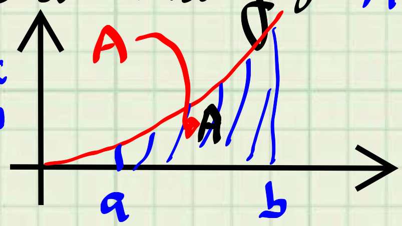
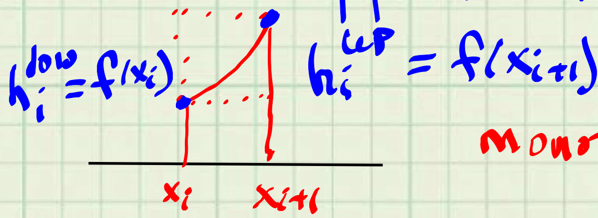
Summary: Archimedes Approx. Principle

monotonic function





monotonic function

$\Delta x := \dfrac{b-a}{n}$ $\qquad a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b$

$\boxed{n+1 \text{ points to define } n\text{-rectangles}}$

$A_i^{low} := h_i^{low} \cdot \Delta x \qquad A_i^{up} := h_i^{up} \cdot \Delta x$

$$\text{Area}_n^{low} := \sum_{i=1}^{n} A_i^{low} \leq A \leq \sum_{i=1}^{n} A_i^{up} = \text{Area}_n^{up}$$

IF $\lim_{n\to\infty} \text{Area}_n^{low} = \lim_{n\to\infty} \text{Area}_n^{up}$ exist & finite

THEN we define A as the common value $\square$

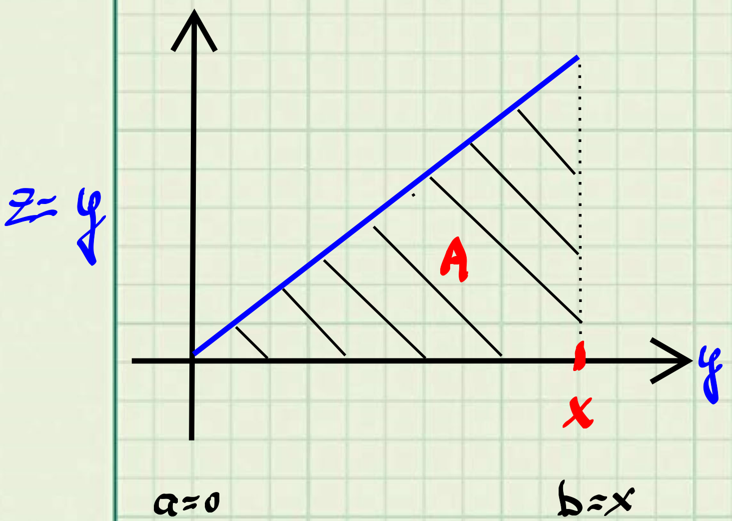
Archimedes $\approx$ 200 BCE
Calculus $\approx$ 1670

$\Big\}$ 1870 year gap!

We had to get through the dark ages and start asking deep questions about the stars and planets.

Sums of Integers to Power k

$$1^k + 2^k + \cdots + n^k = \sum_{i=1}^{n} (i)^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \underbrace{O(n^{k-1})}_{\text{lower-order terms}}$$



$$A = \frac{x^2}{2}$$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$\text{Area}_n^{\text{low}} = \frac{x^2}{2}\left(1 - \frac{1}{n}\right) \xrightarrow[n \to \infty]{} \frac{x^2}{2}$$

$$\text{Area}_n^{\text{up}} = \frac{x^2}{2}\left(1 + \frac{1}{n}\right) \xrightarrow[n \to \infty]{} \frac{x^2}{2}$$

