ROB 201 16 April 2025

Lecture 26

26.1 Fourier Series for Veterans of Linear Algebra

Introduction

Fourier series offer a powerful method for representing periodic functions as infinite sums of simpler oscillatory functions. In this writeup we present three common forms of the Fourier series:

1. The sine–cosine form.
2. The cosine form with a phase shift, i.e., $\cos(k\omega t + \theta)$.
3. The exponential form.

This document begins by framing Fourier series as the decomposition of a periodic function into its orthogonal components—much like expressing a vector in a finite-dimensional space by projecting it onto an orthonormal basis. To ground this abstract perspective, we compute the Fourier coefficients for the familiar signal $|\sin(t)|$, thereby demonstrating how these coefficients naturally emerge as inner products between the function and the three classical basis functions.

Fourier Series as an Expansion in Terms of Inner Products

The Finite-Dimensional Case

In finite-dimensional real vector spaces, consider an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ for $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be an arbitrary vector. Since the v_i form a basis, we can express x as a linear combination:

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \sum_{k=1}^{n} \alpha_k v_k,$$

where the scalars α_k are to be determined.

To find a formula for each α_k, we take the inner product of both sides with v_j, for any $j \in \{1, 2, \ldots, n\}$:

$$\langle x, v_j \rangle = \left\langle \sum_{k=1}^{n} \alpha_k v_k, v_j \right\rangle.$$

Using linearity of the inner product, this becomes:

$$\langle x, v_j \rangle = \sum_{k=1}^{n} \alpha_k \langle v_k, v_j \rangle.$$

Since the basis is orthonormal, we know that

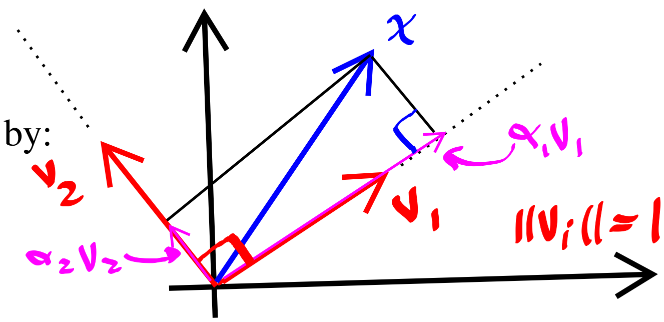
$$\langle v_k, v_j \rangle = \delta_{kj} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases} \qquad \textbf{the Kronecker delta}$$

This means that all terms in the sum vanish except when $k = j$, so:

$$\langle x, v_j \rangle = \alpha_j.$$

Therefore, the coefficients in the expansion are given by:

$$\alpha_j = \langle x, v_j \rangle.$$



We conclude that: any vector $x \in \mathbb{R}^n$ can be expressed as:

$$x = \sum_{k=1}^{n} \langle x, v_k \rangle \, v_k.$$

This expansion expresses x as a combination of its projections onto the orthonormal basis vectors v_k, and forms the foundation for understanding Fourier series in function spaces.

Corollary: A linearly independent set $\{v_1, v_2, \ldots, v_m\}$ is a basis for R^n if, and only if, the only $x \in \mathbb{R}^n$ satisfying $< x, v_j >= 0$ for all $1 \le j \le m$ is $x = 0_{n\times 1}$.

Inner Product for Functions

For piecewise continuous functions defined on an interval $[0, T]$ (or equivalently, on $[-\frac{T}{2}, \frac{T}{2}]$), we define an inner product for functions f and g by

$$\langle f, g \rangle := \int_0^T f(t)\, g(t)\, dt.$$

This inner product induces a norm

$$\|f\| := \sqrt{\langle f, f \rangle} \iff \|f\|^2 = \langle f, f \rangle,$$

and provides a framework for discussing orthonormality in function spaces.

Exercise: Apply Gram-Schmidt to make $\{u_1 = t, u_2 = t^2\}$ into an orthonormal set of vectors on the interval $[0, 1]$.

