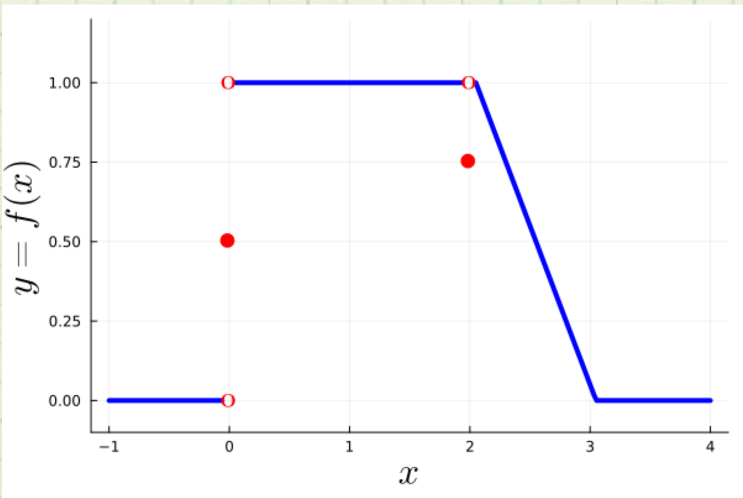
ROB 201 — 03 Feb. 2025

Summary: Completed a few applications of integration

- Maps velocity/acceleration to change in position/velocity (Project 1)
- Total mass & Center of Mass
- Moment of Inertia about z-axis

Started Chapter 4: Properties of Functions, Left and Right Limits, Types of Continuity, and Generalizations of Max and Min

Toughest, Dryest Chapter in the Book



Preliminary Definitions:

$$\lim_{x \to x_0^+} f(x) := \lim_{n \to \infty} f\left(x_0 + \frac{1}{n}\right)$$ Limit from the RIGHT

$$\lim_{x \to x_0^-} f(x) := \lim_{n \to -\infty} f\left(x_0 + \frac{1}{n}\right)$$ Limit from the LEFT

Wolfram Alpha Pro (Free): Sent Friday

Today [video for later]

Formal and more common definitions

$1/n$, $n \to \infty$ replaced with $h \to 0^+$
($h>0$, h small)

$1/n$, $n \to -\infty$ replaced with $h \to 0^-$
($h<0$, $|h|$ small)

<u>Def.</u> (One-sided limits at a point) Suppose $a< x_0<b$ and f defined on (a,b), except possibly at x_0 itself. Let $L \in \mathbb{R}$. Then,

$$\lim_{x \to x_0^+} f(x) := \lim_{h \to 0^+} f(x_0+h) = L$$

if, for all $\varepsilon > 0$ (no matter how small), there exists $\delta_\varepsilon > 0$ such that

$$|f(x_0+h) - L| \leq \varepsilon \quad \text{for all } 0<h\leq\delta_\varepsilon$$

(f approaches L as $h \to 0^+$)

Similarly,

$$\lim_{x \to x_0^-} f(x) := \lim_{h \to 0^-} f(x_0+h) = L$$

if for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$|f(x_0+h) - L| \leq \varepsilon \quad \text{for all } -\delta_\varepsilon \leq h < 0$$

(f approaches L as $h \to 0^-$)

Important: Never evaluate f at x_0 when computing limits.

```
eta = 100:1:1e4
h_Right = 1.0 ./eta
k = 2
y_Right = f.(h_Right, k)
p1 = plot(h_Right, y_Right, linewidth=4, color=:blue, label=false, guidefont=20)
plot!(xlabel=L"$x$", ylabel=L"$y=f(x)$")
h_Left = -h_Right
y_Left = f.(h_Left, k)
p1 = plot!(h_Left, y_Left, linewidth=4, color=:green, label=false, guidefont=20)
if k !=3
    annotate!(0, f(0,k), text(L"$\bullet$", :red, :center, 20))
end
ylims!(p1, (minimum(y_minus), maximum(y_Right)))
```

Examples

a) $f(x) = \sqrt{x-3}$ working with real numbers

Thought Process:

We observe that $\lim_{x \to 3^-} f(x)$ is undefined because $\sqrt{y}$ is undefined for $y < 0$.

