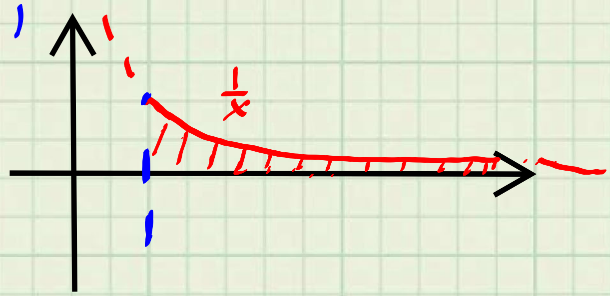
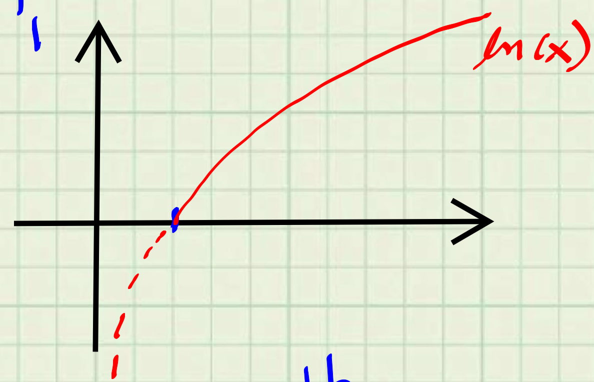
Summary

Type-I Improper Integral

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx \longleftrightarrow \begin{cases} \text{finite} & \text{convergent} \\ \pm\infty & \text{divergent} \\ \text{undefined / does not exist} & \end{cases}$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(x)\Big|_1^b = +\infty$$



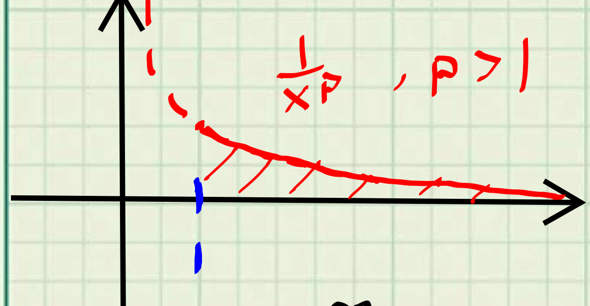


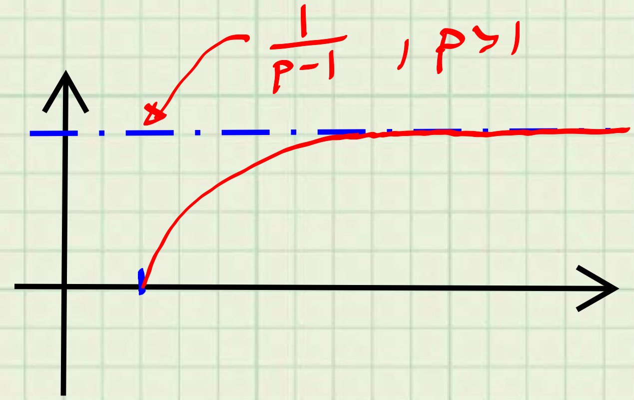
$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx = \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^b$$

$p \neq 1$

$$= \lim_{b\to\infty} \frac{1}{-p+1}\left(b^{-p+1} - 1\right) = \begin{cases} \infty & -p+1>0 \\ \frac{-1}{-p+1} & -p+1<0 \end{cases}$$

