ROB 201 - Calculus for the Modern Engineer
HW #8
(This HW Set Cannot be Dropped)

Prof. Grizzle

Check Canvas for due date and time

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete and turn in.

1. (a) Create a "Cheat Sheet" for Chapters 7, 8, and 9 of the textbook. You'll receive the same score for a handwritten solution as a typeset solution. Here is an example from ROB 101. We're at the end of the term, so keep it reasonably short; focus on the key material.

   (b) Note any material where you found the explanation confusing or difficult to master.

2. Work **one of the following two** problems involving improper integrals.

Prob. I A logarithmic spiral is defined in polar coordinates by $(r, \theta) = (e^{a\theta}, \theta)$, where $a \in \mathbb{R}$ is a constant and $\theta \geq 0$ parameterizes the spiral. Find the length of the spiral from $\theta = 0$ to $\theta = \infty$ (this is the same as the curve's arc length or path length).

Prob. II The **harmonic sum** is the sum of the reciprocals of the first n positive integers, expressed as

$$H_n := \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n},$$

and the **harmonic series** is defined as

$$H := \lim_{n\to\infty} H_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

Because the terms being added become arbitrarily small, one wants to believe (often in the worst way) that the series converges to a finite number. **Alas, the harmonic series is famously divergent!** This problem guides you in using the Comparison Test for functions to show that the harmonic series is divergent.

Define the "staircase function" $f_H : [0, \infty) \to (0, 1]$ by

$$f_H(x) := \begin{cases} 1 & 0 \leq x \leq 1 \\ \frac{1}{k} & (k-1) < x \leq k, \quad k \in \mathbb{N}, k \geq 2. \end{cases}$$

You are not required to sketch the function, but it will likely be helpful to do so.

(a) State briefly why $f_H(x)$ is piecewise continuous, and hence its Riemann integral exists over closed, bounded intervals.

(b) For $n \in \mathbb{N}$, evaluate $\int_0^n f_H(x)\, dx$.

(c) For $x \geq 0$, compare $f_H(x)$ to the function $g(x) := \frac{1}{x+1}$; you are not required to sketch $g(x)$, but overlaying it on a sketch of the staircase function may be helpful.

(d) Deduce that the Harmonic series diverges.

3. The Lotka-Volterra **predator-prey model** is given by

$$\begin{aligned}\dot{x} &= x(\alpha - \beta y)\\ \dot{y} &= -y(\gamma - \delta x),\end{aligned}$$

where $x(t)$ and $y(t)$ represent the populations of the prey and predator at time t, respectively, and $\alpha, \beta, \gamma, \delta$ are positive constants. Should you be interested, you can read about the model's Biological interpretations here.

(a) From an initial condition (x_0, y_0), with $x_0 \geq 0$ and $y_0 \geq 0$, will solutions exist and be unique over a sufficiently small time interval? Explain why or why not.

(b) Using results in the textbook, can you guarantee that solutions exist on an infinite time interval, $[t_0, \infty)$? Explain why or why not.

(c) Determine the equilibrium points of the system. There are two of them. It is important to remember that $\alpha, \beta, \gamma, \delta$ are positive constants, and hence all are nonzero.

4. We set the parameters for the Lotka-Volterra predator-prey model as, $\alpha = 0.1, \beta = 0.02, \gamma = 0.3$ and $\delta = 0.01$. With these values, the non-trivial equilibrium becomes

$$\begin{bmatrix} x_e \\ y_e \end{bmatrix} = \begin{bmatrix} 30.0 \\ 5.0 \end{bmatrix} = \begin{bmatrix} \text{prey, those that are eaten} \\ \text{predators, the consumers of prey} \end{bmatrix}$$

in made-up units of tens of animals per square kilometer. Moreover, the Jacobians needed for linearized models at the two equilibria are

$$J(0.0, 0.0) = \begin{bmatrix} 0.1 & -0.0 \\ 0.0 & -0.3 \end{bmatrix} \quad \text{and} \quad J(30.0, 5.0) = \begin{bmatrix} 0.00 & -0.60 \\ 0.05 & 0.00 \end{bmatrix}.$$

(a) Using eigenvalues, analyze the behavior of the linearized predator-prey model near the trivial equilibrium point. You are not asked to compute solutions; instead, give qualitative properties. You can use software to compute the eigenvalues.

(b) Continuing with the trivial equilibrium point, in (a), you will have found that one of the animal populations explodes while the other vanishes. Yikes! Show how the eigenvectors allow you to distinguish which animal population explodes and which vanishes.

