ROB 201 - Calculus for the Modern Engineer
HW #1

Prof. Grizzle

Check Canvas for due date and time

No late HW is accepted. Instead, we drop your two lowest HW grades. We are using Gradescope for turning in HW; see relevant information on the course Canvas site. Gradescope will always be set to accept HW solutions until 11:59 PM = 23:59 ET (same time zone as NY City). The three-hour difference is a grace period. Technically, after 9 PM ET you are late, and if you try to turn in your HW solution at midnight Ann Arbor time, or later, we have no guilt in refusing it.

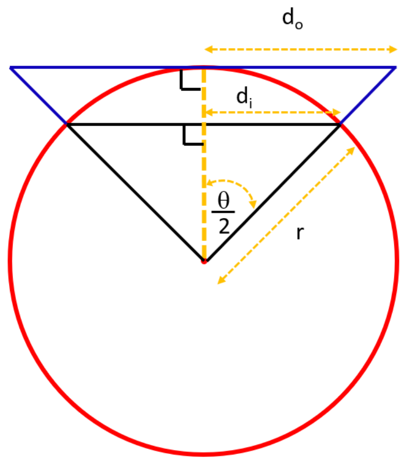
Uploading HW to Gradescope: PLEASE watch this video on how to upload your HW (written) solutions to GradeScope `https://youtu.be/nksyA0s-Geo`. We need you to associate problem solutions with pages.

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete and turn in.

1. Read Chapters 1 and 2 of our ROB 201 Textbook, `Calculus for the Modern Engineer`; you will find a copy on our Canvas site, in the `file` folder. Based on your reading of the chapters, summarize in your own words:
   (a) the purpose of Chapter 01;
   (b) two things you found the most interesting;
   (c) the purpose of Chapter 02;
   (d) and, two things you found the most interesting.

   There are no "right" or "wrong" answers, but no answer means no points. The goal is to reflect a bit on what you are learning and why.

2. Archimedes' computation of π in Figure 1.1 of our textbook (shown below) depends on the following inequalities for the lower and upper bounds,
$$d_i \leq \frac{r\,\theta}{2} \leq d_o,$$
   where $r\theta$ is the length of the arc of the circle bounded by the triangle. Develop a geometric proof showing that $\frac{r\,\theta}{2} \leq d_o$. We are taking it as "obvious to the most casual observer" that $2d_i \leq r\,\theta$ (because a straight line is the shortest path between two points).



3. For $\epsilon > 0$, find all $\left\{\delta > 0 \,\middle|\, |e^{\delta} - 1| < \epsilon\right\}$.

4. Use induction to show that for all integers $n \geq 0$, $1^2 + 2^2 + 4^2 + \cdots + (2^n)^2 = \frac{4^{n+1}-1}{3}$.

   Note: $1^2 + 2^2 + 4^2 + \cdots + (2^n)^2 = \sum_{k=0}^{n} \left(2^k\right)^2 = \sum_{k=0}^{n} \left(2^{2k}\right) = \sum_{k=0}^{n} \left(4^k\right)$.

5. Which of the following compositions are valid? You do NOT have to show your work. You only need to answer Yes or No.

   (a) $f : [-4, 4] \to \mathbb{R}$ by $f(x) = \sin(x)$ and $g : [0, \infty) \to \mathbb{R}$ by $g(x) = \frac{1}{1+x^2}$.

   $\boldsymbol{f \circ g}$ **exists? (Y/N)** $\qquad$ $\boldsymbol{g \circ f}$ **exists? (Y/N)**

   (b) $f : [1, 2] \to [3, 4]$ is surjective and $g : [0, \infty) \to (0, 1]$ is surjective.

   $\boldsymbol{f \circ g}$ **exists? (Y/N)** $\qquad$ $\boldsymbol{g \circ f}$ **exists? (Y/N)**

   (c) $f : \mathbb{R} \to \mathbb{R}$ is surjective and $g : [0, 1] \to [-2, 1]$ by $g(x) = 2x - 1$.

