COLLISION HANDLING

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Dedicated to Homer the dog.

ABSTRACT. This part focuses on collision handling, especially formulas used to calculate collision impulses. These formulas can be generally presented in the linear form $A \, F \, \mathrm{d} t = B$. Particular object types will supply A and B with different coefficients. It is shown in the article how to obtain them.

This article is a part of documentation of the \mathbf{GPX} library. It describes methods, algorithms and derivations of formulas used in \mathbf{GPX} .

1. One-dimensional material body in two body system with single collision point

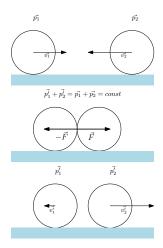


FIGURE 1. Colliding bodies.

Whenever two material bodies collide they exert forces on each other for a period of a collision. Newton's third law states that these forces are always exact in magnitude, but opposite in direction. In real time simulations we often expect collision time to be very short, that is collision is performed in a time shorter than a single step of simulation. We are not interested in exact distribution of collision force over that time. Considering the above and denoting collision time as dt, momentum of the first body as $\vec{p_1}$ and momentum of the second body as $\vec{p_2}$, we can define equations for post-collision momentums:

$$\begin{cases} \vec{p_1'} = \vec{p_1} - \vec{F} dt \\ \vec{p_2'} = \vec{p_2} + \vec{F} dt. \end{cases}$$

Now we would like to find the impulse \vec{Fdt} . Assuming that the direction of impulse \vec{Fdt} is known, let's focus on its magnitude. There are two edge cases we could find Fdt for. These are:

- (a) perfectly elastic collisions,
- (b) perfectly inelastic ones.

In terms of physics, perfectly elastic collisions are those in which kinetic energy of colliding bodies is conserved¹. Perfectly inelastic collisions are those in which kinetic energy loss is maximal. That happens when velocities are equalized after the collision, what is intuitively perceived as objects being sticked together. Note that no matter how we choose the magnitude, the total momentum of the system is preserved².

1.1. **Perfectly elastic collision.** We are considering one-dimensional case, so there is only one type of kinetic energy, which is associated with translational motion along the only available axis of freedom. Conservation of the kinetic energy in the system can be expressed as below:

$$\begin{split} E_{k1}' + E_{k2}' &= E_{k1} + E_{k2}, \\ \frac{{p_1'}^2}{2m_1} + \frac{{p_2'}^2}{2m_2} &= \frac{p_1^2}{2m_1} + \frac{(-p_2)^2}{2m_2}, \\ \frac{(p_1 - F dt)^2}{2m_1} + \frac{(-p_2 + F dt)^2}{2m_2} &= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}. \end{split}$$

¹There is distinction between something being *preserved* and *conserved*. By calling something conserved we mean it does not change at the output of some process, yet we don't care what happens in process duration. On the other hand, something is preserved when its state is conserved all the time. The building may have been damaged during the war, then restored to its previous state and we would say that it is conserved. However, if building was untouched by the war, we may say it is preserved (http://forum.wordreference.com/showthread.php?t=795689).

 $^{^2}$ See 1.

We have assumed that $\vec{p_2}$ vector is pointing in direction opposite to $\vec{p_1}$ and put minus sign near its scalar value. After simple transformations we may receive result in the following form.

(1.1)
$$Fdt\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{2p_1}{m_1} + \frac{2p_2}{m_2}$$

1.2. **Perfectly inelastic collision.** Perfectly inelastic collision can be expressed straightly as equalization of post-collision velocities:

$$v'_1 = v'_2,$$

$$\frac{p_1 - F dt}{m_1} = \frac{-p_2 + F dt}{m_2}.$$

Again, equation can be transformed to take the following form.

(1.2)
$$Fdt\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{p_1}{m_1} + \frac{p_2}{m_2}$$

1.3. **Elasticity.** When we compare equations (1.1) and (1.2), we will point out that they differ only by factor near momentums on the right-hand side, which in case of perfectly elastic collision is equal 2. This factor may be considered as a measure of elasticity and it can parametrize equation.

$$(1.3) Fdt\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{elasticity_1 \ p_1}{m_1} + \frac{elasticity_2 \ p_2}{m_2}$$

Without referring to particular body, may $elasticity_1 = elasticity_2 = elasticity$. Edge cases for perfectly elastic and perfectly inelastic collisions are defined respectively by $elasticity \in \{2,1\}$. Intermediate collision types, that are neither perfectly elastic nor perfectly inelastic are defined by $elasticity \in (1,2)$. Values other than above make less sense in classical mechanics. If $elasticity \in (2,\infty)$, it would mean that system gained some energy during collision. The effect of $elasticity \in (0,1)$ would be objects tunneling each other. In a case of elasticity = 0 the system would behave like there had been no collision. Negative values would reverse the force action. Whenever $elasticity_1 \neq elasticity_2$ some kind of intermediate collision type is performed.

