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MATH434

HW5

1.

a.

①

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

a)

Let $x = \tan\theta = \dfrac{\sin\theta}{\cos\theta} \Rightarrow \theta = \tan^{-1}x$

$$\Rightarrow dx = \frac{\cos\theta\cos\theta - \sin\theta(-\sin\theta)}{\cos^2\theta}\,d\theta = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\,d\theta = \frac{1}{\cos^2\theta}\,d\theta$$

$$\Rightarrow \int^1 \frac{1}{1+\frac{\sin^2\theta}{\cos^2\theta}} \cdot \frac{1}{\cos^2\theta}\,d\theta = \int^1 \frac{1}{\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}} \cdot \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int \frac{1}{1/\cos^2\theta} \cdot \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int 1\,d\theta = \theta$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1}(x)]_0^1 = \pi/4$$

b.

b)

Infinite geometric series

$$S = 1 + r + r^2 + \dots + r^k + \dots = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad |r| < 1$$

Consider: $$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^n x^{2n}$$

Hence, $$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^1 (1 - x^2 + x^4 - x^6 + x^8 + \dots)\,dx$$

$$\tan^{-1}(1) - \tan^{-1}(0) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots$$

$$\Rightarrow \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots$$

c.

c) From a) $\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$

b) $\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

$\Rightarrow \pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$

$= 4\left[\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)}\right]$

2.

a.

(2)

a) $f(x) = \begin{cases} 1 & , 0 < x < \pi/2, \\ 0 & , \pi/2 < |x| < \pi, \\ -1 & , -\pi/2 < x < 0. \end{cases}$

The Fourier series is given by

$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{L} nx\right) + b_n \sin\left(\frac{2\pi}{L} nx\right)$

Since L is the period of the function to be approximated

$\Rightarrow f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(4nx) + b_n \sin(4nx)$ Here, $L = \pi$

$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi n}{L} . x\right) dx$

$= \frac{2}{\pi} \int_{-\pi/2}^{\pi} f(x) \cos(2n x) dx$

$= \frac{2}{\pi}\left[\int_{-\pi/2}^{0} -1 \cos(2nx) dx + \int_0^{\pi/2} 1 . \cos(2nx) dx + \int_{\pi/2}^{\pi} 0 \cos(2nx) dx\right]$

$= \frac{2}{\pi}\left[-\frac{\sin(2nx)}{2n}\Big|_{-\frac{\pi}{2}}^{0} + \frac{\sin 2nx}{2n}\Big|_0^{\pi/2} + 0\right]$

$= \frac{2}{\pi}\left[0 + 0 + 0\right] = 0$

$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi n}{L} . x\right) dx$

$= \frac{2}{\pi} \int_{-\pi/2}^{\pi} f(x) \sin(2nx) dx$

$= \frac{2}{\pi}\left[\int_{-\pi/2}^{0} -1 \sin(2nx) dx + \int_0^{\pi/2} 1 \sin(2nx) dx + \int_{\pi/2}^{\pi} 0 \sin(2nx) dx\right]$

$$= \frac{2}{\pi}\left[-\frac{\cos(2nx)}{2n}\Big|_{-\pi/2}^{0} + \frac{\cos(2nx)}{2n}\Big|_{0}^{\pi/2} + 0 \right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2n} - \frac{(-1)^{-n}}{2n} + \frac{1}{2n} - \frac{(-1)^{n}}{2n} \right]$$

$$= \frac{2}{\pi}\left[\frac{2 - (-1)^n - (-1)^{-n}}{2n} \right]$$

$$= \frac{2 - (-1)^n - (-1)^{-n}}{n\pi}$$

$$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left[\int_{-\pi/2}^{0} -1\,dx + \int_{0}^{\pi/2} 1\,dx + \int_{\pi/2}^{\pi} 0\,dx \right]$$

$$= \frac{2}{\pi}\left[-x\Big|_{-\pi/2}^{0} + x\Big|_{0}^{\pi/2} \right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2} + \frac{\pi}{2} \right]$$

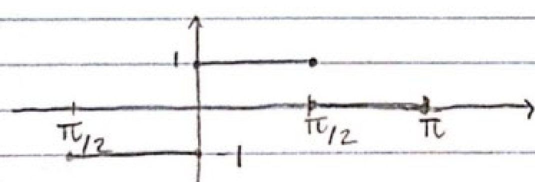
$$= 0$$

$\Rightarrow$ Fourier series: $f(x) = 0 + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left(2-(-1)^n-(-1)^{-n}\right)\sin(4nx)$

b.

