ROB 201 - Calculus for the Modern Engineer
HW #2

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Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete.

1. (a) Create a "Cheat Sheet" for the first two chapters of the textbook. Here is an example from ROB 101.

   (b) Note any material where you found the explanation confusing or difficult to master.

2. For the strictly decreasing function, $f(x) = 8 - x - x^2$, determine lower and upper bounds for the area under the function on the interval $[0, 3]$, using $n := 3$, $\Delta x := \frac{3-0}{3} = 1$, and $x_i := (i-1) \cdot \Delta x$, $1 \le i \le 4$.

   (a) Give a numerical value for $\mathrm{Area}_3^{\mathrm{Low}}$.

   (b) Show your steps for computing $\mathrm{Area}_3^{\mathrm{Low}}$.

3. A bouncing ball is dropped from a height of 20 meters. Each time it bounces, it reaches half the height of the previous bounce.

   (a) What is the height of the 6th bounce?

   (b) Calculate the total vertical distance the ball has traveled by the time it hits the ground for the 6th time.

4. Compute the following limits using the "Easy Way" as explained in the textbook. You can just give an answer; you do not need to show your work.

   (a) $\displaystyle\lim_{x\to\infty} \frac{5x^4 + 7x^2 - 3}{2x^4 - x + 6}$

   (b) $\displaystyle\lim_{x\to\infty} \frac{2x^3 + 4x - 5}{7x^5 - 3x^2 + 1}$

   (c) $\displaystyle\lim_{x\to\infty} \frac{4x - 1}{x^{\frac{1}{2}} + 5}$

   (d) $\displaystyle\lim_{x\to-\infty} \frac{6x^5 + 2x^3 - 8}{-4x^4 + x - 1}$

   (e) $\displaystyle\lim_{x\to-\infty} 3^{-x} \frac{6x^5 + 2x^3 - 8}{-4x^4 + x - 1}$

   (f) $\displaystyle\lim_{x\to\infty} e^{0.02\,x} \frac{5x^3 + 2x - 4}{6x^2 + x + 9}$

5. This problem focuses on the Binomial Theorem from Chapter 1.

   (a) Verify that $(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$ using the Binomial Theorem.

   (b) Using your result, evaluate $(0.9)^4$ by substituting $x = 0.1$.

6. **The Infinite Library of Zeta:** Deep in the desert lies the legendary Infinite Library of Zeta. This vast archive contains only books whose titles are finite strings made from the 26 letters of the English alphabet.

Description:

- Each book has a title consisting of one or more letters.
- There are no restrictions on length (as long as it's finite), so titles can be "A," "HELLO," "ZEBRAS," or even very long combinations like "QWERTYUIOPASDFGHJKLZXCVBNM."
- There are no duplicate titles in the library—each distinct title appears exactly once.

The Questions:

(a) Is the set of all book titles in the Infinite Library of Zeta **countable** or **uncountable**? Explain your reasoning clearly.

(b) Suppose we add a rule that titles may only use the letters "A" and "B" but can still be any finite length (1 letter, 2 letters, etc.). Is the set of possible titles under this rule countable or uncountable? Explain.

(c) What about the set of all *infinite* sequences of letters from the alphabet (A–Z)? Is that set countable or uncountable? Briefly explain.

Hints

Prob. 1 Nothing to add.

Prob. 2 Since $f(x)$ is strictly decreasing on $[0,3]$, the left-endpoint rule gives the lower estimate.

- Identify the left endpoints: $x_0 = 0$, $x_1 = 1$, $x_2 = 2$.
- For each, compute $f(x_i)$.
- Multiply each $f(x_i)$ by $\Delta x = 1$.
- Sum these areas to find $\mathrm{Area}_3^{\mathrm{Low}}$.

Prob. 3 Check out geometric sums.

Prob. 4 None given.

Prob. 5 Recall that the Binomial Theorem says

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

For part (a), let $a = 1$ and $b = -x$. For part (b), just plug in $x = 0.1$ into your expansion.

Prob. 6 You have many resources at your disposal.

