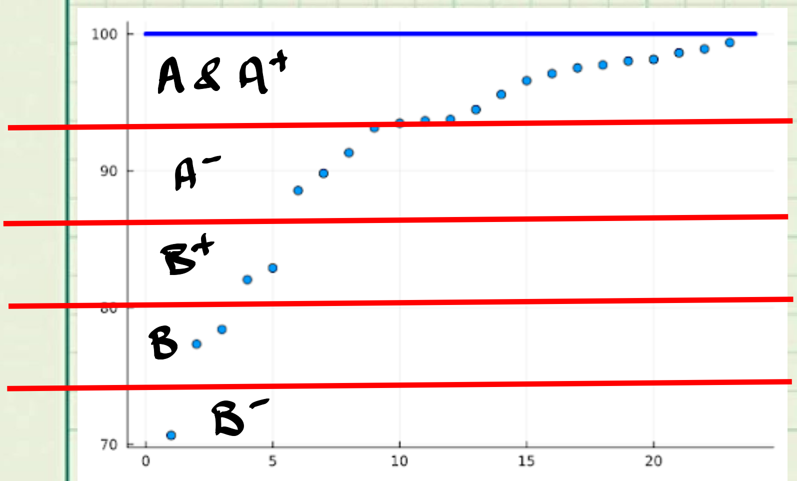
ROB 201 | Prof. Grizzle | 08 JAN 2025

Calculus for the Modern Engineer
"Computational Calculus"

- We teach the parts of Math 115, 116, and 216 used by (robotics) engineers.

- We develop Calculus from scratch more thoroughly than the Math Dept.

Read Chap. 01 (Pre-Calc) on your own
- Quiz 01 is open for 2^+ weeks
- Key notation in Chap. 1.2
- ∞ is a concept and NOT a number
- Grading explained on Canvas



Fall 2024

Course Evaluations Posted on Canvas

Today's Lecture

π and Archimedes' Approx. Principle

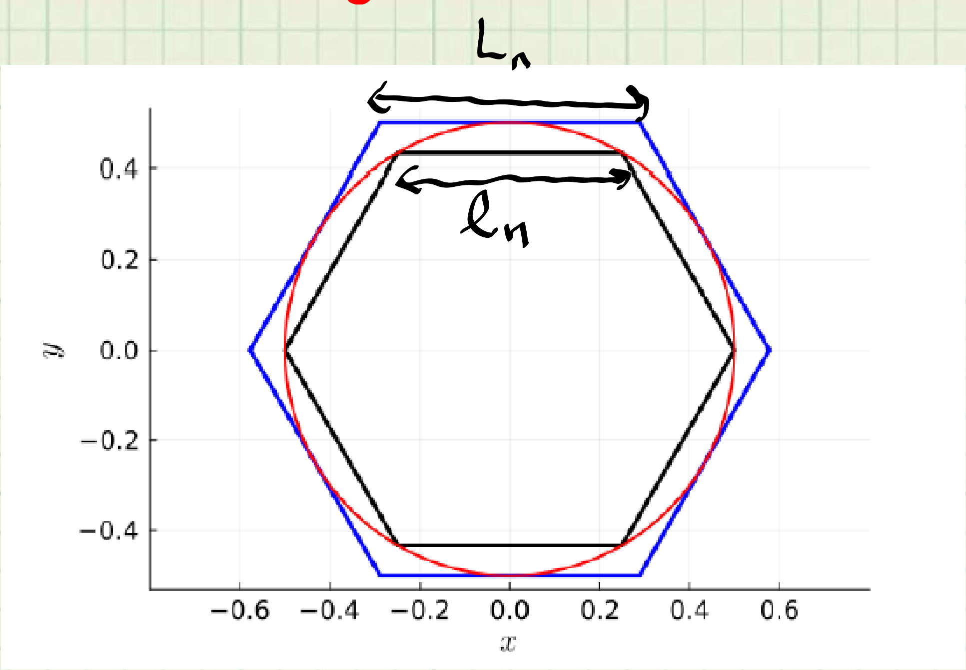
$$\pi := \frac{C}{D} = \frac{\text{Circumf.}}{\text{Diam.}}$$

$$\pi \approx 3.14159\ldots$$

How was it computed in the first place?