$n\pi$

b)



The Fourier series converge to f when there is continuity. Here, $(-\pi/2 < x < \pi)$ on this interval the Fourier series is not converge.

c.

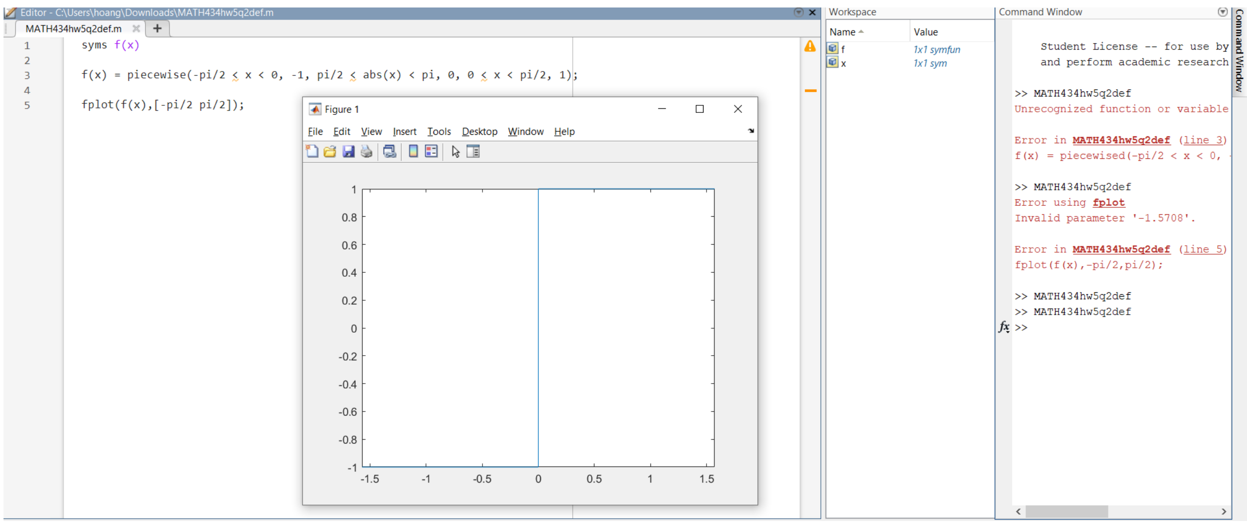
c)

Fourier series $f(x) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \left[2-(-1)^{n}-(-1)^{-n}\right] \sin(4nx)$