Orthonormal Basis for Functions

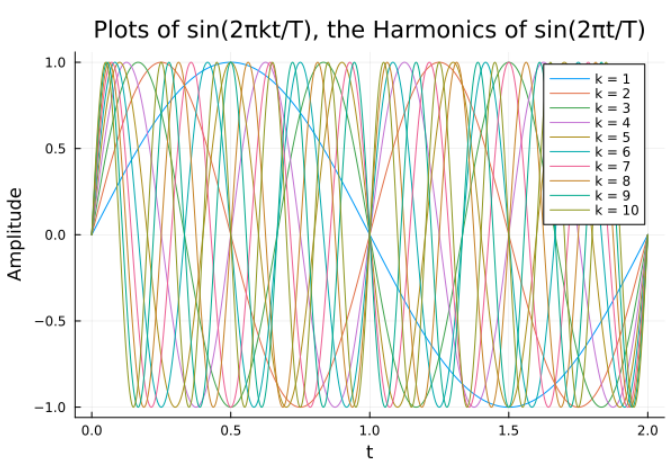
$L^2([0,T]) := \{f : [0,T] \to \mathbb{R} \mid f$ p.w. cont. and $\int_0^T f^2(\tau)\, d\tau < \infty\}$ (functions with finite "energy").

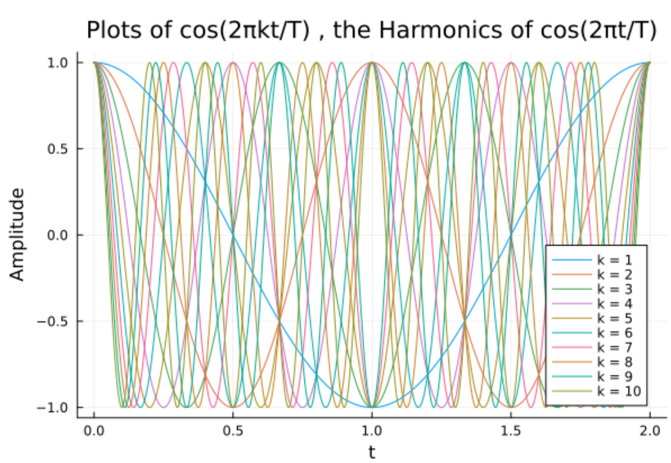
Def. The countably infinite set $\{\varphi_1, \varphi_2, \ldots\}$ is a complete orthonormal basis for $L^2[0,T]$ if

- $\langle \varphi_j, \varphi_k \rangle = \delta_{jk}$ (orthonormal)
- The only piecewise continuous function $f : [0,T] \to \mathbb{R}$ satisfying $\langle f, \varphi_j \rangle = 0$ for all $1 \le j < \infty$ is the zero function (aka, $f(t) = 0$ for all $0 \le t \le T$).

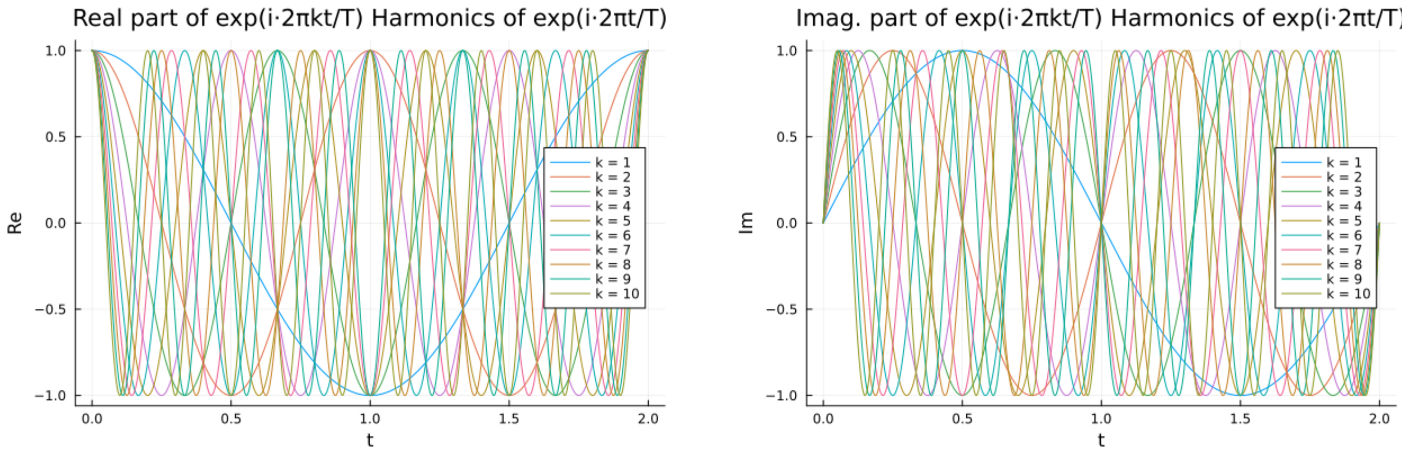
Example Complete Orthonormal Bases on $[0,T]$