In 200 BCE. invented a means to compute π to arbitrary accuracy.

Let $n \geq 3$ be a natural number (aka counting numbers) $\mathbb{N} = \{1, 2, 3, \ldots\}$



$n = 6$

$$\pi_n^{up} := n L_n$$
$$\pi_n^{low} := n \ell_n$$

note $\pi_n^{low} \leq \pi \leq \pi_n^{up}$

Step 1 Lower (inner) approx for C: Inscribe a "regular n-gon" (equal sides)

Step 2 Upper (outer) Circumscribe (outside) a regular n-gon

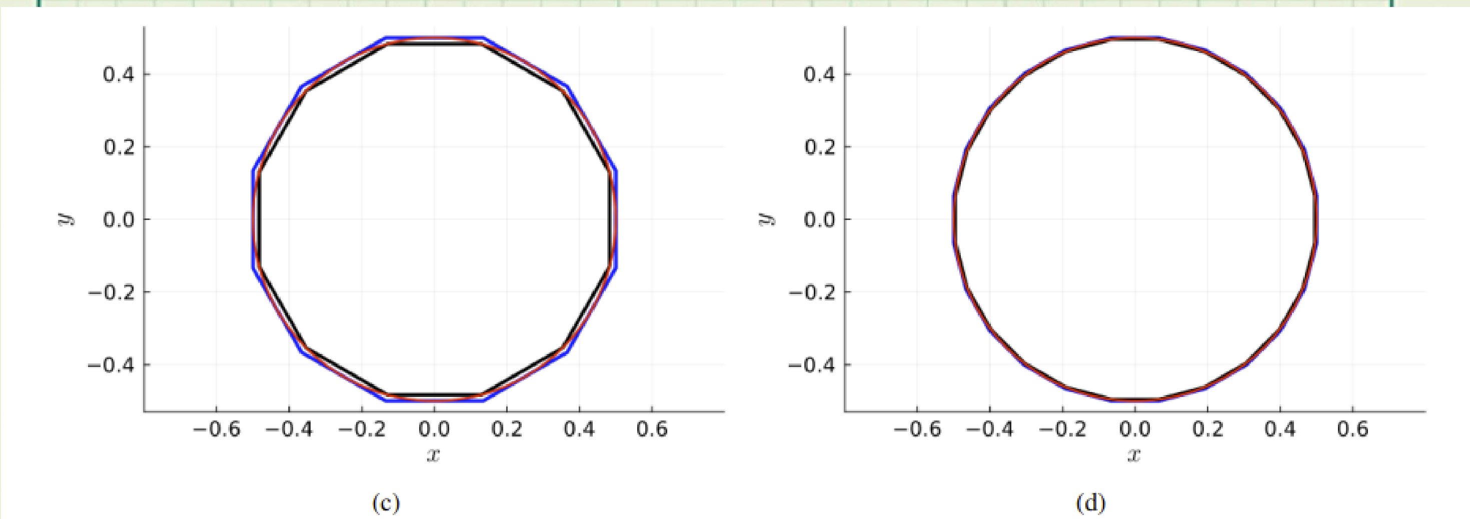
Step 3: Evaluate $\pi_n^{up} := n L_n$

$\pi_n^{low} := n \ell_n$

Define: $\pi_n^{est} := \dfrac{\pi_n^{up} + \pi_n^{low}}{2}$ $\triangleq$

$\pi_n^{error} := \dfrac{\pi_n^{up} - \pi_n^{low}}{2}$

$$\boxed{\pi = \pi_n^{est} \pm \pi_n^{error}}$$



(c)

(d)

n	π^{low}	π^{est}	π^{up}	$\pm\pi^{\text{error}}$
6	3.0000	3.2320	3.4641	± 0.2320
24	3.1326	3.1461	3.1597	± 0.0135
128	3.1416	3.1416	3.1416	± 1.3e-5
512	3.1416	3.1416	3.1416	± 8.2e-7

