ROB 201 - Calculus for the Modern Engineer
HW #6

Prof. Grizzle

Check Canvas for due date and time

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete and turn in. **Red Flag:** Problems 2, 3, and 4 are quite short, while Problems 5 and 6 are much longer. We highlight this so that when you have completed Problems 1 through 4, you do not think you are almost done with the HW set.

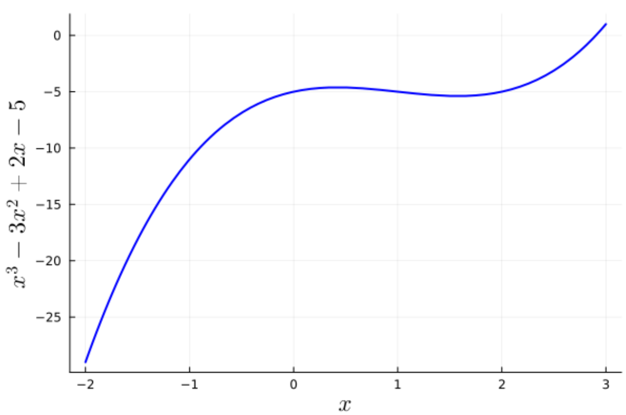
1. (a) Create a "Cheat Sheet" for Chapters 5 and 6 of the textbook. You'll receive the same score for a handwritten solution as a typeset solution. Here is an example from ROB 101.

   (b) Note any material where you found the explanation confusing or difficult to master.

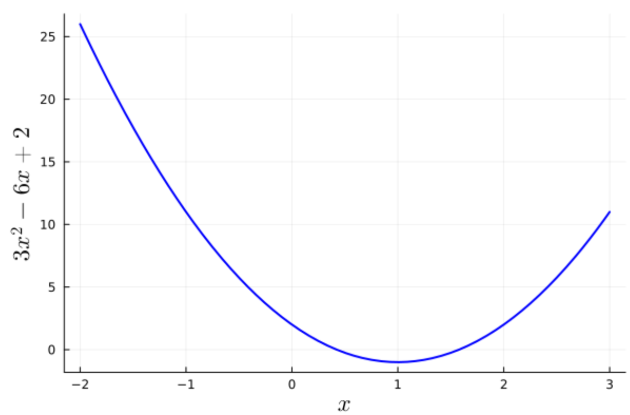
2. Use the two graphs below if you find them helpful. The problem does not require the plots; they can provide a "sanity check" on your answers.

   (a) For the function $q(x) = x^3 - 3x^2 + 2x - 5$, find all of the critical (aka, extreme or stationary) points and classify each critical point as a local minimum, local maximum, or inflection point.

   (b) Find the absolute minimum and absolute maximum of $q(x)$ on the closed interval $[-2, 3]$. Do you have to consider more than just the critical points? Explain.



(a)



(b)

3. Mostly thinking, very little computing:

   (a) Let $q : \mathbb{R} \to \mathbb{R}$ by $q(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ be a **cubic polynomial**, with $a_3 \neq 0$. Is it possible for there to be three distinct points where one is a local minimum, one a local maximum, and yet another a **stationary point of inflection**[1]: Yes or No? Explain briefly your answer.

   (b) For $C \in \mathbb{R}$ a constant and $f : (a, b) \to \mathbb{R}$ a differentiable function, explain why $g(x) := f(x) + C$ has the same critical points as $f(x)$.

[1] Check the textbook for the definition.

(c) Suppose that $f : (a, b) \to \mathbb{R}$ is a differentiable function and $f(x) \neq 0$ for all $x \in (a, b)$. Relate the critical points of $\frac{1}{f(x)}$ to those of $f(x)$.

4. A company manufactures two types of products, A and B, in separate facilities. The profit functions (mega-dollars earned per unit of product produced) for products A and B are given by $P_A(x) = -x^2 + 10x - 15$ and $P_B(x) = -2x^2 + 16x - 20$, respectively, where x is the number of units produced. Their shapes can be explained by, if you produce nothing, you still have to pay rent, your workers, material storage, etc., and if you produce too much, a flooded market suppresses prices.

