1. (1 point) Please identify anyone, whether or not they are in the class, with whom you discussed your homework. This problem is worth 1 point, but on a multiplicative scale (multiplied by the rest). Collaboration is encouraged.

2. (5 points) Consider the quadratic function $f : \mathbb{R}^2 \to \mathbb{R}$, for

$$\boldsymbol{x} = (x_1, x_2)^\top, \qquad f(\boldsymbol{x}) = x_1^2/6 + x_2^2/4 + x_1 x_2.$$

   (a) Write $f(\boldsymbol{x}) = \boldsymbol{x}^\top A \boldsymbol{x}$.

   (b) Compute ∇f and $\nabla^2 f$.

3. (5 points) Consider the function $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\boldsymbol{x}) = \left(\boldsymbol{a}^\top \boldsymbol{x}\right)\left(\boldsymbol{b}^\top \boldsymbol{x}\right)$$

   where $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top$ are n-dimensional vectors.

   (a) Find ∇f.

   (b) Find $\nabla^2 f$.

4. (5 points) Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\top \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 6$$

   (a) Find the gradient and Hessian of f at $\boldsymbol{x} = (1, 1)^\top$.

   (b) Find the directional derivative of f at $\boldsymbol{x} = (1, 1)^\top$ with respect to a unit vector in the direction of maximum rate of increase.

   (c) Find a point that satisfies the FONC for f. Does this point satisfy the SONC?

5. (5 points) Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \begin{bmatrix} 2 & 5 \\ -1 & 1 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\top \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7$$

   (a) Find the directional derivative of f at $\boldsymbol{x} = (0, 1)^\top$ in the direction of $(1, 0)^\top$.

   (b) Find all points that satisfy the FONC for f. Does f have a minimizer? If it does, then find all minimizer(s); otherwise, explain why it does not.

6. (5 points) (Optional) *Linear regression in* $\mathbb{R}^2$. Let $\{(x_i, y_i)^\top\}_{i=1}^n$ $(n > 2)$ be a set of points in $\mathbb{R}^2$. To find the straight line of "best fit" through these points ("best" in the sense that the average squared error is minimized); that is, find $a, b \in \mathbb{R}$ to minimize:

$$f(a,b) = \frac{1}{n}\sum_{i=1}^{n}\left(ax_i + b - y_i\right)^2.$$

   (a) Let

$$\begin{aligned}
\overline{X} &= \frac{1}{n}\sum_{i=1}^{n} x_i, \\
\overline{Y} &= \frac{1}{n}\sum_{i=1}^{n} y_i, \\
\overline{X^2} &= \frac{1}{n}\sum_{i=1}^{n} x_i^2, \\
\overline{Y^2} &= \frac{1}{n}\sum_{i=1}^{n} y_i^2, \\
\overline{XY} &= \frac{1}{n}\sum_{i=1}^{n} x_i y_i,
\end{aligned}$$

   show that $f(a,b)$ can be written in the form of $\boldsymbol{z}^\top Q\boldsymbol{z} - 2\boldsymbol{c}^\top\boldsymbol{z} + d$ where $\boldsymbol{z} = (a,b)^\top$, $Q \in \mathbb{R}^{2\times 2}$ is a symmetric matrix, $\boldsymbol{c} \in \mathbb{R}^2$, and $d \in \mathbb{R}$. Find the expression for Q, $\boldsymbol{c}$, and d in terms of $\overline{X}, \overline{Y}, \overline{X^2}, \overline{Y^2}$ and $\overline{XY}$.

   (b) Assume that the x_i, $i = 1, \ldots, n$ are not all equal. Find the minimizer $(a^*, b^*)^\top$ of $f(a,b)$ in terms of $\overline{X}, \overline{Y}, \overline{X^2}, \overline{Y^2}$ and $\overline{XY}$. Show that $(a^*, b^*)^\top$ is the only local minimizer of $f(a,b)$. *Hint*: $\overline{X^2} - (\overline{X})^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2$.