ROB 201 - Calculus for the Modern Engineer
HW #2

Prof. Grizzle

Check Canvas for due date and time

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete and turn in.

1. (a) Create a "Cheat Sheet" for the first two chapters of the textbook. You'll receive the same score for a handwritten solution as a typeset solution. Here is an example from ROB 101.

   (b) Note any material where you found the explanation confusing or difficult to master.

2. For the strictly increasing function, $f(x) = \frac{x}{2} + 2\,x^2$, determine lower and upper bounds for the area under the function for the interval $[0, 2]$, using $n := 4$, $\Delta x := \frac{2-0}{4} = 0.5$, and $x_i := (i-1)\cdot\Delta x$, $1 \le i \le 5$.

   (a) Give a numerical value for $\mathrm{Area}_4^{\mathrm{Low}}$.

   (b) Show your steps for computing $\mathrm{Area}_4^{\mathrm{Low}}$.

3. A 3D printer is building a 3D object by depositing layers of material. The thickness of each layer is designed to be half that of the previous layer. The first layer is 4 mm thick. The 3D printer is programmed to stop after depositing 10 layers.

   (a) What is the thickness of the 10th layer?

   (b) Calculate the total thickness of the object after the 10th layer has been deposited.

4. Compute the following limits using the "Easy Way" as explained in the textbook. You can just give an answer; you do not need to show your work.

   (a) $\lim\limits_{x\to\infty} \dfrac{7x^5 - 2x^3 + x - 4}{3x^5 + x^2 + 1}$

   (b) $\lim\limits_{x\to\infty} \dfrac{3x^2 - x + 1}{4x^4 + 2x^3 - x + 3}$

   (c) $\lim\limits_{x\to\infty} \dfrac{2 + x}{x^{\frac{1}{3}} - 3}$

   (d) $\lim\limits_{x\to-\infty} \dfrac{4x^4 - x^2 + 5}{3x^3 + 2x + 1}$

   (e) $\lim\limits_{x\to-\infty} 2^{-x}\dfrac{4x^4 - x^2 + 5}{3x^3 + 2x + 1}$

   (f) $\lim\limits_{x\to\infty} \pi^{0.01\,x}\dfrac{3x^2 - x + 1}{4x^4 + 2x^3 - x + 3}$

5. This problem focuses on the Binomial Theorem from Chapter 1.

(a) For the first six binomial coefficients, where k ranges from 0 to 5, give their formulas and their values.

(b) Compute the expansion of $(1 + \frac{1}{x})^5$.

6. **The Cantorian Clan's Eternal Craftsmanship:** The Cantorian Clan is renowned across the galaxy for their unparalleled craftsmanship. Each day, they manufacture **Harmonic Orbs**, fantastical objects with weights measured precisely in grams. The Cantorians have a unique tradition: every day, they craft one orb for each of their beloved *harmonic fractions*, that is, for every number of the form:

$$\frac{1}{i}, \quad \text{where } i = 2, 3, 4, \ldots$$

For example:

- On the first day, they craft orbs with weights $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ grams.
- On the second day, they craft orbs with the same weights, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ grams, and so on.

The Cantorians' lineage is eternal, and their production continues forever. At the end of time, their collection of all Harmonic Orbs is complete.

The Questions:

(a) Is the set of Harmonic Orbs produced by the Cantorian Clan on Day 1 **countable** or **uncountable**? Explain why.

(b) How does the total number of Harmonic Orbs produced after two days (i.e., adding up both days' production) compare to the number of Harmonic Orbs produced after one day? Give an answer and then very briefly explain it.

(c) How does the total number of Harmonic Orbs produced at the end of time (i.e., adding up the total production over days $1, 2, 3, \ldots$) compare to the number of Harmonic Orbs produced after one day? Give an answer and then very briefly explain it.

Remark: In HW #8, we will compute the total weight in grams of the Cantorian Clan's daily orb manufacturing. For **zero bonus points:** Can you guess the total weight of each day's orbs to within one gram?

Hints

Prob. 1 Nothing to add.

Prob. 2 Below is the upper bound worked out. You should follow a similar pattern for the lower bound. Don't be shocked when your lower bound is far from the upper bound of $8\frac{3}{4}$. For $n = 500$, the two answers are (lowerSum = 6.315344000000007, upperSum = 6.351344000000007, estIntegral = 6.333344000000007, pmError = 0.018000000000000238), but of course, you would never compute that by hand.

