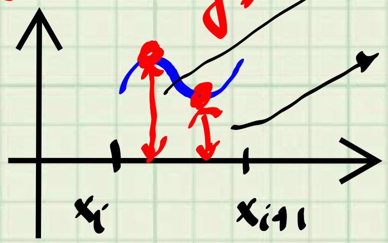
Summary:



$$h_i^{up} = \max_{x_i \le x \le x_{i+1}} f(x) \qquad A_i^u = h_i^{up} \cdot \Delta x$$

$$h_i^{low} = \min_{x_i \le x \le x_{i+1}} f(x) \qquad A_i^{low} = h_i^{low} \cdot \Delta x$$

$$\Delta x = \frac{b-a}{n} \xrightarrow[n \to \infty]{} dx$$

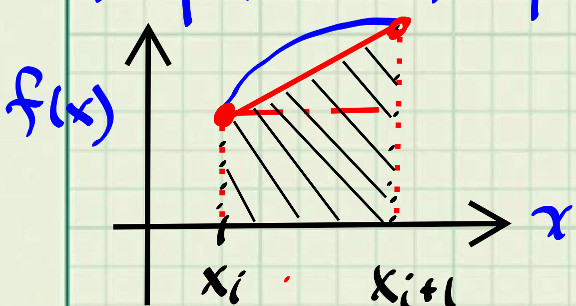
$$\int_a^b f(x)\,dx := \begin{cases} \lim_{n\to\infty} \sum_{i=1}^{n} h_i^{low} \Delta x \\ \lim_{n\to\infty} \sum_{i=1}^{n} h_i^{up} \Delta x \end{cases}$$

both limits exist, finite, EQUAL

Compact, expressive notation!

Prop. $f:[a,b] \to \mathbb{R}$ cont. $\Rightarrow$ Riemann integral exists

TrapZ or Trapezoidal Rule



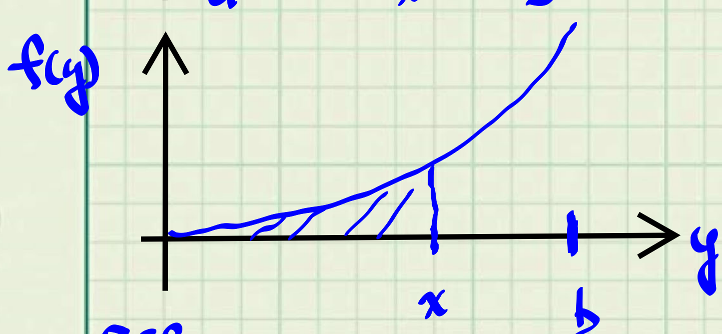
$$\text{trap}A_i = \frac{f(x_{i+1}) + f(x_i)}{2} \cdot \Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \text{trap}A_i$$

when f is continuous.

Indefinite integral

$$g(x) = \int_a^x f(y)\,dy$$

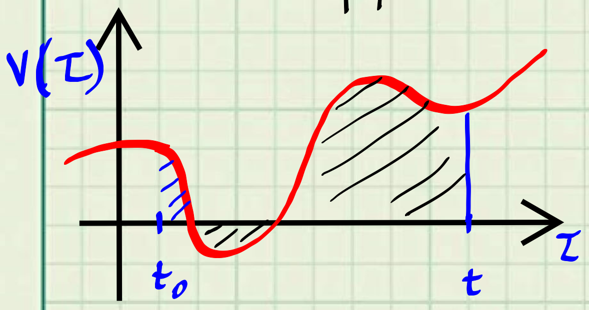


$$g(x) = \int_0^x y^k\,dy = \frac{x^{k+1}}{k+1}$$

"Lovely closed-form solution"

