Socially Optimal Non-Discriminatory Restrictions for Continuous-Action Games (Appendix)

Main Track, Paper ID: 6770

A Equilibrium Oracle for Quadratic Utilities

For a quadratic utility function u and a restriction R, the best response $\mathcal{B}_1|_R(x_2)$ can be found by a straight-forward case analysis: Let u be defined as

$$u(x_1, x_2) = ax_1^2 + bx_2^2 + cx_1x_2 + dx_1 + ex_2 + f \ ,$$

and define five "candidate points" $x_l := \min_{x\in R} x$, $x_u := \max_{x\in R} x$, $x^* := \frac{cx_2+d}{-2a}$, $x_- := \max_{x\in R, x<x^*} x$, and $x_+ := \min_{x\in R, x>x^*} x$. Then

- if u is constant in x_1 (i.e., $a = 0$ and $cx_2 + d = 0$), $\mathcal{B}_1(x_2) = R$
- if u is linear in x_1 with positive slope (i.e., $a = 0$ and $cx_2 + d > 0$), $\mathcal{B}_1(x_2) = \{x_u\}$
- if u is linear in x_1 with negative slope (i.e., $a = 0$ and $cx_2 + d < 0$), $\mathcal{B}_1(x_2) = \{x_l\}$
- if u is convex in x_1 (i.e., $a > 0$), $\mathcal{B}_1(x_2) = \arg\max_{x\in\{x_l, x_u\}} u(x)$
- if u is concave in x_1 (i.e., $a < 0$) and $x^* \in R$, $\mathcal{B}_1(x_2) = \{x^*\}$
- if u is concave in x_1 (i.e., $a < 0$) and $x^* \notin R$, $\mathcal{B}_1(x_2) = \arg\max_{x\in\{x_-, x_+\}} u(x)$.

Note that $\mathcal{B}_1(x_2)$ is not necessarily unique (or even a finite set). To find the NE, observe that the unrestricted best response functions $\mathcal{B}_1(x_2) = -\frac{c_1x_2+d_1}{2a_1}$ and $\mathcal{B}_2(x_1) = -\frac{c_2x_1+e_2}{2b_2}$ lead to the unique unrestricted NE

$$\boldsymbol{x}^* = (x_1^*, x_2^*) = \left(\frac{c_1e_2 - 2d_1b_2}{4a_1b_2 - c_1c_2}, \frac{c_2d_1 - 2e_2a_1}{4a_1b_2 - c_1c_2}\right) \ .$$

If this point exists and is allowed by R, i.e., $4a_1b_2 - c_1c_2 \neq 0$ and $\boldsymbol{x}^* \in R^2$, then $\mathcal{N}|_R = \{\boldsymbol{x}^*\}$. Otherwise, we use fictitious play (i.e., successive mutual best responses) to find the fixed points, repeatedly calling the restricted best response functions while maintaining a list of candidate solutions.

B Expected Results for the Cournot Game

For the (unrestricted) PCG, the unique best responses are $\mathcal{B}_1(q_2) = \frac{\lambda - q_2}{2}$ and $\mathcal{B}_2(q_1) = \frac{\lambda - q_1}{2}$. Therefore, we get

$$\begin{aligned} D(\boldsymbol{q}) &= \left(q_1 - \frac{\lambda - q_2}{2}\right)^2 + \left(q_2 - \frac{\lambda - q_1}{2}\right)^2 \\ &= \frac{5}{4}(a^2 + b^2) + 2ab - \frac{3}{2}\lambda(a + b) + \frac{1}{2}\lambda^2 \end{aligned}$$

which has a unique global minimum $\boldsymbol{q}^* = (\frac{\lambda}{3}, \frac{\lambda}{3})$ with $D(\boldsymbol{q}^*) = 0$.

If we allow interval union restrictions for the PCG, best responses are not unique anymore, but still follow a simple pattern: If the unrestricted best response q^* is not part of an allowed interval, the restricted best responses are the closest allowed actions on either one or both sides of q^*.

