ROB 201 - Calculus for the Modern Engineer
HW #8

Prof. Grizzle

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete.

1. (a) Create a "Cheat Sheet" for Chapters 7, 8, and 9 of the textbook. Here is an example from ROB 101.

   (b) Note any material where you found the explanation confusing or difficult to master.

2. Work **one of the following two** problems involving improper integrals.

Prob. I A logarithmic spiral is defined in polar coordinates by $(r, \theta) = (2^{a\theta}, \theta)$, where $a \in \mathbb{R}$ is a constant and $\theta \geq 0$ parameterizes the spiral. Find the length of the spiral from $\theta = 0$ to $\theta = \infty$ (this is the same as the curve's arc length or path length).

Prob. II The **harmonic sum** is the sum of the reciprocals of the first n positive integers, expressed as

$$H_n := \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n},$$

and the **harmonic series** is defined as

$$H := \lim_{n\to\infty} H_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

Because the terms being added become arbitrarily small, one wants to believe (often in the worst way) that the series converges to a finite number. **Alas, the harmonic series is famously divergent!** This problem guides you in using the Comparison Test for functions to show that the harmonic series is divergent.

Define the "staircase function" $f_H : [1, \infty) \to (0, 1]$ by

$$f_H(x) := \left\{ \frac{1}{k} \quad (k-1) < x \leq k, \quad k \in \mathbb{N},\ k \geq 2. \right.$$

You are not required to sketch the function, but it will likely be helpful to do so.

(a) State briefly why $f_H(x)$ is piecewise continuous, and hence its Riemann integral exists over closed, bounded intervals.

(b) For $n \in \mathbb{N}$, evaluate $\int_1^{n+1} f_H(x)\, dx$.

(c) For $x \geq 1$, compare $f_H(x)$ to the function $g(x) := \frac{1}{x+2}$; you are not required to sketch $g(x)$, but overlaying it on a sketch of the staircase function may be helpful.

(d) Deduce that the Harmonic series diverges.

3. The Lotka-Volterra **predator-prey model** is given by

$$\begin{aligned} \dot{x} &= x(a - by) \\ \dot{y} &= -y(c - dx), \end{aligned}$$

where $x(t)$ and $y(t)$ represent the populations of the prey and predator at time t, respectively, and a, b, c, d are positive constants. Should you be interested, you can read about the model's Biological interpretations here.

(a) From an initial condition (x_0, y_0), with $x_0 \geq 0$ and $y_0 \geq 0$, will solutions exist and be unique over a sufficiently small time interval? Explain why or why not.

(b) Using results in the textbook, can you guarantee that solutions exist on an infinite time interval, $[t_0, \infty)$? Explain why or why not.

(c) Determine all equilibrium points of the system. Write down the conditions for equilibrium, solve them, and interpret the meaning of each equilibrium in the context of the model.

4. We set the parameters for the Lotka-Volterra predator-prey model as, $a = 0.1, b = 0.02, c = 0.3$ and $d = 0.01$. With these values, the non-trivial equilibrium becomes

$$\begin{bmatrix} x_e \\ y_e \end{bmatrix} = \begin{bmatrix} 25.0 \\ 4.0 \end{bmatrix} = \begin{bmatrix} \text{prey, those that are eaten} \\ \text{predators, the consumers of prey} \end{bmatrix}$$

in made-up units of tens of animals per square kilometer. Moreover, the Jacobians needed for linearized models at the two equilibria are

$$J(0.0, 0.0) = \begin{bmatrix} 0.1 & -0.0 \\ 0.0 & -0.3 \end{bmatrix} \quad \text{and} \quad J(25.0, 4.0) = \begin{bmatrix} 0.00 & -0.60 \\ 0.05 & 0.00 \end{bmatrix}.$$

(a) Using eigenvalues, analyze the behavior of the linearized predator-prey model near the trivial equilibrium point. You are not asked to compute solutions; instead, give qualitative properties. You can use software to compute the eigenvalues.

(b) Continuing with the trivial equilibrium point, in (a), you will have found that one of the animal populations explodes while the other vanishes. Show how the eigenvectors allow you to distinguish which animal population explodes and which vanishes.

(c) Using eigenvalues, analyze the behavior of the linearized predator-prey model near the nontrivial equilibrium point. You are not asked to compute solutions; instead, give qualitative properties. Moreover, you are not asked to look at the eigenvectors.

(d) For $A := J(25.0, 4.0) = \begin{bmatrix} 0.00 & -0.60 \\ 0.05 & 0.00 \end{bmatrix}$, the matrix exponential $e^{A \cdot 5} = \exp(5A) = \begin{bmatrix} 0.648 & -2.639 \\ 0.220 & 0.648 \end{bmatrix}$ has been computed for you. Based on this information, if the initial animal population at time $t_0 = 0$ is

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 90 \\ 6 \end{bmatrix},$$

what population will be predicted by the linearized model at time $t = 5$?

5. You are given that the matrix $A := \begin{bmatrix} -2.00 & -1.00 & 0.00 & 1.00 \\ -0.25 & -1.00 & -0.25 & 1.25 \\ 1.25 & 0.00 & -0.75 & -1.25 \\ -1.25 & -1.00 & -1.25 & 0.25 \end{bmatrix}$ has eigenvalues and eigenvectors

$$\lambda_1 = 2 + 3\,\mathbf{i}, \quad v_1 = \begin{bmatrix} 1 \\ \mathbf{i} \\ 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 2 - 3\,\mathbf{i}, \quad v_2 = \begin{bmatrix} 1 \\ -\mathbf{i} \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = -1, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \lambda_4 = -4, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Because A is a real matrix, the complex eigenvalues and eigenvectors occur in complex conjugate pairs.

Compute the following functions analytically, by hand. Of course, you can check them in Julia if you want.

(a) $e^{At} v_2$

(b) $x(t)$ when $x(0) = x_0 = v_1 + 3v_2$

(c) Express the complex exponential $e^{\lambda_1 t}$ in the form $e^{at} \cdot (\cos(\omega t) + \mathbf{i}\sin(\omega t))$

(d) $e^{At}v_4$

(e) $x(t)$ when $x(0) = x_0 = \text{real}(v_1)$, where $\text{real}(v_1)$ is the real part of the eigenvector v_1. Note: all of the components of $x(t)$ will be real functions of t.

6. For the linear input-output model $\dddot{y}(t) + 2\ddot{y}(t) - 3\dot{y}(t) + 4y(t) = 6u(t) + 2\dot{u}(t) + e^{-t}u(t)$, determine the following:

(a) Find the transfer function $G(s) = \frac{Y(s)}{U(s)}$, assuming zero initial conditions.

(b) Determine the poles and zeros of $G(s)$.

(c) Is the system BIBO stable? Explain your reasoning.

