Summary: $\dot{x} = f(x), \quad x(t_0) = x_0 \in \mathbb{R}^n$

- Existence & Uniqueness of Solutions
  - Local: $J_f(x) := \frac{\partial f}{\partial x}(x)$ cont. near $x_0 \Rightarrow$ exists $\delta > 0$ and $\varphi : [t_0, t_0 + \delta) \to \mathbb{R}^n$ s.t. $\varphi(t_0) = x_0$ and $\dot{\varphi}(t) = f(\varphi(t))$.
  - Global: If there exists $L < \infty$ s.t. for all $x \in \mathbb{R}^n$, $\left| \frac{\partial f_i}{\partial x_j}(x) \right| \leq L$, $\Rightarrow$ for all $x_0 \in \mathbb{R}^n$, exists $\varphi : [t_0, \infty) \to \mathbb{R}^n$ s.t. $\varphi(t_0) = x_0$ and $\dot{\varphi}(t) = f(\varphi(t))$.

Matrix Exponential: e^{At}

$A = n \times n$

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots$
- $e^{at} = 1 + a\frac{t}{1!} + a^2\frac{t^2}{2!} + a^3\frac{t^3}{3!} + \cdots + a^k\frac{t^k}{k!} + \cdots$

- Suppose $A = n \times n$

$$e^{At} := I_n + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots + A^k\frac{t^k}{k!} + \cdots$$

This is a Calculus-born concept.

You can do similar things with other functions:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(At) = At - \frac{A^3t^3}{3!} + \frac{A^5t^5}{5!} - \frac{A^7t^7}{7!} + \cdots$$

Behavior of $\lim_{N\to\infty} \sum_{k=0}^{N} f^{(k)}(0)\, A^k \frac{t^k}{k!}$ is studied in Math 451. For the exponential function, the limit exists and is finite for all $n \times n$ matrices A.

$$\frac{d}{dt} e^{At} = A e^{At}$$

$$\boxed{\frac{d}{dt} e^{at} = a e^{at}}$$

$$e^{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

$$\exp\left(\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} t \right) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

Today:

- Solution $\dot{x} = Ax,\ x(t_0) = x_0$
- Eigenvalues & Eigenvectors for the win!
- Stability
- Linearization of NL ODEs

Claim: $\varphi(t) := e^{A(t-t_0)} x_0$ is the (unique) solution to $\dot{x} = Ax,\ x(t_0) = x_0 \in \mathbb{R}^n$

Pf.

I.C. $\varphi(t_0) = \underbrace{e^{A(\cancel{t_0 - t_0})}}_{I_n} x_0 = I_n \cdot x_0 = x_0$ ✓

O.D.E. $\dot{\varphi}(t) = \frac{d}{dt} e^{A(t-t_0)} x_0 = A \underbrace{e^{A(t-t_0)} \cdot x_0}_{\varphi(t)}$

$\dot{\varphi}(t) = A \cdot \varphi(t)$ ✓ Satisfies the ODE.

$\square$

Julia $\exp(A \cdot t)$ very different from $\exp_\circ(A \cdot t)$

$\exp(0_n) = I_n,\quad \exp_\circ(0_n) = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{n \times n}$

Big Question: Can we easily analyze properties of e^{At}?

Recall: Consider $A = n \times n$. If

$Av = \lambda v$, $v \neq 0_{n\times 1}$, the λ is called

an **eigenvalue of A** and v is called

an eigenvector associated with λ.

$F = \text{eigen}(A)$

$F.\text{values} \longleftrightarrow \lambda_1, \dots, \lambda_n$

$F.\text{vectors} \longleftrightarrow v_1, v_2, \dots, v_n$

Exercise: Suppose $Av = \lambda v$.