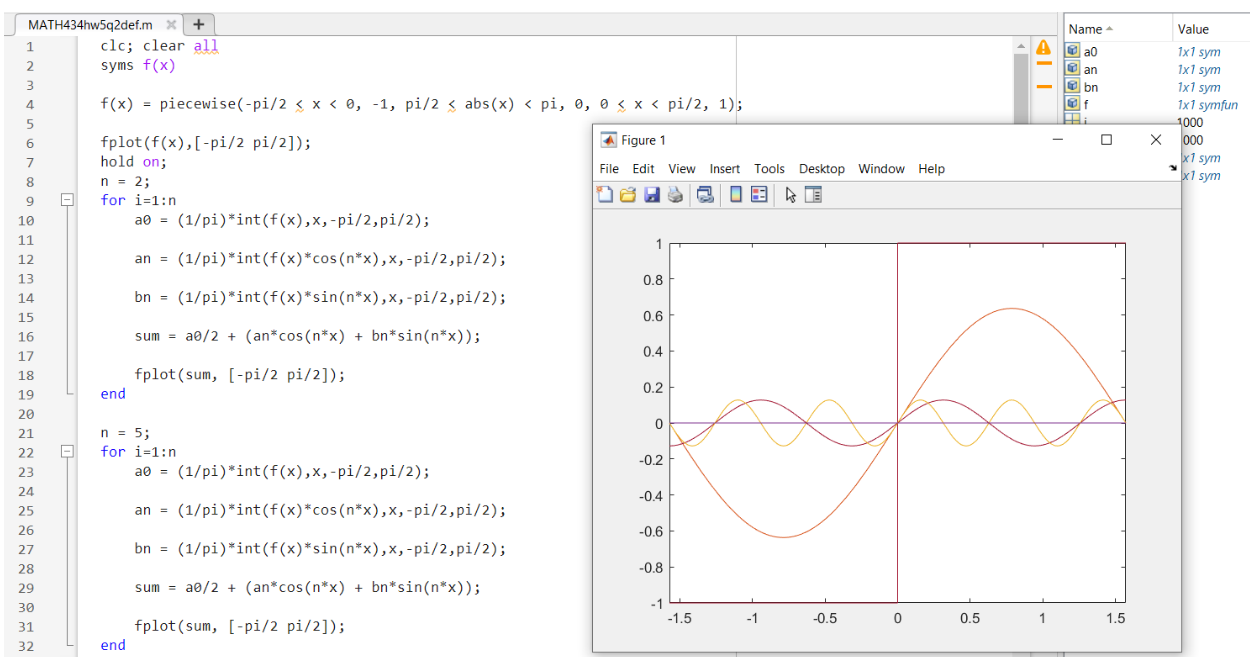
$$\Rightarrow f(x=\pi/2) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \left[2-(-1)^{n}-(-1)^{-n}\right] \underbrace{\sin(2\pi n)}_{0}$$