(d) Suppose $u(t) = 1 - e^{-2t}$. Find the Laplace transform $U(s)$, and compute the corresponding output $Y(s)$ under zero initial conditions.

(e) Find the time-domain expression for $y(t)$ using inverse Laplace transform.

Hints

Prob. 1 Write approximately 15 or more words for each part of the question.

Prob. 2 From Chapter 6 of our textbook, the arc length of a curve defined parametrically can be found by applying

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In your problem, compute the derivatives with respect to θ instead of t. Simplify under the square root, and check the conditions under which the integral converges.

Prob. 3 No hints are provided, beyond, know the textbook!

Prob. 4 To obtain the eigenvalues and eigenvectors, you can use Julia or your favorite LLM. You are not expected to find them by hand. Focus on the qualitative behavior of solutions.

Prob. 5 Solve everything by hand, but do not compute the matrix exponential directly. Use the fact that $e^{At}v = e^{\lambda t}v$ for eigenvectors. Complex solutions will naturally involve sines and cosines.

Prob. 6 You may use software to solve any polynomial roots that arise. Be systematic with Laplace transforms and algebraic steps.

Solutions HW 08

Prob. 1 (a) Included at the end of the solution set

(b) These will vary by person, but some of the more challenging topics may have been:

- Challenging topics from Chapter 7 were discussed in the solutions to HW #7.

Chapter 8 Discussion: Understanding Improper Integrals

- **Grasping the Concept of Improper Integrals**: The notion of integrating functions over unbounded intervals or dealing with integrands that approach infinity at certain points can be quite perplexing at first. It's a significant leap from the definite integrals of bounded functions over finite intervals, which are more intuitive. The extension of the integral concept to embrace these "infinite" situations often requires a mental shift and a deeper understanding of limits and their behavior at infinity.
- **Applying Numerical Methods to Improper Integrals**: While analytical solutions provide a clean, elegant way to solve integrals, they aren't always possible. This is where numerical methods step in, such as using Julia's `QuadGK` to approximate the values of integrals. The transition from a symbolic, formula-driven mindset to one that accepts approximations—necessary yet imperfect—can be tricky. Moreover, understanding the limitations and potential pitfalls of numerical integration, like convergence issues and the importance of choosing correct parameters for the algorithms, adds another layer of complexity.

Chapter 9 Discussion: Understanding ODEs

- **Understanding Linear Systems of ODEs**: Linear systems of first-order ordinary differential equations (ODEs) serve as the backbone of many mathematical models in physics, engineering, and other sciences. The step from simple, one-dimensional ODEs to multidimensional linear systems can be significant. The inclusion of matrix operations, such as the matrix exponential and its properties, introduces a layer of abstract algebra that might not be intuitive for everyone. Particularly, grasping how the matrix exponential, denoted as e^{At}, interacts with vectors and affects system behavior over time requires a solid understanding of both linear algebra and differential equations.
- **Grasping the Concept of Eigenvectors, Eigenvalues, and Their Role in Stability**: The dynamics of linear systems heavily rely on the concepts of eigenvectors and eigenvalues. Understanding that $Av = \lambda v \implies e^{A(t-t_0)}v = e^{\lambda(t-t_0)}v$ and how this relationship influences the system's response over time involves not only algebraic manipulation but also a conceptual understanding of how these mathematical entities encode the system's inherent dynamics. The stability criteria, which involve the real parts of eigenvalues, can be particularly tricky—linking these abstract concepts to the practical question of whether a system will settle down or blow up as time progresses.
- **Numerical Methods for Non-Analytic Solutions**: When students first encounter numerical methods, the transition from analytical solutions to numerical approximations can be challenging. It's one thing to solve an equation on paper where every step follows from a mathematical law, and quite another to iterate a solution using a computer, accepting that sometimes an approximation is the only available answer.

Not Required: Chapter 10 Discussion: Designing Feedback Controllers for Linear SISO Models

- **Grasping the Concept and Mathematical Theory of Laplace Transforms**: The introduction to Laplace transforms is often a significant leap for students. Laplace transforms convert complex time-domain functions into simpler s-domain representations, which is conceptually challenging. Understanding why and how this transformation simplifies differential equations requires a solid grasp of integral calculus and complex functions. Students may struggle with the abstract notion that these transforms change differential equations, which describe system dynamics over time, into algebraic equations that are typically easier to manipulate and solve.
- **Understanding and Applying Inverse Laplace Transforms**: While learning about Laplace transforms is one thing, grasping inverse Laplace transforms is another layer of complexity. The process of finding an inverse involves techniques like partial fraction decomposition, which itself is a topic that can be tricky to master. The ability to decompose a complex rational expression into simpler, solvable parts and then apply the inverse Laplace transform

to each part to reconstruct the original time-domain function requires not only analytical skills but also a good deal of practice and conceptual understanding.

- **Defining and Deriving Transfer Functions for Linear Time-Invariant Systems**: Transfer functions are fundamental in control theory and signal processing, providing a powerful tool for understanding systems in the frequency domain. The derivation of these functions from differential equations using Laplace transforms demands a thorough understanding of both the physical system being modeled and the underlying mathematical principles. Students must be able to not only perform the mathematical manipulations but also interpret what these functions represent about the system's behavior, such as stability and responsiveness. This dual requirement of deep theoretical knowledge and practical application often makes this topic challenging.

Prob. 2 Prob. I: Logarithmic Spiral

Ans. We are given $r(\theta) = 2^{a\theta}$, $\theta \geq 0$. To compute the arc length from $\theta = 0$ to ∞, we first convert the spiral to Cartesian coordinates:

$$x(\theta) = r(\theta)\cos(\theta) = 2^{a\theta}\cos(\theta), \quad y(\theta) = r(\theta)\sin(\theta) = 2^{a\theta}\sin(\theta)$$

Using the formula for the parametrized path length,

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta,$$

we calculate $dx/d\theta$ and $dy/d\theta$ for $x(\theta) = 2^{a\theta}\cos(\theta)$ and $y(\theta) = 2^{a\theta}\sin(\theta)$:

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(2^{a\theta}\cos\theta\right) = a\ln 2 \cdot 2^{a\theta}\cos\theta - 2^{a\theta}\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(2^{a\theta}\sin\theta\right) = a\ln 2 \cdot 2^{a\theta}\sin\theta + 2^{a\theta}\cos\theta$$

Substituting $dx/d\theta$ and $dy/d\theta$ into the arc length formula gives

$$\begin{aligned} S &= \int_0^\infty \sqrt{(a\ln 2 \cdot 2^{a\theta}\cos\theta - 2^{a\theta}\sin\theta)^2 + (a\ln 2 \cdot 2^{a\theta}\sin\theta + 2^{a\theta}\cos\theta)^2}\, d\theta \\ &= \int_0^\infty 2^{a\theta} \cdot \sqrt{(a\ln 2)^2 + 1}\, d\theta = \sqrt{(a\ln 2)^2 + 1}\int_0^\infty 2^{a\theta}\, d\theta \end{aligned}$$