What is $A^k \cdot v$ for $k = 1, 2, \dots$

Ans: $A^k v = \lambda^k v$ **Induction: Base Case $k=1$.**

Induction Step $A^{k+1} \cdot v = A[\underbrace{A^k \cdot v}_{\lambda^k \cdot v}] = A \lambda^k \cdot v$

$= \lambda^k A \cdot v = \lambda^k \cdot \lambda v = \lambda^{k+1} \cdot v$ $\square$

$[A^2 \cdot v = A(Av) = A(\lambda v) = \lambda Av = \lambda \lambda v = \lambda^2 v$

Suppose $f(x) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!}$

$$f(A) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{A^k}{k!}$$

Exercise: Given $A^k \cdot v = (\lambda)^k \cdot v$, compute

$f(A) \cdot v$

Ans. $f(A) \cdot v = f(\lambda) \cdot v$

$$f(A) \cdot v = \sum_{k=0}^{\infty} f^{(k)}(0) A^k \cdot v = \underbrace{\left[\sum_{k=0}^{\infty} f^{(k)} \lambda^k \right]}_{f(\lambda)} \cdot v$$

Corollary $Av = \lambda v \Rightarrow e^{A} \cdot v = e^{\lambda} \cdot v$

$$e^{At} \cdot v = e^{\lambda t} \cdot v$$

Exercise: What is the solution to $\dot{x} = Ax$, $x(t_0) = v$ when $Av = \lambda v$?

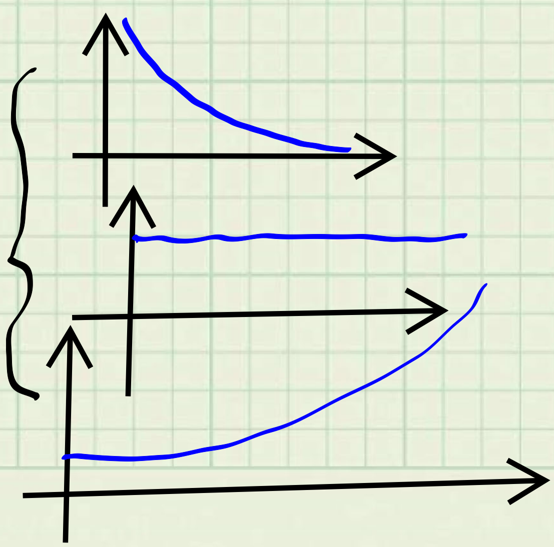
Ans. $x(t) = e^{\lambda(t-t_0)} \cdot v$ because

$$x(t) = e^{A(t-t_0)} \cdot v = e^{\lambda(t-t_0)} \cdot v \qquad \square$$

From here on, take $t_0 = 0$ for simplicity.

Suppose λ is a real number.

$e^{\lambda t} = \Big\{$



$\lambda < 0$

$\lambda = 0$

$\lambda > 0$

$$e^{\lambda t} \xrightarrow[t\to\infty]{} \begin{cases} 0 & \lambda<0 \\ 1 & \lambda=0 \\ \infty & \lambda>0 \end{cases}$$

Fact From Linear Algebra.

Suppose the eigenvalues of A are real and distinct, $\lambda_1, \lambda_2, \dots, \lambda_n$, $\lambda_i \neq \lambda_j$ for $i \neq j$. Then the corresponding eigenvectors $\{v_1, v_2, \dots, v_n\}$ $(Av_i = \lambda_i v_i)$ are linearly independent, and hence form a basis for $\mathbb{R}^n$.

$\therefore$ For all $x_0 \in \mathbb{R}^n$, there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ such that

$$x_0 = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

Hence

$$x(t) = e^{At} x_0 = \alpha_1 e^{At} v_1 + \alpha_2 e^{At} v_2 + \dots + \alpha_n e^{At} v_n$$

$$= \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 + \cdots + \alpha_n e^{\lambda_n t} v_n$$

$$\underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{\det \neq 0} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = x_0$$

We stop writing and try to focus 100% on understanding.

Claim: Consider the linear system of ODEs, $\dot{x} = Ax$, $x(t_0) = v \in \mathbb{R}^n$, where $v \neq 0_{n\times 1}$, $Av = \lambda v$, $\lambda \in \mathbb{R}$. Then, the solution is $\varphi(t, t_0, v) = e^{\lambda(t-t_0)} \cdot v$, and

$$\lim_{t\to\infty} \| e^{\lambda(t-t_0)} v \| = \begin{cases} 0 & \lambda < 0 \\ \infty & \lambda > 0 \\ \|v\| & \lambda = 0 \end{cases}$$

$\blacksquare$

Remark: Solutions that "converge" to the origin are associated with "asymptotic stability", while solutions that "blow up" are associated with "instability".