   (a) For each product, find the number of units that should be produced to maximize profit. They do not have to be whole numbers.

   (b) What is the maximum total profit?

5. **Lagrange Multipliers and The Quest for Balanced Joy in STEM:** You have been studying STEM for a long time, totally winging the allocation of study time and socialization so as to "have a life" and yet not disappoint yourself or those close to you. In your pursuit of happiness and academic excellence, you've decided to formulate a constrained optimization problem whose solution will give you an irreproachable basis[2] to allocate your time between studying and socializing. Here we go!

   Data:

   - Your week has disposable time which you can divide between studying (S hours) and socializing (Z hours).
   - Your Joy (J) is quantified as $J = 60 - S + 2Z$, meaning you are an atypical engineer who likes people (lol).
   - However, your GPA, G, still needs to stay at a respectable level. After much soul searching and talking to friends, you decide that your GPA model is $G = 3.0 + \frac{\sqrt{S}}{10} - \frac{Z^2}{800}$ and you are aiming for a GPA of exactly 3.6 so that you look totally competent to employers without setting the bar too high.

   Optimization Problem: Solve:

$$\begin{array}{rl} \text{Maximize} & J(S, Z) := 60 - S + 2Z \\ \text{subject to} & g(S, Z) = 0.0, \end{array}$$

   where $g(S, Z) := G(S, Z) - 3.6 = 3.0 + \frac{\sqrt{S}}{10} - \frac{Z^2}{800} - 3.6$

   Solve the problem using Lagrange Multipliers and hand computations. Document your thought process. It is OK if you prefer to set up a minimization problem. You would then minimize negative joy; makes sense, right? It's a glass-half-empty vs half-full view of life.

6. **While this problem looks super long, the calculations at each stage are very similar, and you can take advantage of that. This will provide you with a refresher on Riemann Integration being a fancy way to compute sums, while helping you to understand the concept of moment of inertia**.

   Kinetic Energy of Bodies with Distributed Mass (means, does not consist of a finite number of point masses): Figure 2 defines two sets of coordinates on a uniform bar in the plane, modeled as a rectangle. Its total length is L, and its total mass is M. From this, we compute a uniform (linear) density of $\rho := \frac{M}{L}$ and that each segment has mass $\frac{M}{n}$, where n is the number of segments in the bar. As in the textbook, the kinetic energy of the bar will be a quadratic function of $\dot{\theta}$, namely

$$KE = \frac{1}{2} I_z \left(\dot{\theta}\right)^2,$$

   where I_z is called the "moment of inertia about the z-axis " (i.e., the axis coming out of the plane and passing through the origin). In this problem, we'll see how I_z changes depending on how we model the bar. You do not need to know anything about moments of inertia in order to work the problem.

   (a) Compute the center of mass of the bar, (x_c, y_c), in the world frame coordinates (always assuming uniform density).

[2] The problem and all numbers in it are figments of the imagination of your instructor. They bear no resemblance to any person living or dead.