Factor out $2^{a\theta}$ inside the square root:

$$= \int_0^\infty 2^{a\theta} \cdot \sqrt{(a\ln 2\cos\theta - \sin\theta)^2 + (a\ln 2\sin\theta + \cos\theta)^2}\, d\theta$$

Now simplify the expression under the square root:

$$= \int_0^\infty 2^{a\theta} \cdot \sqrt{(a\ln 2)^2 + 1}\, d\theta$$

So the integrand becomes:

$$S = \sqrt{(a\ln 2)^2 + 1}\int_0^\infty 2^{a\theta}\, d\theta$$

To evaluate the integral $\int_0^\infty 2^{a\theta}\, d\theta$, observe:

- If $a < 0$, the function decays and the integral converges

- If $a \geq 0$, the function grows and the integral diverges

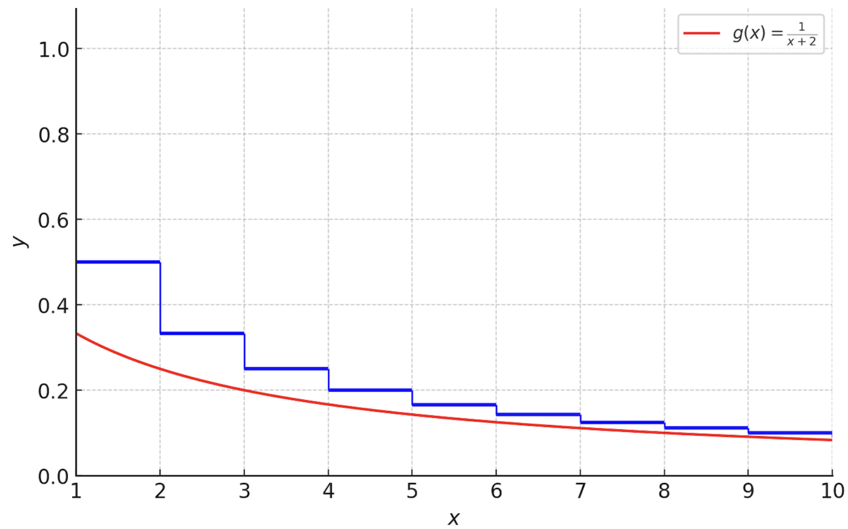
For $a < 0$, we compute:

$$\int_0^\infty 2^{a\theta}\, d\theta = \frac{1}{a\ln 2}\cdot 2^{a\theta}\Big|_0^\infty = \frac{1}{-a\ln 2}$$

Hence the total arc length is:

$$S = \sqrt{(a\ln 2)^2 + 1}\cdot\frac{1}{-a\ln 2}, \quad \text{for } a < 0$$

Prob. II: Harmonic Series



(a) Piecewise Continuity of $f_H(x)$: The function has jump discontinuities at integer values of $x \in (1,\infty)$, and hence over any bounded interval $[a,b]$, it has a finite number of discontinuities. Between the jump discontinuities, the function is constant, hence continuous on intervals of the form $((k-1),k)$. Therefore, $f_H(x)$ is a prototypical piecewise continuous function.

(b) To evaluate $\int_1^{n+1} f_H(x)\,dx$ for $n \in \mathbb{N}$, observe the definition of $f_H(x)$. The integral over each interval $(k-1,k]$, for $k = 2$ to $n+1$, is simply the area of a rectangle with height $\frac{1}{k}$ and width 1, hence the area is $\frac{1}{k}$. Summing these areas from $k = 2$ to $n+1$, we obtain

$$\int_1^{n+1} f_H(x)\,dx = \sum_{k=2}^{n+1}\frac{1}{k} = H_{n+1} - 1.$$

This means the integral of $f_H(x)$ from 1 to $n+1$ equals the $(n+1)$st harmonic number minus 1.

(c) Comparison of $f_H(x)$ and $g(x) = \frac{1}{x+2}$: For $x \geq 1$, consider $g(x) = \frac{1}{x+2}$. Notice that within each interval $(k-1,k]$, we have $\frac{1}{k} \geq \frac{1}{x+2}$ because $x+2 \in [k+1,k+2)$, making $\frac{1}{x+2}$ less than or equal to $\frac{1}{k}$. Hence, $f_H(x) \geq g(x)$ for all $x \geq 1$.

(d) Divergence of the Harmonic Series: Given the comparison $f_H(x) \geq g(x)$ for all $x \geq 1$, and knowing the integral of $g(x)$ over $[1,\infty)$ is,

$$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{dx}{x+2} = \ln(x+2)\Big|_1^\infty = \infty,$$

we deduce that if the integral of the smaller function $(g(x))$ diverges, then the integral of the larger function $(f_H(x))$ must also diverge. Since $\int_1^{n+1} f_H(x)\,dx = H_{n+1} - 1$ and $H_{n+1} \to \infty$ as $n \to \infty$, the divergence of the integral of $f_H(x)$ over $[1,\infty)$ implies the divergence of the harmonic series. Therefore, despite the terms of the harmonic series becoming arbitrarily small, the sum itself grows without bound, illustrating the series' divergence.

Note: There are many other proofs of the divergence of the Harmonic series. Here is a YouTube Playlist. Another interesting series is built from the reciprocals of prime numbers. Surely, that must converge, right? But No! Prime numbers are scarce, but not scarce enough to yield convergence. You have to do something like the reciprocals of twin primes (primes that differ by two) to find a harmonic-like series that converges. None of this is relevant to Engineering, but yet, it's kind of fascinating to contemplate.

Prob. 3 To solve the first two parts of this problem, we use results from the textbook on **Existence and Uniqueness of Solutions.** The last part needs the definition of an equilibrium point from the section on **Linearization of Nonlinear ODE Models**.

(a) Local Existence and Uniqueness of Solutions: **Ans.** Yes, we can assure local existence and uniqueness because the Jacobian exists and is continuous.

Given the initial condition (x_0, y_0) where $x_0 \geq 0$ and $y_0 \geq 0$, we need to examine if solutions to the Lotka-Volterra equations exist and are unique over a sufficiently small time interval.
The Lotka-Volterra equations are:

$$\dot{x} = x(a - by)$$
$$\dot{y} = -y(c - dx),$$

where a, b, c, d are positive constants.
To apply the Local Existence and Uniqueness theorem, we need to consider the Jacobian matrix of the system, $\frac{\partial f(x)}{\partial x}$, where $f(x)$ represents the right-hand side of the system. The Jacobian of the Lotka-Volterra system is:

$$J = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}.$$

Since the entries of J are all finite for any $x, y \geq 0$, and considering x, y in an open ball $B_r(x_0, y_0)$ with $r > 0$, the Jacobian exists and is finite for all $(x, y) \in B_r(x_0, y_0)$. Therefore, according to the Local Existence and Uniqueness theorem, there exists a $\delta > 0$ such that the Lotka-Volterra system has a unique solution on the interval $[t_0, t_0 + \delta]$.