Fact from Linear Algebra

(See Appendix of ROB 101 texbook) Suppose that A is an $n \times n$ real matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and eigenvectors $v_1, v_2, \ldots, v_n$ $(Av_i = \lambda_i v_i,\ v_i \neq 0_{n\times 1})$. If the eigenvalues are distinct $(\lambda_i \neq \lambda_j$ for $i \neq j)$, then the eigenvectors are linearly independent. Consequently $\{v_1, v_2, \ldots, v_n\}$ forms a basis for $\mathbb{R}^n$. ■

Application to solution of $\dot{x} = Ax$, $x(t_0) = x_0$

If $\lambda_i \neq \lambda_j$ for $i \neq j$, we can always write

$$x_0 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \;, \text{ for } \alpha_i \in \mathbb{R}$$

unique. Hence,

$$\begin{aligned}
\varphi(t, t_0, x_0) &:= e^{A(t-t_0)} x_0 \\
&= e^{A(t-t_0)} \left[\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \right] \\
&= \alpha_1 e^{A(t-t_0)} v_1 + \alpha_2 e^{A(t-t_0)} v_2 + \cdots + \alpha_n e^{A(t-t_0)} v_n
\end{aligned}$$

$$= \alpha_1 e^{\lambda_1(t-t_0)} v_1 + \alpha_2 e^{\lambda_2(t-t_0)} v_2 + \cdots + \alpha_n e^{\lambda_n(t-t_0)} v_n$$

a vector sum of scalar exponentials.

Questions

1) When is it true that $\lim_{t\to\infty} \varphi(t,t_0,x_0) = 0_{n\times 1}$

for all initial conditions $x_0 \in \mathbb{R}^n$?

Ans. If, and only if, $\lambda_i < 0 \quad 1 \le i \le n$; all eigenvalues are negative.

Why? Suppose $\lambda_i \ge 0$ and let $x_0 = v_i$.

$$\therefore e^{A(t-t_0)} x_0 = e^{\lambda_i(t-t_0)} v_i \underset{t\to\infty}{\not\longrightarrow} 0_{n\times 1}.$$

2) When does there exist an initial condition $x_0 \in \mathbb{R}^n$ such that

$$\lim_{t\to\infty} \|\varphi(t,t_0,x_0)\| = \lim_{t\to\infty} \|e^{A(t-t_0)} x_0\| = \infty ?$$

ANS. If, and only if, there is at least one positive eigenvalue.

3) Suppose $\lambda_i > 0$. What initial conditions will result in $\lim_{t \to \infty} ||\varphi(t, t_0, x_0)|| = \infty$?

ANS. Any (all) $x_0 = \alpha_1 v_1 + \cdots + \alpha_i v_i + \cdots + \alpha_n v_n$ with $\alpha_i \neq 0$.

Julia HW 8 Extends to Complex Eigenvalues

Key Ideas: $A = n \times n$ real matrix

- $Av = \lambda v$, $v \neq 0_{n \times 1}$ $\begin{cases} \lambda = a + i\omega \\ v = v_R + i v_I \end{cases}$ "$i = \sqrt{-1}$" $\omega \neq 0$

where $a = \text{Re}\{\lambda\}$ real part of λ

$\omega = \text{Im}\{\lambda\}$ imaginary part of λ

$v_R = \mathrm{Re}\{v\} \in \mathbb{R}^n$ real part of v

$v_I = \mathrm{Im}\{v\} \in \mathbb{R}^n$ imag. part of v

Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$\therefore\ e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

$$\boxed{\therefore\ e^{(a+i\omega)t} = e^{at} \cdot e^{i\omega t} = e^{at}\cos(\omega t) + i\, e^{at}\sin(\omega t)}$$

<u>Def.</u> $e^{(a+i\omega)t}$ is a complex exponential.