Different collision types were glued together by elasticity parameter, but let us generalize equation (1.3) one step further and replace its parts with completely abstract components, so that:

$$Fdt (a_1 + a_2) = b_1 + b_2.$$

Or even more symbolically

$$A F dt = B$$
.

Obviously, impulse magnitude

$$F dt = \frac{B}{A}$$
.

We derived the formula for one-dimensional material bodies colliding at single point. In further paragraphs we will expand our model to more dimensions, more axes of freedom and multiple collision points. We will find out that form A F dt = B is very convenient for the purpose of collision engine. The only requirement is that A must have dimension of $\lceil 1/m_{ass} \rceil$ and B must have dimension of $\lceil velocity \rceil$.

2. Two-dimensional material body in two body system with single collision point

In two-dimensional space, momentum of material body can be described by two-dimensional vector. If we choose orthogonal base, coordinates of this vector will correspond to two independent degrees of freedom defined by base. As a reference we use coordinate system of screen³, denoting horizontal direction by base vector $\vec{x} = [1,0]$ and vertical direction by vector $\vec{y} = [0,1]$. Momentum of material body is then defined as $\vec{p} = [p_x, p_y]$ by the means of this base.

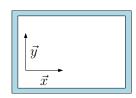


Figure 2. Screen coordinates.

Now let us introduce normal vector $\vec{n} = [n_x, n_y]$.

This vector is perpendicular to the shape of a bumper⁴ at given collision point. Inverse of this vector $-\vec{n}$ will be assigned to the wedge⁵, so that both objects will have a normal for the collision point (see figure 3). In both cases normal is pointing outward from the associated object.

Normal vector can form a new orthogonal base if it is used with another vector perpendicular to itself. Momentum of material body can be expressed in this base as $\vec{p} = [p_n, p_p]$. Let us call these coordinates respectively: "normal" and "tangent". As it was mentioned at the beginning each coordinate corresponds to a degree of freedom. By proper selection of base they are now adjusted to match the shape of colliding objects. Currently we will focus only on normal coordinate. This can be seen as a degradation to one-dimensional system and in fact this reduction is on the other side at the root of two-dimensional expansion.

To calculate normal impulse for two-dimensional system of material bodies one has to find p_n for each body and follow similar path as in one-dimension. This can be done using transformation matrix or simply by casting \vec{p} onto normal vector. Casting operator may be defined as

$$cast \ \vec{v} \equiv \cdot \frac{\vec{v}}{\|\vec{v}\|}.$$

Since norm of \vec{n} equals 1, $cast \ \vec{n}$ is simply a dot product. By applying casting operator on \vec{p} we get

$$p_n = \vec{p} \cdot \vec{n} = p_x n_x + p_y n_y.$$

Having calculated p_n we can use it like in one-dimensional case. There are however differences. Firstly, thanks to appropriate selection of normal we do not need to worry about a sign. Scalar value p_n will be positive if body is moving in the direction of normal and negative otherwise (or zero in case of no motion). Secondly, because we are operating on two dimensions, impulse must be vectorized. To vectorize impulse we multiply its magnitude by inverted normal.

$$\vec{Fdt} = -\vec{n} \cdot Fdt = [-n_x Fdt, -n_y Fdt]$$

³More precisely OpenGL coordinate system.

⁴This is a **GPX** collision detection concept. It is a body, which area was violated by a vertex belonging to another body. This will be described in more detail in another article or paragraph.

⁵Complementary to bumper, wedge is a body which violated the area of another object by one of its vertices.

Normal is inverted because impulse is acting in direction opposite to the direction of motion. Above could be formally achieved from transformation matrix and extensive commentary when moving between coordinates and scalar values, but for the purpose of documentation I think it is enough obvious and intuitive to leave this behind.

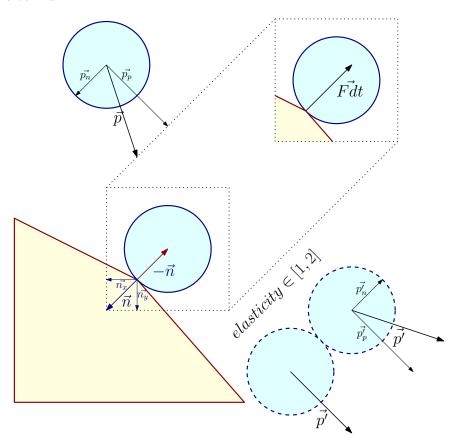


FIGURE 3. Colliding bodies and symbols used in this section. Yellow shape is a wedge, blue shape is a bumper. Depending on elasticity blue body will bounce away or stay close to the yellow shape.