   $\boldsymbol{f \circ g}$ **exists? (Y/N)** $\qquad$ $\boldsymbol{g \circ f}$ **exists? (Y/N)**

6. Compute, **the hard way**, $\lim\limits_{x\to\infty} f(x)$ for the following functions. For each answer, give the following information:

   - State whether the limit exists or not.
   - If the limit exists, give its value, which we'll call $f_\infty := \lim_{x\to\infty} f(x)$.
   - If the limit exists **and is finite**, for $\epsilon = 0.1$, find $0 < M < \infty$ such that $|f_\infty - f(x)| \leq 0.1$ for all $x \geq M$. The smaller your M, the more points you'll earn.
   - If the limit exists **and is unbounded** or **does not exist**, you do not need to do anything further.

   Remark: We are using $x \geq M$ instead of $x \geq N$ because, below, we use $N(x)$ to denote the numerator of a ratio of two functions.

   (a) $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \frac{x^2}{x^2+1}$.

   (b) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x^3}{x^2+1}$.

   (c) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x^3+3}{1+x^8}$.

Note: We work the example $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \frac{x}{x^2+1}$ so that you are clear on what to do.

- The limit exists.
- $f_\infty = 0.0$.
- $|f_\infty - f(x)| \leq 0.1$ for all $x \geq 10$. In other words, $M = 10.0$.

This is true because

$$\begin{aligned} |f_\infty - f(x)| &= \left|0.0 - \frac{x}{x^2+1}\right| \\ &= \frac{x}{x^2+1} \\ &= \frac{\frac{1}{x}}{1+\frac{1}{x^2}} \text{ (divide through by largest term in the denominator).} \end{aligned}$$

From the Hint: To find an upper bound for a fraction, $\frac{n(x)}{d(x)}$, you replace the numerator with an upper bound and the denominator with a lower bound for the inequality to hold. In symbols, suppose that $0 \leq n(x) \leq N(x)$. Then,

$$\frac{n(x)}{d(x)} \leq \frac{N(x)}{d(x)},$$

because having a larger number in the numerator gives a larger fraction. Next, suppose that $0 < D(x) \leq d(x)$. Then,

$$\frac{n(x)}{d(x)} \leq \frac{N(x)}{d(x)} \leq \frac{N(x)}{D(x)},$$

because having a smaller number in the denominator yields a larger fraction. **While these are “obvious ” facts, applying them successfully when you are first learning limits can be hard to master!**

Going back to our problem: For $x \geq 10$, $\frac{1}{x} \leq 0.1$ and $1 + \frac{1}{x^2} \geq 1.0$. Hence, for $x \geq 10$,

$$\frac{\frac{1}{x}}{1+\frac{1}{x^2}} \leq \frac{1}{10}.$$

As stated above, if $0 < n(x) \leq N(x)$ and $d(x) \geq D(x) > 0$, then $\frac{n(x)}{d(x)} \leq \frac{N(x)}{D(x)}$. In our case, $n(x) = \frac{1}{x} \leq N(x) := 0.1$ holds for $x \geq 10$, and so does $d(x) = 1 + \frac{1}{x^2} \geq 1.0 =: D(x)$. **If you are struggling with this idea, please attend office hours.**

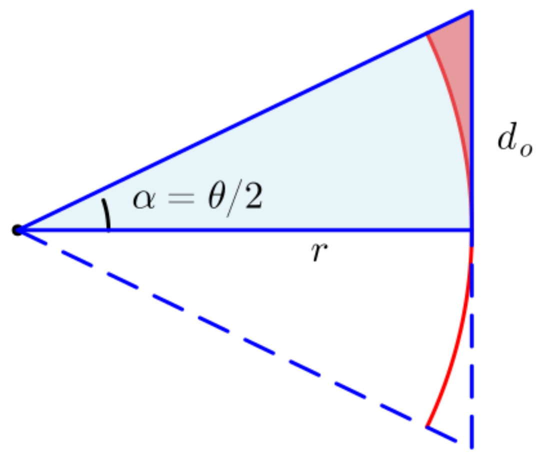
This is the end of the drill problems. The second part of the HW set will introduce you to the Julia Programming Language via `jupyter notebooks`. **Please go to** the course Canvas Site and find the assignment titled "Homework 1 (Julia)". This assignment will be completed in Vocareum, a cloud-hosted jupyter notebook environment for the Julia programming language. If you have questions on Julia programming, you can ask them in recitation and office hours.

Hints

Prob. 1 Write approximately 15 or more words for each part of the question.

Prob. 2 Young students find proofs hard because they have not practiced them much. Your author finds the sport of Cricket hard because he's not practiced it much. Like anything, proofs become easier with practice.

