Summary: $L \in \mathbb{R}$

$\lim\limits_{x\to\infty} f(x) = L \iff$ for all $\varepsilon > 0$, there exists $N_\varepsilon < \infty$ s.t. $x \ge N_\varepsilon \Rightarrow |f(x) - L| \le \varepsilon$

$\lim\limits_{x\to\infty} f(x) = \infty \iff$ for all $K > 0$, there exists $N_K < \infty$ s.t. $x \ge N_K \Rightarrow f(x) \ge K$

$\lim\limits_{x\to\infty} f(x)$ does not exist otherwise.

$$\lim_{x\to\infty} \frac{b_m x^m + \cdots + b_0}{a_n x^n + \cdots + a_0} = \lim_{x\to\infty} \frac{b_m}{a_n} \frac{x^m}{x^n}$$

$$= \begin{cases} \dfrac{b_m}{a_n} & m = n \\ \operatorname{sign}\left(\dfrac{b_m}{a_n}\right)\cdot\infty & m > n \\ 0 & m < n \end{cases}$$

Exponentials dominate monomials

$\lim\limits_{x\to\infty} \dfrac{a^x}{x^m} = \infty$ $(a > 1)$ $\qquad \lim\limits_{x\to\infty} x^m a^x = 0$ $(0 < a < 1)$

$\infty - \infty$ is an indeterminate form because

$$\lim_{x\to\infty} \begin{cases} x - x^2 & = -\infty \\ x - \dfrac{x^3}{x^2+1} & = 0 \\ x^2 - x & = +\infty \end{cases}$$

Video link for geometric sums

Today: More on Algebra of Limits

Prop. For $1 \le i \le n$, if $\lim_{x\to\infty} f_i(x)$ is finite, then, for coeff's $\alpha_i \in \mathbb{R}$,

$$\lim_{x\to\infty} \sum_{i=1}^{n} \left(\alpha_i f_i(x)\right) = \sum_{i=1}^{n} \lim_{x\to\infty} \alpha_i f_i(x).$$

Example:

$$\lim_{x\to\infty} \left[\underbrace{\frac{x^2}{3-x^3}}_{0} + \underbrace{\frac{4+x+3x^2}{x^2-6x+11}}_{3} \right] = 0 + 3 = 3$$

Did not have to place over a common denominator.

More Intuition

- For $\alpha > 0$, $\alpha \cdot \infty = +\infty$
- For $\alpha < 0$, $\alpha \cdot \infty = -\infty$
- However, $0 \cdot \infty$ is undefined and is another of our INDETERMINATE Forms.

Why:

$$\lim_{x\to\infty}\Big[\underbrace{\frac{1}{x^2+1}}_{0}\cdot\underbrace{x^4}_{\infty}\Big] = \infty$$

$$\lim_{x\to\infty}\Big[\underbrace{\frac{1}{x^2+1}}_{0}\cdot\underbrace{x^2}_{\infty}\Big] = 1$$

$$\lim_{x\to\infty}\Big[\underbrace{\frac{1}{x^2+1}}_{0}\cdot\underbrace{x}_{\infty}\Big] = 0$$

Similarly: $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are also indeterminate forms.

Prop. Let $f:(0,\infty)\to\mathbb{R}$ and $g:(0,\infty)\to\mathbb{R}$ be two functions, and suppose $\lim_{x\to\infty} g(x)$ exists, is finite, and non-zero. Then,

1. $\lim\limits_{x\to\infty}\left[f(x)\cdot g(x)\right] = \left[\lim\limits_{x\to\infty} f(x)\right]\cdot\left[\lim\limits_{x\to\infty} g(x)\right]$

2. $\lim\limits_{x\to\infty}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim\limits_{x\to\infty} f(x)}{\lim\limits_{x\to\infty} g(x)}$

$\square$

Examples

1. $\lim\limits_{x\to\infty}\left(\underbrace{\dfrac{x^2}{3-x^4}}_{0}\cdot\underbrace{\dfrac{4+3x^2}{x^2-6x+11}}_{3}\right) = 0\cdot 3 = 0$

with

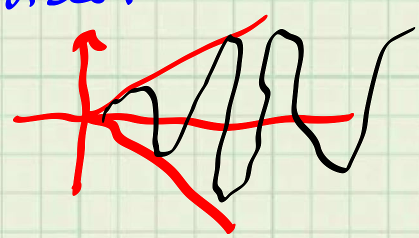
$$f(x) = \frac{x^2}{3-x^4}$$

$$g(x) = \frac{4+3x^2}{x^2-6x+11}$$

2. $\lim\limits_{x\to\infty}\left(\dfrac{x\cdot\sin(x)}{1+e^{-x}+e^{-2x}}\right) =$ undefined

with

$$f(x) = x\cdot\sin(x) \xrightarrow[x\to\infty]{} \text{undefined}$$



$$g(x) = 1+e^{-x}+e^{-2x} \xrightarrow[x\to\infty]{} 1$$

However

$$f(x) = \sin(x) \xrightarrow[x\to\infty]{} \text{undefined}$$

$$g(x) = \frac{x}{1+e^{-x}+e^{-2x}} \xrightarrow[x\to\infty]{} \infty$$

Moral: In math as in life, one can make Good and Bad Choices.