3. Two-dimensional rigid body in two body system with single collision point

Rigid body in comparison to material object, in two dimensions, has additional axis of freedom - the rotation. Although rigid body itself has three axes of freedom each point on its edge still has only two. Point can perform only translational motion, however motion of each point belonging to rigid body depends on momentum and angular momentum of rigid body. Briefly

$$\vec{v_v} = [v_{vx}, v_{vy}],$$

$$v_{vx} = f(L, p_x),$$

$$v_{vy} = f(L, p_y),$$

where $\vec{v_v}$ is velocity of a point, L is angular momentum, p_x and p_y are coordinates of momentum \vec{p} . Edge points collide, not rigid body as a whole, thus collision formula for rigid body should be deduced in terms of its edges.

Taking those premises into account, we will now repeat steps taken in section 1 and see how equations are affected. To remind it, we will consider two edge types of collisions:

- (a) perfectly elastic,
- (b) perfectly inelastic.

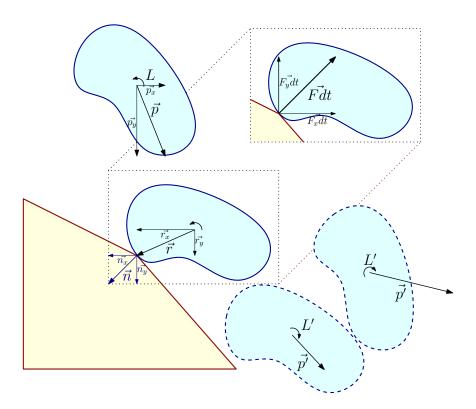


FIGURE 4. Rigid body (blue shape) and symbols used in this section.

3.1. Perfectly elastic collision. Total kinetic energy of rigid body E_k is defined as a sum of rotational and translational energies.

(3.1)
$$E_{k} = E_{r} + E_{t}$$

$$E_{r} = \frac{L^{2}}{2I}$$

$$E_{t} = \frac{\|\vec{p}\|^{2}}{2m} = \frac{p_{x}^{2} + p_{y}^{2}}{2m}$$

Post-collision energies are given by following equations.

(3.2)
$$E'_{k} = E'_{r} + E'_{t}$$

$$E'_{r} = \frac{(L + \det(\vec{r}, \vec{Fdt}))^{2}}{2I} = \frac{(L + (r_{x}F_{y}dt - r_{y}F_{x}dt))^{2}}{2I}$$

$$E'_{t} = \frac{\|\vec{p} + \vec{Fdt}\|^{2}}{2m} = \frac{(p_{x} + F_{x}dt)^{2} + (p_{y} + F_{y}dt)^{2}}{2m}$$

In above \vec{r} is a position vector measuring distance of the colliding point from the rigid body origin. It must be also said what $F_x dt$ and $F_y dt$ stand for. If \vec{n} is a normal vector they expand to the following.

$$\vec{n} = [n_x, n_y]$$

$$F_x dt = -n_x F dt$$

$$F_y dt = -n_y F dt$$

In one-dimensional case we explicitly wrote the energy of another object. But in two dimensions we have already two types of objects - rigid bodies and material bodies. We will be more abstract and denote energy of another object as E_o and E'_o (respectively before and after collision). This will bring sketchy equation for the conservation of kinetic energy in the system.

$$(3.4) E_k + E_o = E_k' + E_o'$$

By combining (3.4) with (3.3), (3.2) and (3.1) one may obtain

$$E_o + \frac{L^2}{2I} + \frac{p_x^2 + p_y^2}{2m} = E_o' + \frac{L^2 - 2LFdt(r_x n_y - r_y n_x) + (Fdt)^2(r_x n_y - r_y n_x)^2}{2I} + \frac{p_x^2 - 2Fdt n_x p_x + (Fdt)^2 n_x^2 + p_y^2 - 2Fdt n_y p_y + (Fdt)^2 n_y^2}{2m}.$$

After subtraction of repeating elements and division by $\frac{Fdt}{2}$ we will have:

$$Fdt\left(\frac{(r_{x}n_{y}-r_{y}n_{x})^{2}}{I}+\frac{n_{x}^{2}+n_{y}^{2}}{m}\right)+\frac{2\Delta E_{o}}{Fdt}=\frac{2L(r_{x}n_{y}-r_{y}n_{x})}{I}+\frac{2(n_{x}p_{x}+n_{y}p_{y})}{m},$$

where $\Delta E_o = E'_o - E_o$. Equation can be written in more compact form as below.

$$(3.5) \qquad Fdt\left(\frac{\det(\vec{r},\vec{n})^2}{I} + \frac{\|\vec{n}\|^2}{m}\right) + \frac{2\Delta E_o}{Fdt} = \frac{2L\det(\vec{r},\vec{n})}{I} + \frac{2\vec{p}\cdot\vec{n}}{m}$$

We almost have a formula in familiar form A F dt = B. Remaining part $\frac{2\Delta E_o}{F dt}$ may be calculated for rigid body or material object and split into additions to A and B in the very same way as we did for the rigid body of interest. Therefore let

us denote additions to formula for two-dimensional rigid body performing perfectly elastic collision at given collision point as $A_{elastic}$ and $B_{elastic}$.