- **Harmonic Trigonometric Functions:** $\left\{\sqrt{\frac{1}{T}}, \sqrt{\frac{2}{T}} \sin\left(\frac{2k\pi}{T} t\right), \sqrt{\frac{2}{T}} \cos\left(\frac{2k\pi}{T} t\right) \,\middle|\, k = 1, 2, \ldots\right\}$





- **Harmonic Complex Exponential Functions:** $\{\sqrt{\frac{1}{T}}e^{\frac{2k\pi \mathrm{i}}{T}t} \,\Big|\, -\infty < k < \infty\}$



General Fourier Series

Let $\{\varphi_k(t)\}_{k=1}^{\infty}$ be a complete orthonormal basis for $L^2([0,T])$. Then any piecewise continuous function $f : [0,T] \to \mathbb{R}$ can be represented (in the L^2-sense) as

$$f(t) \sim \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle\, \varphi_k(t),$$

where the inner product is

$$\langle f, g \rangle = \int_0^T f(t)\, g(t)\, dt.$$

This parallels the finite-dimensional result:

$$x = \sum_{k=1}^{n} \langle x, v_k \rangle\, v_k.$$

Key takeaways:

- $\alpha_k := \langle f, \varphi_k \rangle \;= \int_0^T f(t)\varphi_k(t)\, dt$ are called **Fourier Coefficients**.
- The Fourier series is simply the projection of a function onto a complete orthonormal basis, just as a vector is projected onto a finite orthonormal basis in $\mathbb{R}^n$.

- $\lim_{N\to\infty} \|f(t) - \sum_{k=1}^{N} \langle f, \varphi_k \rangle \varphi_k(t)\| = 0.$

Special Case 1: Trigonometric Fourier Series

Suppose $f : (-\infty, \infty) \to \mathbb{R}$ is piecewise continuous and T-periodic ($f(t+T) = f(t)$ for all $t \in \mathbb{R}$), then the sine–cosine form of the Fourier series is,

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi k}{T} t\right) + b_k \sin\left(\frac{2\pi k}{T} t\right) \right].$$

The Fourier coefficients are given by

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2\pi k}{T} t\right) dt, \quad k \geq 0,$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2\pi k}{T} t\right) dt, \quad k \geq 1.$$

Remark: The above is the traditional way to write the series, where, with the exception of a_0, a_k and b_k differ from $\langle f, \varphi_k \rangle$ by a factor of $\sqrt{\frac{2}{T}}$

Special Case 2: Cosine Form with Phase Shift

An alternative representation expresses $f(t)$ solely as a sum of cosine functions with phase shifts:

$$f(t) \sim A_0 + \sum_{k=1}^{\infty} A_k \cos\left(\frac{2\pi k}{T} t + \theta_k\right),$$

where the amplitude and phase are defined by

$$A_k = \sqrt{a_k^2 + b_k^2}, \quad \theta_k = -\arctan\left(\frac{b_k}{a_k}\right).$$