<u>Fact:</u>

$$
\begin{aligned}
e^{At} \cdot v &= e^{\lambda t} v \\
&= e^{(a+i\omega)t} (v_R + i\, v_I) \\
&= \left(e^{at}\cos(\omega t)\, v_R - e^{at}\sin(\omega t)\, v_I\right) + \\
&\quad + i \cdot \left(e^{at}\cos(\omega t)\, v_I + e^{at}\sin(\omega t)\, v_R\right)
\end{aligned}
\quad (*)
$$

Facts

$$e^{At} v_R = e^{At}\,\mathrm{Re}\{v\}$$
$$= \mathrm{Re}\{e^{At} v\}$$
$$= \mathrm{Re}\{\text{" * "}\}$$
$$= e^{at}\cos(\omega t)\, v_R - e^{at}\sin(\omega t)\, v_I$$

$$e^{At} v_I = e^{At}\,\mathrm{Im}\{v\}$$
$$= \mathrm{Im}\{e^{At} v\}$$
$$= \mathrm{Im}\{\text{" * "}\}$$
$$= e^{at}\cos(\omega t)\, v_I + e^{at}\sin(\omega t)\, v_R$$

Fact $x_0 \in \mathrm{span}\{v_R, v_I\}$, $\omega \neq 0$

$$\lim_{t\to\infty} e^{At} x_0 = \begin{cases} 0_{n\times 1} & a<0 \\ \text{bounded oscillations} & a=0 \\ \text{unbounded oscillations} & a>0 \end{cases}$$

Proposition 9.59: Exponential Stability of Linear Systems of ODEs

Let A be a real $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and consider the linear system of ODEs,

$$\dot{x} = Ax.$$

asymptotically

If $\text{real}(\lambda_i) < 0$ for $1 \leq i \leq n$ (aka, the real parts of all eigenvalues are negative), then the **origin** is globally exponentially stable.

Note: The proof is given in Chapter 9.9 for the case of distinct eigenvalues. The general case is part of EECS 560 Linear Systems and uses something called a Jordan Normal Form for square matrices. While it's an awesome thing to know, saving a few things for grad school is okay.

Equilibrium Points and Stability Properties

Let A be a real $n \times n$ matrix and consider the linear system of ODEs $\dot{x} = Ax$. We specialize Def. 9.30 to the case of linear systems.

Definition 9.56. *$x_e \in \mathbb{R}^n$ is an **equilibrium point** of $\dot{x} = Ax$ if $Ax_e = 0$. Equivalently, the constant vector $\varphi(t, t_0, x_e) := x_e$ is a solution of the ODE.*

Note: For a linear ODE, the origin is always an equilibrium point.

Definition 9.57. *$x_e \in \mathbb{R}^n$ is a **globally asymptotically stable equilibrium point** of $\dot{x} = Ax$ if for all $x_0 \in \mathbb{R}$,*

$$\lim_{t \to \infty} e^{At} x_0 = 0.$$

Definition 9.58. *$x_e \in \mathbb{R}^n$ is a **globally exponentially stable equilibrium point** of $\dot{x} = Ax$ if there exist $\kappa > 0$ and $\gamma < \infty$ such that, for all $x_0 \in \mathbb{R}$,*

$$||e^{At} x_0|| \leq \gamma \cdot e^{-\kappa t} \cdot ||x_0||.$$

Note: For a linear system of ODEs, Global **exponential** stability of an equilibrium point and global **asymptotic** stability of an equilibrium point are equivalent concepts. For a nonlinear system of ODEs, they are different notions of stability. In this course, understanding that each notion implies that solutions converge to zero as time marches to infinity is enough. The rate at which the convergence takes place is less relevant.

Basis of Much Technology

x_e an equilibrium point of $\dot{x} = f(x)$ is locally exponentially stable if, and only if, the origin is a globally exponentially stable equilibrium of the linearized model $\delta\dot{x} = J_f(x_e) \cdot \delta x$.

$\{ \delta\dot{x} = A \cdot \delta x, \quad A = J_f(x_e) \}$

$\boxed{\text{Illustration in Julia.}}$

See below for linearization of the Robot Equations

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = 0_{n\times 1}$$

Key Pts: $x_e = \begin{bmatrix} q_e \\ \dot{q}_e \end{bmatrix}$ s.t. $G(q_e) = 0_{n\times 1}$
$\dot{q}_e = 0_{n\times 1}$