- Check out the textbook's Table of Contents and click on it to go to the section on Countable Sets (see Chapter 2).
- If the textbook does not help you to have a general idea of what it means for a set to be countable, provide the following **prompt to your favorite LLM:**

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Give me a basic introduction to countable sets w/o using the word injective.
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Solutions HW 02

Prob. 1 (a) Included at the end of the solution set

(b) These will vary by person, but some of the more challenging topics may have been:

i. Chap 01: The Approximation Principle
ii. Chap 01: Euler's number
iii. Chap 01: Binomial Theorem
iv. Chap 01: Shifting and scaling operations on functions
v. Chap 01: The level of rigor used in stating the results
vi. Chap 02: Exactly what is a proof?
vii. Chap 02: Countable sets
viii. Chap 02: Proofs by induction
ix. Chap 02: Epsilon-delta notions of limits at infinity

Prob. 2 Here is the lower bound:

$$\begin{aligned}
\text{Area}_3^{\text{Low}} &= \sum_{i=1}^{3} f(x_i) \cdot \Delta x \\
&= \sum_{i=1}^{3} \left(f((i-1) \cdot \Delta x) \right) \cdot \Delta x \\
&= \sum_{i=1}^{3} \left(8 - (i-1) \cdot \Delta x - ((i-1) \cdot \Delta x)^2 \right) \cdot \Delta x \\
&= \sum_{i=1}^{3} \left(8 - (i-1) \cdot 1 - ((i-1) \cdot 1)^2 \right) \cdot 1 \\
&= \left[8 - 0 - 0^2 \right] + \left[8 - 1 - 1^2 \right] + \left[8 - 2 - 2^2 \right] \\
&= [8] + [8 - 1 - 1] + [8 - 2 - 4] \\
&= 8 + 6 + 2 \\
&= 16.
\end{aligned}$$

Prob. 3 You can solve this problem using the geometric progression formula.

(a) The height of the 6th bounce can be calculated using the formula for the n-th term of a geometric sequence, which is $h_n = h_1 \cdot r^{(n-1)}$. Here, the initial height $h_1 = 10$ meters, the common ratio $r = 0.5$, and $n = 6$.

Therefore, the height of the 6th bounce is:

$$h_6 = 10 \cdot (0.5)^5 = 10 \cdot \frac{1}{32} = 0.3125 \text{ meters} \tag{1}$$

(b) The total vertical distance traveled by the ball by the time it hits the ground for the 6th time can be calculated by summing:

- the initial drop (20 meters),
- and twice the sum of the first 5 bounce heights (because each bounce has an up and down travel).

The sum of the first 5 bounce heights is a geometric series with first term $h_1 = 10$ and common ratio $r = 0.5$:

$$S_5 = h_1 \cdot \frac{1 - r^5}{1 - r} = 10 \cdot \frac{1 - (0.5)^5}{0.5} = 10 \cdot \frac{1 - \frac{1}{32}}{0.5}$$

$$= 10 \cdot \frac{\frac{31}{32}}{0.5} = 10 \cdot \frac{31}{16} = 19.375 \text{ meters.}$$

Therefore, the total distance is:

$$\text{Total Distance} = 20 + 2 \cdot 19.375 = 20 + 38.75 = 58.75 \text{ meters.} \tag{2}$$

Prob. 4 (a) $\lim_{x\to\infty} \frac{5x^4 + 7x^2 - 3}{2x^4 - x + 6} = \lim_{x\to\infty} \frac{5x^4}{2x^4} = \frac{5}{2} = 2.5$

(b) $\lim_{x\to\infty} \frac{2x^3 + 4x - 5}{7x^5 - 3x^2 + 1} = \lim_{x\to\infty} \frac{2x^3}{7x^5} = \frac{2}{7x^2} \to 0$

(c) $\lim_{x\to\infty} \frac{4x - 1}{x^{\frac{1}{2}} + 5} = \lim_{x\to\infty} \frac{4x}{x^{1/2}} = \lim_{x\to\infty} 4x^{1/2} = \infty$ because the numerator grows much faster than the denominator.

(d) $\lim_{x\to-\infty} \frac{6x^5 + 2x^3 - 8}{-4x^4 + x - 1} = \lim_{x\to-\infty} \frac{6x^5}{-4x^4} = \lim_{x\to-\infty} -\frac{3x}{2} = \infty$ because x is negative, so $-3x/2$ goes to $+\infty$ as $x \to -\infty$.

(e) $\lim_{x\to-\infty} 3^{-x} \frac{6x^5 + 2x^3 - 8}{-4x^4 + x - 1} = \lim_{x\to-\infty} 3^{-x} \frac{6x^5}{-4x^4} = \lim_{x\to-\infty} -\frac{3x}{2} \cdot 3^{-x}$. We note that $3^{-x} = 3^{|x|}$ grows exponentially as $x \to -\infty$, and $-3x/2$ is positive and linear, so the product goes to ∞.