$$= \begin{cases} \infty & p<1 \\ \frac{1}{p-1} & p>1 \end{cases}$$





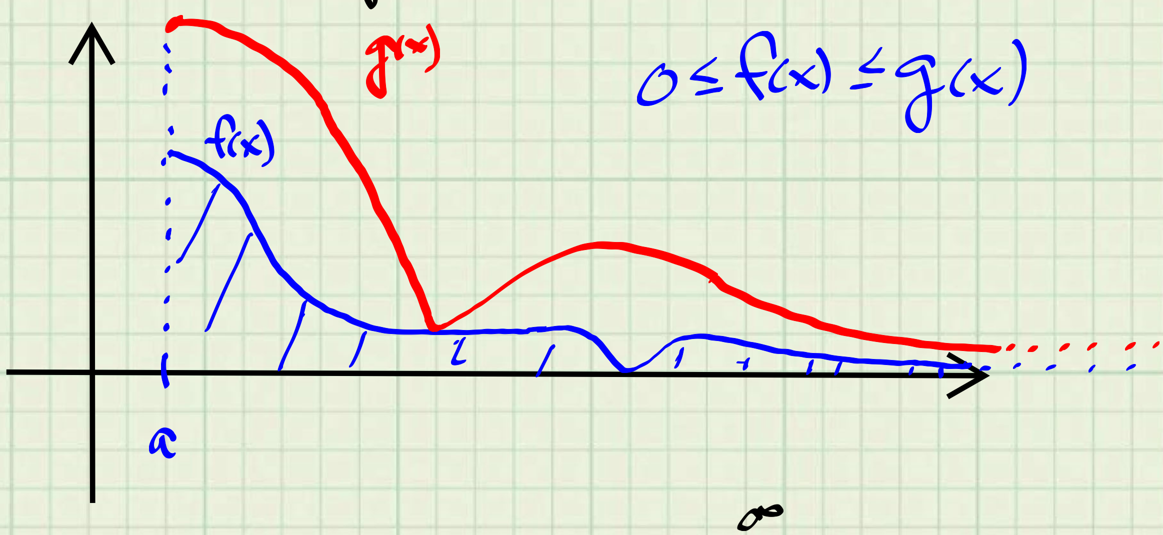
Note: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \frac{1}{2-1} = \frac{1}{1} = 1$

Video 3-Link Video Robot

Today: Comparison Principle

Consider $f:[a,\infty)\to\mathbb{R}$, $g:[a,\infty)\to\mathbb{R}$

both piecewise continuous [Riemann integral exists and is finite on $[a,b]$, for $b<\infty$]. In addition, we assume $0\le f(x)\le g(x)$ for all $x\ge a$, as in the image below



$$\int_0^\infty g(x)\,dx<\infty \implies \int_0^\infty f(x)\,dx<\infty$$

Conversely,

$$\int_0^\infty f(x)\,dx=\infty \implies \int_0^\infty g(x)\,dx=\infty.$$

$\square$

"**Test Functions**": $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \infty & p \le 1 \\ \text{Finite} & p > 1 \end{cases}$

Example: Determine whether

$$\int_1^\infty \frac{x^3+4}{x^5-3x+21}\,dx$$

is convergent or divergent

You are given that has a real root $\approx -1.93 \pm 0.001$ and four complex roots,

$\therefore$ Integrand is cont. on $[1, \infty)$.

Intuition: $\displaystyle f(x) = \frac{x^3+x^4}{x^5-3x+21} \approx \frac{1}{x^2}$

for x large. And we know that

$$\int_1^\infty \frac{1}{x^2}\,dx = 1 < \infty.$$

It $\int^\infty f(x)\,dx$ should be convergent.

How can we use this idea?

Let's divide num. and den. by x^5

$$f(x) = \frac{\frac{1}{x^2} + \frac{4}{x^5}}{\underbrace{1 - \frac{3}{x^4} + \frac{21}{x^5}}_{d(x)}} =: \frac{n(x)}{d(x)}$$

