^{\text{up}}$$

IF $\lim_{n \rightarrow \infty} \text{Area}_n^{\text{low}} = \lim_{n \rightarrow \infty} \text{Area}_n^{\text{up}}$ exist & finite

THEN we define A as the common value \square

Archimedes \approx 200 BCE

Calculus \approx 1670

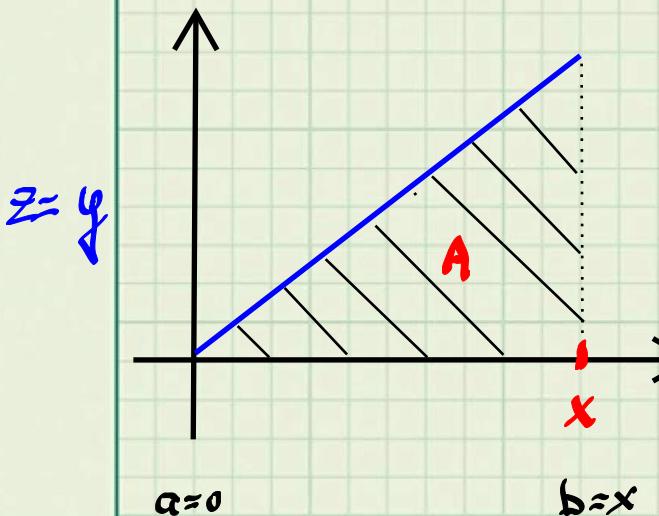
1870 year gap

We had to get through the dark ages and start asking deep questions about the stars and planets.

Sums of Integers to Power k

$$1^k + 2^k + \dots + n^k = \sum_{i=1}^n (i)^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + O(n, k-1)$$

lower-order terms



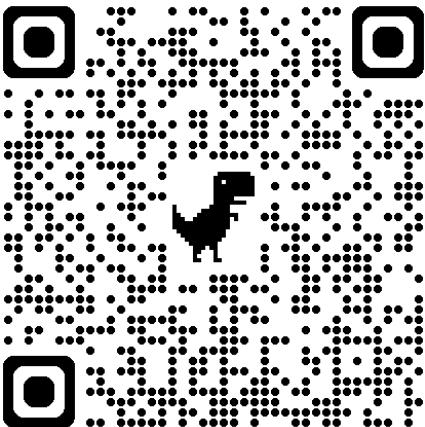
$A = \frac{x^2}{2}$

$1 + 2 + \dots + n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$

$$\text{Area}_{n, \text{low}} = \frac{x^2}{2} \left(1 - \frac{1}{n}\right) \xrightarrow{n \rightarrow \infty} \frac{x^2}{2}$$

$$\text{Area}_{n, \text{up}} = \frac{x^2}{2} \left(1 + \frac{1}{n}\right) \xrightarrow{n \rightarrow \infty} \frac{x^2}{2}$$

Today: Chap. 03 Definite Integration
as the Signed Area Under a Curve

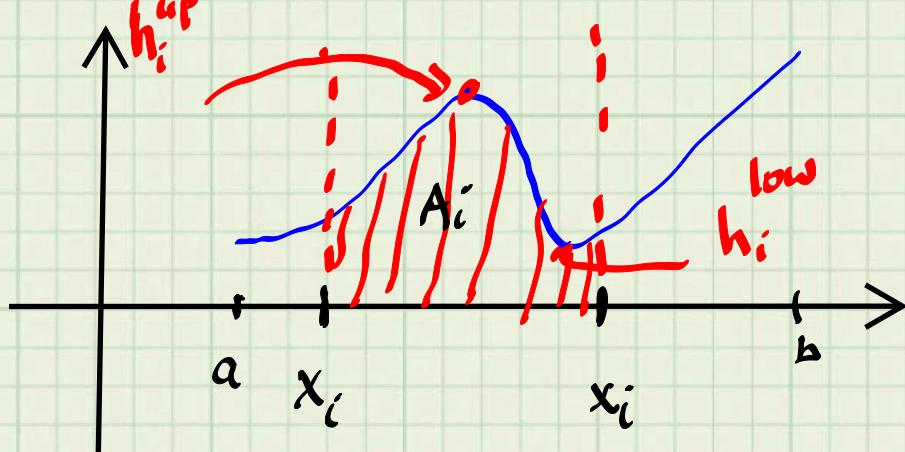


Three Types of Integrals

- definite integrals } This Chapter
- indefinite integrals }
- antiderivatives } After differentiation

95% of the integrals needed by engineers do not work via the antiderivatives.

