ROB 201 - Calculus for the Modern Engineer
HW #7

Prof. Grizzle

Remark: There are six (6) HW problems plus a *Jupyter notebook* to complete.

1. Read Chapter 7 of our ROB 201 Textbook, `Calculus for the Modern Engineer`. Based on your reading of the Chapter, summarize in your own words:

   (a) the purpose of Chapter 07;

   (b) two things you found the most DIFFICULT.

   There are no "right" or "wrong" answers. The goal is to reflect a bit on what you are learning and why.

2. Let $f(x) = \sqrt{x+2}$ and define $F(x) = \int_1^x f(t)\,dt = \int_1^x \sqrt{t+2}\,dt$.

   (a) Find $F'(x)$.
   (b) Compute $F(2)$ exactly.

3. For the following two functions, find an antiderivative $F(x)$ of $f(x)$ by hand, and include the constant of integration.

   (a) $f(x) = x^2 \sin(x^3)$.
   (b) $f(x) = \sin^2(x)\cos(x)$.

4. By hand, determine the following integral. Show your work and/or explain your reasoning as the problem may require. If you are using a method highlighted in the section of the textbook, **The Art of the Antiderivative: Inverting Differentiation Rules to Find Antiderivatives**, call it out.

$$\int_0^{\pi/2} x^2 \sin(x)\,dx$$

5. By hand, determine the following integrals. Show your work and/or explain your reasoning as the problem may require. If you are using a method highlighted in the section of the textbook, **The Art of the Antiderivative: Inverting Differentiation Rules to Find Antiderivatives**, call it out.

   (a) $\int x\,\cos(x)\,dx$
   (b) $\int \frac{3}{x^2+5x+6}\,dx$

6. By hand, evaluate the improper integral, or explain why it diverges:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

Hints

Prob. 1 Write approximately 15 or more words for each part of the question.

Prob. 2 Apply the First Fundamental Theorem of Calculus to part (a). For part (b), integrate using a basic substitution.

Prob. 3 For (a), try a u-substitution: $u = x^3$. For (b), you are on your own!

Prob. 4 Use **integration by parts**, and you will need to apply it twice.

Prob. 5 Part (a) requires integration by parts. Part (b) can be handled by partial fraction decomposition.

Prob. 6 Check for a vertical asymptote at $x = 0$. Write as a limit.

Solutions HW 07

Prob. 1 Your answers may vary.

(a) The purpose of Chapter 07 is to bridge the understanding between differential calculus and integral calculus by introducing the concept of an antiderivative. This chapter aims to:

- Illuminate the inverse relationship between differentiation and integration through the Fundamental Theorems of Calculus.
- Demonstrate that while antidifferentiation and definite integration are distinct operations, they are closely related and, often to the peril of young learners, conflated in calculus education.
- Highlight the historical context and the modern computational tools that have made manual computation of antiderivatives less critical, except in specific scenarios.
- Prepare you for further studies in calculus, including indefinite integration, improper integrals, and differential equations, by establishing a solid foundation in understanding antiderivatives.

(b) Two things you may have found DIFFICULT in Chapter 07 are:

- Understanding when and how to apply the various methods for computing antiderivatives, especially distinguishing between situations that call for different techniques. This difficulty stems from the need for significant experience to make these judgments accurately.
- Dealing with the frustration associated with the multi-step process of computing antiderivatives. Errors in calculation, such as sign mistakes or confusion over arithmetic operations, can lead to incorrect results, discouraging learners and learning. This challenge is compounded by the fact that, outside of academic exercises, the practical application of manually computed antiderivatives is limited in modern engineering and mathematical practice.