Retain If we wish to estimate a quantity x,

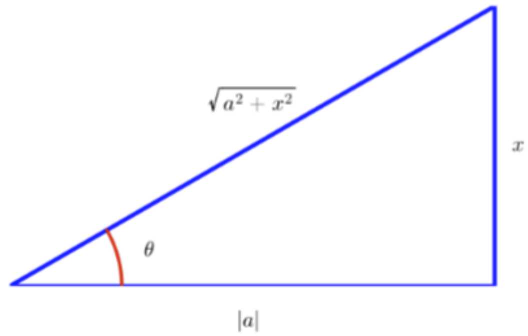
- Form lower (x_n^{low}) and upper (x_n^{up}) approximations for x with (of course)

$$x_n^{low} \le x \le x_n^{up}$$

- If for LARGE n, x_n^{up} and x_n^{low} "approach" a common value, then we have a way to compute x with arbitrary accuracy.

"Approach" gives rise to Limits in Chapter 2.

Do we recall trigonometry?



- Compute $\sin(\theta)$ as a function of x and a. $= \frac{x}{\sqrt{a^2+x^2}}$
- Compute $\tan(\theta)$ as a function of x and a. $= \frac{x}{|a|}$
- **Harder ??**: Find $\sin\left(\arctan\left(\frac{x}{|a|}\right)\right)$.

$\tan(\theta) = \frac{x}{|a|}$

$\theta = \arctan\left(\frac{x}{|a|}\right)$

$\sin(\theta) = \frac{x}{\sqrt{a^2+x^2}}$

Chapter 2: Calculus Foundations:

Proofs, Finite Sums, Limits at Infinity and Geometric Sums

To be pasted in

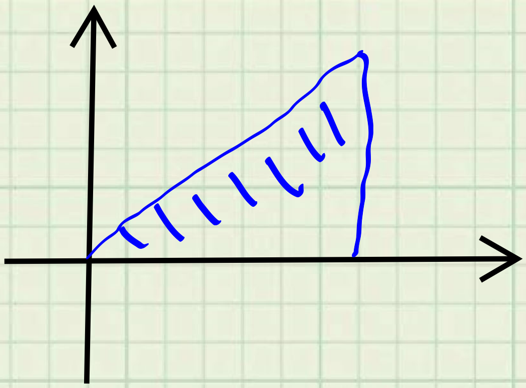
- We build Calculus from scratch
- Almost everything is proved in the textbook, though a few are references
- You are NOT tested on proofs
- When we do a proof in class, our aim is to promote understanding. Calculus formulas do not come from thin air.

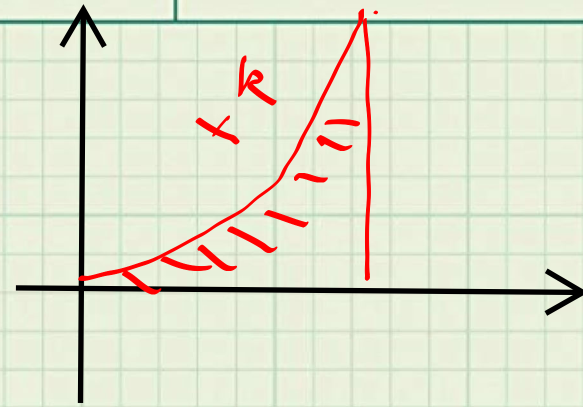
A mathematical proof is a logical argument that establishes the truth of a mathematical statement. It's a demonstration that a given set of hypotheses imply a conclusion.

In plain words, proofs separate the "intuitively obvious" from the "correct".

Finite Sums of Powers of Integers and Proofs by Induction.

Why: Provide means to determine area under monomials (x^k), $k \geq 1$





Objective: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Method: Proof by Induction

Let $P(n)$ be a statement about the counting numbers with the following properties:

(a) Base Case: $P(1)$ is true

(b) Induction Hypothesis: For all $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is also true.

Conclusion: $P(n)$ is true for all $n \geq 1$.

$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots$

We apply this to our objective.

Here is how we define $P(n)$

- $L(n) := 1+2+3+\cdots+n$ (Left side of objective)
- $R(n) := \frac{n(n+1)}{2}$ (Right side of objective).