$k = 1, 2, \ldots$

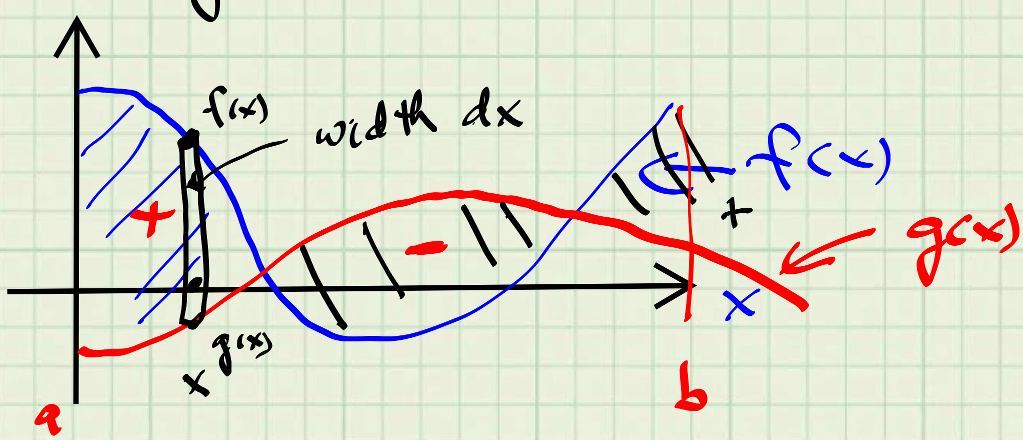
First Application of Integration



Velocity to Position

$$p(t) = \underline{p(t_0)} + \underbrace{\int_{t_0}^{t} v(\tau)\, d\tau}$$

New Material: The power of the rectangle: Area between two functions



What is the area defined by $f(x) - g(x)$

Area = height × width

height of the rectangle $= f(x) - g(x)$

width of " " $= dx$

differential area

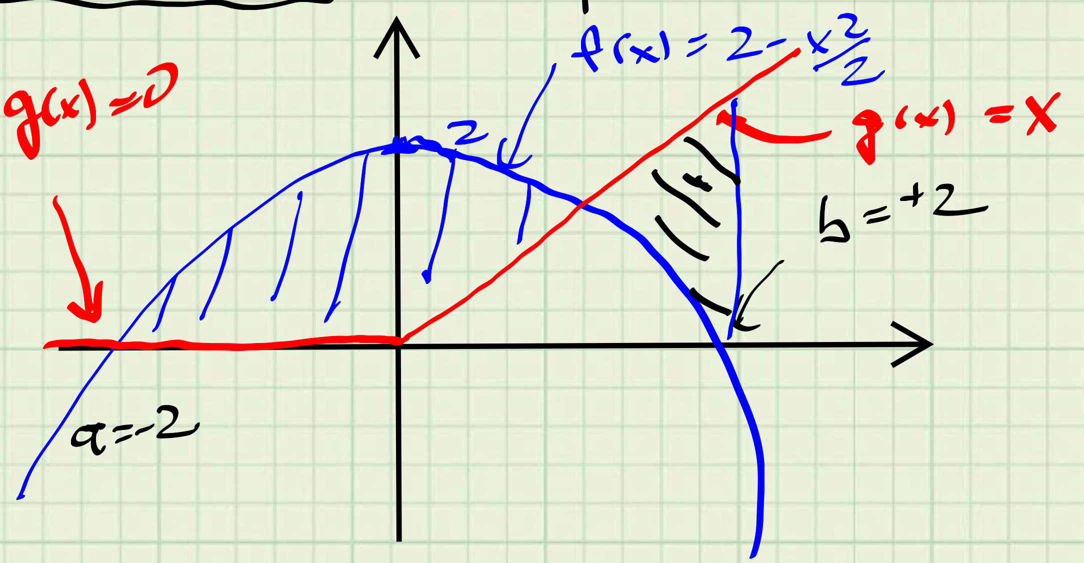
$$dA(x) := \underbrace{[f(x) - g(x)]}_{\text{height}} \cdot \underbrace{dx}_{\text{differential width}}$$

Total area is the sum of the

differential areas:

$$A = \int_a^b dA(x) = \int_a^b [f(x) - g(x)]\, dx$$

Example Compute the Area Below



$$g(x) = \begin{cases} 0 & x \le 0 \\ x & x > 0 \end{cases}$$

$$A = \int_{-2}^{2} [f(x) - g(x)]\, dx$$

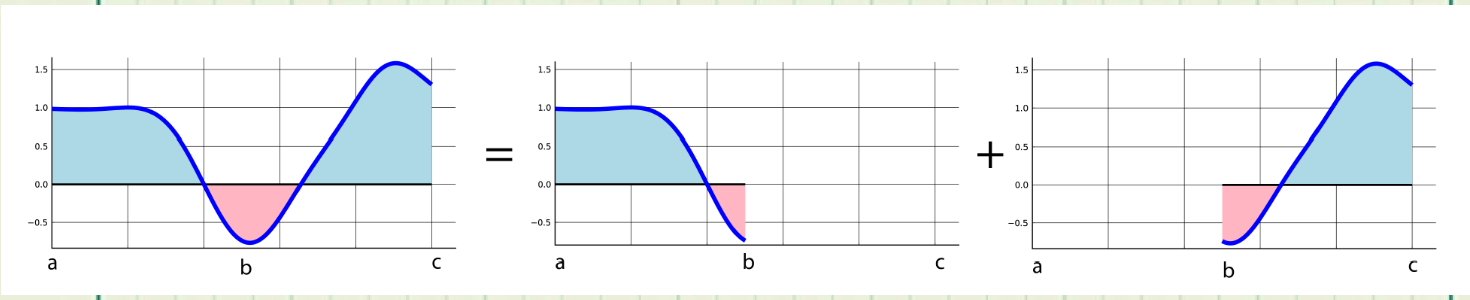
$$\stackrel{?}{=} \int_{-2}^{0} [f(x) - g(x)]\, dx + \int_{0}^{2} [f(x) - g(x)]\, dx$$

$$\stackrel{?}{=} \int_{-2}^{0} [2 - \tfrac{1}{2}x^2]\, dx + \int_{0}^{2} [2 - \tfrac{1}{2}x^2 - x]\, dx$$

- Can we break the integral

into two non-overlapping pieces? Y

- Is the integral of a sum the sum of the integrals?? Y



$$a < b < c$$

$$\int_a^b h(x)\,dx + \int_b^c h(x)\,dx = \int_a^c h(x)\,dx$$

<u>Prop.</u> (Linearity of the Riemann Integral)
Suppose $a < b$ and functions $f_i : [a,b] \to \mathbb{R}$, $1 \le i \le n$. Then for all constants $\alpha_i \in \mathbb{R}$,

$$\int_a^b [\alpha_1 f_1(x) + \dots + \alpha_n f_n(x)]\,dx =$$

$$\alpha_1 \int_a^b f_1(x)\,dx + \dots + \alpha_n \int_a^b f_n(x)\,dx$$

Remark: Polynomial = weighted sum of monomials.

True: Because the limit of a sum is the sum of the limits.

Our First Integral Table

$$\int_a^b 1\,dx = x\Big|_a^b := b - a$$

$$h(x)\Big|_a^b := h(b) - h(a)$$

$$\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2}$$

$$\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}$$

$$A_1 = \int_{-2}^{0} \left[2 - \tfrac{1}{2}x^2\right] dx$$

$$= 2\int_{-2}^{0} 1\,dx - \tfrac{1}{2}\int_{-2}^{0} x^2\,dx$$

$$= 2\,x\Big|_{-2}^{0} - \frac{1}{2}\,\frac{x^3}{3}\Big|_{-2}^{0}$$

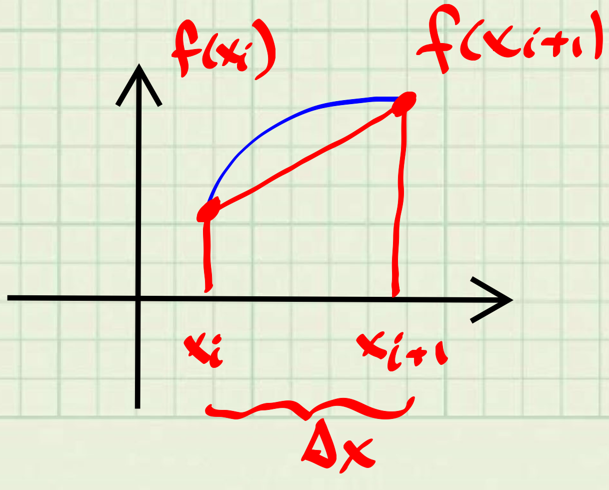
$$= 2[0-(-2)] - \frac{1}{2}\left[0 - \frac{(-2)^3}{3}\right]$$