(3.6)
$$A_{elastic} = \frac{\det(\vec{r}, \vec{n})^2}{I} + \frac{\|\vec{n}\|^2}{m}$$
$$B_{elastic} = \frac{2L \det(\vec{r}, \vec{n})}{I} + \frac{2\vec{p} \cdot \vec{n}}{m}$$

3.2. **Perfectly inelastic collision.** For inelastic collisions we are looking for equalization of post-collision velocities. Particular point of rigid body is moving with a velocity being a superposition of rotational and translational motion. That is

$$\vec{v_v} = \vec{v_r} + \vec{v_t} = \omega[-r_y, r_x] + \vec{v} = \frac{L}{I}[-r_y, r_x] + \frac{\vec{p}}{m}.$$

After collision, velocity is affected by an impulse. This is reflected by following equations.

(3.7)
$$\vec{v_{v'}} = [v'_{vx}, v'_{vy}] = \frac{(L + \det(\vec{r}, \vec{F}dt))}{I} [-r_y, r_x] + \frac{\vec{p} + \vec{F}dt}{m}$$

$$v'_{vx} = -r_y \frac{L - Fdt(r_x n_y - r_y n_x)}{I} + \frac{p_x - Fdt n_x}{m}$$

$$v'_{vy} = r_x \frac{L - Fdt(r_x n_y - r_y n_x)}{I} + \frac{p_y - Fdt n_y}{m}$$

We are interested in normal component of velocity, thus we cast $\vec{v_v}'$ onto normal.

(3.8)
$$v'_{vn} = \vec{v_v}' \cdot \vec{n} = v'_{vx} n_x + v'_{vy} n_y$$

Equations (3.7) and (3.8) bring

$$v'_{vn} = \frac{L - Fdt(r_x n_y - r_y n_x)}{I}(r_x n_y - r_y n_x) + \frac{p_x n_x + p_y n_y - Fdt(n_x^2 + n_y^2)}{m} = \frac{L - Fdt \det(\vec{r}, \vec{n})}{I} \det(\vec{r}, \vec{n}) + \frac{\vec{p} \cdot \vec{n} - Fdt \|\vec{n}\|^2}{m}.$$

Now let us suppose that post-collision velocity of point belonging to another object is $\vec{v_o}'$ and v'_{on} was found analogously to v'_{vn} . Perfectly inelastic collision means simply that $v'_{vn} = v'_{on}$. After substituting with previously calculated value and with some basic regrouping we have final formula for perfectly inelastic collision impulse of rigid body (in two body system and one collision point).

(3.9)
$$Fdt\left(\frac{\det(\vec{r}, \vec{n})^2}{I} + \frac{\|\vec{n}\|^2}{m}\right) + v'_{on} = \frac{L\det(\vec{r}, \vec{n})}{I} + \frac{\vec{p} \cdot \vec{n}}{m}$$

We can assume that v'_{on} can be also represented in form A F dt = B. Anyway, for perfectly inelastic collision of rigid body A and B coefficients are given as follows.

(3.10)
$$A_{inelastic} = \frac{\det(\vec{r}, \vec{n})^2}{I} + \frac{\|\vec{n}\|^2}{m}$$
$$B_{inelastic} = \frac{L \det(\vec{r}, \vec{n})}{I} + \frac{\vec{p} \cdot \vec{n}}{m}$$

3.3. **Elasticity.** If we compare above results with those for perfectly elastic collision (3.6), we will see that only B differs. In case of perfectly elastic collision it contains factor 2. These results are compatible with one-dimensional case, hence discussion about *elasticity* parameter from subsection 1.3 is still applicable.

3.4. **Solving.** If remaining parts of equations (3.5) and (3.9), that shall be calculated for the second collision participant, were transformed into form A F dt = B, equation for collision impulse can be written as:

$$Fdt(A_1 + A_2) = B_1 + B_2.$$

Pairs (A_1, B_1) , (A_2, B_2) may belong to material body, rigid body or some other entity and they can contain factors for any type of collision. Achieved flexibility reveals why efforts were taken to present collision in form AFdt = B.

4. Two-dimensional material body in many-body system with multiple collision points

So far we were considering collisions occurring between two objects at single collision point. Now we will expand our models to handle multiple collision points and many collision participants. This transition consists in introduction of system of equations. Each collision point is given an equation, so that for N points one will have N equations. This is the basic idea, but it does not come without some issues, which must be resolved. For the clarity one can at first assume that collision occurs between two bodies, because some details are obscured in sacrifice for fluent expression of main plot.