In the figure below, we have simply rotated Figure 1.1 so that you might better understand the relationship between the outer right triangle defining d_0 and the circle of radius r.



(a) The area of the (light-blue) shaded arc of the circle is $\frac{\alpha}{2}r^2$, when α is in radians.

(b) The area of the shaded triangle (light-blue plus light-red) is larger than the area of the shaded arc of the circle.

We are sure that you know all of these facts, but due to inexperience with proofs, you would not think to put them together in order to show that $r\alpha = r\,\frac{\theta}{2} \le d_o$.

Write down what you want to show. Write down what you know (from the hint). Observe that you are like one line away from being done with the proof. If you get stuck, reach out on Piazza or talk to us in Office Hours. If the hour is not propitious for that, try UMGPT or Maizey (aka, if the timing doesn't work, try UMGPT or Maizey).

Prob. 3 **Notation:** Some of you are struggling with college-level notation. It's OK; we all have to start somewhere. In Chapter 1 of our textbook, you can look for the section titled **"Set Notation and How to Make Sense of It"**.

What does this mean as a subset of the real numbers,

$$\left\{\delta > 0 \,\middle|\, |e^{\delta} - 1| < \epsilon\right\}?$$

First of all, you see that only real numbers $\delta > 0$ are being considered. You can think of this part of the definition of the set as being similar to specifying the domain of a function. The symbol $\big|$ means "such that"; it indicates that among all of the positive real numbers, the set only contains those that satisfy the additional requirement, $|e^{\delta} - 1| < \epsilon$, which you can also write as

$$\begin{aligned} -\epsilon < & e^{\delta} - 1 < \epsilon \quad (\text{because } |y| < a \iff -a < y < -a) \\ 1 - \epsilon < & \quad e^{\delta} \quad < 1 + \epsilon \quad (\text{after adding one to both sides and the middle}). \end{aligned}$$

If you don't see this, ask Maizey or UM GPT for clarification.

By definition, $f : (0, \infty) \to \mathbb{R}$ is **strictly increasing** if, and only if, $x_1 < x_2 \implies f(x_1) < f(x_2)$. The natural logarithm is a strictly increasing function.

Prob. 4 Three proofs by induction are worked out in the textbook. If you get stuck, here is a second hint:

Prompt for UM GPT:

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Give a proof by induction for: 1^2 + 2^2 + 4^2 + \cdots +  (2^n)^2 =
\frac{4^{n+1} - 1}{3}
```

Remark: If you go this route, give the proof in "your own words". Show us that, after seeing the proof, you understand it adequately to add insightful comments, or you are able to re-write the proof using the second style presented in the textbook. Solutions that appear to be straight out of an LLM, with no human commentary, will not earn full credit.

Prob. 5 In each of the problems, you are given the **domains** of the functions (i.e., the functions' allowed "inputs"). To solve the problems, you must determine the **range** of each function (i.e., the "outputs" of the function). **Remember, in general, the co-domain is not the same as the range.**

Prob. 6 **Intuition:** To find an upper bound for a fraction, $\frac{n(x)}{d(x)}$, you can replace the numerator with an upper bound and/or the denominator with a lower bound for the inequality to hold. In symbols, suppose that $0 \leq n(x) \leq N(x)$. Then,

$$\frac{n(x)}{d(x)} \leq \frac{N(x)}{d(x)},$$

because having a larger number in the numerator gives a larger fraction. Next, suppose that $0 < D(x) \leq d(x)$. Then,

$$\frac{n(x)}{d(x)} \leq \frac{N(x)}{d(x)} \leq \frac{N(x)}{D(x)},$$

because having a smaller number in the denominator yields a larger fraction. **While these are "obvious" facts, applying them successfully when you are first learning limits can be hard to master!**

Conversely, to find a lower bound for a fraction, $\frac{n(x)}{d(x)}$, you replace the numerator with a lower bound and the denominator with an upper bound for the inequality to hold.

In other words, if $n(x) \geq N(x)$ and $0 < d(x) \leq D(x)$, then $\frac{n(x)}{d(x)} \geq \frac{N(x)}{D(x)}$. This is useful for showing a limit is equal to infinity.

Solutions HW 01

Prob. 1 (a) **The purpose of Chapter 01:** Chapter 01 reviews in detail how mathematics is written using good notation. It introduces what is likely a new concept for you, the Approximation Principle, which shows how to estimate quantities with firm lower and upper bounds. We'll come back to this over and over in the early part of learning Calculus. The remainder of Chapter 01 reviews basic facts about algebra, functions, trigonometry, powers, radicals, exponentials, and logarithms. Only a few of you will have seen Euler's formula during your HS career. The Binomial Theorem should be familiar to you, though you may be rusty at using it.

(b) **Two things I found the most interesting:** The history behind some of the mathematical inventions is cool. How one goes about defining x^y when y is an irrational number is pretty neat. If I were to choose a third idea, it would be Euler's formula.

(c) **The purpose of Chapter 02:** Chapter 02 begins by underlining that mathematics is hard to create. Cantor discovered that the concept of infinity comes in more than one size, an idea so shocking that he was ostracized by the mathematical community of his day. Yet, we were able to go through a very simple proof that showed Cantor was correct. Proofs are how mathematicians try to avoid letting errors slip into their body of work. In ROB 201, you will see some proofs in lectures, some you will read in the textbook, and yet others are too challenging for your level of mathematical training. Chapter 02 introduced (or maybe reviewed for some of you) how to construct a proof by induction. It then brought us to our first contact with the notion of a limit.

(d) **Two things I found the most interesting:** The first time I saw it, I was super surprised that rational numbers can be put into one-to-one correspondence with the counting numbers. Cantor's Diagonal Argument is super elegant, and hence, I found that interesting. Limits are the workhorse of Calculus. I did not really understand them until I took the equivalent of Michigan's Math 451 Advanced Calculus I. My hope is that our textbook does a better job than the one I had for Calculus I and II back in the day.

Prob. 2