To upperbound $f(x) = \frac{n(x)}{d(x)}$

$$\frac{1}{3} < \frac{1}{2}$$

We need a $\boxed{\text{lower bound}}$ for $d(x)$

$$d(x) = 1 - \frac{3}{x^4} + \underbrace{\frac{21}{x^5}}_{\geq 0} \geq 1 - \frac{3}{x^4} \quad \text{for } x \geq 1$$

$$\geq 1 - \frac{3}{10^4} \quad \text{for } x \geq 10$$

$$\geq 0.99 \quad \text{for } \underline{x \geq 10}$$

$$\int_1^\infty \frac{x^3+4}{x^5-3x+21}\,dx = \int_1^{10} \frac{x^3+4}{x^5-3x+21}\,dx + \int_{10}^\infty \frac{x^3+4}{x^5-3x+21}\,dx$$

$$\leq \underbrace{\int_1^{10} \frac{x^3+4}{x^5-3x+21}\,dx}_{\text{finite by cont. and } [1,10] \text{ bdd.}} + \underbrace{\frac{1}{0.99}\int_{10}^\infty \left[\frac{1}{x^2} + \frac{4}{x^5}\right] dx}_{\text{finite by } \frac{1}{x^p},\ p>1}$$

$\therefore$ Convergent integral.

Easy Way of Doing Improper Integrals

Proposition 8.9: Families of Convergent and Divergent Improper Integrals: Take 2

Consider a rational function $f(x) = \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}$ and assume it is not the zero function. If the denominator $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has any real roots, choose $c \geq 0$ strictly greater than the largest root; otherwise, set $c = 0$. Then the function $\underline{f : [c, \infty)} \to \mathbb{R}$ is continuous and the following hold:

(a) For all $\alpha > 0$, $\int_c^\infty e^{-\alpha x} \cdot \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}\, dx$ is convergent.

(b) For all $\beta > 0$, $\int_c^\infty e^{\beta x} \cdot \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}\, dx$ is divergent.

(c) When there is no exponential term,

$$\int_c^\infty \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}\, dx \quad \begin{cases} \text{is convergent} & n \geq m+2 \\ \text{is divergent} & \text{otherwise.} \end{cases}$$

Note: When there is no exponential term present, the comparison function $\frac{1}{x^p}$ applies, as we essentially demonstrated in the solution to Example 8.7-(d). When the degree of the denominator is at least two greater than the degree of the numerator, the integrand behaves as $\frac{1}{x^p}$ for $p \geq 2$, which is convergent; otherwise, the integrand behaves as $\frac{1}{x^p}$ for $p \leq 1$, which is divergent.

When an exponential term is present, the numerator and denominator degrees can be anything; for example, $\int_1^\infty x^{1000} \cdot e^{-0.001x}\, dx$ is convergent and $\int_1^\infty \frac{1}{x^{1000}} \cdot e^{0.001x}\, dx$ is divergent. If you try these cases with a numerical integrator, your results may vary due to numerical inaccuracy. The theory is useful.