$$= 4 - \frac{4}{3}$$

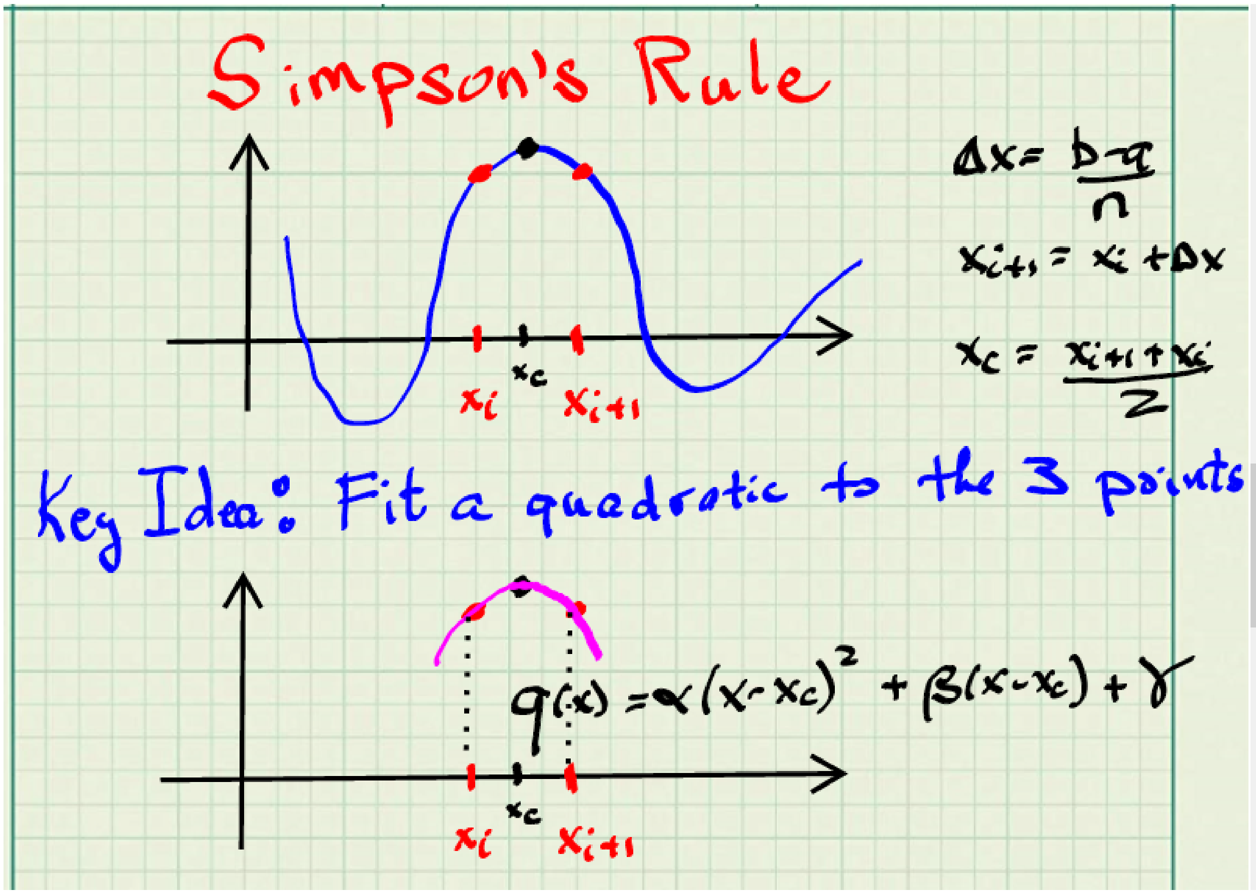
$$= \frac{8}{3}$$

$$\begin{aligned}
A_2 &:= \int_0^2 \left[2 - \frac{1}{2}x^2 - x\right] dx \\
&= 2\int_0^2 1\,dx - \frac{1}{2}\int_0^2 x^2\,dx - \int_0^2 x\,dx \\
&= 2x\Big|_0^2 - \frac{1}{2}\frac{x^3}{3}\Big|_0^2 - \frac{x^2}{2}\Big|_0^2 \\
&= 2\,[2-0] - \frac{1}{2}\left[\frac{2^3}{3} - \frac{0^3}{3}\right] - \left[\frac{2^2}{2} - \frac{0^2}{2}\right] \\
&= 4 - \frac{1}{2}\left(\frac{8}{3}\right) - \frac{1}{2}\,(4) \\
&= \frac{12}{3} - \frac{4}{3} - \frac{6}{3} \\
&= \frac{12-4-6}{3} \\
&= \frac{2}{3}.
\end{aligned}$$

$$A = A_1 + A_2 = \frac{8}{2} + \frac{2}{3} = \frac{10}{3} = 3\frac{1}{3}$$



$$A_i = \frac{f(x_i) + f(x_{i+1})}{2} \cdot \Delta x$$



$$f(x) = e^{\sin(x^3 + \sqrt{1+x^2})}$$

$\mathrm{SimpA}_i :=$ Area under the quadratic

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{SimpA}_i$$

$$\int_{x_i}^{x_{i+1}} q(x)\,dx$$

Moreover, the Riemann integral of the interpolating quadratic function is equal to