(b) Can global existence and uniqueness of solutions be guaranteed based on results in the textbook: **Ans.** No. The Jacobian contains entries that grow unbounded as a function of x and y.

To guarantee solutions exist on an infinite time interval $[t_0, \infty)$, we look for a constant $0 \leq L < \infty$ such that for all $x \in \mathbb{R}^n$, $||\frac{\partial f(x)}{\partial x}|| \leq L$. However, three of the four entries of the Jacobian of the Lotka-Volterra equations are linear (actually affine) in x or y, and hence their growth is not bounded by a constant independent of x and y. This means the global bound condition fails for these equations, and we cannot guarantee the existence of unique solutions on the interval $[t_0, \infty)$ using results from the textbook.

(c) Equilibrium points occur when $\dot{x} = 0$ and $\dot{y} = 0$. Setting the right-hand sides of the Lotka-Volterra equations to zero gives us two equations to solve:

$$x(a - by) = 0$$
$$-y(c - dx) = 0.$$

From these equations, we identify two equilibrium points:

- The trivial equilibrium when $x = 0$ and $y = 0$. This occurs because if either population is 0, the growth rate of the other population is also 0.
- A non-trivial equilibrium occurs when $x = \frac{c}{d}$ and $y = \frac{a}{b}$. This is found by setting the inside of each parenthesis to zero and solving for x and y, respectively.

These equilibrium points represent states where the populations of predators and prey do not change over time. The first represents the absence of both species, while the second represents a stable coexistence point determined by the system parameters.

Prob. 4 (a) **Behavior Near the Trivial Equilibrium:** The Jacobian at the trivial equilibrium

$$J(0.0, 0.0) = \begin{bmatrix} 0.1000 & -0.0000 \\ 0.0000 & -0.3000 \end{bmatrix},$$

has eigenvalues are $\lambda_1 = -0.3$ and $\lambda_2 = 0.1$ with corresponding eigenvectors

$$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The eigenvalues indicate that in one direction, the solution will expand, and in another direction, it will contract. Specifically, the negative eigenvalue λ_1 suggests "stability (exponential contraction) in the direction of v_1," and the positive eigenvalue λ_2 indicates "instability (exponential expansion/explosion) in the direction of v_2".

(b) **Which animal population explodes and which vanishes:** The eigenvectors help us understand the direction of the population changes. Since $\begin{bmatrix}1\\0\end{bmatrix}$ is associated with the positive eigenvalue, this direction (entirely in the prey population) tends to increase, showing that the prey populations goes wild (grows exponentially, or multiplies like rabbits), while the direction $\begin{bmatrix}0\\1\end{bmatrix}$ (entirely in the predator population) associated with the negative eigenvalue suggests the predator population decreases to zero (extinction or starvation, poor guys).

(c) **The behavior of the linearized predator-prey model near the nontrivial equilibrium point:** For the nontrivial equilibrium $(30.0, 5.0)$, the eigenvalues are complex conjugates $0.0 \pm 0.17320508075688773i$, indicating oscillatory behavior. This means the system near this equilibrium point does not diverge or converge to the equilibrium point directly but oscillates around it.

Not required: In population terms, if the prey increases greatly in number, the predators have more food and reproduce more rapidly, causing the prey population to crash to a small number. This, in turn, starves out the predators, allowing the prey to recover. In this model, there is no steady-state equilibrium.

(d) **Solution five units of time in the future:** **Ans.** $\begin{bmatrix}x(5)\\y(5)\end{bmatrix} = \begin{bmatrix}42.486\\23.688\end{bmatrix}$.

Given $A = J(25.0, 4.0)$ and the initial condition

$$\begin{bmatrix}x_0\\y_0\end{bmatrix} = \begin{bmatrix}90\\6\end{bmatrix},$$

the predicted population at $t = 5$ using $e^{A\cdot 5} = \exp(5A) = \begin{bmatrix}0.648 & -2.639\\0.220 & 0.648\end{bmatrix}$ is calculated as

$$\begin{bmatrix}x(5)\\y(5)\end{bmatrix} = \exp(5A)\cdot\begin{bmatrix}90\\6\end{bmatrix} = \begin{bmatrix}0.648 & -2.639\\0.220 & 0.648\end{bmatrix}\cdot\begin{bmatrix}90\\6\end{bmatrix} = \begin{bmatrix}42.486\\23.688\end{bmatrix}.$$

Prob. 5 (a) **Ans.** $e^{At}v_2 = e^{(2-3i)t}\begin{bmatrix}1\\-i\\1\\0\end{bmatrix}$

Given $\lambda_2 = 2-3i$ and $v_2 = \begin{bmatrix}1\\-i\\1\\0\end{bmatrix}$, we use the fact that for any eigenvector v of A with eigenvalue λ, the matrix exponential satisfies:

$$e^{At}v = e^{\lambda t}v.$$

Substituting in, we get:

$$e^{At}v_2 = e^{(2-3i)t}\begin{bmatrix}1\\-i\\1\\0\end{bmatrix}.$$

(b) **Ans.** $x(t) = e^{(2+3i)t}\begin{bmatrix}1\\i\\1\\0\end{bmatrix} + 3e^{(2-3i)t}\begin{bmatrix}1\\-i\\1\\0\end{bmatrix}$

We seek to compute the function $x(t)$ for the initial condition $x(0) = v_1 + 3v_2$, where v_1 and v_2 are eigenvectors corresponding to eigenvalues $\lambda_1 = 2+3i$ and $\lambda_2 = 2-3i$, respectively.

The solution to the system $\dot{x} = Ax$ with initial condition x_0 can be written as a linear combination of solutions of the form $e^{\lambda t}v$. Therefore,

$$x(t) = e^{\lambda_1 t}v_1 + 3e^{\lambda_2 t}v_2,$$

with

$$v_1 = \begin{bmatrix} 1 \\ i \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -i \\ 1 \\ 0 \end{bmatrix}.$$

(c) **Ans.** For $\lambda_1 = 2 + 3i$, we have:

$$e^{\lambda_1 t} = e^{2t}\left(\cos(3t) + i\sin(3t)\right).$$

This follows from Euler's identity:

$$e^{(a+i\omega)t} = e^{at}\left(\cos(\omega t) + i\sin(\omega t)\right),$$

where $a = 2$, $\omega = 3$.