Special Case 3: Complex Exponential Fourier Series

The Fourier series can also be expressed in complex exponential form:

$$f(t) \sim \sum_{k=-\infty}^{\infty} c_k \, e^{i\frac{2\pi k}{T}t},$$

where

$$c_k = \frac{1}{T}\int_0^T f(t)\, e^{-i\frac{2\pi k}{T}t}\, dt.$$

Remark: The above is the traditional way to write the series, where the coefficients c_k differ from $\langle f, \varphi_k \rangle := \langle f, \sqrt{\frac{1}{T}} e^{\frac{2k\pi\,\mathrm{i}}{T}t} \rangle$ by a factor of $\sqrt{\frac{1}{T}}$

Orthogonality of Complex Exponentials

Let k and m be distinct integers. We now show that the complex exponentials

$$e^{-i\frac{2\pi k}{T}t} \quad \text{and} \quad e^{-i\frac{2\pi m}{T}t}$$

are orthogonal on the interval $[0, T]$ with respect to the inner product

$$\langle f, g \rangle = \frac{1}{T}\int_0^T f(t)\, \overline{g(t)}\, dt,$$

where $\overline{g(t)}$ denotes the complex conjugate of $g(t)$. To prove this, compute:

$$\langle e^{-i\frac{2\pi k}{T}t}, e^{-i\frac{2\pi m}{T}t} \rangle = \frac{1}{T}\int_0^T e^{-i\frac{2\pi k}{T}t}\, \overline{e^{-i\frac{2\pi m}{T}t}}\, dt.$$

Since the complex conjugate of $e^{-i\frac{2\pi m}{T}t}$ is $e^{i\frac{2\pi m}{T}t}$, this becomes

$$\langle e^{-i\frac{2\pi k}{T}t}, e^{-i\frac{2\pi m}{T}t} \rangle = \frac{1}{T}\int_0^T e^{-i\frac{2\pi k}{T}t}\, e^{i\frac{2\pi m}{T}t}\, dt = \frac{1}{T}\int_0^T e^{i\frac{2\pi(m-k)}{T}t}\, dt.$$

For $k \neq m$, we have $m - k \neq 0$. Then, evaluating the integral yields

$$\int_0^T e^{i\frac{2\pi(m-k)}{T}t}\,dt = \left.\frac{e^{i\frac{2\pi(m-k)}{T}t}}{i\frac{2\pi(m-k)}{T}}\right|_{t=-T/2}^{T/2} = \frac{T}{i\,2\pi(m-k)}\Big(e^{i\pi(m-k)} - e^{-i\pi(m-k)}\Big).$$

Using the identity

$$e^{i\pi(m-k)} - e^{-i\pi(m-k)} = 2i\sin\big(\pi(m-k)\big),$$

this simplifies to

$$\int_0^T e^{i\frac{2\pi(m-k)}{T}t}\,dt = \frac{T}{i\,2\pi(m-k)}\cdot 2i\sin\big(\pi(m-k)\big) = \frac{T\,\sin\big(\pi(m-k)\big)}{\pi(m-k)}.$$

Since $m - k$ is a nonzero integer, $\sin\big(\pi(m-k)\big) = 0$. Hence,

$$\int_0^T e^{i\frac{2\pi(m-k)}{T}t}\,dt = 0,$$

and consequently,

$$\langle e^{-i\frac{2\pi k}{T}t}, e^{-i\frac{2\pi m}{T}t}\rangle = \frac{1}{T}\cdot 0 = 0.$$

Thus, the complex exponentials $e^{-i\frac{2\pi k}{T}t}$ and $e^{-i\frac{2\pi m}{T}t}$ are orthogonal when $k \neq m$.

Examples

We now compute all three forms of the Fourier series for the function

$$f(t) = |\sin(t)|,$$

which has fundamental period $T = \pi$. Note that $|\sin(t)|$ is even and piecewise smooth on $[0,\pi]$, which simplifies several computations.