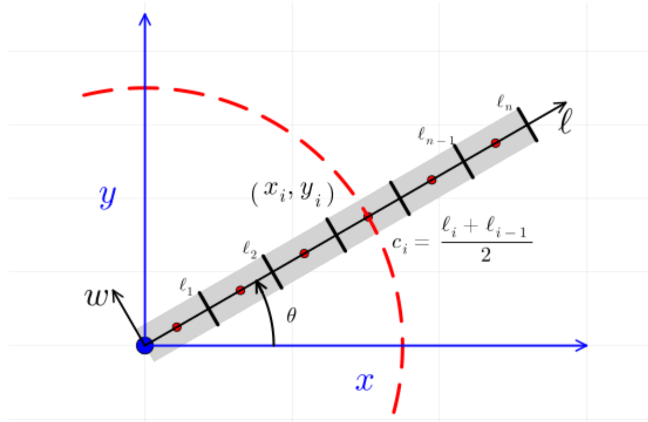


Figure 2: **(For Prob. 6)** A uniform bar in the plane, of length L and total mass M. (x, y) are standard Cartesian coordinates called **world coordinates** in mechanics, while (ℓ, w) are **body coordinates**, that is, coordinates measured in a coordinate frame attached to the bar itself (meaning, if the bar rotates, the coordinates rotate with the bar). We will ignore the width coordinate, w, and focus on the length coordinate, ℓ, to tell us where we are along the length of the bar. The figure shows the bar partitioned into n **segments** of equal length, $\Delta\ell := \frac{L}{n}$. It follows that $\ell_i := i\Delta\ell = \frac{i}{n}L$ and $\ell_0 = 0$. Each red dot is in the center of the corresponding segment. The red dashed line shows that the path traced out by each segment's center is identical to that of a bob (aka, a point mass on the end of a pendulum); see Section "*Kinetic and Potential Energy of a Point Mass in the Plane*" of the textbook.

(b) In the body frame coordinates, compute a simple formula for the position of the center of each segment of the bar, denoted c_i in the Figure. Because the w-coordinate is zero, you can ignore it and only use the length coordinate, ℓ. Your formula should contain L, n, and i. As a check, if you plug in $L = 12$ and $n = 24$, you should obtain $c_6 = 3 - \frac{1}{4} = 2.75$ (yes, $i = 6$). **Suggestion:** Put everything over a common denominator of $2n$.

(c) Compute the kinetic energy KE of the bar, assuming all of the mass is lumped at the center of mass. This means you replace the distributed mass of the bar with a single point mass, having mass equal to the total mass of the bar. Equivalently, you view the bar as having $n = 1$ segments and use your formula from the previous sub-problem. You can find a simple formula for KE of a point pass attached to a pivot in Section "*Kinetic and Potential Energy of a Point Mass in the Plane*" of the textbook.

(d) Compute the kinetic energy KE of the bar when it is divided into $n = 3$ segments of equal length. For each segment, lump its mass of $\frac{M}{3}$ at c_i. **As part of your solution, give the formula for KE_i, $1 \le i \le 3$.** KE will be the sum of the three individual kinetic energies, KE_i. Simplify it as much as you can. The algebra just involves fractions. **Suggestion:** Put the c_i over a common denominator before squaring them.

(e) Compute the kinetic energy KE of the bar when it is divided into $n > 1$ segments of equal length. For each segment, lump its mass of $\frac{M}{n}$ at c_i. **As part of your solution, give the formula for KE_i.** KE will be the sum of the n individual kinetic energies, KE_i. You can use the hint to obtain a nice formula for KE.

(f) Compute the kinetic energy of the bar in the limit as the number of segments $n \to \infty$. Obviously, you can take the limit of the previous problem. Alternatively, you can use your knowledge of Riemann integration to pass directly to an integral instead of taking the limit in the previous problem. It's your choice, and each approach will obtain the same credit. We'll provide both solutions.

(g) Compare the formula for KE when lumping the mass at the center of the bar against the "real" KE when considering a distributed mass. In fact, compute the ratio,

$$\frac{KE_{\text{lumped}}}{KE_{\text{distributed}}}.$$

One used to say, "10% is close enough for engineering work!" Is the relative error within this folkloric margin?

Fact: When using 3-segments,

$$\frac{KE_{\text{lumped}}}{KE_{\text{distributed}}} = \frac{35}{36},$$

which means the relative error is one part in 36, or less than 3%. That's pretty incredible!

We repeat: While this problem looks super long, the calculations at each stage are very similar, and you can take advantage of that. This will provide you with a refresher on Riemann Integration being a fancy way to compute sums.

Hints

Prob. 1 Nothing more to add.

Prob. 2 Nothing further offered.

Prob. 3 Nothing further offered.

Prob. 4 Nothing further offered.

Prob. 5 While the problem is a bit of a grind, it is a straightforward application of the Method of Lagrange Multipliers.

(a) Write down the stationary conditions as in the textbook. The first two equations, which are partial derivatives with respect to the decision variables, are sufficiently nice that you can solve for S and Z in terms of λ.