We suspect $\lim_{x \to 3^+} f(x) = 0$. Hence, we set $x_0 = 3$ and $L = 0$. Seek, for all $\varepsilon > 0$, a $\delta_\varepsilon > 0$ satisfying the definition

Given

$$|f(x_0+h) - L| \le \varepsilon \quad \text{for} \quad 0 < h \le \delta_\varepsilon$$

to be found

$$\left|\sqrt{(\cancel{3}+h) - \cancel{3}} - 0\right| \le \varepsilon \qquad 0 < h \le \delta_\varepsilon$$

$$|\sqrt{h}| \le \varepsilon \qquad 0 < h \le \delta_\varepsilon$$

$$h \le \varepsilon^2 \qquad 0 < h \le \delta_\varepsilon$$

$$\Downarrow$$

$$\boxed{\delta_\varepsilon = \varepsilon^2}$$

(b) $f(x) = 2 + e^{-\frac{1}{(x+1)}}$. What happens near $x = -1$?

Start with limit from the right

$x \to -1^+$

$x_0 = 1$, L = unknown

Thought Process: Examine $f(x_0+h)$ for $\underline{h>0}$, h small.

$$f(-1+h) = 2 + e^{\frac{-1}{(\cancel{-1}+h\cancel{+1})}} = 2 + e^{-1/h}$$

$h>0$, h small

$$\lim_{x\to -1^+} f(-1+h) = \lim_{h\to 0^+} 2+e^{-1/h} = 2$$

We next look at $x \to -1^-$ [from the LEFT]

$$f(-1+h) = 2 + e^{-1/h}$$

$h<0$, $|h|$ small

i) $h<0 \Rightarrow -1/h > 0$

ii) $h \to 0^- \Rightarrow -\frac{1}{h} \to +\infty$

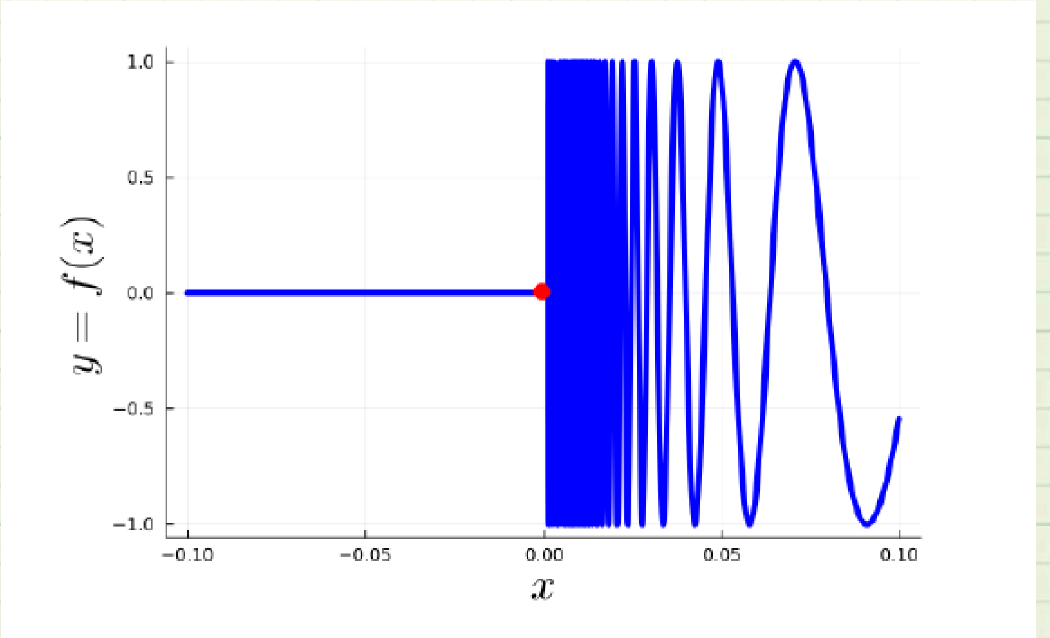
iii) $\lim_{h\to 0^-} 2+e^{-1/h} = +\infty$