So for $\lambda_1 = 2 + 3i$, we write it in trigonometric form as:

$$e^{\lambda_1 t} = e^{2t}(\cos(3t) + i\sin(3t)).$$

(d) **Ans.** $e^{At}v_4 = e^{-4t}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Given $\lambda_4 = -4$, and $v_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, we again apply the result that:

$$e^{At}v = e^{\lambda t}v.$$

This gives:

$$e^{At}v_4 = e^{-4t}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(e) **Ans.** $x(t) = e^{2t}\begin{bmatrix} \cos(3t) \\ -\sin(3t) \\ \cos(3t) \\ 0 \end{bmatrix}$

From the textbook, given a complex eigenvalue $\lambda = a + i\omega$ and corresponding eigenvector $v = v_R + iv_I$, we can write the real-valued solution when the initial condition is $x(0) = \mathrm{Re}(v) = v_R$ as:

$$x(t) = e^{at}\left(\cos(\omega t)v_R - \sin(\omega t)v_I\right).$$

For the specific case where $v_1 = \begin{bmatrix} 1 \\ i \\ 1 \\ 0 \end{bmatrix}$, we have:

$$v_R = \mathrm{Re}(v_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_I = \mathrm{Im}(v_1) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda = 2 + 3i.$$

Substituting into the formula:

$$x(t) = e^{2t}\left(\cos(3t)\begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \sin(3t)\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = e^{2t}\begin{bmatrix}\cos(3t)\\-\sin(3t)\\\cos(3t)\\0\end{bmatrix}.$$

Prob. 6 (a) **Ans.** Taking the Laplace transform of both sides of the equation, assuming zero initial conditions, we obtain

$$(s^2+2s+1)Y(s) = \left(6+2s+\frac{1}{s+1}\right)U(s).$$

Therefore, the transfer function $G(s)$ is given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{6+2s+\frac{1}{s+1}}{s^2+2s+1}.$$

(b) **Ans.** The zeros of $G(s)$ are the roots of the numerator, which are the solutions to

$$6+2s+\frac{1}{s+1} = 0.$$

Solving this yields

$$2s^2+8s+7=0 \quad\Longrightarrow\quad s=-2\pm\frac{\sqrt{2}}{2}.$$

The poles of $G(s)$ are the roots of the denominator, which are the solutions to $s^2+2s+1=0$, leading to a double pole at $s=-1$.

(c) **Ans.** No.

For a system to be BIBO (Bounded Input, Bounded Output) stable, all poles of its transfer function $G(s)$ must have negative real parts. Although the pole at $s=-1$ has a negative real part, it is a repeated pole at the boundary of stability. Therefore, the system is NOT BIBO stable.

(d) **Ans.** Given $u(t)=1-e^{-2t}$, we have

$$U(s) = \frac{2}{s(s+2)}.$$

Then,

$$Y(s) = G(s)U(s) = \frac{6+2s+\frac{1}{s+1}}{s^2+2s+1}\cdot\frac{2}{s(s+2)}.$$

Hence, $Y(s)$ is the Laplace transform of the output $y(t)$.

(e) **Ans.** Simplifying the transfer function:

$$G(s) = \frac{2s^2+8s+7}{(s+1)^3}.$$

Thus,

$$Y(s) = \frac{2s^2+8s+7}{(s+1)^3}\cdot\frac{2}{s(s+2)}.$$

The time-domain expression $y(t)$ can be obtained by applying partial fraction decomposition and inverse Laplace transform.

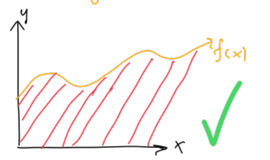
ROB 201 CHAPTER 7 STUDY GUIDE

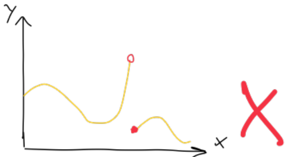
Integration Guide

Antiderivative $\int f(x)\,dx$	Function $f(x)$	Derivative $f'(x)$
C	0	0
$kx + C$	k	0
$\frac{1}{3}x^3 + C$	x^2	$2x$
$e^x + C$	e^x	e^x
$-\cos(x) + C$	$\sin(x)$	$\cos(x)$
$x\ln(x) - x + C$	$\ln(x)$	$\frac{1}{x}$
$\ln\lvert x\rvert + C$	$\frac{1}{x}$	$-\frac{1}{x^2}$
$-\ln\lvert\cos(x)\rvert + C$	$\tan(x)$	$1 + \tan^2(x)$

Fundamental Theorems

#1: Every continuous function has an antiderivative





#2: All definite integrals can be evaluated through antiderivatives

$\int_a^b f(x)\,dx = F(b) - F(a)$

Ex: ① $\int_0^2 (3x^2 + 2x)\,dx = F(2) - F(0)$

② $F(x) = x^3 + x^2 + C$

③ $F(2) = 2^3 + 2^2 = 12 \quad F(0) = 0^3 + 0^2 = 0 \Rightarrow \boxed{F(2) - F(0) = 12}$

Power Rule for Integration

Power Rule for integration:

- for $n \neq -1$: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

Ex: $\int 3x^6\,dx = 3\int x^6\,dx = \frac{x^{6+1}}{6+1} = \frac{x^7}{7} \Rightarrow 3\frac{x^7}{7}$

u-Substitution

Invert the chain rule by using integration by substitution.

*since $f(x) = g(h(x))$, we substitute $h(x)$ for u and integrate a simpler function.

Ex: $\int 3x\cos(x^2+1)\,dx$

① Choose a substitution: let $u = x^2 + 1$.

② Compute du: differentiate u w/ respect to x to find du.

$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

③ Solve for dx: $dx = \frac{du}{2x}$

④ Substitute into the integral: $\int 3x\cos(x^2+1)\,dx = \int 3x\cos(u)\cdot\frac{du}{2x}$

⑤ Cancel out x terms: $\int 3/2\cos(u)\,du$

⑥ Integrate: $\frac{3}{2}\int\cos(u)\,du = \frac{3}{2}\sin(u) + C$

⑦ Plug in $u = x^2+1$: $\frac{3}{2}\sin(x^2+1) + C$

Integration by Parts

Integration by parts: $\int u\,dv = uv - \int v\,du$

Ex: $\int xe^x\,dx$: ① Choose u and dv:

$u = x \quad dv = e^x dx$

② Compute du and v:

- differentiate u to get du: $u = x \Rightarrow du = dx$
- integrate dv to get v: $dv = e^x dx \Rightarrow v = e^x$

③ Integration by parts formula: $\int xe^x dx = xe^x - \int e^x dx$

④ integrate remaining integral: $\int e^x dx = e^x$

⑤ Combine results: $\int xe^x dx = xe^x - e^x + C \Rightarrow \boxed{e^x(x-1) + C}$

