Summary:

- Taylor Polynomials

$$f(x) \approx f(x_0) + f'(x_0)\frac{(x-x_0)}{1!} + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$$

- Matches first n derivatives.
- Works because $\frac{d}{dx^k}\left(\frac{x^n}{n!}\right) = \frac{x^{n-k}}{(n-k)!}$
  $(1 \le k \le n)$

- Taylor series $n \to \infty$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$$

Important for the study of ODEs

Wow! $A = n \times n$, $I_n = n \times n$ Identity

$$e^A := I_n + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$ [Calculus Construct]

Distinct From: $x^y = \lim_{n \to \infty} x^{r_n}$

where $\lim_{n \to \infty} r_n = y$, and $r = \frac{p}{q} \Rightarrow x^r = \sqrt[q]{x^p}$

Nothing super special about "e"

$$3^x = e^{\ln 3^x} = e^{x\ln(3)} \quad \Big\| \quad e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots$$

$$\therefore 3^x = 1 + \frac{x\cdot\ln(3)}{1!} + \frac{[x\ln(3)]^2}{2!} + \frac{[x\ln(3)]^3}{3!} + \dots$$

$$3^A = I_n + \frac{\ln(3)\cdot A}{1!} + \frac{(\ln(3)\cdot A)^2}{2!} + \dots$$

Because $\ln(e) = 1$, it's simpler to use!

Partial Derivatives: $f : \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f}{\partial x_1}(x_1, x_2) := \lim_{h\to 0} \frac{f(x_1+h, x_2) - f(x_1, x_2)}{h}$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2) := \lim_{h\to 0} \frac{f(x_1, x_2+h) - f(x_1, x_2)}{h}$$

"Differentiation one variable at a time"

"Hold other variables constant"

$$f(x_1, x_2) = x_1 + x_1 x_2 + (x_2)^2 \quad \begin{cases} \dfrac{\partial f}{\partial x_1} = 1 + x_2 + 0 \\ \dfrac{\partial f}{\partial x_2} = 0 + x_1 + 2x_2 \end{cases}$$

Today: Warm-up problem. Compute the partial derivatives of

$$h(x_1, x_2) := \cos\left(x_1 (x_2)^2\right)$$

$$\frac{\partial h}{\partial x_1}(x_1, x_2) = -\sin\left(x_1 (x_2)^2\right) \cdot (x_2)^2$$

$$\frac{\partial h}{\partial x_2}(x_1, x_2) = -\sin\left(x_1 (x_2)^2\right) \cdot \left(\underbrace{x_1 (2) x_2}_{2x_1 x_2}\right)$$

Extension to $f: \mathbb{R}^n \to \mathbb{R}^m$?

Needs better notation:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ \vdots \\ x_{0,n} \end{bmatrix}, \quad f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}$$

Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}$$

Natural basis vectors (aka, canonical)

$x_0 + he_i$, $|h|$ small perturbation in the i-th component of x_0

Everything constant except the i-th component: essence of partial derivatives.

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{h \to 0} \frac{f(x_0+he_i) - f(x_0)}{h}$$

Second take

$$g_i(h) := f(x_0 + he_i)$$

$$\frac{\partial f}{\partial x_i}(x_0) = \underbrace{\frac{d}{dt} g_i(h)}_{\text{function of a single variable}}\Big|_{h=0} = \lim_{h \to 0} \frac{g_i(h) - g(0)}{h}$$

Linearization

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $x_0 \in \mathbb{R}^n$

$$\underbrace{f(x)}_{m\times 1} \approx f(x_0) + \underset{m\times n}{A}\underbrace{(x-x_0)}_{n\times 1}$$

what is A?

Linear Algebra from ROB 101

$$\underset{m\times n}{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}_{m\times n}$$

columns

$A_i \in \mathbb{R}^m$

$A \cdot e_i = ?$

$$\boxed{A \cdot e_i = A_i}$$

$$\begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$$

$$f(x) \approx f(x_0) + A(x - x_0)$$

$$\rightarrow x = x_0 + h e_i$$

$$f(x_0 + h e_i) \approx f(x_0) + A(\cancel{x_0} + h e_i \cancel{- x_0})$$
$$\approx f(x_0) + A \cdot h \cdot e_i$$

$$\frac{f(x_0 + h e_i) - f(x_0)}{h} \approx A \cdot e_i = A_i$$

$$A_i = \lim_{h \to 0} \frac{f(x_0 + h e_i) - f(x_0)}{h} = \frac{\partial f}{\partial x_i}(x_0)$$

$$A = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x_1}(x_0) & \frac{\partial f}{\partial x_2}(x_0) & \cdots & \frac{\partial f}{\partial x_n}(x_0) \end{bmatrix}_{m \times n}}_{\text{Jacobian of } f \text{ at } x_0}$$