More formally: Let $q^* := \frac{\lambda}{3}$ be the unrestricted optimal quantity, and define, for a given restriction $R \subseteq [0, \lambda]$, the two closest allowed quantities $q^+ := \min_{q\in R}(\{q > q^*\})$ and $q^- := \max_{q\in R}(\{q < q^*\})$. Setting $\Delta^+ := q^+ - q^*$ and $\Delta^- := q^* - q^-$, the Nash Equilibria $\mathcal{N}|_R$ of the restricted PCG are:

$$\mathcal{N}|_R = \begin{cases} \{(q^+, q^+)\} & \text{if } \Delta^+ < \frac{1}{2}\Delta^- \\ \{(q^+, q^+), (q^+, q^-), (q^-, q^+)\} & \text{if } \Delta^+ = \frac{1}{2}\Delta^- \\ \{(q^+, q^-), (q^-, q^+)\} & \text{if } \frac{1}{2}\Delta^- < \Delta^+ < 2\Delta^- \\ \{(q^-, q^-), (q^+, q^-), (q^-, q^+)\} & \text{if } \Delta^+ = 2\Delta^- \\ \{(q^-, q^-)\} & \text{if } \Delta^+ > 2\Delta^- \end{cases} .$$

This suggests the following sequence of successive restrictions for the SOAR algorithm:

- Identify $\frac{\lambda}{3}$ as the unique relevant action of the unrestricted game and therefore exclude $\overline{R} := \left[\frac{\lambda}{3} - \epsilon, \frac{\lambda}{3} + \epsilon\right)$ from the action space
- Identify both boundary actions as relevant and exclude one of them, increasing the excluded region $\overline{R}$ around $\frac{\lambda}{3}$
- Whenever $\overline{R}$ becomes imbalanced by a factor of > 2 around $\frac{\lambda}{3}$, a symmetric equilibrium appears at one end of it
- Finally, $\overline{R}$ is large enough to produce the symmetric equilibrium $(\frac{\lambda}{4}, \frac{\lambda}{4})$
- This occurs when $\overline{R} = [\frac{\lambda}{4}, \frac{\lambda}{2})$, and therefore $R = [0, \frac{\lambda}{4}) \cup [\frac{\lambda}{2}, \lambda)$

- The algorithm goes on to enlarge $\overline{R}$ until the set of allowed actions becomes empty
- Since no further restriction produces a socially better stable solution, the largest (i.e., least restrictive) R with $(\frac{\lambda}{4}, \frac{\lambda}{4}) \in \mathcal{N}|_R$ is finally returned as the optimal restriction R^*
- The resulting degree of restriction is $\mathfrak{r}(R^*) = 25\%$

The optimal restriction R^* has the unique equilibrium $(\frac{\lambda}{4}, \frac{\lambda}{4})$ which gives the SSU $\mathcal{S}(R^*) = \mathfrak{u}(\frac{\lambda}{4}, \frac{\lambda}{4}) = \frac{1}{4}\lambda^2$. In contrast, the unrestricted game produces a unique equilibrium of $(\frac{\lambda}{3}, \frac{\lambda}{3})$, such that $\mathcal{S}(A) = \mathfrak{u}(\frac{\lambda}{3}, \frac{\lambda}{3}) = \frac{2}{9}\lambda^2$. The resulting relative improvement is $\Delta_{rel} = \frac{1}{8}$.

C Number of Oracle Calls in the Cournot Game

To show that the number of oracle calls for SOAR's solution of the Cournot Game grows quadratically rather than exponentially, let us fit the two curves $f_1(\lambda) = ae^{b\lambda} + c$ and $f_2(\lambda) = a\lambda^2 + b\lambda + c$ to the data and check their deviation. Recall that the experimental data is $f(10) = 912$, $f(11) = 1095$, $f(12) = 1294$, $f(13) = 1513$, and so on (the full data set can be reproduced using the Colab notebook in the supplementary material).

As can be seen from Figure 1, the quadratic interpolation polynomial f_2 gives a close-to-perfect fit with parameters $a = 8.33$, $b = 8.00$, and $c = -1.45$. In contrast, the exponential fit with f_1 produces the degenerate parameter values $a = 0.00$, $b = 1.00$, and $c = 1.12 \cdot 10^{62}$.