1. Sine–Cosine Form

Since $f(t)$ is even, all sine coefficients vanish:

$$b_k = \frac{2}{\pi} \int_0^{\pi} \sin(t) \cdot \sin(2kt)\, dt = 0.$$

The cosine coefficients are:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin(t)\, dt = \frac{4}{\pi},$$

and for $k \geq 1$, use the identity:

$$\sin(t)\cos(2kt) = \frac{1}{2}[\sin((2k+1)t) + \sin((2k-1)t)],$$

to compute:

$$a_k = \frac{2}{\pi} \int_0^{\pi} \sin(t)\cos(2kt)\, dt = \frac{-4}{\pi(4k^2-1)}.$$

So the Fourier series is:

$$|\sin(t)| \sim \frac{2}{\pi} - \sum_{k=1}^{\infty} \frac{4}{\pi(4k^2-1)} \cos(2kt).$$

2. Cosine Form with Phase Shift

Since $f(t) = |\sin(t)|$ is even, all sine coefficients are zero ($b_k = 0$). Therefore, the cosine-with-phase-shift form becomes:

$$|\sin(t)| \sim A_0 + \sum_{k=1}^{\infty} A_k \cos\left(\frac{2\pi k}{\pi} t + \theta_k\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty} |a_k| \cos(2kt),$$

since $\theta_k = 0$ or π and $\cos(2kt + \pi) = -\cos(2kt)$, we can absorb the sign into a_k directly:

$$|\sin(t)| \sim \frac{2}{\pi} - \sum_{k=1}^{\infty} \frac{4}{\pi(4k^2 - 1)} \cos(2kt).$$

Same as before, but conceptually obtained from magnitude–phase viewpoint.

3. Complex Exponential Form

We use the exponential form:

$$|\sin(t)| \sim \sum_{k=-\infty}^{\infty} c_k \, e^{i2kt},$$

where

$$c_k = \frac{1}{\pi} \int_0^{\pi} \sin(t) \, e^{-i2kt} \, dt.$$

Because $|\sin(t)|$ is even, the Fourier coefficients satisfy:

$$c_{-k} = c_k,$$

and using Euler's identity, we have:

$$c_k = \frac{1}{2}(a_k - ib_k).$$

Since $b_k = 0$, this becomes:

$$c_k = \frac{1}{2} a_k = \frac{-2}{\pi(4k^2 - 1)}, \quad k \neq 0,$$

$$c_0 = \frac{1}{\pi} \int_0^{\pi} \sin(t) \, dt = \frac{2}{\pi}.$$

So the exponential form of the series is:

$$|\sin(t)| \sim \sum_{k=-\infty}^{\infty} c_k\, e^{i2kt} = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{-2}{\pi(4k^2-1)} \left(e^{i2kt} + e^{-i2kt}\right).$$

Using the identity $e^{i2kt} + e^{-i2kt} = 2\cos(2kt)$, we recover the original cosine series:

$$|\sin(t)| \sim \frac{2}{\pi} - \sum_{k=1}^{\infty} \frac{4}{\pi(4k^2-1)} \cos(2kt).$$

Orthonormal Functions Based on Monomials

The monomials $1, t, t^2, t^3, \ldots$ form a natural basis for function spaces like $C[a,b]$, but they are not orthogonal under the usual L^2 inner product. However, there exist several classical families of orthogonal (and orthonormal, once properly scaled) polynomials that are derived from monomials via procedures such as the Gram–Schmidt process. Below are the most well-known examples.

1. Legendre Polynomials

The *Legendre polynomials* are defined on the interval $[-1,1]$ and are orthogonal with respect to the weight function $w(t) = 1$:

$$\int_{-1}^{1} P_n(t)\, P_m(t)\, dt = 0 \quad \text{if } n \neq m.$$

The first few Legendre polynomials are:

$$P_0(t) = 1, \quad P_1(t) = t, \quad P_2(t) = \frac{1}{2}(3t^2 - 1), \quad \ldots$$