Geometric Sums

<u>Def.</u> For $a \in \mathbb{R}$ and $r \in \mathbb{R}$,

$$S_n := a + ar + ar^2 + \dots + ar^n = \sum_{i=0}^{n} ar^i$$

is called a <u>geometric sum</u> and r is called the <u>common ratio.</u>

<u>Examples</u>

- $1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} \Rightarrow a = 1,\ r = \frac{1}{2}$
- $2 + 6 + 18 + \dots [2 \cdot 3^n] \Rightarrow a = 2,\ r = 3$

Prop. For $r \neq 1$, $S_n = \dfrac{a(1 - r^{n+1})}{1 - r}$

and if $|r| < 1$, then $\lim\limits_{n \to \infty} S_n = \dfrac{a}{1-r}$. $\square$

Video proof!

Proof by Induction (can be pasted into the notes and discussed quickly):

$$L(n) := \sum_{i=1}^{n} ar^i \quad R(n) = \frac{a(1-r^{n+1})}{1-r}$$

$$P(n): \quad L(n) = R(n) \quad \text{all} \quad n \geq 1$$

Bases Case: $L(1) \stackrel{?}{=} R(1)$

$$\begin{aligned} L(1) &= a + ar \\ R(1) &= \frac{a(1-r^2}{1-r} = a\left[\frac{(1-r)(1+r)}{1-r}\right] \\ &= a + ar \\ &= L(1) \end{aligned}$$

and hence, the base case holds.

Induction Hypothesis: $L(k) = R(k) \stackrel{?}{\Longrightarrow} L(k+1) = R(k+1)$

$$R(k+1) = \frac{a(1-r^{k+2})}{1-r}$$

$$L(k+1) = \underbrace{\sum_{i=1}^{k} ar^i}_{L(k)=R(k)} + ar^{k+1} \quad \text{(uses induction hypothesis)}$$

$$= a\left[\frac{1-r^{k+1}}{1-r} + r^{k+1}\right]$$

$$= a\left[\frac{1-r^{k+1}}{1-r} + r^{k+1}\frac{1-r}{1-r}\right]$$

$$= a\left[\frac{1-r^{k+1}}{1-r} + \frac{r^{k+1}-r^{k+2}}{1-r}\right] \quad \text{(cancel like terms)}$$

$$= R(k+1)$$

Problem Prove by induction that

$$\sum_{i=1}^{n} 2^i = 2(2^n - 1).$$

Proof $L(n) = \sum_{i=1}^{n} 2^i$, $R(n) = 2(2^n - 1)$

$P(n)$: $L(n) = R(n)$ all $n \geq 1$

Base Case $(n=1)$

- $L(1) = 2^1 = 2$
- $R(1) = 2(2^1-1) = 2\cdot 1 = 2$

} Base case Holds

Induction Step: $L(k)=R(k) \overset{?}{\Longrightarrow} L(k+1)=R(k)$

Let's find out!

- $R(k+1) = 2(2^{k+1}-1)$
- $L(k+1) = \underbrace{\sum_{i=1}^{k} 2^i}_{L(k)} + 2^{k+1}$

By the induction hypothesis

$$L(k) = R(k) = 2(2^k-1)$$

$$L(k+1) = \underbrace{2(2^k-1)}_{L(k)=R(k)} + 2^{k+1}$$

(Rest is algebra, hard part)

$$= 2(2^k-1) + 2\cdot 2^k$$

$$= 2[2^k - 1 + 2^k]$$

$$= 2[2^k+2^k-1]$$

$$= 2[2^{k+1}-1] = R(k+1) \checkmark$$

$\Box$

Def. A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the only solution to the equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0_{n \times 1}$$

is the trivial solution, $c_1 = c_2 = \cdots = c_k = 0$.

Otherwise, the set is dependent.

Prop. Let $\{\underline{v_1, v_2, \ldots, v_k}\}$ be linearly indep.. If $\{\underline{x}, \underline{v_1}, \underline{v_2, \ldots}, \underline{v_k}\}$ is linearly dependent, then there exist unique coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ such that $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$.

Proof.