Examples

a) $\displaystyle\int_0^\infty \frac{x+1}{x^{24}+1}\, dx$ Convergent

b) $\displaystyle\int_0^\infty \frac{x+1}{x^{24}-1}\, dx$ The above result

does not apply because $x^{24}-1=0$

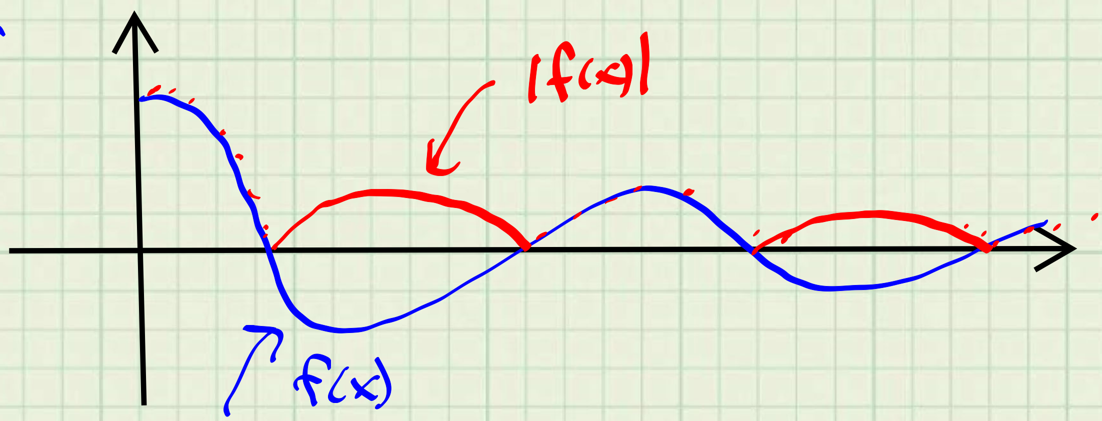
has a root at $x=+1$, and

$0 < +1 < \infty$.

$\square$

What if $f(x)$ takes on both positive and negative values?

Def. A piecewise cont. function is absolutely integrable if
$$\int_a^\infty |f(x)|\,dx < \infty.$$



Prop. If f is absolutely integrable, then $\int_a^\infty f(x)\,dx$ (no absolute value) exists and is finite.

The most common application of absolute integrability is to

$f(t) = t^k e^{at} \sin(\omega t)$, $t \geq 0$.

$|f(t)| = t^k e^{at} |\sin(\omega t)|$

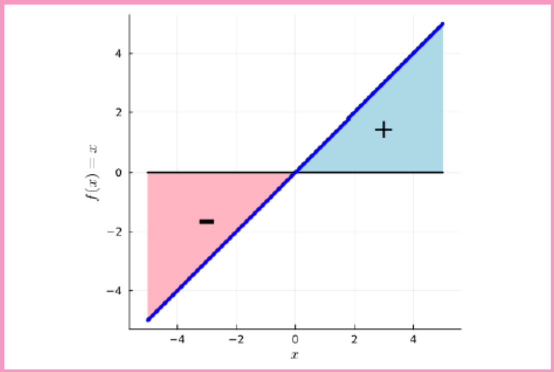
$\leq t^k e^{at}$ because $|\sin(x)| \leq 1$

Secrets of the Arcane 8.3: A Word of Caution for Doubly-infinite Integrals

In general, $\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{c\to\infty} \int_{-c}^{c} f(x)\,dx$. **You must compute the two limits,** $\lim_{c\to\infty} \int_{-c}^{0} f(x)\,dx$ **and** $\lim_{c\to\infty} \int_{0}^{c} f(x)\,dx$, **SEPARATELY!!!**. Why? Consider

$$\int_{-c}^{c} x\,dx.$$

Then, as illustrated in the figure below, the magnitude of the negative area is exactly equal to the positive area, and hence for any interval $[-c, c]$ that is symmetric about the origin, $\int_{-c}^{c} x\,dx = 0.0$, i.e., the areas cancel out, exactly. However, $\int_{-\infty}^{0} x\,dx = -\infty$ and $\int_{0}^{\infty} x\,dx = +\infty$, which do NOT cancel out; the their sum is undefined. Moreover, if we computed $\lim_{c\to\infty} \int_{-c}^{c+1} x\,dx$, we'd obtain positive infinity, and if we computed $\lim_{c\to\infty} \int_{-c^3}^{c^2} x\,dx$, we'd obtain negative infinity. In other words, the limit changes if we advance more quickly in one direction than the other. **We avoid this problem by assessing the two limits separately and then analyzing whether their sum makes sense (or not). The sum will make sense except when both limits are unbounded and have opposite signs, giving rise to the notorious, "∞ − ∞".**



Note: The symmetric limit, $\lim_{c\to\infty} \int_{-c}^{c} f(x)\,dx$, has a name: the **Cauchy Principal Value**; see Wikipedia for more information.

Chapter 9 O.D.E.