(c) Using eigenvalues, analyze the behavior of the linearized predator-prey model near the nontrivial equilibrium point. You are not asked to compute solutions; instead, give qualitative properties. Moreover, you are not asked to look at the eigenvectors.

(d) For $A := J(30.0, 5.0) = \begin{bmatrix} 0.00 & -0.60 \\ 0.05 & 0.00 \end{bmatrix}$, the matrix exponential $e^{A \cdot 5} = \exp(5A) = \begin{bmatrix} 0.648 & -2.639 \\ 0.220 & 0.648 \end{bmatrix}$ has been computed for you. Based on this information, if the initial animal population at time $t_0 = 0$ is

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 4 \end{bmatrix},$$

what population will be predicted by the linearized model at time $t = 5$?

5. You are given that the matrix $A := \begin{bmatrix} -2.00 & -1.00 & 0.00 & 1.00 \\ -0.25 & -1.00 & -0.25 & 1.25 \\ 1.25 & 0.00 & -0.75 & -1.25 \\ -1.25 & -1.00 & -1.25 & 0.25 \end{bmatrix}$ has eigenvalues and eigenvectors

$$\lambda_1 = -2.00, \quad v_1 = \begin{bmatrix} 1.00 \\ 0.00 \\ -1.00 \\ 0.00 \end{bmatrix}, \quad \lambda_3 = -1.00 + 1.00\,\mathbf{i}, \quad v_3 = \begin{bmatrix} \mathbf{i} \\ 1.00 \\ 0.00 \\ \mathbf{i} \end{bmatrix}$$

$$\lambda_2 = 0.500, \quad v_2 = \begin{bmatrix} 0.00 \\ 1.00 \\ -1.00 \\ 1.00 \end{bmatrix}, \quad \lambda_4 = -1.00 - 1.00\,\mathbf{i}, \quad v_4 = \begin{bmatrix} -\mathbf{i} \\ 1.00 \\ 0.00 \\ -\mathbf{i} \end{bmatrix}.$$

Because A is a real matrix, the complex eigenvalues and eigenvectors occur in complex conjugate pairs.

Compute the following functions analytically, by hand. Of course, you can check them in Julia if you want.

(a) $e^{At}v_1$

(b) $x(t)$ when $x(0) = x_0 = v_1 + 2v_2$

(c) Express the complex exponential $e^{\lambda_3 t}$ in the form $e^{at} \cdot (\cos(\omega t) + \mathbf{i}\sin(\omega t))$

(d) $e^{At}v_3$

(e) $x(t)$ when $x(0) = x_0 = \mathrm{real}(v_3)$, where $\mathrm{real}(v_3)$ is the real part of the eigenvector, v_3. Note: all of the components of $x(t)$ will be real functions of t.

6. For the linear input-output model $\dddot{y} - 2\ddot{y} - 3\dot{y} + 10y = 5u - \dot{u}$, determine the following:

   (a) The transfer function, $G(s)$.

   (b) The poles and zeros of $G(s)$.

   (c) Is the system BIBO stable?

   (d) Suppose that $U(s) := \mathcal{L}\{u(t)\} = \frac{1}{s+1}$. Compute $Y(s)$, the Laplace transform of the output when all initial conditions are zero.

Hints

Prob. 1 Write approximately 15 or more words for each part of the question.

Prob. 2 From Chapter 6 of our textbook, for a curve defined parametrically by $x(t)$ and $y(t)$, where t ranges from t_1 to t_2, the arc length is given by,

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

It is important to note that t in the above is a dummy variable. In your problem, compute the derivatives with respect to θ; then do the easy algebra to simplify the expression under the square root. To compute the path length, set up the improper integral that represents the length of the curve and discuss the conditions under which this integral converges versus when it diverges. You will find that it always exists, but it will not always be finite.

No hints are given for the harmonic series problem because it is already broken down into small steps.

Prob. 3 No hints are provided, beyond, know the textbook!

Prob. 4 To obtain the eigenvalues and eigenvectors, you can use Julia or your favorite LLM. You are not expected to find them by hand.

Prob. 5 You are supposed to solve the problem without computing the matrix exponential, e^{At}. See Chapter 9, **Eigenvalues and Eigenvectors to the Rescue**. At a certain point, you may even think the problem is too easy (except for the very last part). That is a good thing! No one is trying to trick you.

Prob. 6 You should use software to find the roots of any polynomials that may show up in the problem.