Today: Chap. 03 Definite Integration as the Signed Area Under a Curve

Three Types of Integrals

- definite integrals
- indefinite integrals

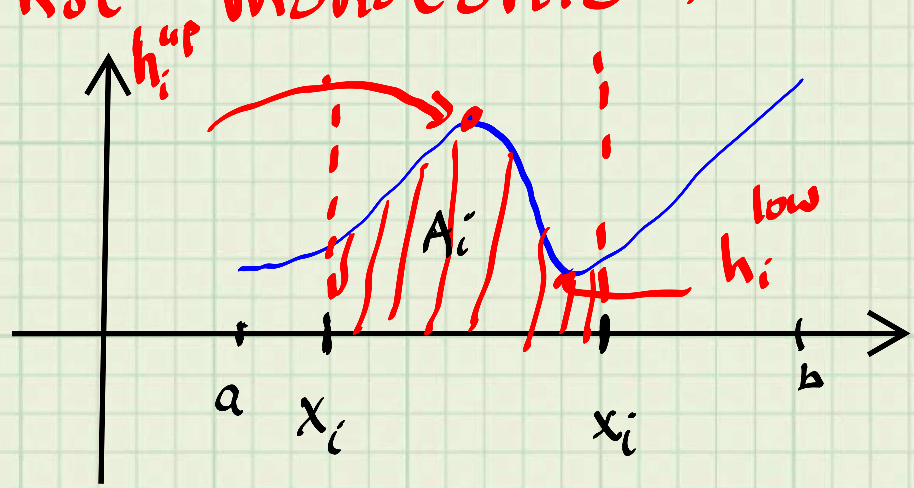
This Chapter

- antiderivatives

After differentiation

$$\boxed{\text{95\% of the integrals needed by engineers do not work via the antiderivatives.}}$$

How to handle functions that are not monotonic?



$\Delta x = \frac{b-a}{n}$

n # rectagles

$$h_i^{low} := \min_{x_i \le x \le x_{i+1}} f(x)$$

$$h_i^{up} := \max_{x_i \le x \le x_{i+1}} f(x)$$

always exist for continuous functions

$A_i^{low} := h_i^{low} \cdot \Delta x$ $\qquad$ $A_i^{low} \le A_i \le A_i^{up}$

$A_i^{up} := h_i^{up} \cdot \Delta x$

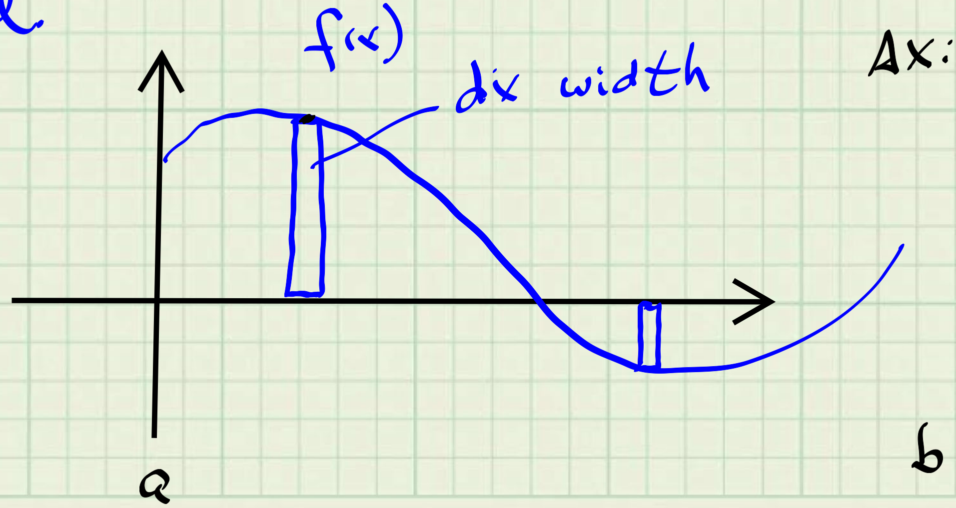
<u>Def.</u> (Riemann-Darboux) Suppose $f:[a,b] \to \mathbb{R}$ that admits max and min exist over all $[x_i, x_{i+1}] \subset [a,b]$. Then the <u>Riemann integral</u> of $f(x)$ is

Lazy S

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \begin{cases} \sum_{i=1}^{n} A_i^{low} \\ \sum_{i=1}^{n} A_i^{up} \end{cases}$$

provided both limits exist, are finite, and are equal.