Ordinary Differential Equations

These are Ordinary D.E.s

$$\dot{x}(t) + 3x(t) = \sin(t)$$

$$\frac{d^2x(t)}{dt^2} + 14\sin(x(t)) = 0$$

Derivative w.r.t. a single variable

This a Partial Diff. Eqn

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad \text{(Heat Equation)}$$

Derivatives w.r.t time (t) and "space", x

Two or more variables that are

used as derivatives

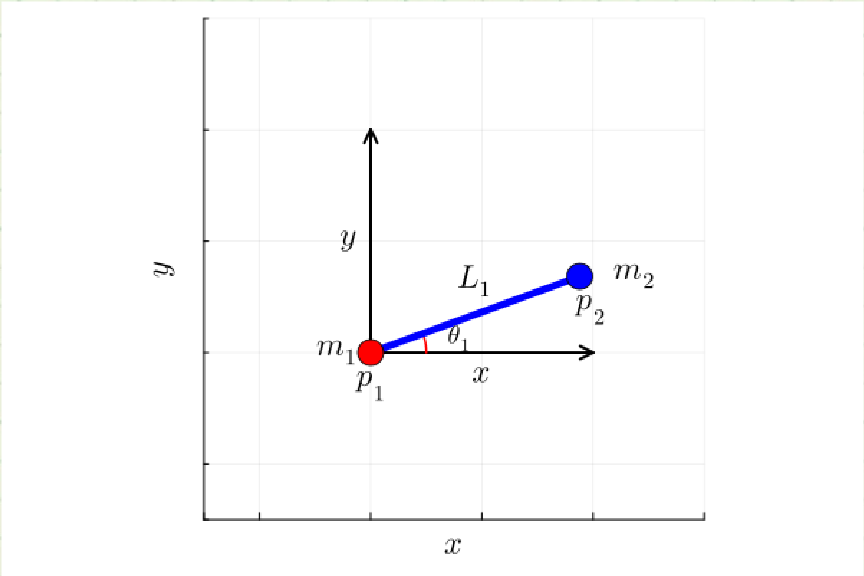
We stick to O.D.E. s.

```
# Dynamics constraints: F = ma using the Robot Equations
# D*ddq + Cdq + G = B*tau
for i in 1:N+1
    dynamics = dyn_mod_tumbler(q[:, i], dq[:, i])
    # Terms in the Robot Equations
    D, Cdq, G, B = dynamics.D(q[:, i]), dynamics.Cdq(q[:, i], dq[:, i]), dynamics.G(q[:, i]), dynamics.B(q[:, i])

    # Define air drag
    @NLconstraint(model, τ[1, i] == - Kx * abs(sin(q[3, i])) * dq[1, i] * abs(dq[1, i])) # air drag
    @NLconstraint(model, τ[2, i] == - Ky * abs(cos(q[3, i])) * dq[2, i] * abs(dq[2, i])) # air drag
    @NLconstraint(model, τ[3, i] == - Kth * dq[3, i] * abs(dq[3, i])) # air drag rotation
    for j = 1:3
        @NLconstraint(model, sum(D[j,k]*ddq[k,i] for k=1:3) + Cdq[j] + G[j] == sum(B[j,k]*τ[k, i] for k=1:3))
    end

end

# TrapZ integration for position and velocity
for i in 1:N
    @constraint(model, q[:, i+1]  .==  q[:, i] + Δt * ( dq[:, i] + dq[:, i+1])/2 )  # Update position
    @constraint(model, dq[:, i+1] .== dq[:, i] + Δt * (ddq[:, i] + ddq[:, i+1])/2)  # Update velocity
end
```



(a) (b)

Figure 9.2: **Using a chute as a brake.** (a) A drag chute used for braking the Space Shuttle Endeavor; image complements of Almany and (b) a parachutist; image complements of pixabay.

We do 4 Warm-up Models and Solve them.