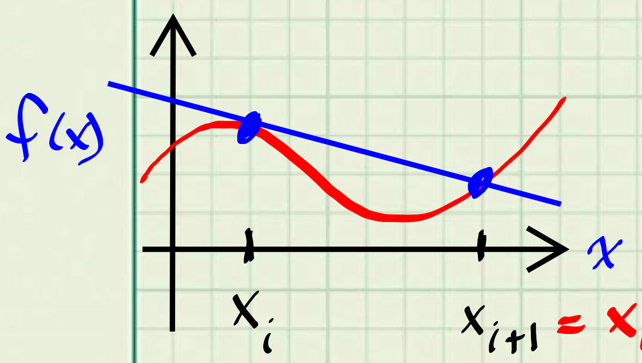
$$\mathrm{SimpA}_i := \int_{x_i}^{x_{i+1}} q(x)\,dx = \alpha \cdot \frac{(\Delta x)^3}{12} + \gamma \cdot \Delta x$$
$$= \underbrace{\frac{\Delta x}{6} \cdot \left(f(x_i) + 4\,f(x_c) + f(x_{i+1})\right)}, \tag{3.37}$$

where $x_c := \frac{x_i + x_{i+1}}{2}$.

$$= \frac{\Delta x}{6}\left[f(x_i) + 4f(x_c) + f(x_{i+1})\right]$$

Recall: $\mathrm{trapA}_i := \frac{\Delta x}{2}\left(f(x_i) + f(x_{i+1})\right)$

How are numerical integration schemes created?



1st-order: Interpolate a line through $f(x_i)$ and $f(x_{i+1})$

$$\boxed{y(x) = m(x - x_i) + b \qquad \text{line}}$$

Interpolation $\Leftrightarrow$ $y(x_i) = f(x_i)$
$y(x_{i+1}) = f(x_{i+1})$

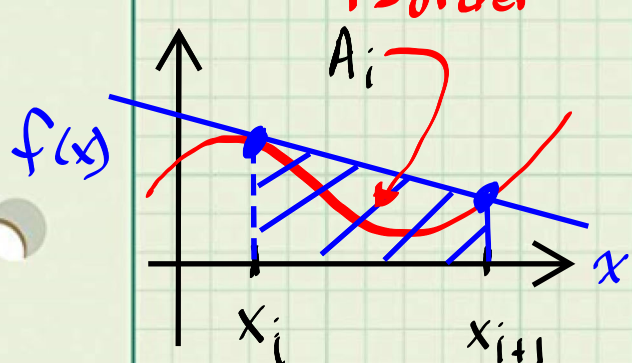
Hence

$$f(x_i) = y(x_i) = m(\underbrace{x_i - x_i}_{\to 0}) + b \qquad \Rightarrow b = f(x_i)$$

$$f(x_{i+1}) = y(x_{i+1}) = m(x_{i+1} - x_i) + b$$

$$= m \cdot \Delta x \quad + b$$

$$\Rightarrow \boxed{m = \frac{f(x_{i+1}) - f(x_i)}{\Delta x}}$$



1st-order

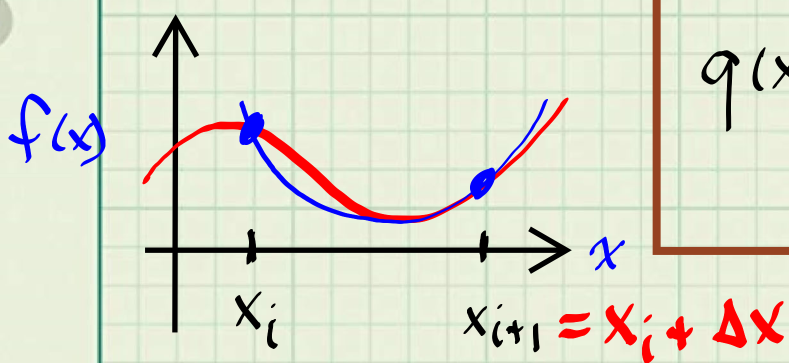
$$A_i := \int_{x_i}^{x_{i+1}} y(x)\, dx$$

Trap-Z! $\longrightarrow$

$$= \left(\frac{f(x_i) + f(x_{i+1})}{2}\right) \cdot \Delta x$$

On your own: Evaluate $\int_{x_i}^{x_{i+1}} y(x)\,dx$ and show you get the Trapezoidal Rule. Warning: The algebra is messier than you'd think!