$\Delta x := \frac{b-a}{n} \xrightarrow[n\to\infty]{} dx$



$$\to \int_a^b f(x)\,dx = \sum f(x)\cdot dx$$

Theorem If $f:[a,b]\to\mathbb{R}$ is continuous, then the Riemann integral exists. $\square$

Suppose: $$\int_0^8 \underbrace{\cos\left(\frac{e^{-x}\cdot\sin(x)}{1+5x+x^2}\right)}_{f(x)}\,dx$$

How to see that $f(x)$ is continuous?

Qualitative view

- $\cos(y)$ is cont all $y\in\mathbb{R}$
- $1+5x+x^2>0$ for all $x\in[0,8]$
- numerator consists of two cont. functions, e^{-x} and $\sin(x)$.
- Nothing becomes unbounded or jumps between two values.

Are all functions Riemann Integrable?

- $f(x) = \frac{1}{x}$ $\quad 1 \le x \le 4$ is OK

  but $0 \le x \le 4$ maybe not.

- $$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \text{ (rational)} \\ 0 & \underbrace{x \in \mathbb{I}}_{x \notin \mathbb{Q}} \text{ (irrational)} \end{cases}$$

then for $x_i < x_{i+1}$

$$\min_{x_i \le x \le x_{i+1}} f(x) = 0$$

$$\max_{x_i \le x \le x_{i+1}} f(x) = 1$$

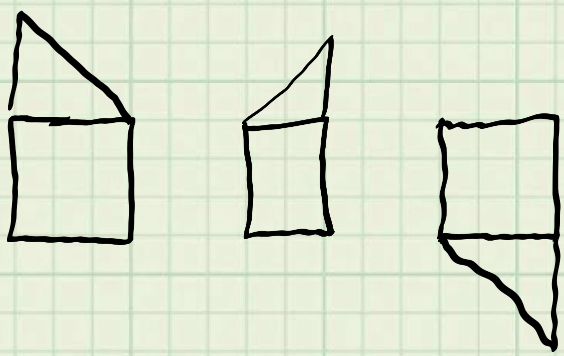
and therefore, for all n, $\text{Area}_n^{low} = 0$

$\text{Area}_n^{up} = 1$ when we integrate over $[0, 1]$. Hence, **NOT** Riemann integrable.

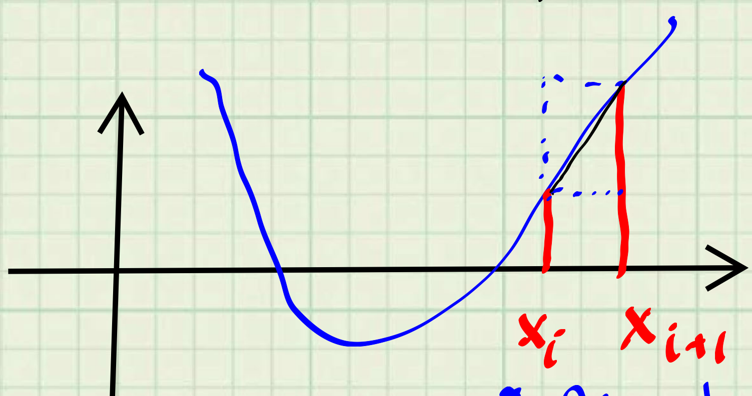
See Demo.

Observation: Rectangles are not very efficient for approximating area under a curve.

