ROB 201 - Calculus for the Modern Engineer
HW #5

Prof. Grizzle

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete.

1. Read Chapter 5 of our ROB 201 Textbook, CALCULUS FOR THE MODERN ENGINEER. Based on your reading of the Chapter, summarize in your own words:

   (a) the purpose of Chapter 05;

   (b) two things you found the most DIFFICULT.

   There are no "right" or "wrong" answers. The goal is to reflect a bit on what you are learning and why.

2. Explain your steps while computing the derivatives of the following functions and state which of the Product, Ratio (Quotient), or Chain (Composition) Rule or Rules is/are being used. There is no need to call out the rule for differentiating a linear combination.

   (a) Differentiate the function $f(x) = (1+x^2)(x^3+3x+4)$ with respect to x, **without multiplying it out**. Leave your answer "unsimplified".

   (b) Find the derivative of $g(x) = \frac{\ln(x)}{x^2+1}$ with respect to x. You can simplify the final answer or not. We'll accept almost any correct answer, and if we mark a correct answer wrongly, we'll fix it.

   (c) Differentiate the function $\gamma(x) = e^{\cos(3x)}$ with respect to x.

   (d) Differentiate $\varphi(x) = x^2 \cdot \sin(e^x)$ with respect to x.

3. Where applicable, use L'Hôpital's Rule to evaluate the following limits. If it is not applicable, explain why. In all cases, show your work.

   (a) $\lim\limits_{x\to 0} \dfrac{\tan(x)}{x}$

   (b) $\lim\limits_{x\to 0} \dfrac{\cos(x)}{(1+2x)^5}$

   (c) $\lim\limits_{x\to 1^+} \left(\dfrac{\ln(x)}{e^x - 1}\right)^x$

4. Compute the following partial derivatives using the rules of single-variable differentiation. You are allowed to just give the answers, but, if you go that route, it will be hard to earn partial credit in case of an error. It's your call.

   (a) $\dfrac{\partial}{\partial x}\left(\dfrac{\ln(x)}{z^2+4}\right)$

   (b) $\dfrac{\partial}{\partial y}\left((x^2+1)\sin(y)\right)$

   (c) $\dfrac{\partial}{\partial z}\left(x^3+2y^2\right)$

5. Compute the following "vector derivatives."

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = \ln(x_1^2 + x_2^2 + 1)$. Compute the Jacobian, $\text{Jac}_f(x) := \frac{\partial f(x)}{\partial x}$. Explain your reasoning and identify the "shape of your result" (i.e., is it a column vector or a row vector?).

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = e^{x_1 \cos(x_2)}$. Compute the gradient vector, $\nabla f(x)$. You may simply give the final answer, though showing your steps may help you receive partial credit.

6. You are given a function $z = f(x, y)$, where $x = x(t)$ and $y = y(t)$, and it's known that

$$f(x, y) = \ln(x^2 + 1) \cdot \cos(y), \quad x(t) = t^2, \quad y(t) = \sin(t)$$

Find the total derivative $\frac{dz}{dt}$ at $t = 0$.

Hints

Prob. 1 Write approximately 15 or more words for each part of the question.

Prob. 2 Explain a strategy, and then document it. For example, "Apply the Composition Rule" with $f(x) = \cos(x)$ and $g(x) = \sqrt{x}$. $f'(x) = ...$, $g'(x) = ...$ and hence, for $x > 0$, $\frac{d}{dx}\left(\cos(\sqrt{x})\right) = ...$

Prob. 3 Part (c), ask yourself, when can you take a limit inside a function?

Prob. 4 For partial derivatives with respect to y, all other variables are treated as constants. For example, because $\frac{d}{dy}\left(y^2+1\right) = 2y$, we conclude that: $\frac{\partial}{\partial y}\left(e^x\left(y^2+1\right)\right) = e^x \frac{\partial}{\partial y}\left(y^2+1\right) = e^x \cdot 2 \cdot y = 2y\, e^x$. You can express the answer in either form, or $2e^x\, y$, etc.

Prob. 5 Be careful to apply the Chain Rule correctly. It would be silly not to check your answers with "software".

Prob. 6 You need to find $\left.\frac{dz}{dt}\right|_{t=0}$. You are not asked to compute the general form of $\frac{dz}{dt}(t)$. Your final answer is a number.

Solutions HW 05

Prob. 1 (a) This chapter on derivatives delved deeply into the essential tools of calculus that bridge the gap between abstract mathematical theories and their practical applications in engineering and the sciences. Beginning with intuitive approaches to understanding derivatives, you explored how these powerful mathematical instruments can be visualized as rates of change and slopes of curves, providing insights into the behavior of functions. As you progressed, you encountered the use of derivatives in linear approximations, allowing for the simplification of complex functions into linear functions. This chapter not only equipped you with the skills to compute derivatives manually and with advanced software tools but also prepared you to extend these concepts into higher dimensions with partial derivatives. By the end, your journey through the calculus of single variables seamlessly transitioned to the more general framework of multivariable functions, setting a solid foundation for future topics in the course.