$\boxed{\cdot\ P(n):\ L(n) = R(n)}$ Objective: Show this.

<u>Base Case:</u> $P(1)$: Is $L(1) \stackrel{?}{=} R(1)$

$L(1) = 1$ $\quad\quad$ $R(1) = \frac{(1)(1+1)}{2} = 1$

$\therefore L(1) = R(1)$ and hence the base case holds.

<u>Induction Step:</u> Does $L(k) = R(k) \stackrel{?}{\Rightarrow} L(k+1) = R(k+1)$

Let's find out:

- $R(k+1) = \dfrac{(k+1)(k+1+1)}{2} = \dfrac{(k+1)(k+2)}{2}$

- $L(k+1) = \underbrace{1+2+\cdots+k}_{L(k)} + (k+1)$

By the induction hypothesis, $L(k) = R(k) = \dfrac{k(k+1)}{2}$

Know
$$
\begin{aligned}
L(k+1) &= L(k) + (k+1) \\
&= R(k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= (k+1)\left[\frac{k}{2} + 1\right] \\
&= (k+1)\left[\frac{k+2}{2}\right] \\
&= \frac{(k+1)(k+2)}{2} = R(k+1).
\end{aligned}
$$

We are done. $P(n)$ holds for all $n \geq 1$.

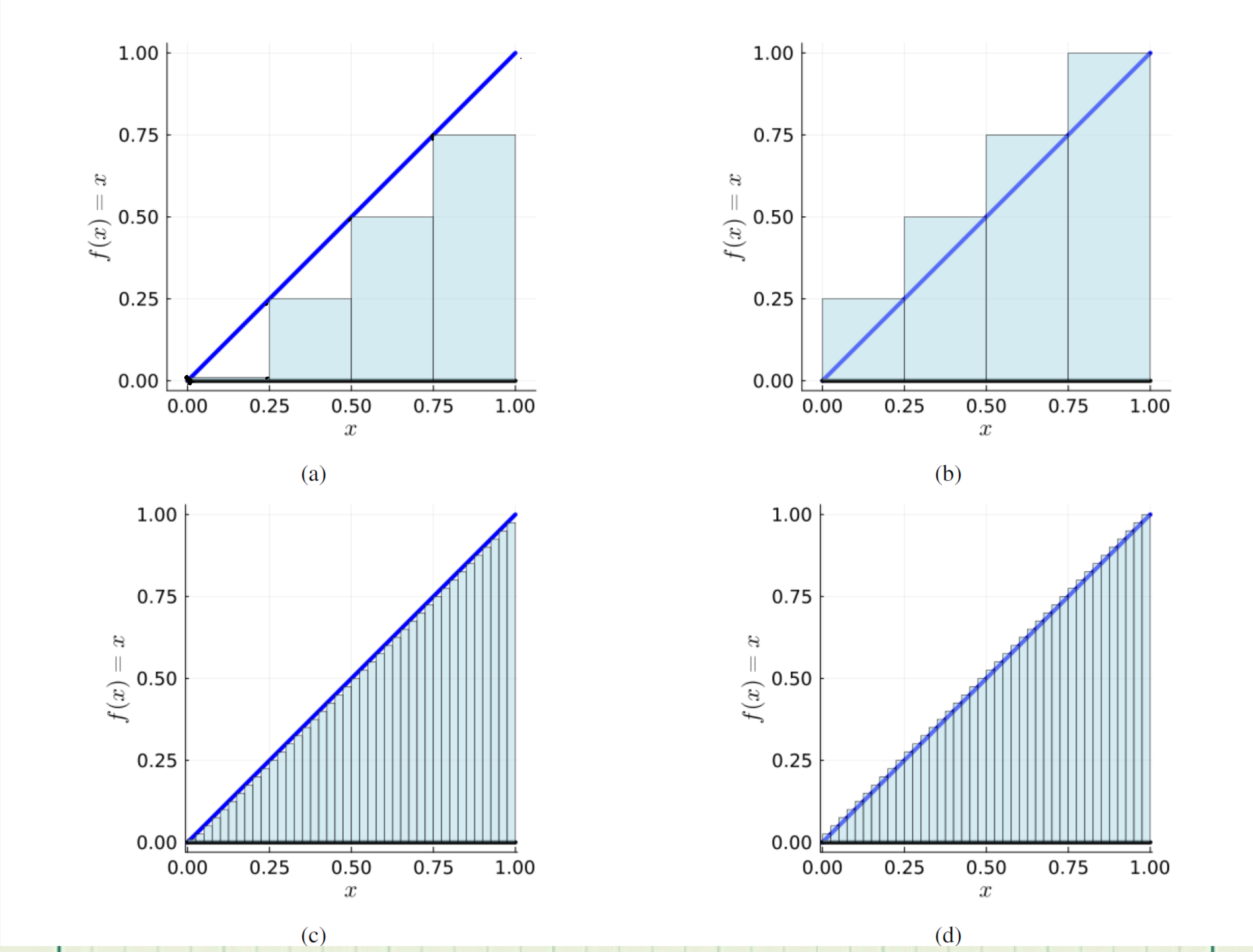
$\square$

The textbook has formulas for