$\{x, v_1, \ldots, v_k\}$ dep. $\Rightarrow$ there exist coeff. $c_0, \ldots, c_k$ not all zero such that

$$c_0 \cdot x + c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0_{n \times 1}$$

$C_0 \neq 0$ because otherwise, one of the c_i $1 \le i \le k$ is non-zero and we have $c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0_{n \times 1}$

$\therefore C_0$ is non-zero.

$$x = \underbrace{\left(-\frac{c_1}{c_0}\right)}_{\alpha_1} v_1 \underbrace{\left(-\frac{c_2}{c_0}\right)}_{\alpha_2} v_2 - \dots - \underbrace{\left(\frac{c_k}{c_0}\right)}_{\alpha_k} v_k$$

I'll include the uniqueness part later.

Definition: A set of vectors $\{v_1, v_2, \dots, v_k\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

implies that $c_1 = c_2 = \cdots = c_k = 0$, where $c_1, c_2, \dots, c_k \in \mathbb{R}$.

Proposition: Let $\{v_1, v_2, \dots, v_k\}$ be a linearly independent set of vectors in the vector space $\mathbb{R}^n$. If $\{x, v_1, v_2, \dots, v_k\}$ is linearly dependent, then there exist unique coefficients

$\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ such that,

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k.$$

Proof:

Since $\{x, v_1, v_2, \ldots, v_k\}$ is linearly dependent, there exist scalars $c_0, c_1, c_2, \ldots, c_k$, not all zero, such that:

$$c_0 x + c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0.$$

We can assume that $c_0 \neq 0$ (since otherwise $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$, contradicting their linear independence). Dividing through by c_0, we can write:

$$x = -\frac{c_1}{c_0} v_1 - \frac{c_2}{c_0} v_2 - \cdots - \frac{c_k}{c_0} v_k.$$

Define $\alpha_i = -\frac{c_i}{c_0}$ for $i = 1, 2, \ldots, m$. Thus, we have:

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k.$$

Now, we show uniqueness. Suppose there exist two representations:

$$x = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k,$$

and

$$x = \gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_k v_k.$$

Subtracting these equations gives:

$$0 = (\beta_1 - \gamma_1)v_1 + (\beta_2 - \gamma_2)v_2 + \cdots + (\beta_k - \gamma_k)v_k.$$

Since $\{v_1, v_2, \ldots, v_k\}$ is linearly independent, it follows that:

$$\beta_1 - \gamma_1 = 0, \quad \beta_2 - \gamma_2 = 0, \quad \ldots, \quad \beta_k - \gamma_k = 0.$$

Thus, $\beta_i = \gamma_i$ for all i, proving that the coefficients are unique.

Conclusion: There exist unique coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that:

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k.$$

$\blacksquare$

In Chapter 3, we will compute the area under a monomial, x^k, use Archimedes' Approx. Principle.

We will care about $\sum_{i=1}^{n} (i)^k$

Humanity has already computed this so that we don't have to!

Observation 2.20: Ratios of Polynomials of n as n tends to Infinity

When approximating our areas with a sum of n rectangles, we were faced with terms of the form

- Triangle ($f(x) = x$): $\frac{1+2+\cdots+n}{n^2} = \frac{an^2+bn}{n^2}$
- Parabola ($f(x) = x^2$): $\frac{1^2+2^2+\cdots+n^2}{n^3} = \frac{an^3+bn^2+cn}{n^3}$

and we can imagine that for the area under x^k, $k > 2$, we'd be faced with a term of the form

$$\frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \frac{an^{k+1} + bn^k + \cdots + \text{ lower powers of } n}{n^{k+1}}.$$

In each case, we want to let n become large so that our scheme of under- and over-approximations through adding up rectangles becomes more accurate. In the limit, we'd like to take an "unbounded" number of "infinitely thin" rectangles, which brings us to the notion of "limits at infinity", the topic of Chapters 2.5 and 2.6. Before we do that, let's address sums of powers of integers for arbitrary $k \geq 1$.

Proposition 2.21: Power Sums

For all $n \geq 1$ and $k \geq 1$, the following holds,

$$\begin{aligned}\sum_{i=1}^{n} i^k &= 1^k + 2^k + 3^k + \cdots + n^k \\ &= \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + O(n, k-1),\end{aligned} \tag{2.25}$$

where $O(n, k-1)$ stands for a polynomial in the variable n with degree less than or equal to $k-1$ (i.e., all of the lower-order terms after n^k).