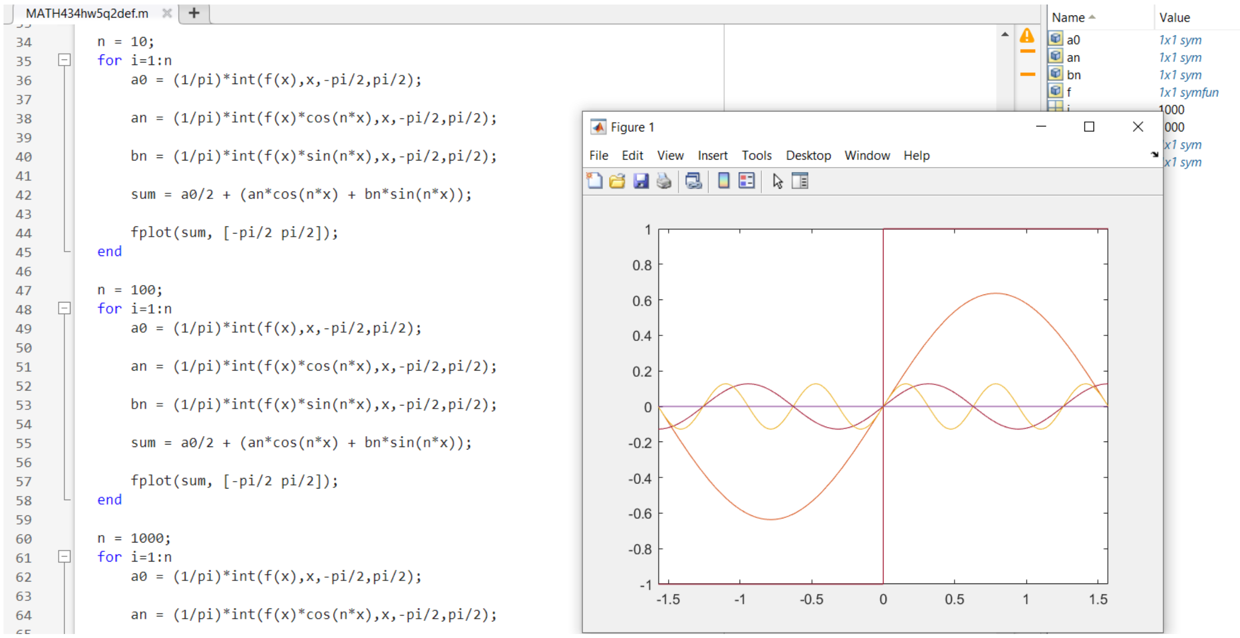
$$= 0$$

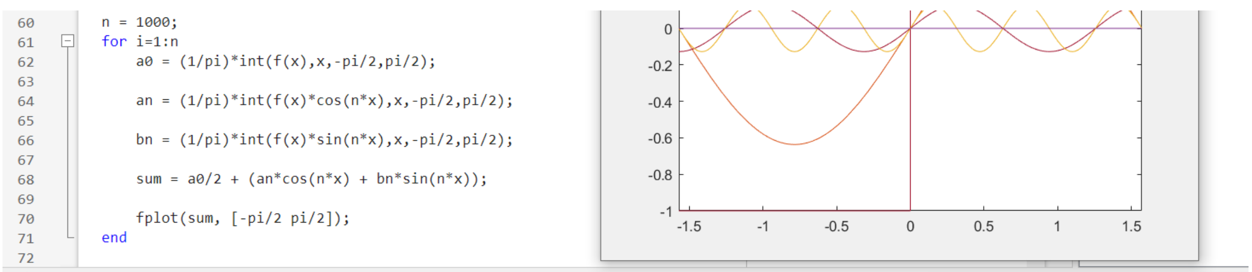
d.



e.







f.

As the n increase the Fourier series converge to 0.

I don't know it this you are looking for. I'm thinking you want to fit the Fourier series into the piecewise function. In that case, as n increase the Fourier series will get closer to the piecewise function, however, no matter how much it increases, there still appear the overshooting phenomenon.

3.

(3)

$f(x) = |\cos x| \quad ; -\pi < x < \pi$

The Fourier series is given by $(L = \pi)$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{L} nx\right) + b_n \sin\left(\frac{2\pi}{L} nx\right)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |\cos(x)| dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos(x) dx + \int_{-\pi/2}^{\pi/2} \cos(x) dx + \int_{\pi/2}^{\pi} -\cos(x) dx \right.$$

$$= \frac{1}{\pi}\left[[-\sin(x)]_{-\pi}^{-\pi/2} + [\sin(x)]_{-\pi/2}^{\pi/2} + [-\sin(x)]_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[1 + 2 + 1 \right]$$

$$= \frac{4}{\pi}$$

Since $f(x)$ is an even function, hence the term $b_n \to 0$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |\cos(x)| \cos(nx) dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} |\cos(x)| \cos(nx) dx$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi/2} 2\cos(nx)\cos(x) dx + \int_{\pi/2}^{\pi} 2.\cos(nx)[-\cos(x)] dx \right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi/2} [\cos(n+1)x + \cos(n-1)x] dx - \int_{\pi/2}^{\pi} [\cos(n+1)x + \cos(n-1)x] dx \right]$$

$$= \frac{1}{\pi}\left[\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1).\pi/2}{n+1} + \frac{\sin(n-1).\pi/2}{n-1} + \frac{\sin(n+1).\pi/2}{n+1} - \frac{\sin(n-1).\pi/2}{n-1} \right]$$

$$= \frac{2}{\pi}\left[\frac{\cos(n\pi/2)}{n+1} - \frac{\cos(n\pi/2)}{n-1} \right]$$

$$= \frac{-4}{\pi(n^2-1)} \cos\left(\frac{n\pi}{2}\right)$$

Now, $a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(x)dx$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos(x)|\cos(x)dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos^2(x)dx - \int_{\pi/2}^{\pi}\cos^2(x)dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2} - \left[x+\frac{\sin 2x}{2}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \frac{\pi}{2}\right]$$

$$= 0$$

=> Fourier series of $|\cos(x)|$

$$|\cos(x)| = \frac{a_0}{2} + a_1\cos(x) + \sum_{n=2}^{\infty} a_n\cos(nx)$$

$$= \frac{2}{\pi} + 0 - \frac{4}{\pi}\left[\frac{1}{2^2-1}\overset{\cos 2x}{\cos\pi} + \frac{1}{3^2-1}\overset{\cos 3x}{\cos\frac{3\pi}{2}} + \frac{1}{4^2-1}\overset{\cos 4x}{\cos\frac{4\pi}{2}} + \dots\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \dots\right]$$

The values of x where the Fourier series is converge to $f(x)$ is $[-\pi, \pi]$

4.

a.