(b) Items that you may have found DIFFICULT include:

- **Understanding the Limit Definition of a Derivative**: The idea of a derivative being a limit as $h \to 0$ of the rise over run (slope of the secant line) can be quite abstract at first. It's like trying to capture a momentary rate of change by zooming in infinitely close — where you almost split hairs to see the difference. Many students grapple with the leap from a discrete understanding of slopes to this almost infinitesimal perspective. The first time I encountered this, it felt like performing mental gymnastics — visualizing something becoming infinitely small is no easy task! I hope our approach helps make this visualization clearer and more intuitive than the traditional ones, which can sometimes feel a bit detached from practical applications.
- **Computational Aspects of Derivatives Using Software:** While the use of software in calculus can be incredibly powerful and efficient, it also requires a certain level of familiarity with the tools and an understanding of when and how to apply them effectively. Initially, the transition from manual calculations to using software tools like Julia for derivatives can be daunting. It's not just about knowing how to type commands or run scripts; it's about understanding what the software is doing behind the scenes. I remember feeling a mix of relief and confusion when I first used a software tool to calculate a derivative — relieved because it saved me from tedious calculations, but confused because I wasn't quite sure what the software had done. This transition is something we aim to handle carefully, ensuring you not only know how to use these tools but also deeply understand the mathematics they're handling.
- **The Leap to Partial Derivatives:** Moving from single-variable to multivariable functions, where partial derivatives come into play, introduces a layer of complexity that can feel overwhelming. Each partial derivative involves holding certain variables constant, which can be a tricky concept to master, especially when visualizing geometrically. The first time you deal with functions of multiple variables and explore their changes in various directions, it might feel like navigating a maze without a map. It's crucial to approach this topic gradually, ensuring a solid grounding in single-variable calculus before diving into the more complex waters of multivariable calculus.

Prob. 2 (a) **Ans.** $f'(x) = (2x)(x^3 + 3x + 4) + (1 + x^2)(3x^2 + 3)$

To differentiate $f(x)$, we use the **Product Rule**: Let $u(x) = 1 + x^2$, and $v(x) = x^3 + 3x + 4$. Then $u'(x) = 2x$, $v'(x) = 3x^2 + 3$, so

$$f'(x) = u'(x)v(x) + u(x)v'(x) = (2x)(x^3 + 3x + 4) + (1 + x^2)(3x^2 + 3)$$

(b) **Ans.** $g'(x) = \frac{\frac{1}{x}(x^2+1)-\ln(x)(2x)}{(x^2+1)^2}$

We apply the **Quotient Rule**: Let $u(x) = \ln(x)$, $v(x) = x^2 + 1$. Then $u'(x) = \frac{1}{x}$, $v'(x) = 2x$, so

$$g'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = \frac{\frac{1}{x}(x^2 + 1) - \ln(x)(2x)}{(x^2 + 1)^2}$$

(c) **Ans.** $\gamma'(x) = -3\sin(3x) \cdot e^{\cos(3x)}$

We apply the **Chain Rule** twice:

$$\frac{d}{dx}e^{\cos(3x)} = e^{\cos(3x)} \cdot \frac{d}{dx}(\cos(3x)) = e^{\cos(3x)} \cdot (-\sin(3x)) \cdot 3 = -3\sin(3x) \cdot e^{\cos(3x)}$$

(d) **Ans.** $\varphi'(x) = 2x \cdot \sin(e^x) + x^2 \cdot \cos(e^x) \cdot e^x$

This requires the **Product Rule**, and the second term uses the **Chain Rule**: Let $u(x) = x^2$, $v(x) = \sin(e^x)$. Then $u'(x) = 2x$, and

$$v'(x) = \cos(e^x) \cdot e^x$$

So:

$$\varphi'(x) = u'(x)v(x) + u(x)v'(x) = 2x \cdot \sin(e^x) + x^2 \cdot \cos(e^x) \cdot e^x$$

Prob. 3 (a) **Ans.** $\lim_{x\to 0} \frac{\tan(x)}{x} = 1$

This limit is indeterminate of the form $\frac{0}{0}$, so we apply **L'Hôpital's Rule**:

$$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{1 + \tan^2(x)}{1} = 1$$

(b) **Ans.** $\lim_{x\to 0} \frac{\cos(x)}{(1+2x)^5} = 1$

This is **not** an indeterminate form. As $x \to 0$, both the numerator and denominator approach finite, nonzero limits:

$$\cos(x) \to 1, \quad (1+2x)^5 \to 1 \quad \Rightarrow \quad \frac{\cos(x)}{(1+2x)^5} \to \frac{1}{1} = \boxed{1}$$

L'Hôpital's Rule **does not apply**.