(f) $\lim_{x\to\infty} e^{0.02\,x} \frac{5x^3 + 2x - 4}{6x^2 + x + 9} = \lim_{x\to\infty} e^{0.02\,x} \frac{5x^3}{6x^2} = \lim_{x\to\infty} e^{0.02\,x} \cdot \frac{5x}{6} = \infty$ because the exponential term dominates any polynomial growth.

Prob. 5 The binomial theorem states that for any positive integer n, the expansion of $(a + b)^n$ is given by,

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where the binomial coefficient $\binom{n}{k}$ (read as "n choose k") is given by the formula,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Armed with this information, let's address both parts of the problem:

(a) To verify the expansion of $(1 - x)^4$, we use the binomial theorem with $a = 1$, $b = -x$, and $n = 4$:

$$(1 - x)^4 = \sum_{k=0}^{4} \binom{4}{k} 1^{4-k} (-x)^k$$

Expanding term by term:

$$= \binom{4}{0} 1^4 (-x)^0 + \binom{4}{1} 1^3 (-x)^1 + \binom{4}{2} 1^2 (-x)^2 + \binom{4}{3} 1^1 (-x)^3 + \binom{4}{4} 1^0 (-x)^4$$

Now using the binomial coefficients:

$$= 1 - 4x + 6x^2 - 4x^3 + x^4$$

$$\boxed{(1 - x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4}$$

(b) Using the result above, to evaluate $(0.9)^4$ we substitute $x = 0.1$ into the expansion:

$$(0.9)^4 = (1 - 0.1)^4$$

$$= 1 - 4(0.1) + 6(0.1)^2 - 4(0.1)^3 + (0.1)^4$$

$$= 1 - 0.4 + 0.06 - 0.004 + 0.0001$$

$$= 0.6561$$

$$\boxed{(0.9)^4 = 0.6561}$$

Prob. 6 Ah yes, the Infinite Library of Zeta:

(a) The set of all book titles in the Infinite Library of Zeta is **countable**. Each title is a finite string made from the 26-letter English alphabet. For each length n, there are exactly 26^n possible titles of that length. Since 26^n is finite for every fixed n, and the set of positive integers $\{1, 2, 3, \dots\}$ indexing the lengths is countable, the collection of all titles can be viewed as a countable union of finite sets. A countable union of finite sets is countable.

Not Required but Good to Know: We can explicitly describe a way to enumerate all titles:
- List all titles of length 1 in alphabetical order.
- Then list all titles of length 2 in lexicographic order.
- Then all titles of length 3, and so on.

This process assigns each title a unique natural number in the enumeration, demonstrating a bijection with $\mathbb{N}$. Therefore, the set of all book titles in the Infinite Library of Zeta is countable.

(b) The set of all titles using only the letters "A" and "B" but of any finite length is also **countable**. The reasoning is similar. Each title is now a finite string over a 2-letter alphabet. For each length n, there are 2^n possible strings. Again, 2^n is finite for each n. The set of all such titles is the union over $n = 1, 2, 3, \dots$ of these finite sets, which is a countable union of finite sets. Thus, the entire set is countable.

Alternative Reasoning: You can also construct an explicit enumeration by:
- Listing all strings of length 1: "A", "B".
- Then all strings of length 2: "AA", "AB", "BA", "BB".
- Then all strings of length 3, and so on.

This procedure creates a bijection with $\mathbb{N}$, proving countability.

(c) The set of all *infinite* sequences of letters from the alphabet (A–Z) is **uncountable**.
Reason: An infinite sequence can be viewed as an infinite string over the 26-letter alphabet:

$$(a_1, a_2, a_3, \dots), \quad a_i \in \{A, B, \dots, Z\}.$$

The set of all such sequences is equivalent to the Cartesian product:

$$\{A, B, \dots, Z\}^{\mathbb{N}}.$$

This is the set of all functions from $\mathbb{N}$ to the 26-letter alphabet. Using Cantor's diagonal argument, even the set of all infinite binary sequences (using just "A" and "B") is uncountable. Since 26 choices per position is even larger, the set remains uncountable.

Alternative Reasoning: There is no way to enumerate all infinite sequences because any proposed listing can be contradicted by constructing a new sequence that differs from the n-th sequence in the n-th letter. Thus, the set of all infinite sequences over the 26-letter alphabet is uncountable.