__Def.__ $J_f(x) := \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) & \cdots & \frac{\partial f}{\partial x_n}(x) \end{bmatrix}$ is the Jacobian. Also denoted $\frac{\partial f}{\partial x}(x)$

Review on Linear Algebra

Suppose $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ and $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}_{m \times n}$

$A_i \in \mathbb{R}^m$ i-th column of A $\left\{ \begin{matrix} \text{has } m \\ \text{rows} \end{matrix} \right\}$

$$A \cdot x = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$$

Hence, $\boxed{A \cdot e_i = A_i}$, $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ i-th spot

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Def. Linearization of f at x_0 is

$$y(x) = \underbrace{f(x_0)}_{m \times 1} + \underbrace{J_f(x_0)}_{m \times n} \cdot \underbrace{(x - x_0)}_{n \times 1}$$

$\square$

Total Derivative of a Function of Several Variables ??

Setting $f(x_1, x_2, \ldots, x_n)$ (function of several variables) and each variable depends on a scalar variable, say t (for time)

$$g(t) = f(x_1(t), x_2(t), \ldots, x_n(t))$$

Want to compute $\frac{dg(t)}{dt}$ without substituting everything in.

Total Derivative is

$$\frac{dg}{dt}(t) = \frac{\partial f}{\partial x_1}(x_1(t), \ldots, x_n(t)) \cdot \frac{dx_1}{dt} +$$

$$+ \frac{\partial f}{\partial x_2}(x_1(t), x_2(t), \ldots, x_n(t)) \cdot \frac{dx_2}{dt} +$$

$$+ \dots + \frac{\partial f}{\partial x_n}(x_1(t), \dots, x_n(t)) \cdot \frac{dx_n}{dt}$$

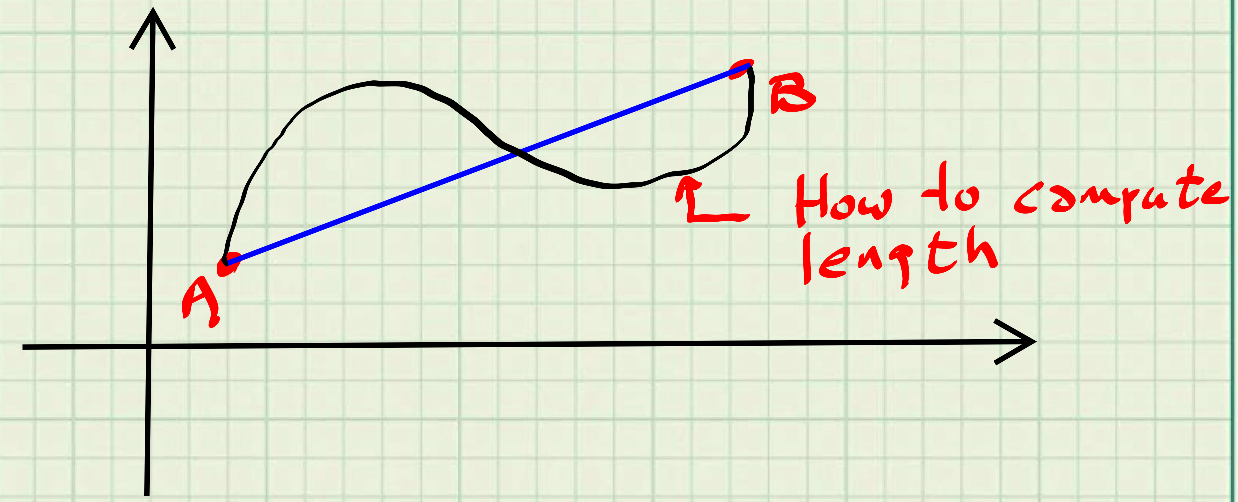
$$\boxed{\begin{gathered} \frac{dg(t)}{dt} = \underbrace{J_f(x(t))}_{m \times n} \cdot \frac{dx}{dt} = m \times 1 \\ \text{where } \frac{dx}{dt} = \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}_{n \times 1} \end{gathered}}$$

Total Derivative in matrix-vector form

End of Chapter 5, because we delay gradients until Wed.