(b) Substituting these values into the constraint gives an equation in only λ. Your equation should contain a constant, a constant times λ, and a constant times $\frac{1}{\lambda^2}$.

(c) Upon multiplying through by λ^2, you will obtain a cubic equation in λ, which has one real solution you can obtain with the Bisection Algorithm or ask your favorite LLM.

(d) If you use the Bisection Algorithm, you need good bracketing points, which you can obtain from a plot.

(e) If you use an LLM, remember to verify the number it gives you is actually a solution.

(f) Now that you know λ^*, look at the first hint again!

Example Problem. You have many others in the textbook. Compute the area of the largest rectangle that can be inscribed inside a circle of radius $r > 0$.

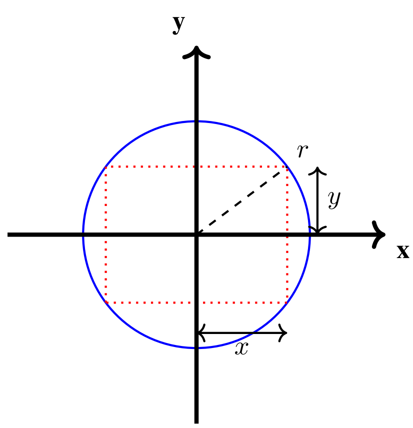


Figure 3: Rectangle inscribed in a circle with labeled dimensions. Note that the width of the rectangle is $2x$ and the height is $2y$.

- Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the area of the rectangle. Then, from the figure,

$$f(x, y) = 4xy.$$

  This is what we want to maximize.

- Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the constraint. We require that the rectangle touches the circle, so

$$g(x, y) = x^2 + y^2 - r^2 = 0.$$

- Using the method of Lagrange multipliers, define

$$L(x,y,\lambda) := f(x,y) + \lambda\, g(x,y) = 4xy + \lambda\,(x^2 + y^2 - r^2).$$

  To find the stationary points, we compute the partial derivatives:

$$\frac{\partial L}{\partial x} = 4y + 2\lambda x = 0,$$
$$\frac{\partial L}{\partial y} = 4x + 2\lambda y = 0,$$
$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - r^2 = 0;$$

  our maximizing values will be one of the solutions to the above equations.

- Solve the first two equations for λ. From

$$4y + 2\lambda x = 0 \quad \Rightarrow \quad \lambda = -\frac{2y}{x},$$

  and

$$4x + 2\lambda y = 0 \quad \Rightarrow \quad \lambda = -\frac{2x}{y}.$$

  Equate these two expressions:

$$-\frac{2y}{x} = -\frac{2x}{y} \quad \Rightarrow \quad \frac{y}{x} = \frac{x}{y} \quad \Rightarrow \quad y^2 = x^2.$$

  Since we are considering the first quadrant ($x, y > 0$), we have

$$y = x.$$

- Substitute $y = x$ into the constraint:

$$x^2 + x^2 = r^2 \quad \Rightarrow \quad 2x^2 = r^2 \quad \Rightarrow \quad x^2 = \frac{r^2}{2} \quad \Rightarrow \quad x^* = \frac{r}{\sqrt{2}}.$$

  Thus,

$$y^* = \frac{r}{\sqrt{2}}.$$

- The maximum area is then computed as:

$$A_{\text{max}} = 4x^* y^* = 4\left(\frac{r}{\sqrt{2}}\right)\left(\frac{r}{\sqrt{2}}\right) = 4 \cdot \frac{r^2}{2} = 2r^2.$$

■

Prob. 6 The sum of the first n ODD integers is equal to $\frac{n(4n^2-1)}{3}$. Indeed, consider,

$$\sum_{i=1}^{n}(2i-1)^2 = \sum_{i=1}^{n}(4i^2 - 4i + 1)$$

$$= 4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$= 4\left(\frac{n(n+1)(2n+1)}{6}\right) - 4\left(\frac{n(n+1)}{2}\right) + n$$

$$= \left(\frac{4n^3}{3} + 2n^2 + \frac{2n}{3}\right) - \left(2n^2 + 2n\right) + n$$

$$= \frac{n(4n^2-1)}{3}$$