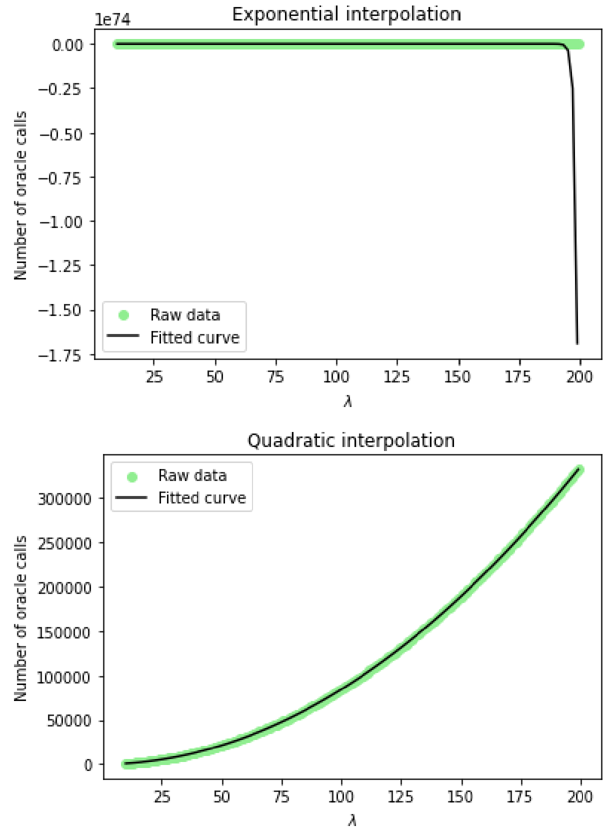

Figure 1: Exponential and quadratic interpolation of the number of oracle calls in the Cournot Game

D Continuous Braess' Paradox

In the original (discrete) version of Braess' Paradox (see Figure 1a in the main paper), each agent has three route options, of which they choose exactly one. The travel time from 0 to 3 is then used as their cost function (i.e., it is to be minimized).

When transforming this into a one-dimensional continuous NFG, we have to address two points: (a) There has to be a continuum of actions, and (b) we need utility functions instead of cost functions. Therefore, we define the action space as $A = [0, 1]$ and give it the following meaning: Agent 1 routes a flow of x_1 through route 0-1-2-3, and the remaining flow of $(1 - x_1)$ through route 0-2-3. Similarly, agent 2 routes a flow of x_2 through route 0-1-2-3, and the remaining flow of $(1 - x_2)$ through route 0-1-3. This means that, for both agents, 0 is the "cooperative" action, while 1 is the "competitive" action. The edge weights are adjusted such that full utilization (which is now a flow of 2 along an edge) gives the same travel time as utilization of 1 in the original setting (see Figure 2).

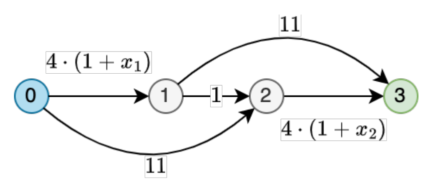

Figure 2: Continuous version of Braess' Paradox

We calculate the expected travel time $c_i(\boldsymbol{x})$ for both agents and subtract them from a virtual baseline of 32 in order to get the utility functions $u_i(\boldsymbol{x})$. The expected travel time along a route is the flow on the route, multiplied by the sum of the edge weights $w_i(\boldsymbol{x})$, given this flow. For agent 1, this calculation is:

$$\begin{aligned} c_1(\boldsymbol{x}) &= x_1(w_{01} + w_{12} + w_{23}) + (1 - x_1)(w_{02} + w_{23}) \\ &= x_1(w_{01} + w_{12}) + (1 - x_1) \cdot w_{02} + w_{23} \\ &= x_1(4(1 + x_1) + 1) + 11(1 - x_1) + 4(1 + x_2) \\ &= 4x_1^2 - 6x_1 + 4x_2 + 15 , \end{aligned}$$

and the corresponding utility function is

$$u_1(x_1, x_2) = -4x_1^2 + 6x_1 - 4x_2 + 17 .$$

In the same way, we get

$$u_2(x_1, x_2) = -4x_2^2 - 4x_1 + 6x_2 + 17 .$$

To generalize this setting, let us assume affine the weight functions $w_{\text{0-2}}(\boldsymbol{x}) = w_{\text{1-3}}(\boldsymbol{x}) = a(x_1 + x_2) + b$ and