Trig Subs for Radicals

*useful when integrand has expressions such as, $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$

Ex: $\int\sqrt{a^2-x^2}\,dx \quad \bullet\ 1-\sin^2(x) = \cos^2(x) \quad \bullet\ \cos(\theta) \geq 0$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\int\left(\sqrt{a^2 - a^2\sin^2(\theta)}\right)\cdot|a|\cos(\theta)\,d\theta = \int\left(\sqrt{a^2\cdot\cos^2(\theta)}\right)\cdot|a|\cos(\theta)\,d\theta$$
$$= \int(|a|\cdot|\cos(\theta)|)\cdot|a|\cos(\theta)\,d\theta$$
$$= a^2\int\cos^2\theta\,d\theta$$

$\Rightarrow$ half angle identity, $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

$\hookrightarrow \frac{a^2}{2}\int(1+\cos(2\theta))\,d\theta = \frac{a^2}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$

$\Rightarrow \bullet\ \sin(\theta) = \frac{x}{|a|} \quad \bullet\ \cos(\theta) = \sqrt{1-\sin^2(\theta)}$

$\hookrightarrow \theta = \arcsin(\frac{x}{a})$, $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \frac{2x}{|a|}\sqrt{1-\left(\frac{x}{|a|}\right)^2}$

$$\Rightarrow \int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\left(\arcsin\left(\frac{x}{|a|}\right) + \frac{x}{|a|}\sqrt{1-\left(\frac{x}{|a|}\right)^2}\right) + C$$

$\hookrightarrow$ For $|x| \leq |a|$, $\int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\arcsin\left(\frac{x}{|a|}\right) + \frac{x}{2}\cdot\sqrt{a^2-x^2} + C$

Partial Fraction Expansions

Ex: $\int\frac{1}{x^2-1}\,dx$

① Decompose to partial fractions:

$$\frac{1}{x^2-1} = \frac{K_1}{(x-1)} + \frac{K_2}{(x+1)}$$

② Compute the coefficients:

- $K_1 = \lim_{x\to r_1}\frac{(x+1)}{(x+1)(x-1)} = \lim_{x\to -1}\frac{1}{(x-1)} = \frac{1}{-1-1} = -1/2$
- $K_2 = \lim_{x\to r_2}\frac{(x-1)}{(x+1)(x-1)} = \lim_{x\to +1}\frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2}$

③ Decompose the Integrand:

$$\frac{1}{x^2-1} = \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}$$

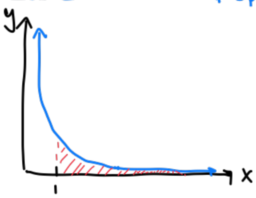
$$\int\frac{1}{x^2-1}\,dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

ROB 201 CHAPTER 8 STUDY GUIDE

Type I Improper Integral

- Improper integrals where the limits of integration extend to $\pm\infty$.
  - One or both of the limits of integration are infinite.

Ex: Evaluate the improper integral $\int_1^\infty \frac{1}{x^2}dx$



① $\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx$

② $\int_1^b \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right)$

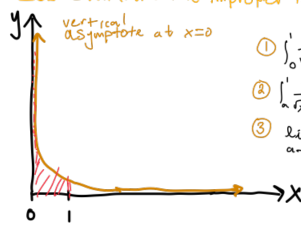
③ $\lim_{b\to\infty}\left(-\frac{1}{b}+1\right)=1$

↳ $\int_1^\infty \frac{1}{x^2}dx = 1$

Type II Improper Integral

- Improper integrals where the integrand has vertical asymptotes within the integral of integration

Ex: Evaluate the improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$



① $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to0^+}\int_a^1 \frac{1}{\sqrt{x}}dx$

② $\int_a^1 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_a^1 = 2-2\sqrt{a}$

③ $\lim_{a\to0^+}(2-2\sqrt{a}) = 2$

↳ $\int_0^1 \frac{1}{\sqrt{x}}dx = 2$

Comparison Test

- Compare one function against another function's known behavior to make conclusions about $\int_a^\infty f(x)\,dx$ by comparing it to $\int_a^\infty g(x)dx$.

Ex: $\int_1^\infty \frac{1}{x^2+1}dx$

① Compare to known convergent integral:
For all $x\ge1$, $\frac{1}{x^2+1}\le\frac{1}{x^2}$

② Analyze integral $\int_1^\infty \frac{1}{x^2}dx$:
$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right)=1$

③ Assess
Since $\int_1^\infty \frac{1}{x^2}dx$ converges to 1 and $\frac{1}{x^2+1}\le\frac{1}{x^2}$ for all $x\ge1$, by the comparison test, $\int_1^\infty \frac{1}{x^2+1}dx$ also converges.

Convergent or Divergent

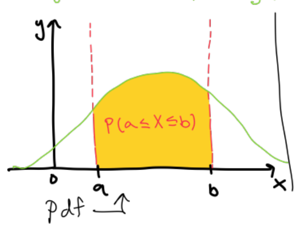
- Assess whether improper integrals are convergent or divergent

Ex: (a) $\int_0^\infty e^{-x}\cdot\frac{x^{24}+2}{x^2+1}dx$ $\Big\{$ Convergent due to decaying exponent and because x^2+1 has no real roots

(b) $\int_0^\infty e^{x}\cdot\frac{x^1+1}{x^{24}+1}dx$ $\Big\{$ Divergent due to exploding exponential out front

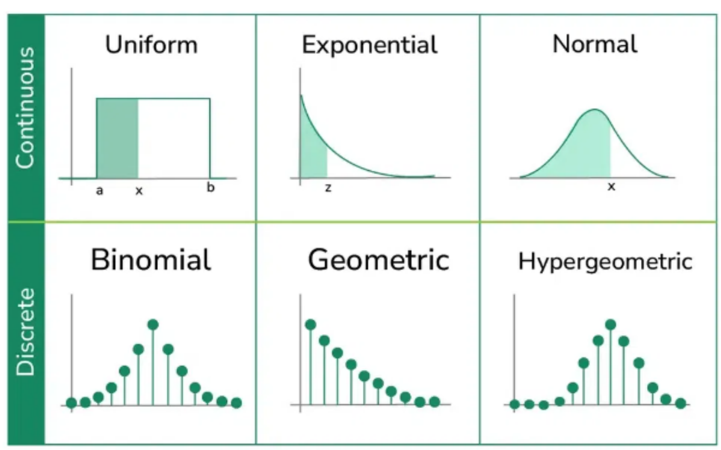
Improper Integrals Through the Lens of Probability

- Function to define the relationship between a random variable and its probability, so that you can find the probability of the variable using the function.