(4)

Given: $\cosh x = \frac{1}{2}(e^{x} + e^{-x})$ "Hyberbolic function"

$f(x) = \cosh x$, $-\pi \le x \le \pi$, $f(x+2\pi) = f(x)$.

a)

Since $f(x)$ is a even function => $b_n = 0$

$$a_n = \frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\cosh(x)\cos(nx)\,dx$$

Apply I.B.P

$u = \cos(nx)$ | $dv = \cosh(x)$

$\Rightarrow du = -n\sin(nx)$ | $\Rightarrow v = \sinh(x)$

$$\Rightarrow \int \cosh(x)\cos(nx)\,dx = \cos(nx)\sinh(x) + n \underbrace{\int \sinh(x)\sin(nx)\,dx}_{\text{I.B.P}}$$

$u = \sin(nx)$ | $dv = \sinh(x)$

$\Rightarrow du = n\cos(nx)$ | $\Rightarrow v = \cosh(x)$

$$\Rightarrow \int \cosh(x)\cos(nx)\,dx = \cos(nx)\sinh(nx) + n\left[\sin(nx)\cosh(x) - n\int \cosh(x)\cos(nx)\,dx\right]$$

move to the LHS

$$= \cos(nx)\sinh(x) + n\sin(nx)\cosh(x) - \underbrace{n^2 \int \cosh(x)\cos(nx)\,dx}$$

$$\Rightarrow (1+n^2)\int \cosh(x)\cos(nx)\,dx = \cos(nx)\sinh(x) + n\sin(nx)\cosh(x)$$

$$\Rightarrow \int \cosh(x)\cos(nx)\,dx = \frac{1}{n^2+1}\left[\cos(nx)\sinh(x) + n\sin(nx)\cosh(x)\right]$$

$$\Rightarrow \int_0^{\pi} \cosh(x)\cos(nx)\,dx = \frac{2}{\pi}\cdot\frac{1}{(n^2+1)}\left[\cos(nx)\sinh(x) + n\sin(nx)\cosh(x)\right]_0^{\pi}$$

$$= \frac{2}{\pi(n^2+1)}\left[[\cos(\pi n)\sinh(\pi) + 0] - [0]\right]$$

$$\Rightarrow a_n = \frac{2}{\pi(n^2+1)}\cos(n\pi)\sinh(\pi)$$

Now,

$$\left.\begin{aligned} a_0 &= \frac{2}{\pi}\sinh(\pi) \\ a_1 &= -\frac{1}{\pi}\sinh(\pi) \\ a_2 &= \frac{2}{5\pi}\sinh(\pi) \end{aligned}\right\} \quad \cos(n\pi) = \begin{cases} +1, & 2n \\ -1, & 2n+1 \end{cases}$$

$\vdots$

$$a_n = \frac{2\sinh(\pi)}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{1}{n^2+1}$$

⇒ Fourier series of $\cosh(x)$ is

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \dots$$

$$= \frac{\frac{2\sinh(\pi)}{\pi}}{2} + \sum_{n=1}^{\infty}\left[\frac{2\sinh(\pi)}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{1}{n^2+1}\right]\cos(nx)$$

b.

b)

For any values of $x \in [-\pi, \pi]$ the Fourier series will converge.

c.

c)