Remark 2.22. *If you want to see the explicit values of the sums, here are the first few:*

$$\begin{aligned}\sum_{i=1}^{n} i &= \frac{1}{2}(n^2+n) = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2} \\ \sum_{i=1}^{n} i^2 &= \frac{1}{6}(2n^3+3n^2+n) = \frac{n^3}{3} + \frac{n^2}{2} + O(n,1) \\ \sum_{i=1}^{n} i^3 &= \frac{1}{4}(n^4+2n^3+n^2) = \frac{n^4}{4} + \frac{n^3}{2} + O(n,2) \\ \sum_{i=1}^{n} i^4 &= \frac{1}{30}(6n^5+15n^4+10n^3-n) = \frac{n^5}{5} + \frac{n^4}{2} + O(n,3) \\ \sum_{i=1}^{n} i^5 &= \frac{1}{12}(2n^6+6n^5+5n^4-n^2) = \frac{n^6}{6} + \frac{n^5}{2} + O(n,4).\end{aligned} \tag{2.26}$$

You can find longer lists online. A general formula was given by Faulhaber.

Proofs separate the "Intuitively Obvious" from the Correct!

Georg Cantor ~ 1874 driven to a "mental health institution" by his peers.

Def. The number of elements in a set is called its cardinality, denoted card(A).

Examples:

(i) $A = \{e, \sqrt{2}, \pi\} \Rightarrow \text{card}(A) = 3$

(ii) $\mathbb{N} = \{1, 2, 3, 4, \ldots\} \Rightarrow \text{card}(\mathbb{N}) = \infty$

(iii) $\mathbb{Q}$ = Rational numbers $\Rightarrow \text{card}(\mathbb{Q}) = \infty$

(iv) $\mathbb{R}$ = real numbers $\Rightarrow \text{card}(\mathbb{R}) = \infty$

Provocative questions: Are there more rational numbers than counting numbers? Are there more real numbers than rational numbers?

Countable (aka, listable) sets

Def.

(a) Set A is finitely countable if its elements can be enumerated as

$$A = \{a_1, a_2, \ldots, a_n\} \text{ for } n \text{ finite.}$$

(b) Set A is infinitely countable if its elements can be enumerated as

$$A = \{a_k \mid k \in \mathbb{N}\} = \{a_1, a_2, a_3, \ldots\}$$

that is, its elements can be placed in one-to-one correspondence with the counting numbers.

(c) Otherwise, A is said to be uncountable (aka, "un-listable").

Examples

(i) $A_1 = \{\underset{a_1}{2}, 4, 8, \underset{a_4}{16}\}$ $\therefore$ finitely countable

(ii) $A_2 = \{\text{odd counting numbers}\}$

$= \{a_k = 2k-1 \mid k \in \mathbb{N}\} = \{1, 3, 5, 7, \cdots\}$

(iii) Same for $A_3 = \{\text{even counting num.}\}$

$= \{a_k := 2k \mid k \in \mathbb{N}\}$

Question: Do uncountable sets exist?

Consider $S := \{(i,j) \mid i \in \mathbb{N}, j \in \mathbb{N}\}$

all pairs of counting numbers.

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	⋯
1	1/1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10	⋯
2	2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	2/9	2/10	⋯
3	3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	3/9	3/10	⋯
4	4/1	4/2	4/3	4/4	4/5	4/6	4/7	4/8	4/9	4/10	⋯
5	5/1	5/2	5/3	5/4	5/5	5/6	5/7	5/8	5/9	5/10	⋯
6	6/1	6/2	6/3	6/4	6/5	6/6	6/7	6/8	6/9	6/10	⋯
7	7/1	7/2	7/3	7/4	7/5	7/6	7/7	7/8	7/9	7/10	⋯
8	8/1	8/2	8/3	8/4	8/5	8/6	8/7	8/8	8/9	8/10	⋯
9	9/1	9/2	9/3	9/4	9/5	9/6	9/7	9/8	9/9	9/10	⋯
10	10/1	10/2	10/3	10/4	10/5	10/6	10/7	10/8	10/9	10/10	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

The positive rational numbers

$\mathbb{Q}^+ := \{ \frac{p}{q} \mid p \in \mathbb{N}, q \in \mathbb{N}$ no common factors$\}$ are a subset of S. Hence, if S is countable so is $\mathbb{Q}^+$!!!

Remark: Every rational number appears infinitely often in S!

Example: $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots, \frac{k}{2k}, \ldots \in S$.

Hence, S seems to be HUGE!

$i \backslash j$	1	2	3	4	5	6	7	8	9	10
1	1	3	6	10	15	21	28	36	45	55
2	2	5	9	14	20	27	35	44	54	65
3	4	8	13	19	26	34	43	53	64	76
4	7	12	18	25	33	42	52	63	75	88
5	11	17	24	32	41	51	62	74	87	101
6	16	23	31	40	50	61	73	86	100	115
7	22	30	39	49	60	72	85	99	114	130
8	29	38	48	59	71	84	98	113	129	146
9	37	47	58	70	83	97	112	128	145	163
10	46	57	69	82	96	111	127	144	162	181

$f: S \to \mathbb{N}$ by $f(i,j) = \frac{(i+j)(i+j-1)}{2} + j$

The textbook shows that $\mathbb{R}$ is uncountable!