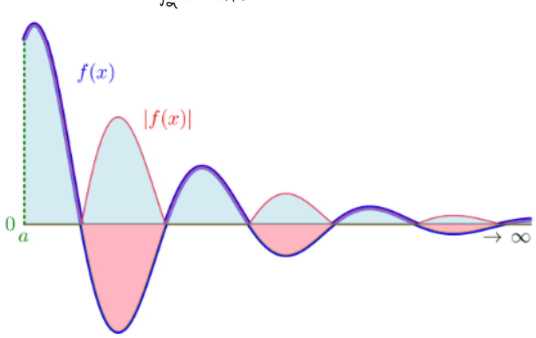
- Must be non-negative
- Normalized to 1
- Probabilistic interpretation

Distribution Examples:



Absolute Integrability

- The absolute value of the function's integrand is integrable over a given interval.
  - A piecewise continuous function $f:[a,\infty)\to\mathbb{R}$ is absolutely integral if, $\int_a^\infty |f(x)|\,dx < \infty$.



ROB 201 CHAPTER 9 STUDY GUIDE

Ordinary Differential Equation (ODE)

- Equation involving functions of one independent variable and their derivatives

- Classification:
  - ⇒ Order: Highest Derivative in equation.
  - ⇒ Linearity: Linear if the function and its derivatives appear linearly.
  - ⇒ Homogeneous vs Non-Homogeneous: Homogeneous if all terms involve the function or its derivatives.

Ex: $\frac{dy}{dx} + y = e^x$ (First-order, linear, non-homogeneous)

Simple Terms

- Describes how a quantity changes with respect to another variable, usually time.

Ex: If we want to study how temperature changes in a room over time, we can use an ODE to model the change.

$\frac{dy}{dt} = ky$ — This ODE says that the rate of change of y, temperature, with respect to time t is proportional to y itself where k is constant.

Finite Time Escape

- Not all ODEs have a solution for all $t \geq 0$.

Ex: $\frac{dx(t)}{dt} = 1 + x^2(t)$, $x(0) = 0$

① Separate variables

$\frac{dx}{1+x^2} = dt$

② Integrate both sides

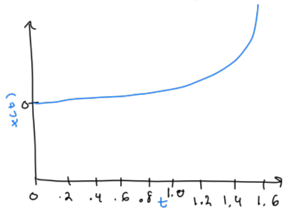
$\int \frac{1}{1+x^2}\,dx = \int dt$

$\hookrightarrow \arctan(x) = t + C$

③ Use initial condition $x(0) = 0$ to find C.

$\arctan(0) = 0 + C \rightarrow C = 0$

$\arctan(x) = t \Rightarrow x(t) = \tan(t)$



Solutions to ODEs

- Some nonlinear ODEs have more than one solution.

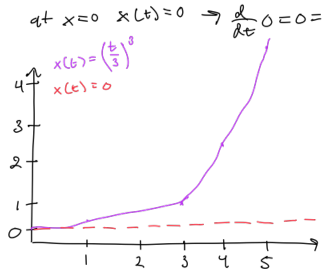
Ex: $\frac{dx(t)}{dt} = (x(t))^{2/3}$, $x(0) = 0$

Sol: $\frac{dx}{x^{2/3}} = dt \rightarrow \int x^{-2/3}\,dx = 3x^{1/3} \rightarrow 3x^{1/3} = t + C$

$\rightarrow x^{1/3} = \left(\frac{t+C}{3}\right)^3 \rightarrow x(0) = 0 \rightarrow \left(\frac{0+C}{3}\right)^3 \rightarrow C = 0$ so $\boxed{x(t) = \left(\frac{t}{3}\right)^3}$ ← Non Trivial sol

OR:

at $x = 0$ $x(t) = 0 \rightarrow \frac{d}{dt} 0 = 0 = 0^{2/3}$ $\boxed{x(t) = 0}$ ← Trivial sol



Higher Order ODEs

- Higher order ODEs involve multiple variables and derivatives to help us solve complex real world systems.

Pendulum Example: consider the differential Equation: $m_p \cdot L_p^2 \cdot \ddot{\theta} + m_p \cdot g \cdot L_p \cdot \sin(\theta) = \tau_p$

A Second order ODE in a single variable θ, due to $\ddot{\theta}$.

- m_p: mass of pendulum
- L_p: length of pendulum
- $\ddot{\theta}$: Second derivative of angle (angular acceleration)
- g: acceleration due to gravity
- τ_p: torque applied

DC Motor Example: DC Motor modeled using Kirchoff's Current and voltage laws.

$L_m \frac{di}{dt} + R_m i = V - b_m \dot{\theta}_m$

motor's output connected to pendulum:

$\theta_m = N\theta$

$\tau_m = K_m \cdot i$

- θ_m: Angle of motor shaft
- L_m: Inductance
- R_m: Resistance
- V: voltage source
- $b_m \dot{\theta}_m$: back electromotive force (emf)
- N: gear ratio
- K_m: motor torque constant

Third order ODE for Motorized Pendulum: the combined system of a DC Motor connected to a pendulum is governed by:

$$m_p \cdot L_m \cdot L_p^2 \cdot \dddot{\theta} + (R_m \cdot m_p \cdot L_p^2) \cdot \ddot{\theta} + (m_p \cdot g \cdot L_m \cdot L_p \cdot \cos(\theta) + N \cdot K_m \cdot b_m) \cdot \dot{\theta} + R_m \cdot m_p \cdot g \cdot L_p \cdot \sin(\theta) = N \cdot K_m \cdot V$$

High-Oder ODE Def

- General Form: An nth order ODE is of the form: $y^{(n)} = f(y^{(n-1)}, \ldots, y', y, t)$.

- Linear Time Invariant Form:

$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$

Ex: Second Order Linear ODE

$\ddot{x} + 5\dot{x} + 6x = 0$

The Robot Equations

$$D(q)\cdot\ddot{q} + C(q,\dot{q})\cdot\dot{q} + G(q) = B\cdot\tau$$

- q: vector of generalized positions
- $\dot{q}$: vector of velocities
- $\ddot{q}$: vector of accelerations
- τ: vector of motor torques
- $D(q)$: $n\times n$ mass inertia matrix
- $G(q)$: $\nabla V(q)$: $n\times 1$ gradient of PE
- $C(q,\dot{q})\cdot\dot{q}$: remaining terms
- B: $n\times m$ torque distribution Matrix

Converting to First-Order ODEs:

Rewrite Equation as first order ODEs, Let:

- $x_1 = q$
- $x_2 = \dot{q}$

$\hookrightarrow$ $\dot{x}_1 = \dot{q} = x_2$

$\dot{x}_2 = \ddot{q} = -D^{-1}(q)\cdot[C(q,\dot{q})\cdot\dot{q} + G(q)] + D^{-1}(q)\cdot B\cdot\tau$

Stack x_1 and x_2 into a $2n\times 1$ vector x;

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -D^{-1}(q)\cdot[C(q,\dot{q})\cdot\dot{q} + G(q)] + D^{-1}(q)\cdot B\cdot\tau\end{bmatrix}$$

$\hookrightarrow$ $\dot{x} = f(x,\tau)$

Existence and Uniqueness

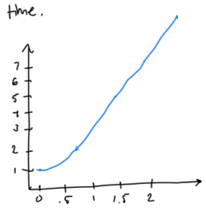
Local Existence and Uniqueness: If the Jacobian $\frac{\partial f(x)}{\partial x}$ exists and is continuous near x_0, a unique solution exists on a small interval around t_0.