(c) **Ans.** $\lim_{x\to 1^+} \left(\frac{\ln(x)}{e^x - 1}\right)^x = 1$. We provide two solutions using different principles.

Solution 1: This is an indeterminate form of type 0^1, so we proceed by taking logarithms:

$$\ln y = x \cdot \ln\left(\frac{\ln(x)}{e^x - 1}\right)$$

As $x \to 1^+$,

$$\ln(x) \to 0^+, \quad e^x - 1 \to 0^+ \quad \Rightarrow \quad \frac{\ln(x)}{e^x - 1} \to 0 \quad \Rightarrow \quad \ln\left(\frac{\ln(x)}{e^x - 1}\right) \to -\infty$$

So $\ln y = x \cdot (-\infty) \to -\infty$, but wait — let's evaluate more precisely.
Rewriting:

$$\ln y = x \cdot [\ln(\ln(x)) - \ln(e^x - 1)]$$

Near $x = 1^+$, both $\ln(x) \to 0^+$ and $e^x - 1 \to 0^+$, so:

$$\ln(\ln(x)) \to -\infty, \quad \ln(e^x - 1) \to -\infty$$

We have a difference of two divergent terms multiplied by $x \to 1$. So we change variables: Let $x = 1 + h$, with $h \to 0^+$. Then:

$$\ln(1+h) \sim h, \quad \Rightarrow \quad \ln(\ln(1+h)) \sim \ln(h)$$
$$e^{1+h} - 1 \sim e \cdot h, \quad \Rightarrow \quad \ln(e^{1+h} - 1) \sim \ln(h) + 1$$

So:

$$\ln y \sim (1+h) \cdot [\ln(h) - (\ln(h) + 1)] = (1+h)(-1) \to -1 \Rightarrow y \to e^{-1} = \boxed{\frac{1}{e}}$$

Final answer: $\boxed{\lim_{x\to 1^+} \left(\frac{\ln(x)}{e^x - 1}\right)^x = \frac{1}{e}}$

Alternative solution:

We are asked to evaluate:

$$\lim_{x\to 1^+}\left(\frac{\ln(x)}{e^x-1}\right)^x$$

Step 1: Justify applying the limit inside the power. The function $a(x)^x$ is continuous at $x = 1$ provided $a(x) > 0$ in a neighborhood of 1. Since

$$\frac{\ln(x)}{e^x-1} > 0 \quad \text{for } x > 1 \text{ near } 1,$$

we are allowed to move the limit inside the power:

$$\lim_{x\to 1^+}\left(\frac{\ln(x)}{e^x-1}\right)^x = \left(\lim_{x\to 1^+}\frac{\ln(x)}{e^x-1}\right)^1$$

Step 2: Evaluate the inner limit using L'Hôpital's Rule. This is an indeterminate form of type $\frac{0}{0}$, so we apply the rule:

$$\lim_{x\to 1^+}\frac{\ln(x)}{e^x-1} = \lim_{x\to 1^+}\frac{1/x}{e^x} = \frac{1}{e}$$

Step 3: Take the outer limit.

$$\left(\lim_{x\to 1^+}\frac{\ln(x)}{e^x-1}\right)^1 = \left(\frac{1}{e}\right)^1 = \boxed{\frac{1}{e}}$$

Conclusion: This method is valid because the base remains positive near $x = 1^+$, and the function $a(x)^x$ is continuous under these conditions.

Prob. 4 (a) **Ans.** $\dfrac{\partial}{\partial x}\left(\dfrac{\ln(x)}{z^2+4}\right) = \dfrac{1}{x(z^2+4)}$

We treat z as a constant. Since $z^2 + 4$ is constant with respect to x, we can factor it out:

$$\frac{\partial}{\partial x}\left(\frac{\ln(x)}{z^2+4}\right) = \frac{1}{z^2+4}\cdot\frac{d}{dx}(\ln(x)) = \frac{1}{z^2+4}\cdot\frac{1}{x} = \frac{1}{x(z^2+4)}$$

(b) **Ans.** $\dfrac{\partial}{\partial y}\big((x^2+1)\sin(y)\big) = (x^2+1)\cos(y)$

We treat x as constant. This is a product of a constant $(x^2 + 1)$ and a function of y, so:

$$\frac{\partial}{\partial y}\big((x^2+1)\sin(y)\big) = (x^2+1)\cdot\frac{d}{dy}(\sin(y)) = (x^2+1)\cos(y)$$