Prob. 2 (a) By the First Fundamental Theorem of Calculus, we have:

$$F'(x) = \frac{d}{dx}\left(\int_1^x \sqrt{t+2}\,dt\right) = \sqrt{x+2}$$

(b) Compute:

$$F(2) = \int_1^2 \sqrt{t+2}\,dt$$

Let $u = t+2$, then $du = dt$. When $t = 1$, $u = 3$; when $t = 2$, $u = 4$:

$$F(2) = \int_3^4 \sqrt{u}\,du = \int_3^4 u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_3^4 = \frac{2}{3}(4^{3/2} - 3^{3/2})$$

$$= \frac{2}{3}(8 - 3\sqrt{3}) = \frac{16}{3} - 2\sqrt{3}$$

Prob. 3 (a) Let $u = x^3$, then $du = 3x^2\,dx$, so $x^2\,dx = \frac{1}{3}du$. Then

$$\int x^2 \sin(x^3)\,dx = \int \sin(u)\cdot\frac{1}{3}\,du = \frac{1}{3}\int \sin(u)\,du = -\frac{1}{3}\cos(u) + C$$

$$= -\frac{1}{3}\cos(x^3) + C$$

(b) Let $u = \sin(x)$, so that $du = \cos(x)\,dx$. Then,

$$\int \sin^2(x)\cos(x)\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sin^3(x)}{3} + C$$

Thus, an antiderivative is $F(x) = \frac{\sin^3(x)}{3} + C$.

Prob. 4 We use **integration by parts**, which inverts the product rule. Let:

$$u = x^2, \quad dv = \sin(x)dx \Rightarrow du = 2xdx, \quad v = -\cos(x)$$

Apply the formula:

$$\int x^2 \sin(x)dx = -x^2\cos(x) + \int 2x\cos(x)dx$$

Now apply integration by parts again to $\int 2x\cos(x)dx$. Let:

$$u = 2x, \quad dv = \cos(x)dx \Rightarrow du = 2dx, \quad v = \sin(x)$$

Then:

$$\int 2x\cos(x)dx = 2x\sin(x) - \int 2\sin(x)dx = 2x\sin(x) + 2\cos(x)$$

Combining both steps:

$$\int x^2\sin(x)dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

Now evaluate the definite integral:

$$\int_0^{\pi/2} x^2\sin(x)dx = \left[-x^2\cos(x) + 2x\sin(x) + 2\cos(x)\right]_0^{\pi/2}$$

At $x = \pi/2$:

$$-\left(\frac{\pi^2}{4}\right)\cdot\cos\left(\frac{\pi}{2}\right) + 2\cdot\frac{\pi}{2}\cdot\sin\left(\frac{\pi}{2}\right) + 2\cdot\cos\left(\frac{\pi}{2}\right) = 0 + \pi + 0 = \pi$$

At $x = 0$:

$$-0^2\cdot\cos(0) + 0\cdot\sin(0) + 2\cdot\cos(0) = 0 + 0 + 2 = 2$$

Final result:

$$\int_0^{\pi/2} x^2\sin(x)dx = \pi - 2$$

Prob. 5 (a) We use **Integration by Parts**, which inverts the product rule:

$$u = x, \quad dv = \cos(x)\,dx \Rightarrow du = dx, \quad v = \sin(x)$$

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C$$

(b) We apply **Partial Fraction Expansion**, as described in the textbook. Factor the denominator:

$$x^2 + 5x + 6 = (x+2)(x+3)$$

Decompose:

$$\frac{3}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

Multiply both sides:

$$3 = A(x+3) + B(x+2)$$

Set $x = -3 \Rightarrow 3 = A(0) + B(-1) \Rightarrow B = -3$
Set $x = -2 \Rightarrow 3 = A(1) + B(0) \Rightarrow A = 3$
So:

$$\int \frac{3}{x^2+5x+6}\,dx = \int\left(\frac{3}{x+2} - \frac{3}{x+3}\right)dx = 3\ln|x+2| - 3\ln|x+3| + C$$
$$= 3\ln\left|\frac{x+2}{x+3}\right| + C$$

Prob. 6 There is a vertical asymptote at $x = 0$, so we write:

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 x^{-1/2}dx = \lim_{\varepsilon\to 0^+}\left[2x^{1/2}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0^+}(2 - 2\sqrt{\varepsilon}) = 2$$

So the integral converges and the value is 2.