How to handle functions that are not monotonic?



$$\Delta x = \frac{b-a}{n}$$

n # rectangles.

$$h_i^{\text{low}} := \min_{x_i \leq x \leq x_{i+1}} f(x)$$

$$h_i^{\text{up}} := \max_{x_i \leq x \leq x_{i+1}} f(x)$$

} always exist for continuous functions

$$A_i^{\text{low}} := h_i \cdot \Delta X$$

$$A_i^{\text{low}} \leq A_i \leq A_i^{\text{up}}$$

$$A_i^{\text{up}} := h_i^{\text{up}} \cdot \Delta X$$

Def. (Riemann-Darboux) Suppose

$f: [a, b] \rightarrow \mathbb{R}$ that admits max and min exist over all $[x_i, x_{i+1}] \subset [a, b]$.

Then the Riemann integral of $f(x)$

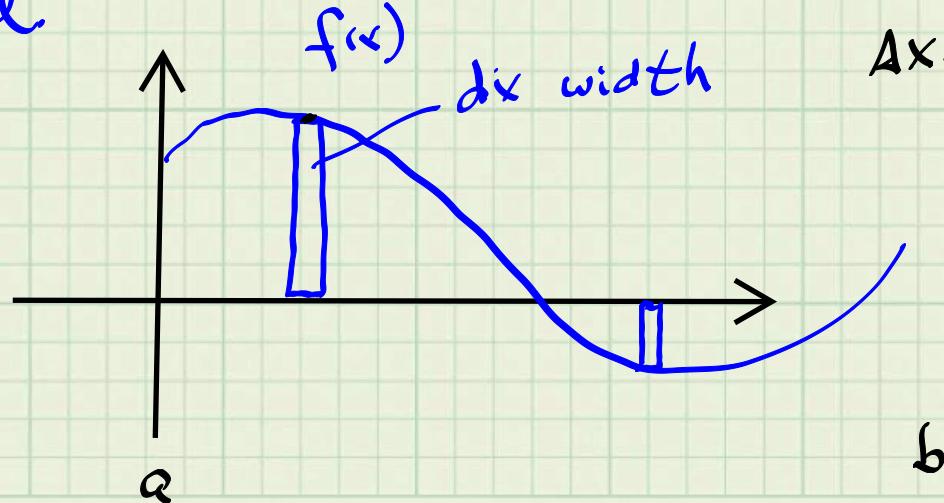
is

Lazy S

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \begin{cases} \sum_{i=1}^n A_i^{\text{low}} \\ \sum_{i=1}^n A_i^{\text{up}} \end{cases}$$

provided

both limits exist, are finite, and are equal



$$\Delta x := \frac{b-a}{n} \xrightarrow{n \rightarrow \infty} dx$$

$$\rightarrow \int_a^b f(x) dx = \sum f(x) \cdot dx$$

Theorem If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then the Riemann integral exists. \square

Suppose: $\int_0^8 \underbrace{\cos\left(\frac{e^{-x} \cdot \sin(x)}{1+5x+x^2}\right)}_{f(x)} dx$

How to see that $f(x)$ is continuous?

Qualitative view

- $\cos(y)$ is cont all $y \in \mathbb{R}$
- $1+5x+x^2 > 0$ for all $x \in [0, 8]$
- numerator consists of two cont. functions, e^{-x} and $\sin(x)$.
- Nothing becomes unbounded or jumps between two values.

Are all functions Riemann Integrable?