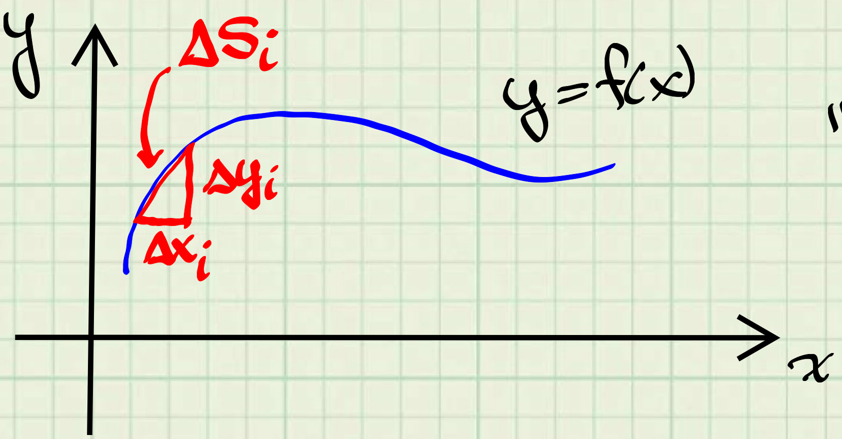
Chapter 6: Applications of the Derivative

Arc Length (Path Length)



Length of a line

$$d(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$



"Zoom"

S = Latin for "spatium" for "space" or "distance"

$$\Delta S_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$S = \sum_{i=1}^{n} \Delta S_i \xrightarrow[n \to \infty]{}$$ Riemann Integral

Case 1 $y = f(x)$

$$\Delta y = f(x+\Delta x) - f(x)$$

$$= \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot \Delta x$$

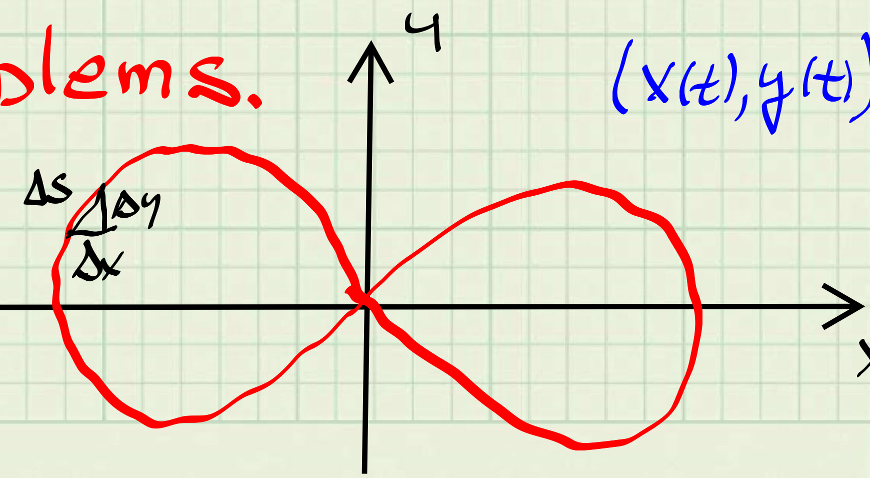
$$= \xrightarrow[\Delta x \to 0]{} f'(x) \cdot dx$$

Hence $\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2} \xrightarrow[\Delta x \to 0]{} \sqrt{(dx)^2 + (f'(x)dx)^2}$

$$= \underbrace{dx\sqrt{1 + (f'(x))^2}}_{ds}$$

$$S = \int_a^b dS = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$ Quad GK

Only compatable by hand for toy problems.



Case 2

$$\Delta x = x(t+\Delta t) - x(t) \xrightarrow[\Delta t \to 0]{} \frac{dx}{dt}(t) \cdot dt$$

$$\Delta y = y(t+\Delta t) - y(t) \xrightarrow[\Delta t \to 0]{} \frac{dy}{dt}(t) \cdot dt$$

$$ds = \sqrt{\left(\frac{dx}{dt}(t)\, dt\right)^2 + \left(\frac{dy}{dt}(t) \cdot dt\right)^2}$$

$$= dt \sqrt{\left[\frac{dx}{dt}(t)\right]^2 + \left[\frac{dy}{dt}(t)\right]^2}$$

$$S = \int_{t_0}^{t_f} dS = \int_{t_0}^{t_f} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Demo