$$\begin{aligned}
\text{Area}_4^{\text{Up}} &= \sum_{i=1}^{4} f(x_{i+1}) \cdot \Delta x \\
&= \sum_{i=1}^{4} \left(f(i \cdot \Delta x)\right) \cdot \Delta x \\
&= \sum_{i=1}^{4} \left(\frac{i \cdot 0.5}{2} + 2 \cdot (i \cdot 0.5)^2\right) \cdot 0.5 \\
&= 0.5 \cdot \left[\left(\frac{0.5}{2} + 2 \cdot (0.5)^2\right) + \left(\frac{2 \cdot 0.5}{2} + 2 \cdot (2 \cdot 0.5)^2\right) + \left(\frac{3 \cdot 0.5}{2} + 2 \cdot (3 \cdot 0.5)^2\right) + \left(\frac{4 \cdot 0.5}{2} + 2 \cdot (4 \cdot 0.5)^2\right)\right] \\
&= 0.5 \cdot \left[(0.25 + 0.5) + (0.5 + 2) + (0.75 + 4.5) + (1 + 8)\right] \\
&= 0.5 \cdot \left[0.75 + 2.5 + 5.25 + 9\right] \\
&= 0.5 \cdot \left[17.5\right] \\
&= 8.75
\end{aligned}$$

Prob. 3 Check out geometric sums.

Prob. 4 None given.

Prob. 5 $\binom{5}{0} = \frac{5!}{0!5!} = 1$

Prob. 6 You have numerous resources at your disposal. In fact, you have so many that we will not discuss this problem in Office Hours so that you learn to take advantage of your resources!

- Check out the textbook's Table of Contents and click on it to go to the section on Countable Sets (see Chapter 2).
- If the textbook does not help you to have a general idea of what it means for a set to be countable, provide the following **prompt to Maizey or UM GPT:**

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  Give me a basic introduction to countable sets w/o using the word injective.
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  Once you understand the most basic definition, you can ask Maizey for a "little more depth". If it jumps too far ahead, ask it to back up. Ask for examples you can work and answer!

- Post on Piazza and seek help from your classmates (recall, we are deliberately not helping you on this problem so that you are motivated to explore various avenues of assistance). Posting and answering questions on Piazza is not considered cheating in CALCULUS FOR THE MODERN ENGINEER.

Solutions HW 02

Prob. 1 (a) Included at the end of the solution set

(b) These will vary by person, but some of the more challenging topics may have been:

i. Chap 01: The Approximation Principle
ii. Chap 01: Euler's number
iii. Chap 01: Binomial Theorem
iv. Chap 01: Shifting and scaling operations on functions
v. Chap 01: The level of rigor used in stating the results
vi. Chap 02: Exactly what is a proof?
vii. Chap 02: Countable sets
viii. Chap 02: Proofs by induction
ix. Chap 02: Epsilon-delta notions of limits at infinity

Prob. 2 Here is the lower bound:

$$\begin{aligned}
\text{Area}_4^{\text{Low}} &= \sum_{i=1}^{4} f(x_i)\cdot\Delta x \\
&= \sum_{i=1}^{4} \left(f((i-1)\cdot\Delta x)\right)\cdot\Delta x \\
&= \sum_{i=1}^{4} \left(\frac{(i-1)\cdot\Delta x}{2} + 2\cdot((i-1)\cdot\Delta x)^2\right)\cdot\Delta x \\
&= \sum_{i=1}^{4} \left(\frac{(i-1)\cdot 0.5}{2} + 2\cdot((i-1)\cdot 0.5)^2\right)\cdot 0.5 \\
&= 0.5\cdot\left[\left(\frac{0\cdot 0.5}{2} + 2\cdot(0\cdot 0.5)^2\right) + \left(\frac{0.5}{2} + 2\cdot(0.5)^2\right) + \left(\frac{2\cdot 0.5}{2} + 2\cdot(2\cdot 0.5)^2\right) + \left(\frac{3\cdot 0.5}{2} + 2\cdot(3\cdot 0.5)^2\right)\right] \\
&= 0.5\cdot[0 + (0.25+0.5) + (0.5+2) + (0.75+4.5)] \\
&= 0.5\cdot[0.75 + 2.5 + 5.25] \\
&= 0.5\cdot[8.5] \\
&= 4.25.
\end{aligned}$$

Prob. 3 You can solve this problem using the geometric progression formula.

(a) The thickness of the 10th layer can be calculated using the formula of the n-th term of a geometric sequence, which is $a_n = a_1 r^{(n-1)}$. Where the first term $a_1 = 4$ mm, the common ratio $r = 0.5$ and $n = 10$.

Therefore, the thickness of the 10th layer is:

$$a_{10} = 4\cdot(0.5)^{10-1} = 4\cdot 0.5^9 \approx 0.0078 \text{ mm} \tag{1}$$

(b) The total thickness of the 3D object after the 10th layer has been deposited can be calculated using the sum of a geometric series $S_n = a_1(1-r^n)/(1-r)$. Here, $a_1 = 4$ mm, $r = 0.5$ and $n = 10$.