• $f(x) = \frac{1}{x}$ $1 \leq x \leq 4$ is OK

but $0 \leq x \leq 4$ maybe not.

$$\bullet f(x) = \begin{cases} 1 & x \in \mathbb{Q} \text{ (rational)} \\ 0 & \underbrace{x \in \mathbb{I}}_{x \notin \mathbb{Q}} \text{ (irrational)} \end{cases}$$

then for $x_i < x_{i+1}$

$$\min_{x_i \leq x \leq x_{i+1}} f(x) = 0$$

$$\max_{x_i \leq x \leq x_{i+1}} f(x) = 1$$

and therefore, for all n , $\text{Area}_n^{\text{low}} = 0$

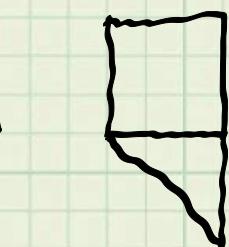
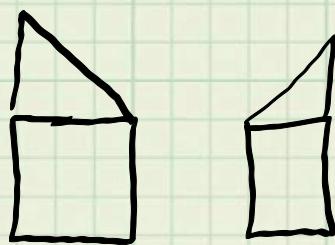
$\text{Area}_n^{\text{up}} = 1$ when we integrate over

$[0, 1]$. Hence, NOT Riemann integrable.

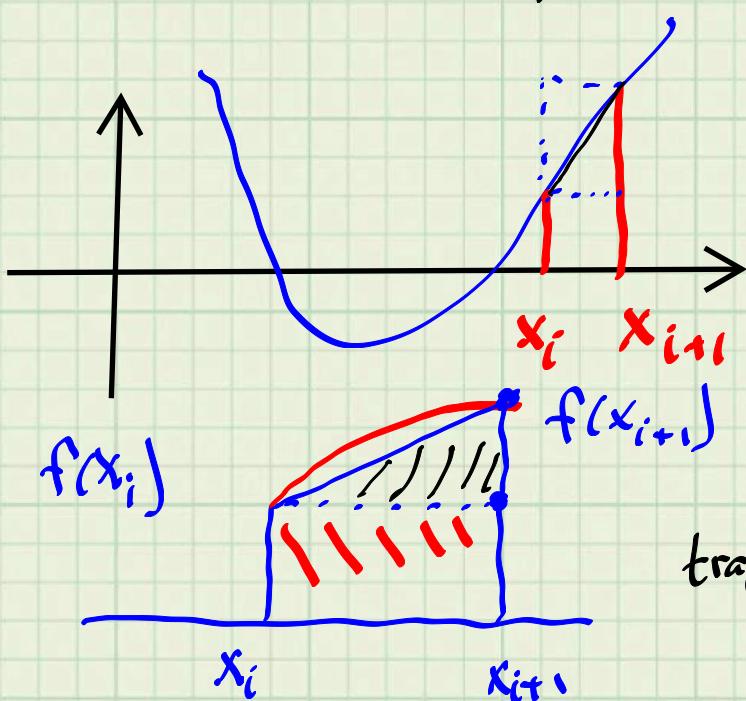
See Demo.

Observation: Rectangles are not very efficient for approximating area under a curve.