2. Chebyshev Polynomials

The *Chebyshev polynomials of the first kind*, denoted $T_n(t)$, are orthogonal on $[-1, 1]$ with respect to the weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$. They are particularly useful in approximation theory:

$$T_0(t) = 1, \quad T_1(t) = t, \quad T_2(t) = 2t^2 - 1, \quad \ldots$$

3. Hermite Polynomials

The *Hermite polynomials* are orthogonal on the real line $(-\infty, \infty)$ with respect to the Gaussian weight $w(t) = e^{-t^2}$. These arise in probability theory and quantum mechanics:

$$H_0(t) = 1, \quad H_1(t) = 2t, \quad H_2(t) = 4t^2 - 2, \quad \ldots$$

4. Laguerre Polynomials

The *Laguerre polynomials* are orthogonal on $[0, \infty)$ with respect to the weight function $w(t) = e^{-t}$. They appear in solutions to differential equations, particularly in quantum physics:

$$L_0(t) = 1, \quad L_1(t) = -t + 1, \quad L_2(t) = \frac{1}{2}(t^2 - 4t + 2), \quad \ldots$$

5. Gram–Schmidt Applied to Monomials

One can also generate orthonormal functions from the monomials $\{1, t, t^2, t^3, \ldots\}$ directly by applying the Gram–Schmidt process on a chosen interval $[a, b]$ using the inner product:

$$\langle f, g \rangle = \int_a^b f(t)\, g(t)\, dt.$$

This approach yields a custom orthonormal basis tailored to the interval and inner product, and is often used in numerical methods and function approximation.

Conclusion

This writeup has reviewed three forms of Fourier series representations for a T-periodic function: the sine–cosine form, the cosine form with phase shift, and the complex exponential form. Using the example $f(t) = |\sin(t)|$ (which has fundamental period π) we computed the Fourier coefficients explicitly in each representation. We then provided a more abstract treatment by framing the Fourier series as the expansion of a function in terms of an orthonormal basis, where the Fourier coefficients arise naturally as inner products. This perspective not only unifies the different forms of the Fourier series but also connects Fourier analysis to foundational concepts in linear algebra and functional analysis.

26.2 Using Optimization to Solve ODEs

We consider the optimal control problem of minimizing the energy of a robotic system over a fixed time horizon $[0, T]$. The dynamics of the robot are governed by the second-order differential equation

$$D(q)\,\ddot{q} + C(q,\dot{q})\,\dot{q} + G(q) = B\tau,$$

where $q \in \mathbb{R}^n$ are the joint positions, $\dot{q}$ are the velocities, and $\Gamma \in \mathbb{R}^n$ are the control inputs (torques). Let $x = (q, \dot{q}) \in \mathbb{R}^{2n}$ denote the state vector.

Optimization Problem

We aim to solve the following optimal control problem, x is the state of the model and $u := \tau$, motor torque.

$$\begin{aligned} \text{minimize} \quad & \int_0^T u^\top(t) u(t)\, dt \\ \text{subject to} \quad & \dot{x}(t) = f(x(t), u(t)) \\ & x(0) = x_0, \quad x(T) = x_{\text{final}} \end{aligned}$$

where the function $f(x,u)$ encodes the second-order robot dynamics as a first-order system.