- From HS Geometry and the image in the Hints, $\text{Area}_{arc} = \frac{1}{2}\alpha\, r^2$, and $\text{Area}_{\triangle} = \frac{1}{2} r\, d_o$.
- Next, we use a fact about the two areas,

$$\begin{aligned}
&\text{Area}_{arc} \le \text{Area}_{\triangle} \\
&\qquad\Updownarrow \\
&\frac{1}{2}\alpha r^2 \le \frac{1}{2} r\, d_0 \\
&\qquad\Updownarrow \quad \text{(after cancelling } \frac{r}{2} \text{ from both sides)} \\
&\alpha r \le d_0,
\end{aligned}$$

which proves the result.

Prob. 3 $\boxed{\textbf{Ans.}\quad \text{For } \epsilon > 0,\ \left\{\delta > 0 \,\middle|\, |e^\delta - 1| < \epsilon\right\} = \left\{0 < \delta < \ln(1+\epsilon)\right\}}$. This set is nonempty for all $\epsilon > 0$.

Reasoning: For all $\delta > 0$, $e^\delta > 1$. Hence,

$$\left(\ |e^\delta - 1| < \epsilon\ \right) \iff \left(\ (e^\delta - 1) < \epsilon\ \right),$$

meaning we can remove the absolute value sign when solving the problem. This simplifies things a lot.

$$\begin{aligned}
&e^\delta - 1 < \epsilon \\
&\qquad\Updownarrow \\
&e^\delta < 1 + \epsilon \ \ \text{(algebra)} \\
&\qquad\Updownarrow \\
&\delta < \ln(1+\epsilon) \ \ \text{(natural log is strictly increasing)}
\end{aligned}$$

Prob. 4 We want to prove that for all integers $n \geq 0$,

$$1^2 + 2^2 + 4^2 + \cdots + (2^n)^2 = \frac{4^{n+1} - 1}{3},$$

where we note that $1^2 + 2^2 + 4^2 + \cdots + (2^n)^2 = \sum_{k=0}^{n} \left(2^k\right)^2 = \sum_{k=0}^{n} \left(2^{2k}\right) = \sum_{k=0}^{n} \left(2^2\right)^k = \sum_{k=0}^{n} 4^k.$

Base Case: For $n = 0$, the sum is

$$\sum_{k=0}^{0} 4^k = 1$$

And $\frac{4^1-1}{3} = \frac{3}{3} = 1$. Thus, the base case holds.

If you also check $n = 1$, that is fine too.

Inductive Step: Assume the statement is true for some integer $n \geq 0$, i.e.,

$$\sum_{k=0}^{n} 4^k = \frac{4^{n+1} - 1}{3}$$

We need to show that the statement holds for $n + 1$, i.e.,

$$\sum_{k=0}^{n+1} 4^k = \frac{4^{n+2} - 1}{3}.$$

Starting from the inductive hypothesis, we add the next term 4^{n+1} to both sides:

$$\begin{aligned}
\sum_{k=0}^{n} 4^k + 4^{n+1} &= \frac{4^{n+1} - 1}{3} + 4^{n+1} \\
&= \frac{4^{n+1} - 1}{3} + \frac{3}{3} 4^{n+1} \\
&= \frac{4 \cdot 4^{n+1} - 1}{3} \\
&= \frac{4^{n+2} - 1}{3}
\end{aligned}$$

Hence, by the principle of mathematical induction, the statement

$$\sum_{k=0}^{n} 4^k = \frac{4^{n+1} - 1}{3}$$

is true for all integers $n \geq 0$.

Second Solution: Here, we use the second approach to proofs by induction, as outlined in the textbook. We seek to show that, for all $n \geq 0$,

$$\sum_{k=0}^{n} 4^k = \frac{4^{n+1} - 1}{3}.$$

We define

$$\begin{aligned}
L(n) &:= \sum_{k=0}^{n} 4^k, \text{ the left side of what we seek to show,} \\
R(n) &:= \frac{4^{n+1} - 1}{3}, \qquad \text{the right side of what we seek to show.}
\end{aligned}$$

Claim (or Step 0): for all counting numbers $n \geq 0$, $P(n) : L(n) = R(n)$ is true (the left and right sides are equal)

- **Step 1:** Check the base case, $P(0)$: For $n = 0$, we have that $L(0) = 1$ and $R(0) = 1$ and hence the base case is true.
- **Step 2:** Show the induction hypothesis is true. That is, if $P(n)$ is true for some $n \geq 0$, then $P(n+1)$ is also true.

When we assume that $P(n)$ is true, we are assuming that

$$L(n) = R(n).$$

To see if this implies $L(n+1) = R(n+1)$, we first evaluate

$$L(n+1) := \sum_{k=0}^{n+1} 4^k = \underbrace{\sum_{k=0}^{n} 4^k}_{L(n)} + 4^{n+1} = L(n) + 4^{v+1}.$$

Using the induction hypothesis, namely, $L(n) = R(n) = (n+1)^2$, we compute

$$\begin{aligned} L(n+1) &= L(n) + 4^{n+1} \\ &= \underbrace{\frac{4^{n+1}-1}{3}}_{L(n)=R(n)} + 4^{n+1} \\ &= \frac{4^{n+1}-1}{3} + \frac{3}{3}4^{n+1} \\ &= \frac{4 \cdot 4^{n+1}-1}{3} \\ &= \frac{4^{n+2}-1}{3} \\ R(n+1) &= \frac{4^{n+2}-1}{3}. \end{aligned} \tag{1}$$

We conclude that $L(n+1) = R(n+1)$ and therefore, by the principle of induction, for all $n \geq 0$, $\sum_{k=0}^{n} 4^k = \frac{4^{n+1}-1}{3}$.

Prob. 5 You were not required to give reasons. We provide them to help you learn the material.

(a) **Ans.** $\boldsymbol{f \circ g}$ **exists (Y)** while $\boldsymbol{g \circ f}$ **does not exist (N)**