$$\lim_{x\to -1^-} f(x) = +\infty.$$

Scan the textbook for

$$\lim_{x\to x_0^+} f(x) = \pm\infty, \quad \lim_{x\to x_0^-} f(x) = \pm\infty$$

Also possible for limits from left or right not to exist

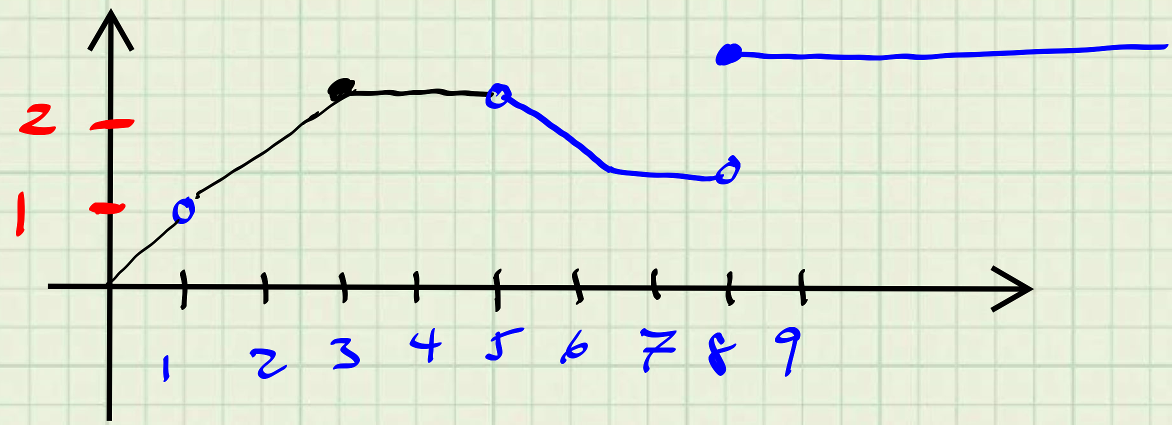


$\sin(\frac{1}{x})$

0

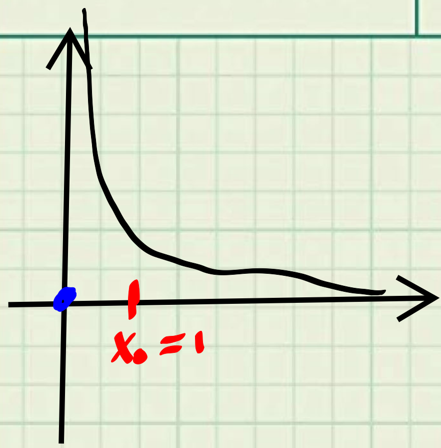
Def. (Continuity at a point x_0)
$f:[a,b] \to \mathbb{R}$, $a < x_0 < b$, is **continuous at x_0** if the limits from the left and right at x_0 are finite, equal, and equal the value of the function at x_0.

$$\boxed{\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0) \qquad \square}$$



Determine if f is continuous at the following points. Answer Y/N and a BRIEF Why?

- $x_0 = 1$ N limits both exist, but the function is not defined.
- $x_0 = 3$ Y limits from L&R = value of $f(x_0)$
- $x_0 = 8$ N limits from L&R do not agree

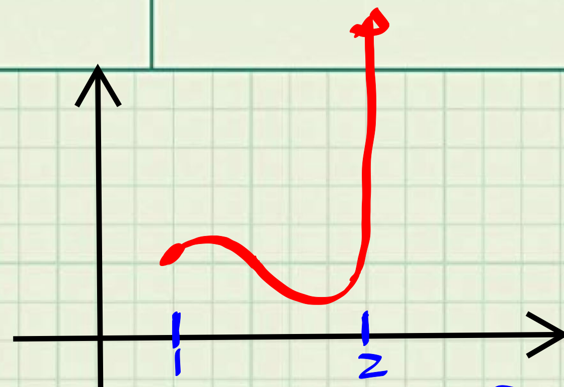


Is the function $f(x) = 1/x$ continuous at $x_0 = 1$? Justify your answer. What about $x_0 = 0$?

Y for $x_0 = 1$.

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

N for $x_0 = 1$. [with $f(0) = 0$ defined] because $\lim_{x \to 0^+} f(x) = +\infty \neq f(0)$



$f : [1, 2) \to \mathbb{R}$

Discuss the continuity of the function shown at the "endpoints"

Y for $x_0 = 1$ because $\lim_{x \to 1^+} f(x) = f(1)$, we ignore $\lim_{x \to 1^-} f(x)$ because f is undefined for $x < 1$.