$1^k + 2^k + \cdots + n^k$ for $k \geq 1$.

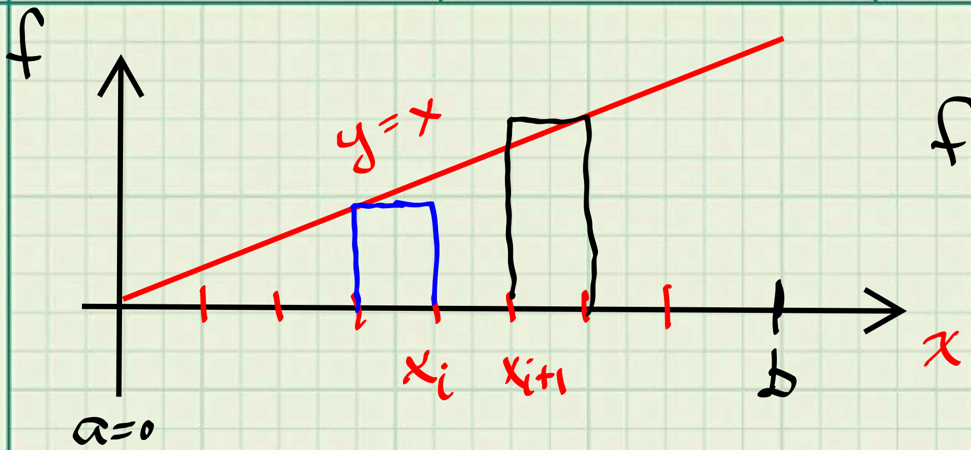
We'll come back to this later.

The Amazing Power of the Rectangle: Finding the area under a curve.



(a) (b) (c) (d)

Let's apply Archimedes' Principle to compute the area.



$f(x) = x$

$n = 2$

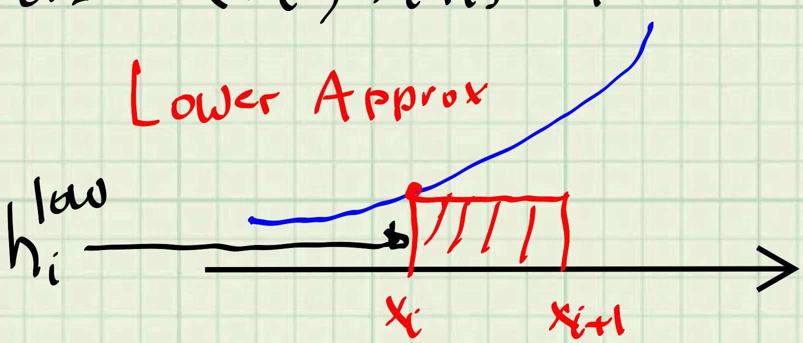
$$\Delta x := \frac{b-a}{n} = \frac{b}{n} \quad (a=0) ,$$

$$0 = a = x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} = b$$

$$x_{i+1} = x_i + \Delta x \quad = (i-1) \cdot \Delta x$$

* multiply

Use (x_i, x_{i+1}) to define rectangles



$$h_i^{low} = f(x_i)$$

There are n-rectangles defined by the $(n+1)$ points $\underset{0}{\overset{\|}{a}} = x_1, x_2, \ldots x_{n+1} = b$

$$A_i^{low} := h_i^{low} \cdot \Delta x$$
$$= f(x_i) \cdot \Delta x$$
$$= x_i \cdot \Delta x \qquad (x_i = (i-1)\cdot \Delta x)$$
$$= (i-1) \cdot (\Delta x)^2$$