- Trapezoidal Rule → TrapZ
- Simpson's Rule



Trapezoids



$$\Delta x = \frac{b-a}{n}$$

$$x_{i+1} = x_i + \Delta x$$

$$x_1 = a$$

$$\text{trap } A_i = \frac{f(x_i) + f(x_{i+1})}{2} \Delta x$$

Area of the rect : $f(x_i) \cdot \Delta x$

Area of the tri. : $\frac{(f(x_{i+1}) - f(x_i))}{2} \cdot \Delta x$

Sum of the above's trapA = $\frac{(f(x_{i+1}) + f(x_i))}{2} \Delta x$

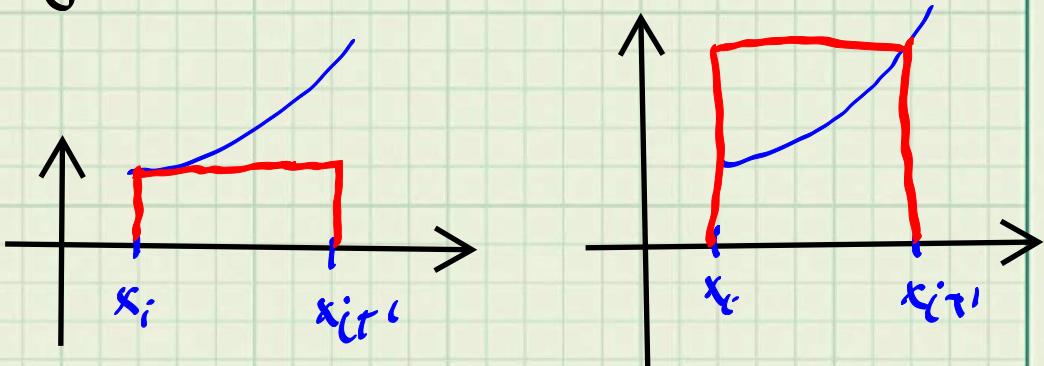
Prop. For $f: [a, b] \rightarrow \mathbb{R}$ continuous,

$$\begin{aligned} \int_a^b f(x) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \text{trapA}_i \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{f(x_{i+1}) + f(x_i)}{2} \Delta x \end{aligned}$$

□

More efficient than rectangles for estimating area.

Remark:
(Monotonic)



$$A_i^{\text{low}} = f(x_i) \cdot \Delta x$$

$$A_i^{\text{up}} = f(x_{i+1}) \cdot \Delta x$$

average =
trapA_i

$$\text{trapA}_i = \frac{f(x_i) + f(x_{i+1})}{2} \cdot \Delta x$$

Indefinite Integral

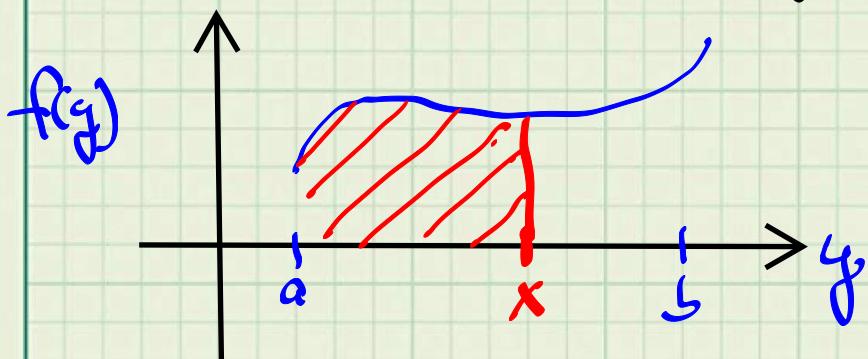
Def. Suppose $f:[a, b] \rightarrow \mathbb{R}$ and

$a \leq x \leq b$. Then

$$g(x) := \int_a^x f(y) dy$$

dummy variable

is an indefinite integral.