4.1. **Perfectly elastic collision.** With single collision point, equation for conservation of kinetic energy was unambiguous. Collision point acted as single "channel" by which energy could be interchanged between bodies. For multiple collision points, each point defines a path upon which collision forces \vec{F} act (see figure 5)⁶.

Energy is defined as $E = \int \vec{F} \cdot d\vec{S}$, so we can derive formula for change of kinetic energy during collision in terms of specific collision point. This will be given by the integral along displacement performed during collision time $\mathrm{d}t$.

(4.1)
$$\Delta E_{vt} = \int_{\mathcal{A}} F_v \, \mathrm{d}S_v$$

According to Newton's second law

$$F_v = \frac{\mathrm{d}p_v}{\mathrm{d}t} = \frac{\sum_i \vec{F_i} dt \cdot \vec{n_v}}{\mathrm{d}t}.$$

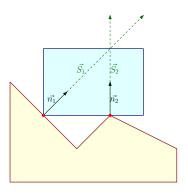


FIGURE 5. Alternative \vec{F} paths.

In above $\vec{F_i}dt$ is an impulse applied to i-th collision point. The path upon which collision forces act is given by

$$S_v = v_v t = \frac{\vec{p} \cdot \vec{n_v} t}{m} = \frac{p_v t}{m}.$$

We assume that m is constant, however velocity changes over time. Therefore product rule has to be applied to differential dS_v .

(4.2)
$$dS_v = dv_v t + v_v dt = \frac{dp_v t}{m} + \frac{p_v dt}{m}$$

⁶Although in material body point of application does not matter and solution could be found from standard equation for kinetic energy of material body. But this approach will be used to explain extra component in equation for rigid body, therefore it is used here to pave the way.

With above in mind we can rewrite and transform equation (4.1) so that:

$$\Delta E_{vt} = \int_{\mathrm{d}t} \sum_{i} \vec{F}_{i} \cdot \vec{n_{v}} \left(\frac{\mathrm{d}p_{v} t}{m} + \frac{p_{v} \, \mathrm{d}t}{m} \right)$$

$$= \int_{\mathrm{d}t} \sum_{i} \vec{F}_{i} \cdot \vec{n_{v}} \frac{t \sum_{i} \vec{F}_{i} \, \mathrm{d}t \cdot \vec{n_{v}}}{m} + \int_{\mathrm{d}t} \sum_{i} \vec{F}_{i} \cdot \vec{n_{v}} \frac{p_{v} \, \mathrm{d}t}{m}$$

$$= \int_{0}^{\mathrm{d}t} \frac{t (\sum_{i} \vec{F}_{i} \cdot \vec{n_{v}})^{2}}{m} \, \mathrm{d}t + \int_{0}^{\mathrm{d}t} \frac{p_{v} \sum_{i} \vec{F}_{i} \cdot \vec{n_{v}}}{m} \, \mathrm{d}t$$

$$= \frac{t^{2} (\sum_{i} \vec{F}_{i} \cdot \vec{n_{v}})^{2}}{2m} \Big|_{0}^{\mathrm{d}t} + \frac{p_{v} t \sum_{i} \vec{F}_{i} \cdot \vec{n_{v}}}{m} \Big|_{0}^{\mathrm{d}t},$$

which finally yields

$$\Delta E_{vt} = \frac{(\sum_{i} \vec{F_i} dt \cdot \vec{n_v})^2 + 2p_v \sum_{i} \vec{F_i} dt \cdot \vec{n_v}}{2m}.$$

Conservation of kinetic energy can be expressed as below.

$$(4.3) \Delta E_{vt} + \Delta E_{vo} = 0$$

In above ΔE_{vo} is energy change of another object at the same collision point. One may find ΔE_{vo} in the same way as we did with ΔE_{vt} or provide different solution. Nevertheless, one term should be always present in obtained solution, that is $\sum_i \vec{F_i} dt \cdot \vec{n_v}$ (remember about Newton's third law, which virtually guarantees it). This is because we have to divide equation (4.3) by it, to achieve results consistent with perfectly inelastic collisions, which will be discussed later. After mentioned division and multiplication by 2 we get:

$$\frac{\sum_{i} \vec{F_{i}} dt \cdot \vec{n_{v}}}{m} + \frac{2p_{v}}{m} + \frac{2\Delta E_{vo}}{\sum_{i} \vec{F_{i}} dt \cdot \vec{n_{v}}} = 0.$$

As it was said earlier $\vec{F_i}$ dt is an impulse acting at i-th collision point. If $\vec{n_i}$ is a normal assigned to i-th point, then impulse can be expressed more verbosely as $-F_i$ dt $\vec{n_i}$. Thus after feeding recent equation with normals we obtain equation for collision impulses for perfectly elastic collision in respect to specific collision point.

