

Summary:Linear drag model shuttle

$$m \dot{v} = F = -kv, v(t_0) = v_0$$

$$v(t) = v_0 e^{-\frac{k}{m}(t-t_0)}$$

Nonlinear drag model shuttle

$$m \dot{v} = F = -kv^2, v(t_0) = v_0$$

$$v(t) = \frac{v_0}{1 + \frac{v_0 \cdot k}{m}(t-t_0)}$$

- $v(t)$  satisfies the I.C.
- $v(t)$  satisfies the ODE



Figure 9.2: Using a chute as a brake. (a) A drag chute used for braking the Space Shuttle Endeavor; image complements of Alamy and (b) a parachutist; image complements of pixabay.

$$\dot{x} + \alpha x = 0, x(t_0) = x_0$$

$$x(t) = x_0 e^{-\alpha(t-t_0)}$$

Separation of Variables

$$\dot{x} + f(x) = 0, x(t_0) = x_0$$

$$\frac{dx}{dt} = -f(x) \quad \boxed{\frac{dx}{f(x)} = -dt}$$

$$\int_{x_0}^{x(t)} \frac{dx}{f(x)} = \int_{t_0}^t -dt = -t \Big|_{t_0}^t = -(t-t_0)$$

③ Today

Video: Spontaneous motion!

Video: ODE Vocabulary

Hidden Figures

③ Parachutist with linear drag

$$m \dot{v} = F = -mg - kv(t), v(t_0) = v_0$$

Solution:  $v(t) = v_0 e^{-\frac{kv}{m}(t-t_0)} - \frac{mg}{k} (1 - e^{-\frac{kv}{m}(t-t_0)})$

initial condition response "Forced" Response

Method: Integrating Factors: Does not out very often, so we skip it.

Detailed in the book. Later, will solve using Laplace transforms.

④ Parachutist with a nonlinear drag model

$$m\ddot{v} = F = -mg - k(v)\cdot v \\ = -mg + k v^2 \quad , \quad v(t_0) = v_0$$

No closed-form solution can be given using Elementary Functions

Numerical Solutions are needed!

Notation For Vector of 1<sup>st</sup>-order

ODEs

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ indep variable} = t, \text{ time}$$

$$\begin{aligned}\dot{x}_1 &= f_1(t, x_1, x_2, \dots, x_n) \\ \dot{x}_2 &= f_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, x_2, \dots, x_n)\end{aligned}$$

$$\begin{aligned}x_1(t_0) &= x_{0,1} \\ x_2(t_0) &= x_{0,2} \\ &\vdots \\ x_n(t_0) &= x_{0,n}\end{aligned}$$

Write as a vector

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix}, \quad \mathbf{f}(t, \mathbf{x}) = \begin{bmatrix} f_1(t, x_1, \dots, x_n) \\ \vdots \\ f_n(t, x_1, \dots, x_n) \end{bmatrix}$$

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$

↑ P

Where do these  $x_i$  come from?

Back to pendulum driven by a DC

motor

$$\ddot{\theta} + a_2 \ddot{\theta} + a_1 \dot{\theta} + g_b \sin(\theta) = b_0 \underbrace{V(t)}_{\substack{\text{Voltage} \\ \text{applied to} \\ \text{motor}}}$$

$\dot{x}_3 \downarrow \quad \dot{x}_2 \downarrow \quad \dot{x} \downarrow$

$\ddot{\theta} + a_2 \ddot{\theta} + a_1 \dot{\theta} + g_b \sin(\theta) = b_0 V(t)$

$x_4?$  Nol

$$\left. \begin{array}{l} x_1 = \theta \\ x_2 = \dot{\theta} \\ x_3 = \ddot{\theta} \end{array} \right\} \text{state variables}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = \ddot{\theta} = -\alpha_2 \theta - \alpha_1 \dot{\theta} - q_0 \sin(\theta) + b_0 V(t)$$

$$= -\alpha_2 x_3 - \alpha_1 x_2 - q_0 \sin(x_1) + b_0 V(t)$$

$$\dot{x} = f(t, x) = \begin{bmatrix} x_2 \\ x_3 \\ -q_0 \sin(x_1) - \alpha_1 x_2 - \alpha_2 x_3 + b_0 V(t) \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$$

the "t" in  $f(t, x)$  comes from  $V(t)$  the applied voltage.



How to Numerically Solve  
an ODE if you do not have  
access to modern packages?