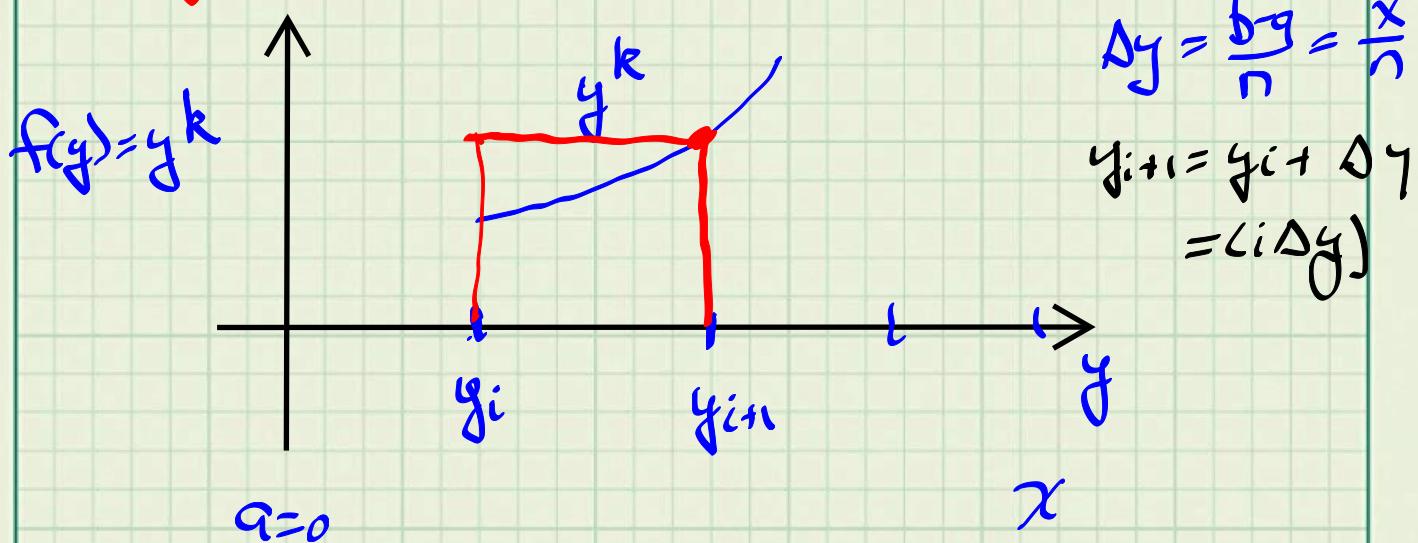


Prop. For all counting numbers $k \geq 1$ and $x > 0$, $\int_0^x y^k dy = \frac{x^{k+1}}{k+1}$

Why: $\sum_{i=1}^n i^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \text{Lower Order Terms}$

Because monomials are cont., we know the Riemann integral exists.

That means lower & upper sums agree, so we only need to compute one of them.



$$\begin{aligned} A_i^{\text{up}} &:= f(y_{i+1}) \cdot \Delta y \\ &= (i\Delta y)^k \cdot \Delta y \\ &= (i)^k (\Delta y)^{k+1} \end{aligned}$$

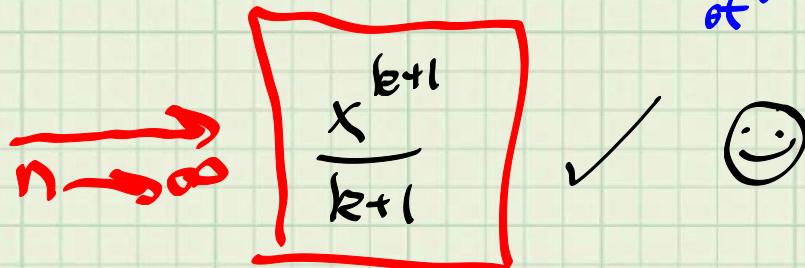
$$\begin{aligned} \text{Area}_n^{\text{up}} &:= \sum_{i=1}^n A_i^{\text{up}} \\ &= \sum_{i=1}^n (i)^k (\Delta y)^{k+1} \end{aligned}$$

$$\Delta y = \frac{x}{n}$$

$$= \left(\frac{x}{n}\right) \sum_{i=1}^{k+1} (i)^k$$

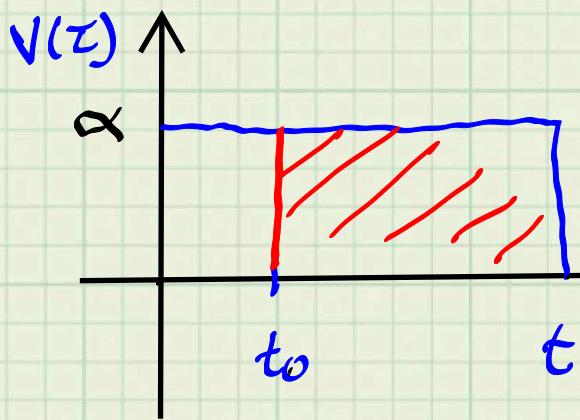
$$= \left(\frac{x}{n}\right)^{k+1} \left[\frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \text{lower order terms} \right]$$

$$= \frac{x^{k+1}}{k+1} + \frac{x^{k+1}}{n \cdot 2} + \frac{x^{k+1}}{\text{higher powers of } n}$$