- Trapezoidal Rule → TrapZ
- Simpson's Rule



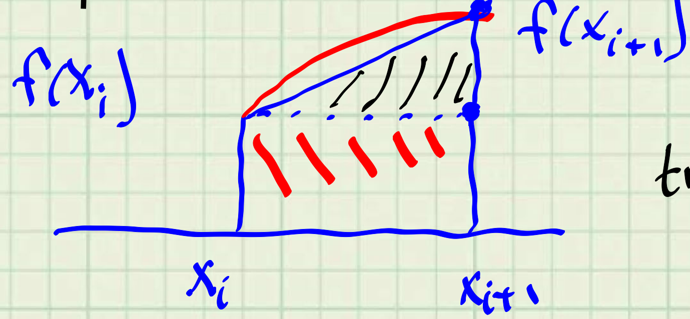
Trapezoids



$$\Delta x = \frac{b-a}{n}$$

$$x_{i+1} = x_i + \Delta x$$

$$x_1 = a$$



$$\text{trap}\,A_i = \frac{f(x_i)+f(x_{i+1})}{2}\Delta x$$

Area of the rect: $f(x_i) \cdot \Delta x$

Area of the tri.: $\dfrac{(f(x_{i+1})-f(x_i)) \cdot \Delta x}{2}$

Sum of the above: trapA = $\frac{(f(x_{i+1})+f(x_i))}{2}\Delta x$

<u>Prop.</u> For $f:[a,b]\to\mathbb{R}$ continuous,

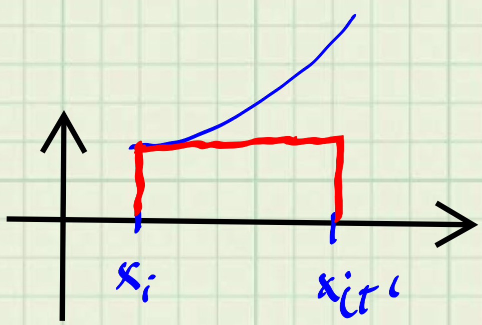
$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} trapA_i$$

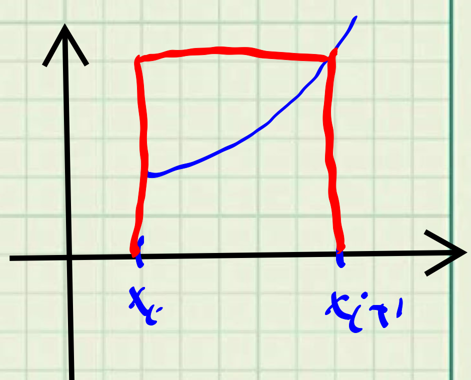
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{f(x_{i+1})+f(x_i)}{2}\Delta x$$

$\square$

More efficient than rectangles for estimating area.

Remark: (Monotonic)





$$\left.\begin{aligned} A_i^{low} &= f(x_i)\cdot\Delta x \\ A_i^{up} &= f(x_{i+1})\cdot\Delta x \end{aligned}\right\}\ \text{average} = trapA_i$$

$$trapA_i = \underbrace{\frac{f(x_i)+f(x_{i+1})}{2}}\cdot\Delta x$$

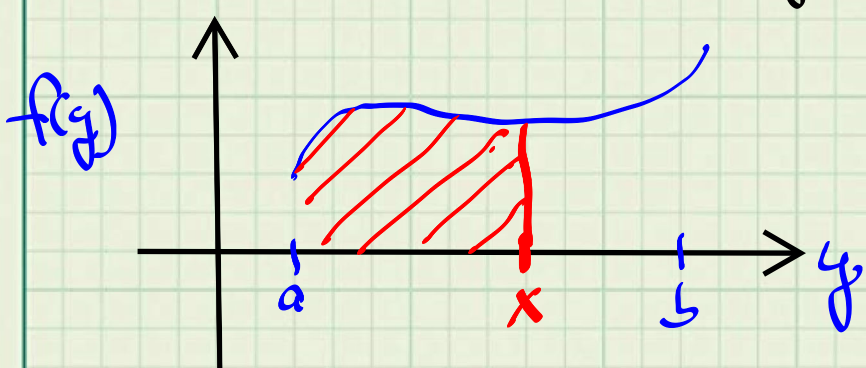
Indefinite Integral

Def. Suppose $f:[a,b]\to\mathbb{R}$ and $a\leq x\leq b$. Then

$$g(x) := \int_a^x f(y)\,dy$$

dummy variable

is an indefinite integral.