Global Existence and Uniqueness: If $\frac{\partial f(x)}{\partial x}$ is bounded globally, a unique solution exists for all time.

Ex: ① Unique Sol:

Consider ODE $\dot{x} = x$, $x(0) = 1$

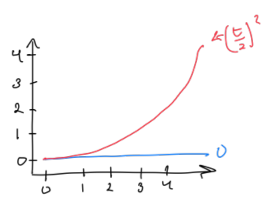
the solution $x(t) = e^t$ is unique for this IC:



② Non Unique Sol:

Consider ODE $\dot{x} = \sqrt{x}$, $x(0) = 0$

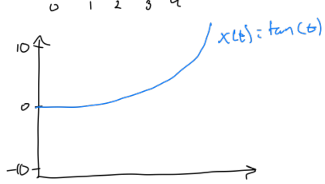
Multiple solutions exist, such as $x(t) = 0$ and $x(t) = \left(\frac{t}{2}\right)^2$.



③ No Sol:

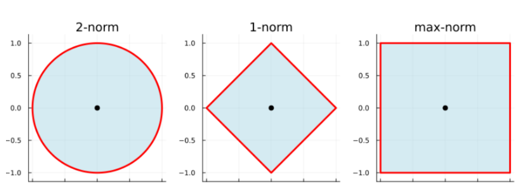
Consider ODE: $\dot{x} = 1 + x^2$, $x(0) = 0$

$x(t) = \tan(t)$ only exists up to $t = \frac{\pi}{2}$ beyond which it blows up. (no global solution).



Norms

Norm: way to measure the "size" or "length" of a vector. Function that takes a vector and returns a non-negative number that represents this size.



First Order Vector ODEs

Solution Def: A function $\varphi: [t_0, T) \to \mathbb{R}^n$ is a solution to the ODE $\dot{x}(t) = f(x(t))$, $x(t_0) = x_0$ if:

① φ is continuous on (t_0, T)

② $\varphi(t_0) = x_0$

③ For all $t\in(t_0, T)$, the derivative $\dot{\varphi}(t)$ exists and $\dot{\varphi}(t) = f(\varphi(t))$.

Ex: For constants $c > 0$,

$$\varphi_c(t) = \begin{cases} 0 & \text{if } t < c \\ \frac{(t-c)^3}{27} & \text{if } t \geq c\end{cases}$$

is a solution to $\dot{x} = \sqrt[3]{x^2}$, $x(0) = 0$

*Check to see if all conditions are satisfied!

Linearization

Consider a simple nonlinear system:

$$\begin{cases}\frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -x_1 + \mu(1 - x_1^2)x_2\end{cases}$$

① Find Equilibrium Points

$$\begin{cases} x_2 = 0 \\ -x_1 + \mu(1-x_1^2)x_2 = 0\end{cases}$$

equilibrium points are $(x_1, x_2) = (0,0)$ and $(x_1, x_2) = (\pm 1, 0)$

② Linearize around $(0,0)$

Compute Jacobian:

$$A = \begin{bmatrix}\frac{\partial f_1}{x_1} & \frac{\partial f_1}{x_1} \\ \frac{\partial f_2}{x_1} & \frac{\partial f_2}{x_2}\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \mu(1 - 3x_1^2)\end{bmatrix}$$

Evaluate at $(0,0)$:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \mu\end{bmatrix}$$

③ Find Eigenvalues

$$\det\begin{bmatrix}-\lambda & 1 \\ -1 & \mu - \lambda\end{bmatrix} = 0 \xrightarrow{\text{eigenvalues}} \lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

$$\lambda^2 - \mu\lambda + 1 = 0$$

- if $\mu^2 < 4$, the eigenvalues are complex with real part $\frac{\mu}{2}$, Indicating a spiral (stable: $\mu < 0$, unstable $\mu > 0$)
- if $\mu^2 > 4$ the eigenvalues are real and distinct, indicating a saddle point.
- if $\mu^2 = 4$ the eigenvalues are real and equal.

Matrix Exponential

$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$ used to solve systems of linear diff Eqs:

$\Rightarrow \frac{dx}{dt} = Ax$

Ex: $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ we want e^{At}

① Find Eigenvalues and Eigenvectors

$\det(A - \lambda I) = 0$

$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda) = 0$

eigenvalues $\Rightarrow \lambda_1 = 2 \quad \lambda_2 = 3$

↳ eigenvectors: $\lambda_1 = 2 \quad (A - 2I)v = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\lambda_2 = 3 \quad (A - 3I)v = 0$

$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

eigenvector $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

② Diagonalize the Matrix

$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$A = PDP^{-1}$

③ Matrix Exponential:

$\boxed{e^{At} = Pe^{Dt}P^{-1}} \Rightarrow e^{Dt} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$

$\Downarrow$

$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{bmatrix}$

↳ $\begin{bmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{3t} - e^{2t} \\ 0 & e^{3t} \end{bmatrix}$

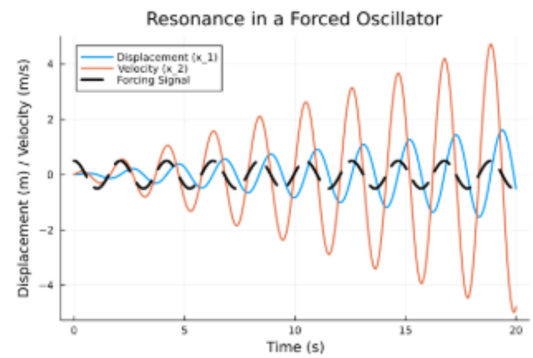
sol to $\frac{dx}{dt} = Ax$ with IC $x(0) = x_0$ is:

$x(t) = e^{At} x_0$

ex: if $x_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$ then

$x(t) = \begin{bmatrix} e^{2t} & e^{3t} - e^{2t} \\ 0 & e^{3t} \end{bmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$

Resonance



When a system oscillates at its natural frequency due to external periodic driving forces at that same frequency.