2^{nd}-order method: Interpolate a quadratic "through" the points.

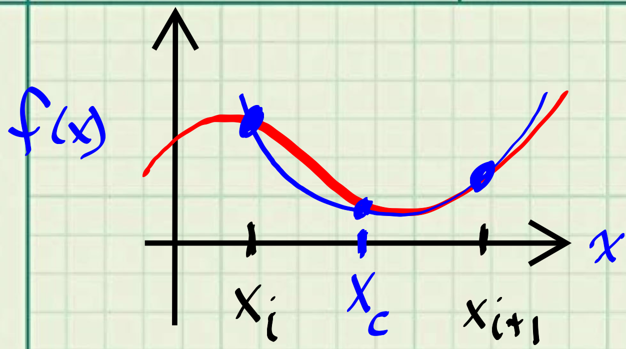


$$\boxed{q(x) = \alpha (x - x_i)^2 + \beta (x - x_i) + \gamma}$$

$$\boxed{\begin{array}{l} \text{3-unknowns} \\ \therefore \text{Need 3-data points} \end{array}}$$

Define $x_c :=$ midpoint $= \dfrac{x_i + x_{i+1}}{2}$

$= x_i + \dfrac{\Delta x}{2}$



$x_i \longrightarrow f(x_i)$

$x_c = x_i + \frac{\Delta x}{2} \longrightarrow f(x_c)$

$x_{i+1} = x_i + \Delta x \longrightarrow f(x_{i+1})$

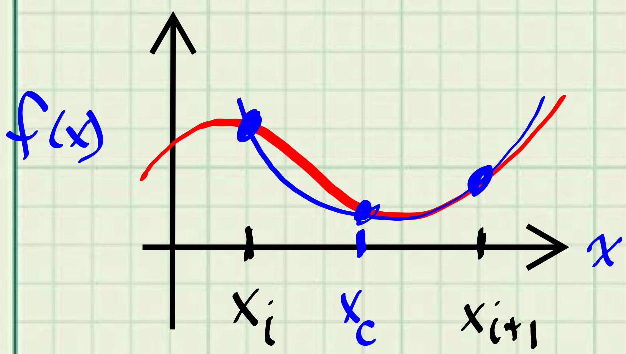
Def. $q(x)$ interpolates $f(x)$ at x_i, x_c, x_{i+1} if $q(x_i) = f(x_i)$, $q(x_c) = f(x_c)$, $q(x_{i+1}) = f(x_{i+1})$.

Exercise: "Solve" for α, β, γ s.t.

$$q(x) := \alpha (x - x_i)^2 + \beta (x - x_i) + \gamma$$

interpolates $f(x)$. [Set up 3-eqns in 3-unknowns]

Solution



$$x_i$$

$$x_c = x_i + \frac{\Delta x}{2}$$

$$x_{i+1} = x_i + \Delta x$$

$$q(x) = \alpha (x - x_i)^2 + \beta (x - x_i) + \gamma$$

$$f(x_i) = q(x_i) = \alpha \cancelto{0}{(x_i - x_i)^2} + \beta \cancelto{0}{(x_i - x_i)} + \gamma$$

$$f(x_c) = q\left(x_i + \frac{\Delta x}{2}\right) = \alpha \left(\frac{\Delta x}{2}\right)^2 + \beta \left(\frac{\Delta x}{2}\right) + \gamma$$

$$f(x_{i+1}) = q(x_i + \Delta x) = \alpha (\Delta x)^2 + \beta \Delta x + \gamma$$