Prop. For all counting numbers $k\geq 1$ and $x>0$, $\int_0^x y^k\,dy = \frac{x^{k+1}}{k+1}$

Why: $\sum_{i=1}^{n} (i)^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2}$ + Lower Order Terms

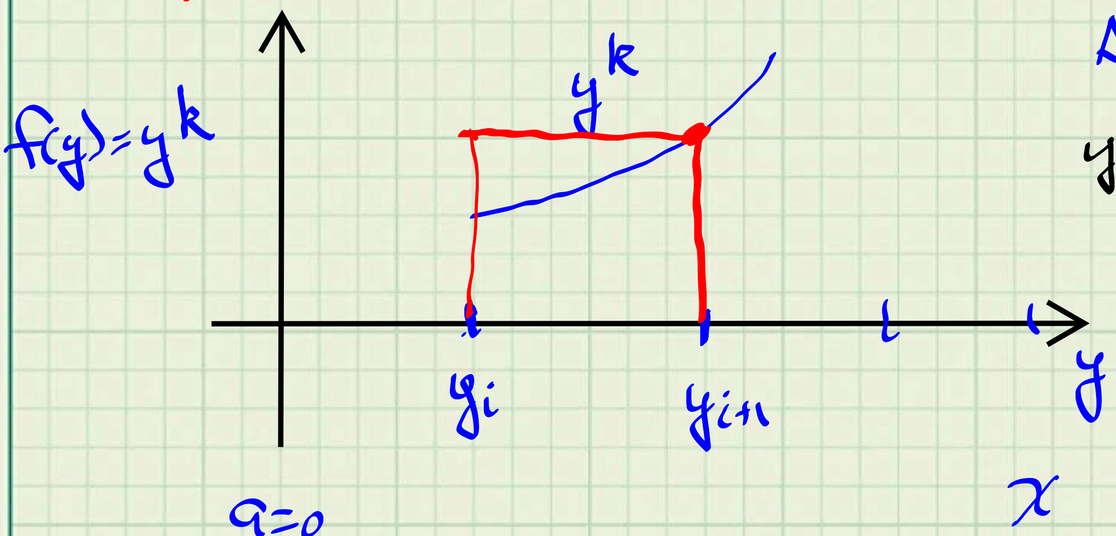
Because monomials are conti, we know the Riemann integral exists.

That means lower & upper sums agree, so we only need to compute one of them.

$\Delta y = \frac{b-a}{n} = \frac{x}{n}$

$y_{i+1} = y_i + \Delta y$

$= (i \Delta y)$



$$A_i^{up} := f(y_{i+1}) \cdot \Delta y$$
$$= (i \Delta y)^k \cdot \Delta y$$
$$= (i)^k \, (\Delta y)^{k+1}$$

$$Area_n^{up} := \sum_{i=1}^{n} A_i^{up}$$
$$= \sum_{i=1}^{n} (i)^k (\Delta y)^{k+1}$$

$\Delta y = \frac{x}{n}$

$$= \left(\frac{x}{n}\right)^{k+1} \sum_{i=1}^{n} (i)^k$$

$$= \left(\frac{x}{n}\right)^{k+1} \left[\frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \text{lower order terms} \right]$$