① Linear drag for the space shuttle

$ma = F \iff m\dot{v}(t) = F(t)$

linear drag $F(t) = -\kappa\, v(t)$ (Kappa)

$\dot{v}(t) = -\frac{\kappa}{m} v(t)$, $\underbrace{v(t_0) = v_0}_{\text{initial condition}}$

Claim $v(t) = v_0 e^{-\frac{\kappa}{m}(t-t_0)}$, $t \geq t_0$ is a solution of the diff. eqn.

Proof:

(a) $v(t)$ satisfies the initial condition

$$v(t_0) = v_0\, e^{-\frac{k}{m}(t_0 - t_0)} = v_0 e^0 = v_0 \quad \checkmark$$

(b) $v(t)$ satisfies the ODE

$$\dot{v}(t) = v_0 e^{-\frac{k}{m}(t-t_0)} \left[-\frac{k}{m}\right]$$

$$= -\frac{k}{m} \underbrace{v_0 e^{-\frac{k}{m}(t-t_0)}}_{v(t)}$$

$$= -\frac{k}{m} v(t)$$

Recall ODE: $\boxed{\dot{v}(t) = -\frac{k}{m} v(t)}$

$\therefore$ $v(t)$ is <u>a</u> solution.

How to find the solution?

Method of Separation of Variables

$$\frac{dv(t)}{dt} = \frac{-k}{m} v(t) \quad , \quad v(t_0) = v_0$$

$\Updownarrow$ (replace dummy variable t by τ)

$$\frac{dv(\tau)}{d\tau} = -\frac{k}{m} v(\tau) \quad , \quad v(t_0) = v_0$$

$\Updownarrow$ (drop explicit dependence on tau, τ)

$$\frac{dv}{d\tau} = -\frac{k}{m} v$$

$\Updownarrow$ (Separate variables to each side)

$$\frac{dv}{v} = -\frac{k}{m} d\tau$$

$\Updownarrow$ (Integrate both sides)

$$\int_{v_0 = v(t_0)}^{v(t)} \frac{1}{v} dv = -\frac{k}{m} \int_{t_0}^{t} d\tau \qquad v(t_0) = v_0$$

$\Updownarrow$

$$\ln(v) \Big|_{v_0}^{v(t)} = -\frac{k}{m} \tau \Big|_{t_0}^{t}$$

$\Updownarrow$

$$\ln(v(t)) - \ln(v_0) = -\frac{k}{m} (t - t_0)$$

$\Updownarrow$

$$\ln\left(\frac{v(t)}{v_0}\right) = -\frac{k}{m} (t - t_0)$$

$\Updownarrow$ (exp. both sides)

$$\frac{v(t)}{v_0} = e^{-\frac{\kappa}{m}(t-t_0)}$$

$$\Updownarrow$$

$$\boxed{v(t) = v_0 e^{-\frac{\kappa}{m}(t-t_0)}}$$

2) Space Shuttle with nonlinear drag

$$m\dot{v}(t) = F(t) \quad , \quad v(t_0) = v_0$$

$$F(t) = -\kappa\, v^2(t)$$

$$m\,\dot{v}(t) = -\kappa\, v^2(t) \; , \quad v(t_0) = v_0$$

Solution: $v(t) = \dfrac{v_0}{1 + v_0(t-t_0)\left(\frac{\kappa}{m}\right)} \quad , t \geq t_0$

At home: Compute $\dot{v}(t)$ and plug in to check it satisfies the ODE.

Can compute the solution once again by Separation of Variables

$$m\dot{v} = -\kappa v^2 \quad , \quad v(t_0) = v_0$$

$$\frac{dv}{d\tau} = -\frac{K}{m} v^2$$

$$\frac{dv}{v^2} = -\frac{K}{m} d\tau$$

$$\int_{v_0}^{v(t)} \frac{1}{v^2} dv = \int_{t_0}^{t} -\frac{K}{m} d\tau$$

$$\left.\frac{v^{-1}}{-1}\right|_{v_0}^{v(t)} = \left.-\frac{K}{m}\tau\right|_{t_0}^{t}$$

Tedious algebra gives the proposed answer

$\square$