$$\frac{\sum_{i} F_{i} \operatorname{dt} \vec{n_{i}} \cdot \vec{n_{v}}}{m} = \frac{2\vec{p} \cdot \vec{n_{v}}}{m} + \frac{\Delta E_{vo}}{2\sum_{i} F_{i} \operatorname{dt} \vec{n_{i}} \cdot \vec{n_{v}}}.$$

Equation (4.4) resembles form A F dt = B, but to give our system final shape N such equations are required, each for one collision point. Then A will be represented by a matrix, whilst F dt and B will become vectors. Let us put this explicitly. Matrix is constructed as follows:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix}.$$

Abstracts B and Fdt are now column vectors:

$$\vec{Fdt} = \begin{bmatrix} F_1 dt \\ F_2 dt \\ \vdots \\ F_N dt \end{bmatrix}, \vec{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}.$$

For perfectly elastic collision of material body, matrix elements a_{vi} and vector elements b_v are defined by equations:

$$(4.5) \quad a_{vi(elastic)} = \frac{\vec{n_i} \cdot \vec{n_v}}{m},$$

$$b_{v(elastic)} = \frac{2\vec{p} \cdot \vec{n_v}}{m}.$$

4.2. **Perfectly inelastic collision.** Perfectly inelastic collisions are much more straightforward than perfectly elastic ones. Just as before, perfectly inelastic collision can be described as equalization of post-collision velocities.

(4.6)
$$v_{vt}^{\vec{r}} = v_{vo}^{\vec{r}} \\ v_{vt}^{\vec{r}} = \frac{\vec{p} + \sum_{i} \vec{F_{i}} dt}{m}$$

We are interested in normal component of velocity, thus we should cast it onto appropriate normal vector associated with specific collision point. Equation for impulses respective to this point can be written in the following form.

(4.7)
$$\frac{\sum_{i} F_{i} \operatorname{d} t \, \vec{n_{i}} \cdot \vec{n_{v}}}{m} = \frac{\vec{p} \cdot \vec{n_{v}}}{m} + \vec{v_{vo}} \cdot \vec{n_{v}}.$$

From the above, components of linear system $\mathbf{A}\vec{F}dt = \vec{B}$ can be extracted (compare with equation (4.4)).

(4.8)
$$a_{vi(inelastic)} = \frac{\vec{n_i} \cdot \vec{n_v}}{m},$$
$$b_{v(inelastic)} = \frac{\vec{p} \cdot \vec{n_v}}{m}.$$

- 4.3. Elasticity. As desired, equations (4.8) and (4.5) differ only by b_v coefficients, which in case of perfectly elastic collisions contain factor 2.
- 4.4. **Solving.** If remaining parts of equations (4.7) and (4.4) belonging to another collision participant were found in similar fashion, they can be also packed into matrix \mathbf{A} and vector \vec{B} . Denoting (matrix, vector) pairs of collision participants respectively as $(\mathbf{A_1}, \vec{B_1})$ and $(\mathbf{A_2}, \vec{B_2})$ we get system of linear equations capable to determine collision impulses, which is given in form:

$$(\mathbf{A_1} + \mathbf{A_2})\vec{F} dt = \vec{B_1} + \vec{B_2}.$$

By solving linear system $\mathbf{A}\vec{F}dt = \vec{B}$ one may find values of column vector $\vec{F}dt$, which consists of impulse magnitudes F_vdt . Impulse then simply satisfies equation:

$$\vec{F_v} dt = \vec{n_v} F_v dt.$$

It is very likely that the system of equations will be dependent or even inconsistent. Dependence can often be a case for material body, because motion of material

body depends on resultant impulse and it does not matter how this resultant impulse is distributed among individual impulses in terms of motion. If so, additional assumption can be made that, if there are many solutions, impulses shall be equal. When the system is inconsistent, currently acceptable solution is to ignore one of equations. Some smarter algorithm may be developed in the future. It is up to solver to deal with it.

4.5. **Many bodies.** Newly constructed model can not only calculate impulses for collisions occurring at multiple points within a two body system. With some tweaks it has also the ability to deal with many objects.

With perfectly inelastic collisions it is quite straightforward, since $\vec{v_{vo}}$ in equation (4.7) can be velocity of any object. For this velocity, related object should bring its own a_{vi} and b_v counterparts.