Consider: $e^x = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

where,

$$c_n = \frac{\int_{-\pi}^{\pi} e^x e^{-inx}\,dx}{2\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-in)}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{x(1-in)}}{1-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(e^{\pi - i\pi} - e^{-\pi + i\pi n}\right)\left(\frac{1}{1-in}\right)$$

$$= \frac{1}{2\pi}\Big(e^{\pi}\underbrace{e^{-i\pi n}}_{\substack{(-1)^{-n} \\ = \\ (-1)^n}} - e^{-\pi}\underbrace{e^{i\pi n}}_{(-1)^n}\Big)\underbrace{\left(\frac{1}{1-i.n}\right)\left(\frac{1+i.n}{1+i.n}\right)}_{1+n^2}$$

$$= \frac{(-1)^n}{\pi}\left(\frac{e^{\pi} - e^{\pi}}{2}\right)\left(\frac{1+i.n}{1+n^2}\right)$$

$$= \frac{(-1)^n}{\pi}\sinh(\pi)\left(\frac{1+i.n}{1+n^2}\right)$$

$$\Rightarrow |c_n|^2 = \frac{1}{\pi^2}\sinh^2(\pi)\frac{(\sqrt{1+n^2})^2}{(1+n^2)^2}$$

$$= \frac{1}{\pi^2}\sinh^2(\pi)\left(\frac{1}{1+n^2}\right)$$

"Parseval thm"
connect l_2-norm to L_2-norm
$$\sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{\sinh^2(\pi)}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{2x}\,dx$$

$$= \frac{1}{2\pi}\left(\frac{e^{2\pi} - e^{-2\pi}}{2}\right)$$

$$= \frac{1}{2\pi} \sinh(2\pi)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = \frac{\pi}{2} \frac{\sinh(\pi)}{\sinh^2(\pi)}$$

Note: $\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} + \frac{1}{1+0^2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2}$

$=$

$$\Rightarrow 2\sum_{n=1}^{\infty} \frac{1}{1+n^2} + 1 = \frac{\pi}{2} \frac{\sinh(2\pi)}{\sinh^2(\pi)}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}\left[\frac{\pi}{2} \frac{\sinh(2\pi)}{\sinh^2(\pi)} - 1\right]$$

5.

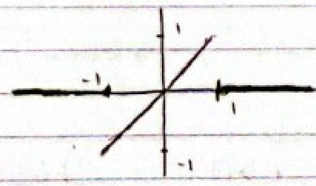
a.

(5)

$$f(x) = \begin{cases} x & , \text{ if } |x| < 1 \\ 0 & , \text{ o.w} \end{cases}$$

a)

Plot



Fourier transform

$$\mathcal{F}[f](\omega) = \hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt$$

$$= \frac{1}{2\pi}\left[\underbrace{\int_{-1}^{0} x e^{i\omega x} dx}_{\text{I.B.P}} + \int_{0}^{1} 0 e^{i\omega x} dx\right]$$

$u = x$, $\Rightarrow du = dx$ | $dv = e^{i\omega x} dx$, $\Rightarrow v = \frac{e^{i\omega x}}{i\omega}$

$$= \frac{1}{2\pi}\left[\frac{x e^{i\omega x}}{i\omega} - \int \frac{e^{i\omega x}}{i\omega} dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{x e^{i\omega x}}{i\omega} - \frac{1}{i\omega}\left[\frac{e^{i\omega x}}{i\omega}\right]\right]_{-1}^{0}$$

$$= \frac{1}{2\pi}\left[0 - \frac{1}{(i\omega)^2} - \left[\frac{-e^{-i\omega}}{i\omega} - \frac{1}{i\omega}\left[\frac{e^{-i\omega}}{i\omega}\right]\right]\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{\omega^2} - \left[\frac{(-e^{-i\omega})}{i\omega} + \frac{e^{-i\omega}}{\omega^2}\right]\right]$$

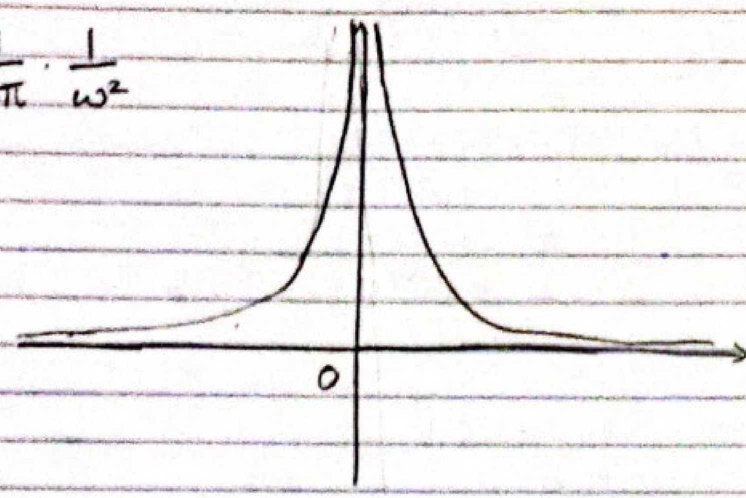
$$= \frac{1}{2\pi}\left[\frac{1}{\omega^2} + \frac{e^{-i\omega}}{i\omega} - \frac{e^{-i\omega}}{\omega^2} \right]$$