$w_{0\text{-}1}(\boldsymbol{x}) = w_{2\text{-}3}(\boldsymbol{x}) = c(x_1 + x_2) + d$, while leaving the constant weight $w_{1\text{-}2}(\boldsymbol{x}) = 1$ unchanged. This gives the parameterized utility functions

$$u_1(\boldsymbol{x}) = -(a+c)x_1^2 + (2a+b-c-1)x_1 - cx_2 + (4c+d+1)$$

and

$$u_2(\boldsymbol{x}) = -(a+c)x_2^2 - cx_1 + (2a+b-c-1)x_2 + (4c+d+1)\ .$$

To obtain a one-dimensional range of experiments, we fix $a = 0$, $c = 4$ and $d = 0$ and vary b (intuitively, we vary the attractiveness of taking the social routes, compared to the selfish route). The parameterized utility functions $u_i(\boldsymbol{x})$ are therefore

$$u_1(\boldsymbol{x}) = -4x_1^2 + (b-5)x_1 - 4x_2 + 17$$

and

$$u_2(\boldsymbol{x}) = -4x_2^2 - 4x_1 + (b-5)x_2 + 17\ .$$

E Expected Results for the Braess Paradox

From u_1 and u_2 as defined above, we can immediately derive the best response functions $\mathcal{B}_i(x_j) = \frac{b-5}{8}$, resulting in $\mathcal{N} = \left\{\left(\frac{b-5}{8}, \frac{b-5}{8}\right)\right\}$. Moreover, since $\mathfrak{u}(\boldsymbol{x}) = u_1(\boldsymbol{x}) + u_2(\boldsymbol{x}) = -4x_1^2 - 4x_2^2 + (b-9)x_1 + (b-9)x_2 + 34$, we get the social optimum $\boldsymbol{x}^* = \left(\frac{b-9}{8}, \frac{b-9}{8}\right)$.

Finally, we conclude from $A = [0,1]$ that, for $b \leq 5$ and $b \geq 17$, $\mathcal{N} = \{\boldsymbol{x}^*\}$ such that the unrestricted and the restricted MESU are equal. For $b \in (5, 17)$, however, the two values differ, such that restricting A can improve the MESU.

Let us first assume that $b \in (5, 9]$. To make $\frac{b-9}{8}$ a best response for a player, we have to exclude any action from A that this player would prefer over $\frac{b-9}{8}$. It is easy to see that the range of actions that needs to be excluded is $(0, \frac{b-5}{4})$, giving the unique optimal restriction $R^* = \{0\} \cup [\frac{b-5}{4}, 1]$.

For $b \in [9, 17)$, a similar analysis yields that $(\frac{b-9}{8}, 1)$ needs to be excluded, and therefore $R^* = [0, \frac{b-9}{8}]$.

From the optimal restriction R^*, we can calculate the unrestricted and restricted MESU as well as the degree of restriction:

$$\mathcal{S}(A) = \begin{cases} 34 & \text{for } b \leq 5 \\ \frac{1}{8}\left(b^2 - 18b + 337\right) & \text{for } b \in [5, 13] \\ 2b + 8 & \text{for } b \geq 13 \end{cases}\ ,$$

$$\mathcal{S}(R^*) = \begin{cases} 34 & \text{for } b \leq 9 \\ \frac{1}{8}(b-9)^2 + 34 & \text{for } b \in [9, 17] \\ 2b + 8 & \text{for } b \geq 17 \end{cases}\ ,$$

and

$$\mathfrak{r}(R^*) = \begin{cases} 0 & \text{for } b \leq 5 \\ \frac{b-5}{4} & \text{for } b \in [5, 9] \\ \frac{17-b}{8} & \text{for } b \in [9, 17] \\ 0 & \text{for } b \geq 17 \end{cases}\ .$$