Therefore, the total thickness of the 3D object is:

$$S_{10} = 4\cdot\left(\frac{1-0.5^{10}}{1-0.5}\right) = 4\cdot\frac{1-0.5^{10}}{0.5} \approx 7.9990 \text{ mm} \tag{2}$$

Prob. 4 (a) $\lim_{x\to\infty} \dfrac{7x^5 - 2x^3 + x - 4}{3x^5 + x^2 + 1} = \lim_{x\to\infty} \dfrac{7x^5}{3x^5} = \dfrac{7}{3} \approx 2.33$

(b) $\lim_{x\to\infty} \dfrac{3x^2 - x + 1}{4x^4 + 2x^3 - x + 3} = \lim_{x\to\infty} \dfrac{3x^2}{4x^4} = 0$

(c) $\lim_{x\to\infty} \dfrac{2 + x}{x^{\frac{1}{3}} - 3} = \lim_{y\to\infty} \dfrac{2 + y^3}{y - 3} = \lim_{y\to\infty} \dfrac{y^3}{y} = \infty$, where we used the substitution $y := x^{\frac{1}{3}}$.

(d) $\lim_{x\to-\infty} \dfrac{4x^4 - x^2 + 5}{3x^3 + 2x + 1} = \lim_{x\to-\infty} \dfrac{4x^4}{3x^3} = \lim_{x\to-\infty} \dfrac{4x}{3} = -\infty$ because we are taking the limit as x goes to negative infinity.

(e) $\lim_{x\to-\infty} 2^{-x}\dfrac{4x^4}{3x^3} = \lim_{x\to-\infty} 2^{-x}\dfrac{4x^4}{3x^3} = \lim_{x\to-\infty} 2^{-x}\dfrac{4x}{3} = -\infty$. We note that $\lim_{x\to-\infty} x \cdot 2^{-x} = \lim_{y\to\infty} (-y)\cdot 2^y = -\infty$ after making the substitution $y = -x$.

(f) $\lim_{x\to\infty} \pi^{0.01\,x} \dfrac{3x^2 - x + 1}{4x^4 + 2x^3 - x + 3} = \lim_{x\to\infty} \pi^{0.01\,x}\dfrac{3x^2}{4x^4} = \lim_{x\to\infty} \pi^{0.01\,x}\dfrac{3}{4x^2} = \infty$ because $\pi > 1$, implying that the exponential term dominates any monomial.

In case the power $0.001x$ is troubling you, let's do the substitution $y = 0.001x$. Then $y \to \infty$ if, and only if, $x \to \infty$. Therefore, $\lim_{x\to\infty} \pi^{0.01\,x}\dfrac{3}{4x^2} = \lim_{y\to\infty} \pi^y \dfrac{3}{4\cdot(100y)^2} = \infty$.

Prob. 5 The binomial theorem states that for any positive integer n, the expansion of $(a + b)^n$ is given by,

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where the binomial coefficient $\binom{n}{k}$ (read as "n choose k") is given by the formula,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Armed with this information, let's address both parts of the problem:

(a) For the first six binomial coefficients $\binom{5}{k}$ where k ranges from 0 to 5, their formulas and values are:

$$\boxed{\begin{aligned} \binom{5}{0} &= \frac{5!}{0!5!} = 1 \\ \binom{5}{1} &= \frac{5!}{1!4!} = 5 \\ \binom{5}{2} &= \frac{5!}{2!3!} = 10 \\ \binom{5}{3} &= \frac{5!}{3!2!} = 10 \\ \binom{5}{4} &= \frac{5!}{4!1!} = 5 \\ \binom{5}{5} &= \frac{5!}{5!0!} = 1 \end{aligned}}$$

(b) To compute the expansion of $(1 + \frac{1}{x})^5$, we use the binomial theorem:

$$(1 + \frac{1}{x})^5 = \sum_{k=0}^{5} \binom{5}{k} 1^{5-k} \left(\frac{1}{x}\right)^k$$

Expanding this gives:

$$(1+\frac{1}{x})^5 = \binom{5}{0}1^5\left(\frac{1}{x}\right)^0 + \binom{5}{1}1^4\left(\frac{1}{x}\right)^1 + \binom{5}{2}1^3\left(\frac{1}{x}\right)^2 + \binom{5}{3}1^2\left(\frac{1}{x}\right)^3 + \binom{5}{4}1^1\left(\frac{1}{x}\right)^4 + \binom{5}{5}1^0\left(\frac{1}{x}\right)^5$$