Euler's Method (for solving ODEs)

$$\dot{x} = f(t, x) \quad , \quad x(t_0) = x_0$$

$$\dot{x}(t) \approx \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} \approx f(t, x)$$

$$x(t+\Delta t) \approx x(t) + f(t, x) \cdot \Delta t$$

$$t_{i+1} = t_i + \Delta t$$

$$x(t_{i+1}) = x(t_i) + f(t_i, x(t_i)) \cdot \Delta t$$

## Interesting Properties of Nonlinear ODEs

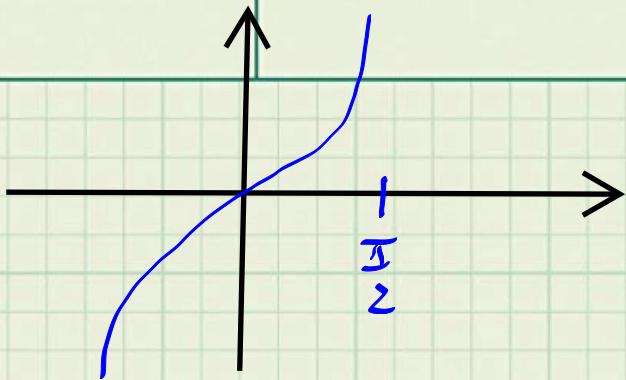
Finite Escape Time: (solution explodes to  $\infty$  in finite time)

$$\frac{dx(t)}{dt} = 1 + x^2(t), \quad t_0 = 0, \quad x_0 = 0$$

$$\frac{dx}{1+x^2} = dt$$

separation of variables

$$x(t) = \tan(t)$$



$$\lim_{t \rightarrow \frac{\pi}{2}^-} x(t) = \infty$$

## Multiple Solutions

$$\dot{x} = x^{\frac{2}{3}} = \sqrt[3]{x^2} \quad x(0) = 0$$

Fact: For all constants  $c > 0$ ,

$$\varphi_c(t) := \begin{cases} 0 & \text{if } 0 \leq t < c \\ \frac{(t-c)^3}{27} & \text{if } t \geq c \end{cases}$$

is a solution of the ODE.

## Famous ODEs with Known Solutions:

In the following,  $x$  is the independent variable and not  $t$ . This is fairly standard in Math.

Hence,  $y' = \frac{dy}{dx}$  and  $y'' = \frac{d^2y}{dx^2}$ , etc.

### 1. Bernoulli Equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

*Solution Method:* Use substitution  $u = y^{1-n}$ .

### 2. Riccati Equation (special cases)

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x)$$

*Solution Method:* Generally unsolvable; special cases solvable when a particular solution is known or via special substitutions.

### 3. Separable Equations

$$\frac{dy}{dx} = f(x)g(y)$$

*Solution Method:* Direct integration:

$$\int \frac{dy}{g(y)} = \int f(x) dx$$

### 4. Homogeneous Nonlinear Equations

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

*Solution Method:* Substitution  $u = \frac{y}{x}$ .

## 5. Autonomous Equations (First-order nonlinear)

$$\frac{dy}{dx} = f(y)$$

*Solution Method:* Direct integration:

$$\int \frac{dy}{f(y)} = x + C$$

## 6. Logistic Equation (Verhulst Equation)

$$\frac{dy}{dt} = r y \left(1 - \frac{y}{K}\right)$$

*Closed-form solution:*

$$y(t) = \frac{K}{1 + Ce^{-rt}}, \quad C = \frac{K - y_0}{y_0}$$

## 7. Gompertz Equation

$$\frac{dy}{dt} = r y \ln\left(\frac{K}{y}\right)$$

*Closed-form solution:*

$$y(t) = K e^{(\ln(y_0/K))e^{-rt}}$$

## 8. Nonlinear Pendulum (Exact solution via elliptic integrals)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0$$

*Closed-form solution* (in terms of Jacobi elliptic functions):

$$\theta(t) = 2 \arcsin \left[ k \operatorname{sn} \left( \sqrt{\frac{g}{L}} t; k \right) \right]$$

## 9. Painlevé Equations (select special solutions)

Certain special solutions of Painlevé equations have closed-form solutions expressed in

terms of classical special functions.

## 10. Lane-Emden Equation (special cases)

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) + y^n = 0$$

Special cases with known closed-form solutions include  $n = 0, 1, 5$ .

## 11. Abel's Equation (special cases)

$$\frac{dy}{dx} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x)$$

Closed-form solutions known in special simplified forms.

## 12. Euler's Equation (Nonlinear forms; see clarification below)

$$x^2 y'' + a x y' + b y = 0$$

*Solution Method:* Often solvable using power-law substitutions  $y = x^m$ .

**Question:** When you get to Euler's equation, explain why it is nonlinear. It looks linear time-varying to me.

**Clarification:** You are correct to question this. The standard Euler–Cauchy (also known as Equidimensional) equation is given by:

$$x^2 y'' + a x y' + b y = 0$$

This equation is indeed *linear and time-varying*, since it satisfies linearity conditions in terms of the unknown function  $y$  and its derivatives. Each term is a linear function of  $y$ ,  $y'$ , or  $y''$  with variable coefficients depending only on the independent variable  $x$ .

The confusion arises because the term “Euler equations” can refer to multiple equations in mathematics and physics:

- **Euler–Cauchy equations** (shown above): Linear ODEs with polynomial coefficients.
- **Euler equations from mechanics and fluid dynamics**: Nonlinear equations describing rigid-body rotation or fluid motion.

The equation presented above is clearly linear and should *not* be listed among nonlinear equations. This distinction is important, and the earlier listing mistakenly conflated the two different types of Euler equations. For a strictly nonlinear list, this Euler–Cauchy equation should be omitted or explicitly identified as linear.