N for $x_0 = 2$ because 2 is not even in the domain of the function

Key Point for Continuity at Endpoints of an interval: only Check one-sided that are in the domain of the function.

Def. $F : I \to \mathbb{R}$, for $I = (a,b)$, $(a,b]$, $[a,b)$, or $[a,b]$ is continuous on I if f is continuous at x_0 for all $x_0 \in I$. Otherwise it is discontinuous. $\square$.

Proposition 4.18: Common Continuous Functions

The following functions are continuous for all $x \in \mathbb{R}$ unless indicated otherwise:

Polynomial Functions

- Constant function: $f(x) = c$
- Linear function: $f(x) = ax + b$
- Quadratic function: $f(x) = ax^2 + bx + c$
- Higher-degree polynomials: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

Trigonometric Functions

- Sine: $f(x) = \sin(x)$
- Cosine: $f(x) = \cos(x)$
- Tangent (continuous except at odd multiples of $\pm\frac{\pi}{2}$): $f(x) = \tan(x)$

Power Function

- Power (continuous for $x > 0$ and $y \in \mathbb{R}$): $f(x) = x^y$

Exponential and Logarithmic Functions

- Exponential: $f(x) = a^x$ (where $a > 1$)
- Natural Exponential: $f(x) = e^x$
- Logarithm (continuous for $x > 0$): $f(x) = \log_a(x)$ (where $a > 1$)
- Natural Logarithm (continuous for $x > 0$): $f(x) = \ln(x)$

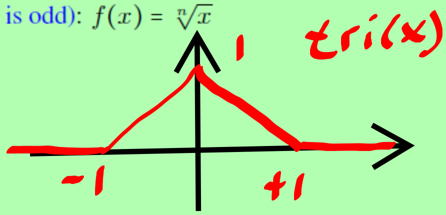
Rational Functions (subject to domain restrictions)

- $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$ (means, rational functions are continuous for all $x \in \mathbb{R}$ such that $q(x) \neq 0$)

Root Functions

- Square Root (continuous for $x \geq 0$): $f(x) = \sqrt{x}$
- Cube Root: $f(x) = \sqrt[3]{x}$
- n-th Root (continuous for $x \geq 0$ if $n \in \mathbb{N}$ is even, all $x \in \mathbb{R}$ if $n \in \mathbb{N}$ is odd): $f(x) = \sqrt[n]{x}$

Other Special Functions



- Absolute Value: $f(x) = |x|$
- Triangle: $f(x) = \text{tri}(x)$
- Radial Basis Function: $f(x) = e^{-\frac{(x-x_c)^2}{s^2}}$
- Gaussian (Normal Distribution) with mean μ and standard deviation σ: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Do discontinuous functions arise in "real" engineering

Yes!

- Algorithms are used as functions. Branching (if-then-else) can lead to discontinuities.
- on/off switches on robots can cause jumps.
- In feedback control, we use step functions as set points.

Def. (Two-sided or double-sided limits)

We say $\lim_{x \to x_0} f(x) = L$ when

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L.$$

Recommend 3Blue1Brown

What is the big payoff of doing limits so carefully?

Many functions are compositions of simpler functions. If $h(x) := f(g(x))$. When is

$$\lim_{x\to x_0} f(g(x)) \stackrel{?}{=} f\left(\lim_{x\to x_0} g(x)\right) \text{ ???}$$

When can you take the limit INSIDE the function.

<u>Prop.</u> If (1) $\lim_{x\to x_0} g(x) = y_0$ and (2) f is continuous at y_0, THEN

$$\lim_{x\to x_0} f(g(x)) = f\left(\lim_{x\to x_0} g(x)\right) = f(y_0)$$

Examples (Easy)

$$\lim_{x \to \frac{\pi}{2}} \frac{3 + \sin(x)}{\sin^2(x) + 4} = ?$$

$g(x) = \sin(x) \Rightarrow \lim_{x \to \frac{\pi}{2}} \sin(x) = +1 = y_0$

$f(y) = \frac{3 + y}{y^2 + 4}$ is this cont. at $y_0 = 1$?

$$\therefore \lim_{x \to \frac{\pi}{2}} f(g(x)) = \frac{3+1}{1+4} = \frac{4}{5}$$

(b) More Challenging Given that $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$. Evaluate

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = ???$$

Can assume $x > 0$.