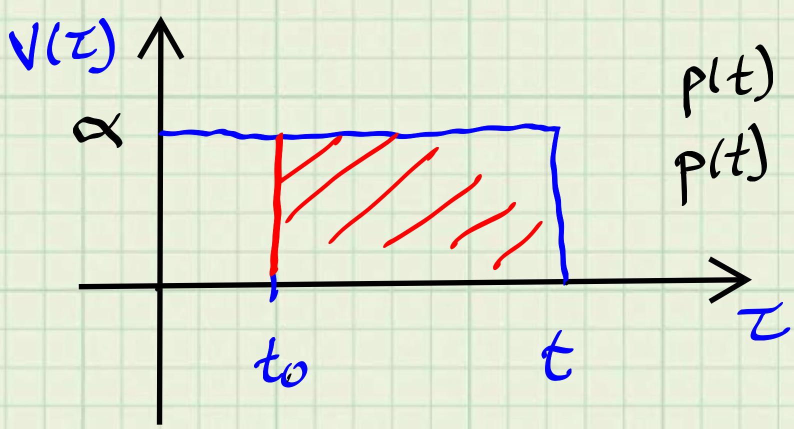
$$= \frac{x^{k+1}}{k+1} + \frac{x^{k+1}}{n \cdot 2} + \frac{x^{k+1}}{\text{higher powers of } n}$$

$$\xrightarrow{n \to \infty} \boxed{\frac{x^{k+1}}{k+1}} \quad \checkmark$$

$$\blacksquare$$

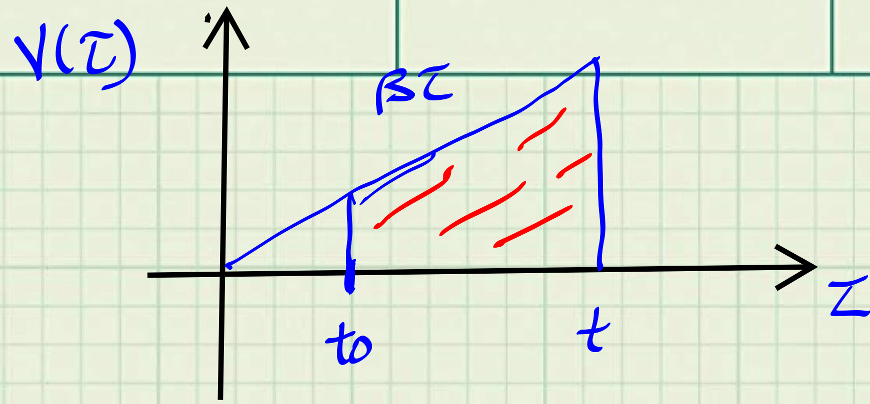
Applications of the Definite Integral

Project 1. From speed to position of a drone & from acceleration to position.

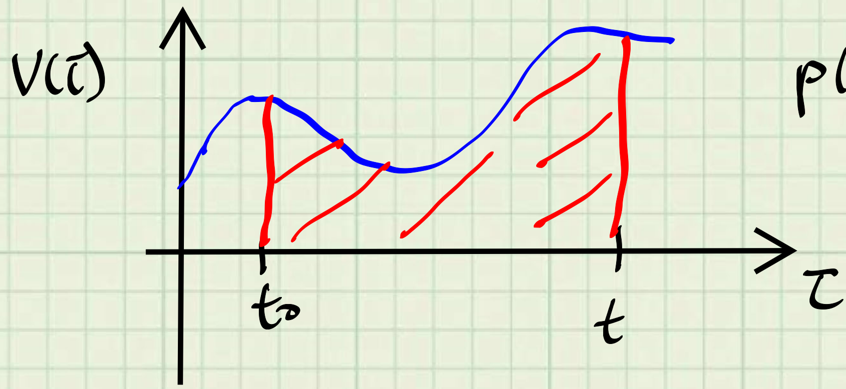


$$p(t) = \alpha \cdot t$$

$$p(t) = \underset{\substack{\uparrow \\ \text{initial position}}}{p(t_0)} + \underbrace{\alpha(t - t_0)}_{\text{change in position}}$$



$$p(t) = p(t_0) + \frac{t^2 - t_0^2}{2}$$



$$p(t) = p(t_0) + \int_{t_0}^{t} V(\tau)\, d\tau$$