Reason: $f : [-4, 4] \to \mathbb{R}$ by $f(x) = \sin(x)$, has
- $\mathrm{dom}(f) = [-4, 4]$
- $\mathrm{range}(f) = [-1, 1]$

while $g : [0, \infty) \to \mathbb{R}$ by $g(x) = \frac{1}{1+x^2}$ has
- $\mathrm{dom}(g) = [0, \infty)$
- $\mathrm{range}(g) = (0, 1]$

$f \circ g$ exists because $\mathrm{range}(g) \subset \mathrm{dom}(f)$

$g \circ f$ fails to exist because $\mathrm{range}(f) \not\subset \mathrm{dom}(g)$

(b) **Ans.** $\boldsymbol{f \circ g}$ **fails to exist (N)** while $\boldsymbol{g \circ f}$ **exists (Y)**

Reason: We are given that $f : [1, 2] \to [3, 4]$ is surjective (aka, onto) and $g : [0, \infty) \to (0, 1]$ is surjective (aka, onto). We conclude from this
- $\mathrm{dom}(f) = [1, 2]$
- $\mathrm{range}(f) = [3, 4]$

and

- $\mathrm{dom}(g) = [0, \infty)$
- $\mathrm{range}(g) = (0, 1]$

$f \circ g$ fails to exist because $\mathrm{range}(g) \not\subset \mathrm{dom}(f)$

$g \circ f$ exists because $\mathrm{range}(f) \subset \mathrm{dom}(g)$

(c) **Ans.** ***f* ∘ *g* exists (Y)** while ***g* ∘ *f* does not exist (N)**

Reason: We are given that $f : \mathbb{R} \to \mathbb{R}$ is surjective, and hence

- $\mathrm{dom}(f) = (-\infty, \infty)$
- $\mathrm{range}(f) = (-\infty, \infty)$

while $g : [0, 1] \to [-2, 1]$ by $g(x) = 2x - 1$ has

- $\mathrm{dom}(g) = [0, 1]$
- $\mathrm{range}(g) = [-1, 1]$

$f \circ g$ exists because $\mathrm{range}(g) \subset \mathrm{dom}(f)$

$g \circ f$ fails to exist because $\mathrm{range}(f) \not\subset \mathrm{dom}(g)$

Prob. 6 (a) $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \frac{x^2}{x^2+1}$.

- The limit exists.
- $f_\infty = 1.0$.
- $|f_\infty - f(x)| \leq 0.1$ for all $x \geq 3$. In other words, $M = 3.0$.

This is true because $|f_\infty - f(x)| = |1 - \frac{x^2}{x^2+1}| = \frac{1}{x^2+1} \leq 0.1$ for $x^2 \geq 9$, which means $x \geq 3$.

(b) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x^3}{x^2+1}$.

- The limit exists.
- $f_\infty = \infty$.
- No other work is required.

For completeness, we note that $f(x)\frac{x^3}{x^2+1} = \frac{x}{1+\frac{1}{x^2}} \geq \frac{x}{2}$ for all $x \geq 0$. Hence, the limit is unbounded

(c) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x^3+3}{1+x^8}$.

- The limit exists.
- $f_\infty = 0.0$.
- $|f_\infty - f(x)| \leq 0.1$ for all $x \geq \sqrt[5]{40}$. In other words, $M = \sqrt[5]{40} \approx 2.091$.

This is true because

$$\begin{aligned}
|f_\infty - f(x)| &= |0 - \frac{x^3+3}{1+x^8}| \\
&= \frac{x^3+3}{1+x^8} \quad \text{(next, we multiply top and bottom by } x^{-8}) \\
&= \frac{x^{-5}+3x^{-8}}{1+x^{-8}} \quad \text{(bounding steps are explained below)} \\
&\leq \frac{4x^{-5}}{1} \quad \text{for all } x \geq 1.
\end{aligned}$$

Then, $\frac{4x^{-5}}{1} \leq \frac{1}{10}$ for $x^5 \geq 40$, that is, $x \geq \sqrt[5]{40} \approx 2.091$

Part(c) was a hard problem. We work it again calling out each step in the following remark from the hints: **Important Note and Intuition:** To find an upper bound for a fraction, $\frac{n(x)}{d(x)}$, you replace the numerator with an upper bound and the denominator with a lower bound for the inequality to hold. In symbols, suppose that $0 \leq n(x) \leq N(x)$. Then,

$$\frac{n(x)}{d(x)} \leq \frac{N(x)}{d(x)},$$

because having a larger number in the numerator gives a larger fraction. Next, suppose that $0 < D(x) \leq d(x)$. Then,

$$\frac{n(x)}{d(x)} \leq \frac{N(x)}{d(x)} \leq \frac{N(x)}{D(x)},$$

because having a smaller number in the denominator yields a larger fraction. **While these are "obvious " facts, applying them successfully when you are first learning limits can be hard to master!**

We have $\frac{x^{-5}+3x^{-8}}{1+x^{-8}}$ and we want a simpler bound that we can wrap our organic brains around. As explained above, we first bound the numerator:

$$x \geq 1 \implies n(x) := x^{-5} + 3x^{-8} \leq x^{-5} + 3x^{-5} = 4x^{-5} =: N(x)$$

so that explains the numerator. Looking at the denominator,

$$x \geq 1 \implies d(x) := 1 + x^{-}8 > 1 := D(x).$$

Putting these together,

$$\frac{x^{-5} + 3x^{-8}}{1 + x^{-8}} \leq \frac{4x^{-5}}{1}.$$