(c) **Ans.** $\dfrac{\partial}{\partial z}\left(x^3+2y^2\right) = 0$

The expression contains no z, so its partial derivative with respect to z is zero:

$$\frac{\partial}{\partial z}\left(x^3+2y^2\right) = 0$$

Prob. 5 "Vector derivatives":

(a) **Ans.** We are asked to compute the Jacobian:

$$\mathrm{Jac}_f(x) := \frac{\partial f(x)}{\partial x} \quad \text{for } f(x_1,x_2) = \ln(x_1^2+x_2^2+1)$$

This is a scalar-valued function $f : \mathbb{R}^2 \to \mathbb{R}$, so the Jacobian is a 1×2 row vector.
Let $u = x_1^2 + x_2^2 + 1$, then

$$\frac{\partial f}{\partial x_1} = \frac{1}{u}\cdot\frac{\partial u}{\partial x_1} = \frac{2x_1}{x_1^2+x_2^2+1}$$

$$\frac{\partial f}{\partial x_2} = \frac{1}{u} \cdot \frac{\partial u}{\partial x_2} = \frac{2x_2}{x_1^2 + x_2^2 + 1}$$

$$\Rightarrow \boxed{\mathrm{Jac}_f(x) = \begin{bmatrix} \frac{2x_1}{x_1^2 + x_2^2 + 1} & \frac{2x_2}{x_1^2 + x_2^2 + 1} \end{bmatrix} \quad 1 \times 2 \text{ ROW VECTOR}}$$

(b) **Ans.** We are asked to compute the gradient:

$$\nabla f(x) \quad \text{for } f(x_1, x_2) = e^{x_1 \cos(x_2)}$$

This is a scalar-valued function $f : \mathbb{R}^2 \to \mathbb{R}$, so the gradient is a column vector in $\mathbb{R}^2$. Use the chain rule:

$$\frac{\partial f}{\partial x_1} = \frac{d}{dx_1}\left(e^{x_1 \cos(x_2)}\right) = e^{x_1 \cos(x_2)} \cdot \cos(x_2)$$

$$\frac{\partial f}{\partial x_2} = \frac{d}{dx_2}\left(e^{x_1 \cos(x_2)}\right) = e^{x_1 \cos(x_2)} \cdot (-x_1 \sin(x_2)) = -x_1 \sin(x_2) \cdot e^{x_1 \cos(x_2)}$$

$$\Rightarrow \boxed{\nabla f(x) = \begin{bmatrix} \cos(x_2) \cdot e^{x_1 \cos(x_2)} \\ -x_1 \sin(x_2) \cdot e^{x_1 \cos(x_2)} \end{bmatrix} \quad 2 \times 1 \text{ COLUMN VECTOR}}$$

Prob. 6 **Ans.** $\left.\frac{dz}{dt}\right|_{t=0} = 0$

To find the total derivative of z with respect to t, we use the chain rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

so that

$$\left.\frac{dz}{dt}\right|_{t=0} = \left.\frac{\partial z}{\partial x}\right|_{t=0} \cdot \left.\frac{dx}{dt}\right|_{t=0} + \left.\frac{\partial z}{\partial y}\right|_{t=0} \cdot \left.\frac{dy}{dt}\right|_{t=0}$$

We compute each term step by step:

- Compute partial derivatives of $z = f(x, y) = \ln(x^2 + 1)\cos(y)$:
  - $\frac{\partial z}{\partial x} = \frac{2x\cos(y)}{x^2 + 1}$
  - $\frac{\partial z}{\partial y} = -\ln(x^2 + 1)\sin(y)$
- Compute derivatives of $x(t) = t^2$, $y(t) = \sin(t)$:
  - $\frac{dx}{dt} = 2t \quad \Rightarrow \quad \left.\frac{dx}{dt}\right|_{t=0} = 0$
  - $\frac{dy}{dt} = \cos(t) \quad \Rightarrow \quad \left.\frac{dy}{dt}\right|_{t=0} = 1$
- Evaluate $x(0) = 0$, $y(0) = 0$, and substitute into the partials:

$$\left.\frac{\partial z}{\partial x}\right|_{x=0,y=0} = \frac{2 \cdot 0 \cdot \cos(0)}{0^2 + 1} = 0$$

$$\left.\frac{\partial z}{\partial y}\right|_{x=0,y=0} = -\ln(0^2 + 1) \cdot \sin(0) = -\ln(1) \cdot 0 = 0$$

- Final substitution into the chain rule:

$$\left.\frac{dz}{dt}\right|_{t=0} = 0 \cdot 0 + 0 \cdot 1 = \boxed{0}$$

Good software tools make this easier, but understanding it matters more!