Hence

$$\begin{bmatrix} 0 & 0 & 1 \\ \left(\frac{\Delta x}{2}\right)^2 & \frac{\Delta x}{2} & 1 \\ (\Delta x)^2 & \Delta x & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} f(x_i) \\ f(x_c) \\ f(x_{i+1}) \end{bmatrix}$$

$$\therefore \quad \gamma = f(x_i)$$

$$\beta = \frac{4f(x_c) - 3f(x_i) - f(x_{i+1})}{\Delta x}$$

$$\alpha = \frac{2f(x_i) + 2f(x_{i+1}) - 4f(x_c)}{(\Delta x)^2}$$

$x_{i+1} = x_i + \Delta x$

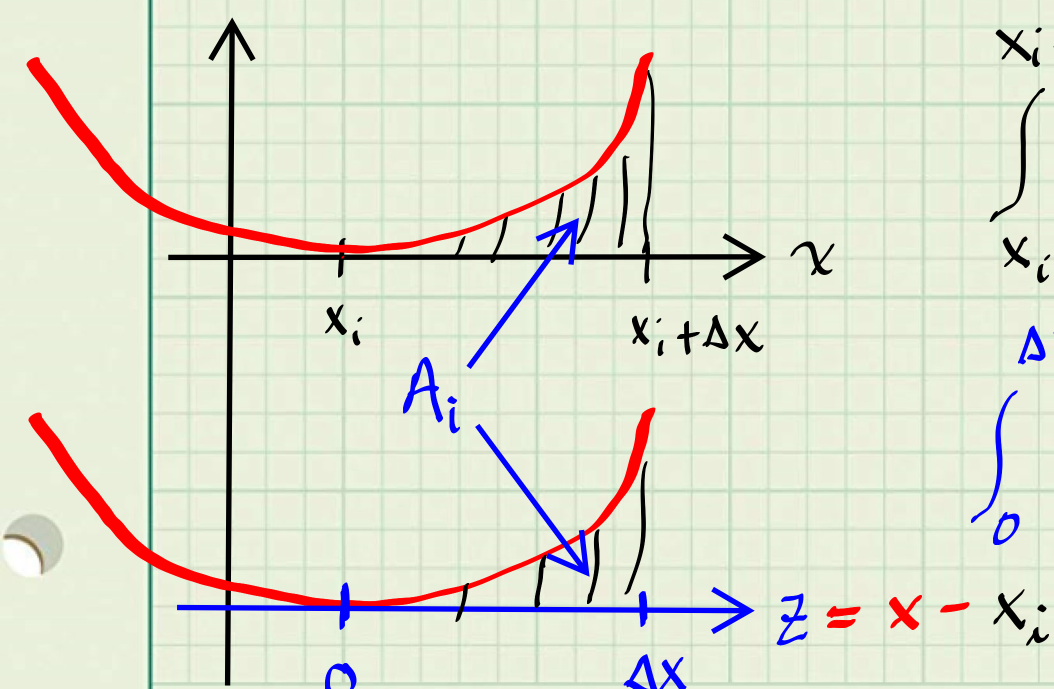
2nd-order

$$A_i := \underbrace{\int_{x_i}^{x_{i+1}} \left[\alpha (x - x_i)^2 + \beta (x - x_i) + \gamma\right] dx}_{\text{Area under the interpolated quadratic}}$$

$$\int_{x_i}^{x_i+\Delta x} (x - x_i)^2\, dx \stackrel{?}{=} \int_0^{\Delta x} z^2\, dz = \left.\frac{z^3}{3}\right|_0^{\Delta x}$$

$$= \frac{(\Delta x)^3}{3}$$

Why?



$$\int_{x_i}^{x_i + \Delta x} (x - x_i)^2\, dx$$

$$\int_0^{\Delta x} z^2\, dz = \left.\frac{z^3}{3}\right|_0^{\Delta x}$$

$$= \frac{(\Delta x)^3}{3}$$

Similarly

$$\int_{x_i}^{x_i+\Delta x} (x-x_i)\,dx = \int_0^{\Delta x} z\,dz = \left.\frac{z^2}{2}\right|_0^{\Delta x} = \frac{(\Delta x)^2}{2}$$

$$\int_{x_i}^{x_i+\Delta x} 1\,dx = \int_0^{\Delta x} 1\,dz = \left. z\right|_0^{\Delta x} = \Delta x$$

Thus

2nd-order

$$A_i = \alpha \frac{(\Delta x)^3}{3} + \beta \frac{(\Delta x)^2}{2} + \gamma\,\Delta x$$

Simpson's Rule (After "some" Algebra)

2nd-order

$$A_i = \frac{\Delta x}{6}\left[f(x_i) + 4f(x_c) + f(x_{i+1})\right]$$

Recall Trapezoidal Rule

$$\text{trap}A_i = \overset{\text{1st-order}}{A_i} = \frac{\Delta x}{2}\left[f(x_i) + f(x_{i+1}) \right]$$

The textbook has a second derivation. You can see which one you like best!