Proposition 9.35: Linearization of the Robot Equations

We initially encountered the robot equations expressed as a set of nonlinear second-order differential equations,

$$D(q) \cdot \ddot{q} + C(q, \dot{q}) \cdot \dot{q} + G(q) = 0_{n\times 1},$$

where we are assuming here that there are no external forces acting on the robot. In Example 9.31, we determined that their equilibrium points correspond to $G(q_e) = 0_{n\times 1}$ and $\dot{q}_e = 0_{n\times 1}$. Their linearization about an equilibrium can be expressed in both second-order and first-order forms, namely,

$$D(q_e) \cdot \delta\ddot{q} + \frac{\partial G(q_e)}{\partial q} \cdot \delta q = 0_{n\times 1}, \tag{9.60}$$

where $\delta q = q - q_e$, and

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0_n & I_n \\ A_{21} & 0_n \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}, \tag{9.61}$$

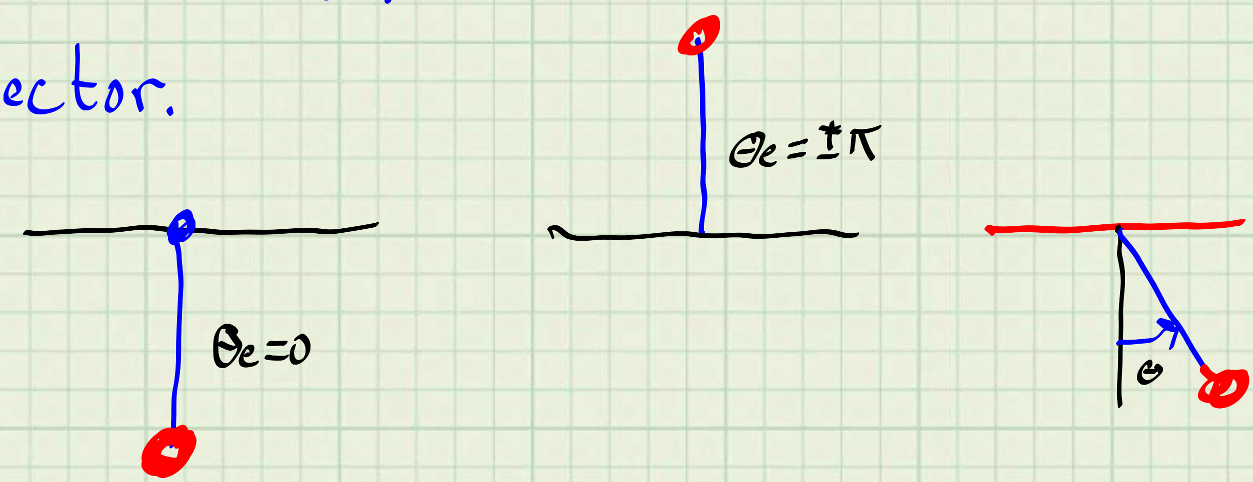
where $A_{21} = -D(q_e) \backslash \frac{\partial G(q_e)}{\partial q}$ because we know not to invert large matrices, I_n is the $n \times n$ identity matrix, $\delta x_1 = q - q_e$, and $\delta x_2 = \dot{q} - 0_{n\times 1}$.

Note: At first, it is surprising that the term $C(q, \dot{q}) \cdot \dot{q}$ does not show up in the linearization. The reason is that it consists of quadratic terms of the form $c_{ij}(q) \cdot \dot{q}_i \cdot \dot{q}_j$, which when linearized about $\dot{q}_e = 0_{n\times 1}$, $1 \leq i \leq n$, all vanish. By taking advantage of this property, the linearization of the robot equations is trivial to compute: you only have to compute one Jacobian, $\frac{\partial G(q_e)}{\partial q}$, and evaluate the mass inertia matrix at q_e.

Linear Approximations of NL ODEs.

$$\dot{x} = f(x), \quad x_0 \in \mathbb{R}^n$$

Equilibrium: $f(x_e) = 0$, means the solution from x_e is a constant vector.



$$f(x_e + \delta x) = \cancelto{0_{n\times 1}}{f(x_e)} + \underbrace{J_f(x_e)}_{A} \cdot \delta x$$

$$\delta x = x - x_e$$

$$\boxed{\dot{\widehat{\delta x}} = A \cdot \delta x}$$

Linear ODE

linear approx of $\dot{x} = f(x)$ near the equilibrium x_e.

If all e-values of A have negative real parts, then for x_0 sufficiently close to the equilibrium x_e, the solution of the NL ODE decays back to the equilibrium.