$$\dot{x}_1 = x_2$$

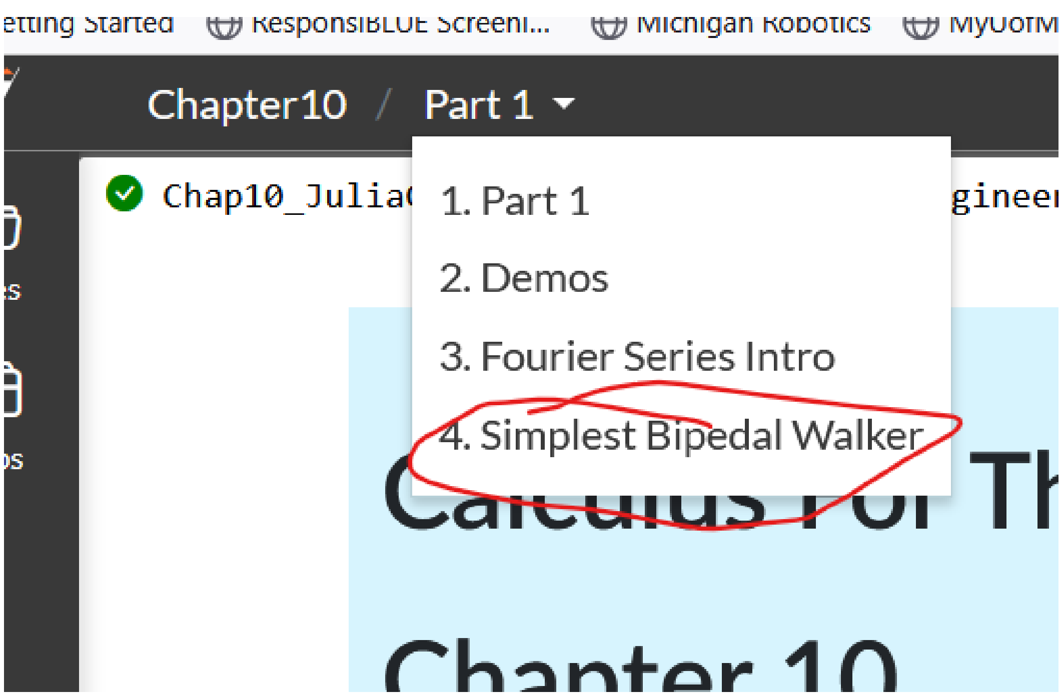
$$\dot{x_2} = D(x_1) \backslash (-C(x_1, x_2) \cdot x_2 - G(x_1) + B\tau)$$

You can even include the initial conditions, x_0, and/or final conditions, x_{final}, as optimization variables.

How to Program in JuMP

- See Project 2, especially the second part. The Diver is a great example. You can find the ODEs in the files
  - diver_model_setup_full.jl
  - diver_model_utilities.jl

  which are in the Work/Data folder
- There is a more complicated example with the Simplest Bipedal Walker in



$u_1 := t : [0,1] \to \mathbb{R}$
$u_2 : t^2 : [0,1] \to \mathbb{R}$

$$v_1 = \frac{u_1}{\|u_1\|} \quad , \quad \|u_1\|^2 = \langle u_1, u_1 \rangle$$

$$= \int_0^1 t \cdot t \, dt$$

$$= \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$= \sqrt{3}\, t \qquad \|u_1\| = \frac{1}{\sqrt{3}}$$

$$\bar{v}_2 = u_2 - \alpha v_1 = t^2 - \alpha \sqrt{3}\, t$$

Choose α s.t. $\langle \bar{v}_2, v_1 \rangle = 0$

$$0 = \langle v_2, v_1 \rangle = \int_0^1 (t^2 - \alpha\sqrt{3}\, t)(\sqrt{3}\, t)\, dt$$

$$= \int_0^1 \sqrt{3}\, t^3 - \alpha \int_0^1 (\sqrt{3}\, t)^2 \, dt$$

$$= \sqrt{3}\, \frac{1}{4} - \alpha 3 \cdot \frac{1}{3}$$

$$\therefore \alpha = \frac{\sqrt{3}}{4}$$

Final step

$$v_2 = \frac{\bar{v}_2}{\|v_2\|}$$

$$\bar{V}_2 = t^2 - \frac{\sqrt{3}}{4}\sqrt{3}\,t = t^2 - \frac{3}{4}t$$

$$\|\bar{V}_2\|^2 = \int_0^1 \left(t^2 - \frac{3}{4}t\right)^2 dt$$

$$= \int_0^1 \left(t^4 - \frac{3}{2}t^3 + \frac{9}{16}t^2\right) dt$$

$$= \frac{1}{5} - \frac{3}{2}\left(\frac{1}{4}\right) + \frac{\cancel{9}^{\,3}}{16}\cdot\frac{1}{\cancel{3}}$$

$$V_2 = \frac{\bar{V}_2}{\|\bar{V}_2\|} = \frac{\bar{V}_2}{\sqrt{\frac{1}{5} - \frac{3}{8} + \frac{3}{16}}}$$