Now plugging in the values of the binomial coefficients:

$$(1+\frac{1}{x})^5 = 1 + 5\left(\frac{1}{x}\right) + 10\left(\frac{1}{x}\right)^2 + 10\left(\frac{1}{x}\right)^3 + 5\left(\frac{1}{x}\right)^4 + 1\left(\frac{1}{x}\right)^5$$

$$\boxed{(1+\frac{1}{x})^5 = 1 + \frac{5}{x} + \frac{10}{x^2} + \frac{10}{x^3} + \frac{5}{x^4} + \frac{1}{x^5}}$$

Prob. 6 Ah yes, the fantastical Cantorian Orbs:

(a) The set of Harmonic Orbs produced by the Cantorian Clan on Day 1 is **countable**. Each orb corresponds to a harmonic fraction of the form $\frac{1}{i}$, where i is a positive integer starting from 2. Since the set $\{2, 3, 4, \ldots\}$ is countable, and there is a one-to-one correspondence between this set and the harmonic fractions, the set of Harmonic Orbs produced on Day 1 is countable.

Not Required but Good to Know: To show that the set $\{2, 3, 4, \ldots\}$ is in one-to-one correspondence with the counting numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, consider the function:

$$f : \mathbb{N} \to \{2, 3, 4, \ldots\}, \quad f(n) = n + 1.$$

This function maps each $n \in \mathbb{N}$ to $n + 1$, which is an element of $\{2, 3, 4, \ldots\}$. It is:

- **Injective (one-to-one):** If $f(n_1) = f(n_2)$, then $n_1 + 1 = n_2 + 1$, which implies $n_1 = n_2$.
- **Surjective (onto):** For every $m \in \{2, 3, 4, \ldots\}$, there exists an $n \in \mathbb{N}$ such that $f(n) = m$, specifically $n = m - 1$.

Thus, f is a bijection, proving that $\{2, 3, 4, \ldots\}$ is in one-to-one correspondence with the counting numbers $\mathbb{N}$.

(b) The total number of Harmonic Orbs produced after two days is the same as the number produced on Day 1. This is because the Cantorians produce the same set of orbs each day. Thus, the collection of orbs after two days is simply the union of two identical countable sets. Since the union of two countable sets is still countable, the total number of Harmonic Orbs after two days remains countable.

Alternative Reasoning: We can also show that the total number of orbs produced over two days has the same cardinality as $\mathbb{N}$ by identifying the orbs produced on the first day with the odd integers and those produced on the second day with the even integers.

- Define a bijection between the orbs produced on the first day and the odd integers $\{1, 3, 5, \ldots\}$ as follows:

$$f_1 : \{\frac{1}{i} \mid i \geq 2\} \to \{1, 3, 5, \ldots\}, \quad f_1\left(\frac{1}{i}\right) = 2i - 3.$$

  This function maps each harmonic fraction $\frac{1}{i}$ to a unique odd integer.
- Similarly, define a bijection between the orbs produced on the second day and the even integers $\{2, 4, 6, \ldots\}$:

$$f_2 : \{\frac{1}{i} \mid i \geq 2\} \to \{2, 4, 6, \ldots\}, \quad f_2\left(\frac{1}{i}\right) = 2i - 2.$$

  This function maps each harmonic fraction $\frac{1}{i}$ to a unique even integer.

Since the odd integers and even integers together form the set of all counting numbers $\mathbb{N}$, the total number of orbs produced over the two days is in one-to-one correspondence with $\mathbb{N}$. Thus, the total number of orbs produced remains countable.

(c) The total number of Harmonic Orbs produced at the end of time (i.e., the union of all daily productions over days $1, 2, 3, \ldots$) is also the same as the number produced on Day 1. No wonder Cantor was attacked by his contemporaries: he broke their brains!

To show that the total number of Harmonic Orbs produced at the end of time is countable, we can use an argument similar to Cantor's proof for the countability of the rational numbers. Each Harmonic Orb corresponds to a weight $\frac{1}{i}$, where $i \geq 2$, produced on Day n, where $n \in \mathbb{N}$.

Consider the set of all Harmonic Orbs:

$$S = \left\{ \frac{1}{i} \mid i \geq 2,\, n \in \mathbb{N} \right\}.$$

We can pair each weight $\frac{1}{i}$ with the day n it was produced, forming the ordered pair (n, i), where $n \in \mathbb{N}$ and $i \geq 2$. This creates a subset of the Cartesian product:

$$S \subseteq \{(n, i) \mid n \in \mathbb{N},\, i \in \mathbb{N},\, i \geq 2\}.$$

Next, note that $\{(n, i) \mid n \in \mathbb{N},\, i \in \mathbb{N}\}$, the set of all ordered pairs of natural numbers, is countable. This follows because we can enumerate all pairs (n, i) by arranging them in a two-dimensional grid and traversing the grid diagonally (Cantor's diagonal argument). Since S is a subset of a countable set, S itself is countable.

Thus, the total number of Harmonic Orbs produced at the end of time is countable.