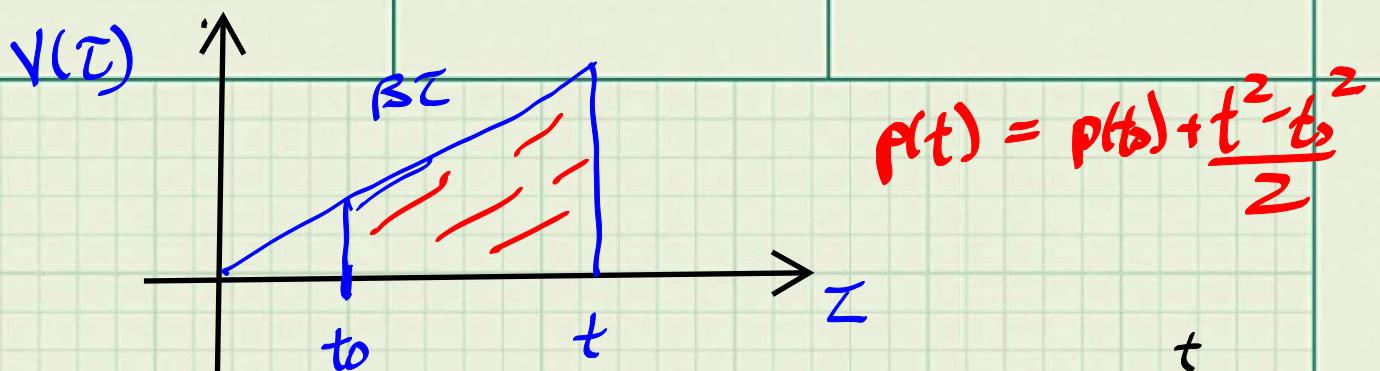


Applications of the Definite Integral

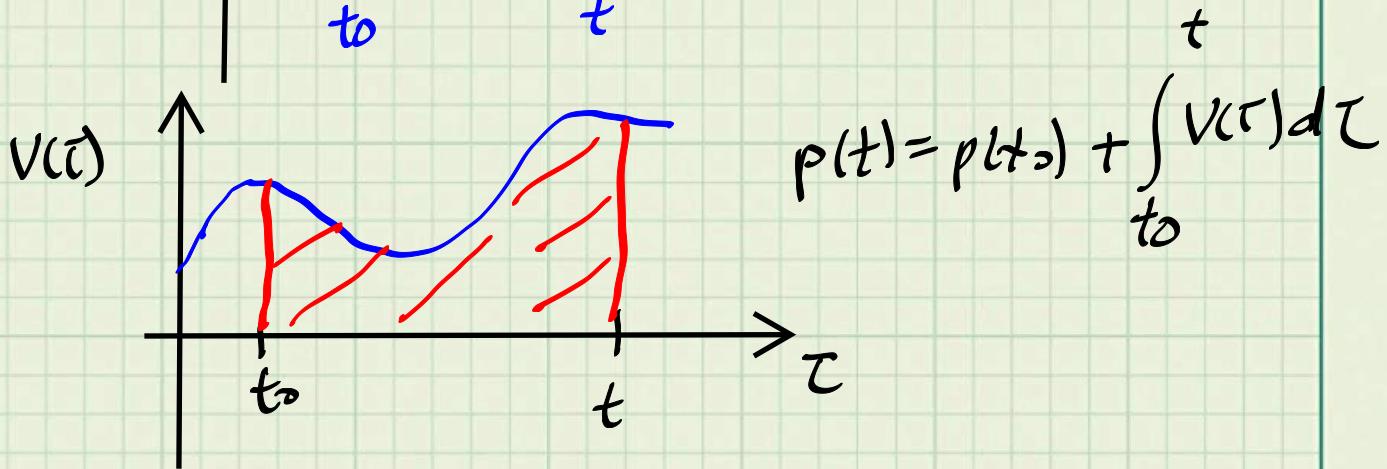
Project 1. From speed to position of a drone & from acceleration to position.



$$\begin{aligned} p(t) &= \alpha \cdot t \\ p(t) &= p(t_0) + \underbrace{\alpha(t-t_0)}_{\text{change in position}} \end{aligned}$$



$$p(t) = p(t_0) + \frac{t^2 - t_0^2}{2}$$



$$p(t) = p(t_0) + \int_{t_0}^t V(\tau) d\tau$$