Calculus:

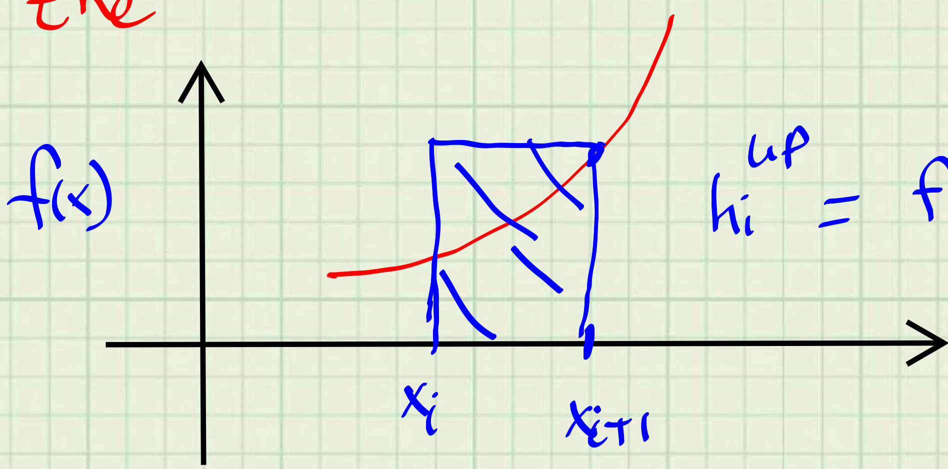
$$Area_n^{low} := A_1^{low} + A_2^{low} + \cdots + A_n^{low}$$
$$= \sum_{i=1}^{n} A_i^{low}$$
$$= \sum_{i=1}^{n} (i-1)(\Delta x)^2$$
$$= (\Delta x)^2 \cdot [0 + 1 + 2 + \cdots + (n-1)]$$
$$= (\Delta x)^2 \sum_{i=1}^{n-1} i$$

algebra

$$= \left(\frac{b}{n}\right)^2 \cdot \frac{(n-1)(n)}{2} \qquad n-1+1$$
$$= \frac{b^2}{n^2} \cdot \left(\frac{n^2-n}{2}\right)$$
$$= \frac{b^2}{2}\left(\frac{n^2-n}{n^2}\right)$$
$$= \frac{b^2}{2}\underbrace{\left(1 - \frac{1}{n}\right)}_{\text{plot it}}$$

$1-\frac{1}{n} \longrightarrow 1$ as n becomes large

For the Upper Approx



$h_i^{up} = f(x_{i+1})$

$$A_i^{up} = h_i^{up} \cdot \Delta x \qquad \Delta x = \frac{b}{n}$$

$$= f(x_{i+1}) \cdot \Delta x \qquad x_{i+1} = i \cdot \Delta x$$

$$= x_{i+1} \cdot \Delta x$$

$$= (i\Delta x) \cdot \Delta x$$

$$= i\,(\Delta x)^2$$

[x_i = array indexed by i]

Will continue on Monday

How to get more practice!

- **Prompt for UMGPT:** Provide four beginner-friendly proofs by induction with detailed explanations. The problems should include:

  1. Showing the formula for the sum of the squares of the first n natural numbers.
  2. Showing the formula for the sum of the cubes of the first n natural numbers.
  3. Proving a formula for the product of terms of the form (k+1)/k from k equals 1 to n.
  4. Proving that the factorial of n is greater than 2 to the power of n, starting from n equals 4.

  Each proof should include a clear and detailed explanation of the base case, inductive hypothesis, inductive step, and a conclusion confirming the proof. The explanations should be thorough and suitable for someone just learning how to write proofs by in-