$$= \frac{1}{2\pi}\left[\frac{1 - e^{-i\omega}}{\omega^2} + \frac{e^{-i\omega}}{i\omega} \right]_{a}$$

b.

b)

$$|\hat{f}| = \frac{1}{2\pi} \cdot \frac{1}{\omega^2}$$



6.

(6)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ t > 0$$ "Heat eq"

$$u(x,0) = \begin{cases} 100, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Fourier transform

$$\mathcal{F}[f](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x}\, dx$$

$$= \frac{1}{2\pi} \int_{-1}^{0} 100\, e^{i\omega x}\, dx$$

$$= \frac{50}{\pi} \left[\frac{e^{i\omega x}}{i\omega} \right]_{-1}^{0}$$

$$= \frac{50}{\pi} \left[\frac{1}{i\omega} - \frac{e^{-i\omega}}{i\omega} \right]$$

$$= \frac{50}{\pi} \left[\frac{1 - e^{-i\omega}}{i\omega} \right]$$

7.

(7)

Proof: $F^{-1}\{F(x-\beta)\}(\omega) = e^{i\omega x} F(\omega)$

Consider: $F^{-1}\{F(x-\beta)\}(\omega) = \int_{-\infty}^{\infty} F(x-\beta) e^{i\omega x} dx$

$$= \int_{-\infty}^{\infty} F(x-\beta) e^{i\omega x} e^{-i\omega\beta} e^{i\omega\beta} dx$$

$$= e^{i\omega\beta} \int_{-\infty}^{\infty} F(x-\beta) e^{i\omega(x-\beta)} dx$$

Let $u = x - \beta \Rightarrow dw = dx$

$$\Rightarrow F^{-1}\{F(x-\beta)\}(\omega) = e^{i\omega\beta} \int_{-\infty}^{\infty} F(u) e^{i\omega u} du$$

$$= e^{i\omega\beta} f(\omega)$$

8.

(8)

Consider: $g(x) = \sin(ax)$, $a > 0$

f is an even function on $\mathbb{R}$

Hence,

$$\int_{-\infty}^{\infty} f(x)\sin(ax)dx = \int_{-\infty}^{\infty} f(-x)\sin(-ax)dx$$

$$= -\int_{-\infty}^{\infty} f(x)\sin(ax)dx$$

$$= 0$$

Now, $\hat{f}(a) = \int_{-\infty}^{\infty} f(x) e^{-iax} dx$

"Euler"

$$= \int_{-\infty}^{\infty} f(x)\cos(ax)dx - i\underbrace{\int_{-\infty}^{\infty} f(x)\sin(ax)dx}_{0}$$

$$= \int_{-\infty}^{\infty} f(x)\cos(ax)dx$$

Then, $f * g(x) = \int_{-\infty}^{\infty} f(t) \cdot g(x-t)dt$

$$= \int_{-\infty}^{\infty} f(t) \cdot \sin[a(x-t)]dt$$

$$= \int_{-\infty}^{\infty} f(t)[\sin(ax)\cos(at) - \cos(ax)\sin(at)]dt$$

$$= 2\pi \sin(ax) \int_{-\infty}^{\infty} f(t)\cos(at)\,dt - \cos(ax) \int_{-\infty}^{\infty} f(t)\sin(at)\,dt$$

$$= 2\pi \sin(ax) \cdot \hat{f}(a) - \cos(ax) \times 0$$

$$\Rightarrow f * g(x) = 2\pi \sin(ax) \cdot \hat{f}(a)$$