Perfectly elastic collisions are a little bit more cumbersome. We have to go back to formula (4.2) and include impulses not involved in collision with actual object to change of momentum. Modified equation will look as follows:

$$\mathrm{d}S_v = \frac{\mathrm{d}p_v\,t}{m} + \frac{p_v\,\mathrm{d}t}{m} = \frac{\sum_i \vec{F_i} \vec{\mathrm{d}}t \cdot \vec{n_v} + \sum_j \vec{F_j} \vec{\mathrm{d}}t \cdot \vec{n_v}}{m} + \frac{p_v\,\mathrm{d}t}{m}.$$

Impulses between former two bodies are indexed by i and remaining impulses are indexed by j. After repeating all previously taken steps, with altered differential dS_v , we get modified version of equation (4.4):

$$\frac{\sum_i F_i \mathrm{d}t\, \vec{n_i} \cdot \vec{n_v} + \sum_j F_j \mathrm{d}t\, \vec{n_j} \cdot \vec{n_v}}{m} = \frac{2\vec{p} \cdot \vec{n_v}}{m} + \frac{\Delta E_{vo}}{2\sum_i F_i \mathrm{d}t\, \vec{n_i} \cdot \vec{n_v}}.$$

As one can see coefficients (4.8) and (4.5) still remain valid for many-body system, although size of matrix \mathbf{A} and vector \vec{B} is determined per object by total number of collision points (now indexed by i and j).

Anyway, each object will supply system with its own square matrix \mathbf{A} and vector \vec{B} . Matrices and vectors shall be combined in respect to appropriate impulses and collision points. Every b_v element corresponds with specific collision point and every a_{vk} (where k may be any of i or j) element corresponds with specific collision point and impulse. This defines a way matrices and vectors shall be added. We can resize smaller matrices to match the largest one, filling them with zeros where needed, or we can define operation to sum matrices of inadequate size. If such sum is marked as \oplus , system of equations containing M colliding objects has the following form:

$$(\mathbf{A_1} \oplus \mathbf{A_2} \oplus \cdots \oplus \mathbf{A_M}) \vec{Fdt} = \vec{B_1} \oplus \vec{B_2} \oplus \cdots \oplus \vec{B_M}.$$

5. Two-dimensional rigid body in many-body system with multiple collision points

Motion of rigid body is a superposition of rotational and translational motion. Because translational motion was explained in previous section, we will now focus on rotational motion. When considering rotational additions, one can imagine rigid body as being pinned at the center of mass⁷.

5.1. **Perfectly elastic collision.** For rigid body, kinetic energy is a sum of rotational and translational additions. Conservation of kinetic energy in collision with another object can be written as below.

$$\Delta E_{vr} + \Delta E_{vt} + \Delta E_{vo} = 0$$

Translational component ΔE_{vt} was found in previous section, ΔE_{vo} is a change of energy of another object and now we will look after rotational part ΔE_{vr} .

Like in case of material body, equation (4.1) is the entry point. Again⁸

$$F_v = \sum_i \vec{F_i} \cdot \vec{n_v}.$$

May $\vec{r_v} = [r_{vx}, r_{vy}]$ be a position vector for considered collision point. Then, for small angles

$$dS_v = [-r_{vu}, r_{vx}] \cdot \vec{n_v} d\alpha = \det(\vec{r_v}, \vec{n_v}) d\alpha,$$

since $S = r\alpha$. On the other hand

$$\alpha = \omega t = \frac{Lt}{I}.$$

Moment of inertia is constant, but angular velocity changes over time, therefore:

$$d\alpha = d\omega t + \omega dt = \frac{dLt}{I} + \frac{Ldt}{I}.$$

Impulses participating in both: collision with actual body and change of angular momentum, are indexed by i. For many-body system these impulses, which do not interact directly with actual collision counterpart must also be included in change of angular momentum. Those will be indexed by j. Sum of them will be written with index k.

$$\mathrm{d}L = \sum_{i} \det(\vec{r_i}, \vec{F_i} \mathrm{d}t) + \sum_{j} \det(\vec{r_j}, \vec{F_j} \mathrm{d}t) = \sum_{k} \det(\vec{r_k}, \vec{F_k} \mathrm{d}t)$$

 $^{^{7}}$ Basically, rotational part is independent and can be used to model rigid body rotating around any point of origin (possibly with a help of Steiner's theorem).

⁸Although this time Newton's second law may look suspicious, so, equation can be recovered from superposition of forces as well.

Index k is introduced to avoid lengthy expressions. Putting all pieces together:

$$\begin{split} \Delta E_{vr} &= \int\limits_{\mathrm{d}t} \sum_{i} \vec{F}_{i} \cdot \vec{n_{v}} \det(\vec{r_{v}}, \vec{n_{v}}) \left(\frac{\mathrm{d}L\,t}{I} + \frac{L\,\mathrm{d}t}{I} \right) \\ &= \det(\vec{r_{v}}, \vec{n_{v}}) \sum_{i} \vec{F}_{i} \cdot \vec{n_{v}} \left(\int\limits_{0}^{\mathrm{d}t} \frac{t \sum_{k} \det(\vec{r_{k}}, \vec{F_{k}})}{I} \, \mathrm{d}t + \int\limits_{0}^{\mathrm{d}t} \frac{L}{I} \, \mathrm{d}t \right) \\ &= \det(\vec{r_{v}}, \vec{n_{v}}) \sum_{i} \vec{F}_{i} \cdot \vec{n_{v}} \left(\frac{t^{2} \sum_{k} \det(\vec{r_{k}}, \vec{F_{k}})}{2I} \Big|_{0}^{\mathrm{d}t} + \frac{Lt}{I} \Big|_{0}^{\mathrm{d}t} \right) \\ &= \det(\vec{r_{v}}, \vec{n_{v}}) \sum_{i} \vec{F}_{i} \cdot \vec{n_{v}} \left(\frac{t^{2} \sum_{k} \det(\vec{r_{k}}, \vec{F_{k}})}{2I} \Big|_{0}^{\mathrm{d}t} + \frac{Lt}{I} \Big|_{0}^{\mathrm{d}t} \right) \\ &= \det(\vec{r_{v}}, \vec{n_{v}}) \sum_{i} \vec{F}_{i} \cdot \vec{n_{v}} \left(\frac{t^{2} \sum_{k} \det(\vec{r_{k}}, \vec{F_{k}})}{2I} \Big|_{0}^{\mathrm{d}t} + \frac{Lt}{I} \Big|_{0}^{\mathrm{d}t} \right) \end{split}$$

Like in case of material body, equation (5.1) is going to be divided by $\sum_{i} \vec{F_i} dt \cdot \vec{n_v}$ and multiplied by 2 to produce form A F dt = B.

$$(5.2) \quad \sum_{k} F_{k} dt \left(\frac{\det(\vec{r_{k}}, \vec{n_{k}}) \det(\vec{r_{v}}, \vec{n_{v}})}{I} + \frac{\vec{n_{k}} \cdot \vec{n_{v}}}{m} \right) = \frac{2L \det(\vec{r_{v}}, \vec{n_{v}})}{I} + \frac{2\vec{p} \cdot \vec{n_{v}}}{m} + \frac{\Delta E_{vo}}{2 \sum_{i} F_{i} dt \, \vec{n_{i}} \cdot \vec{n_{v}}}.$$

Concluding, for perfectly elastic collision, matrix elements a_{vk} and vector elements b_v are given by following equations.

(5.3)
$$a_{vk(elastic)} = \frac{\det(\vec{r_k}, \vec{n_k}) \det(\vec{r_v}, \vec{n_v})}{I} + \frac{\vec{n_k} \cdot \vec{n_v}}{m}$$
$$b_{v(elastic)} = \frac{2L \det(\vec{r_v}, \vec{n_v})}{I} + \frac{2\vec{p} \cdot \vec{n_v}}{m}$$

5.2. **Perfectly inelastic collision.** Velocity of point belonging to rigid body is a sum of rotational addition $\vec{v_{vr}}$ and translational addition $\vec{v_{vt}}$. Post-collision additions are given by following equations.

(5.4)
$$v_{vr}^{\vec{I}} = \frac{L + \sum_{k} \det(\vec{r_k}, \vec{F_k dt})}{I} [-r_{vy}, r_{vx}]$$
$$v_{vt}^{\vec{I}} = \frac{\vec{p} + \sum_{k} \vec{F_k dt}}{m}$$

Our type of collision requires equalization of normal components of velocities:

$$\vec{v_{vr}} \cdot \vec{n_v} + \vec{v_{vt}} \cdot \vec{n_v} = -\vec{v_{vo}} \cdot \vec{n_v}.$$

After substituting with values provided by equation (5.4) we receive following result:

$$\sum_{k} F_k dt \left(\frac{\det(\vec{r_k}, \vec{n_k}) \det(\vec{r_v}, \vec{n_v})}{I} + \frac{\vec{n_k} \cdot \vec{n_v}}{m} \right) = \frac{\vec{p} \cdot \vec{n_v}}{m} + \frac{L \det(\vec{r_v}, \vec{n_v})}{I} + \vec{v_{vo}'} \cdot \vec{n_v}$$

From the above we can extract a_{vk} and b_v components.

(5.5)
$$a_{vk(inelastic)} = \frac{\det(\vec{r_k}, \vec{n_k}) \det(\vec{r_v}, \vec{n_v})}{I} + \frac{\vec{n_k} \cdot \vec{n_v}}{m}$$
$$b_{v(inelastic)} = \frac{L \det(\vec{r_v}, \vec{n_v})}{I} + \frac{\vec{p} \cdot \vec{n_v}}{m}$$

5.3. **Elasticity.** As we compare equations (5.5) and (5.3) we see that *elasticity* parameter can still be used. As before, it takes value 1 for perfectly inelastic collision and 2 for perfectly elastic one.

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