Computational Optimal Transportation

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November 29, 2021

Sinkhorn Algorithm

Regularized optimal plan $P = \min_{P'} \sum_{i,j} C_{i,j} P'_{i,j} - \epsilon H(P)$, where H is entropy.

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Algorithm 2: Sinkhorn Algorithm

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$s, t \in \mathbb{P}(\{1, ..., n\})$ are distributions

$C \in \mathbb{R}^{n \times n}$ is cost matrix

$\epsilon : 0 < \epsilon << 1$

$M \in \mathcal{N}$ is max iterations

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Begin:

$K = exp(-C/epsilon)$; $u = v = ones(n, 1)$

$T = diag(u) \cdot K \cdot diag(v)$

for *i = 1,...,M* **do**

$u = s/(Kv)$

$v = t/(K^T u)$

$T = diag(u) \cdot K \cdot diag(v)$

end

$d = \|C \cdot T\|_{fro}$

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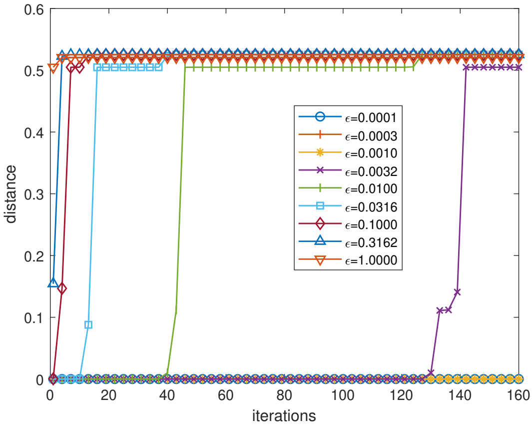

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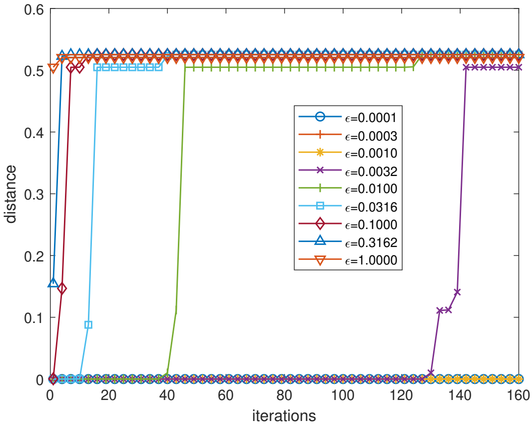

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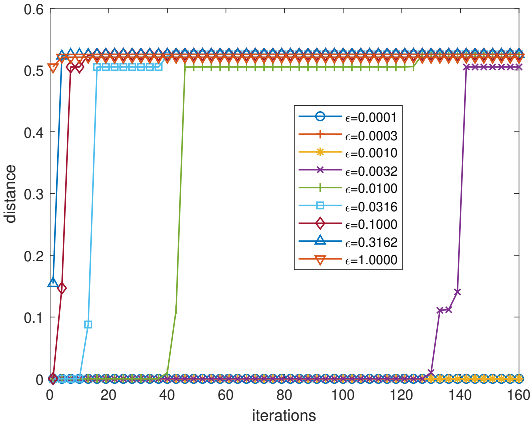

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$$\|x - y\|_1 = 0.8270, \|x - y\|_2 = 0.4964, \|x - y\|_\infty = 0.4135$$

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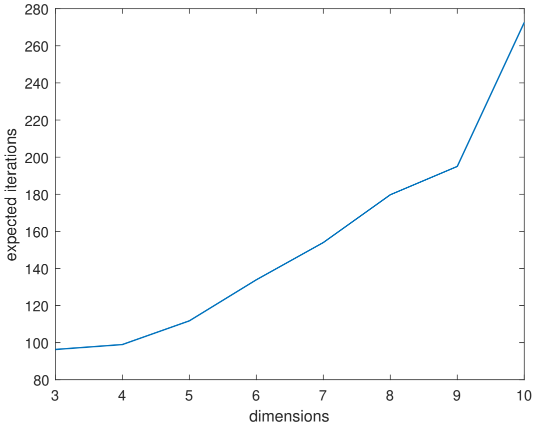

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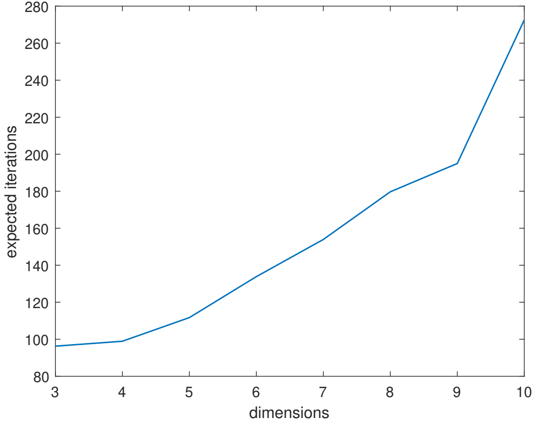

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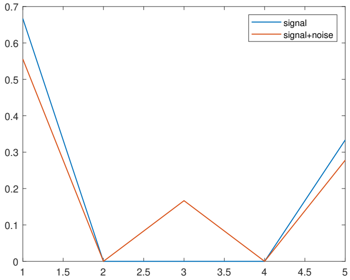

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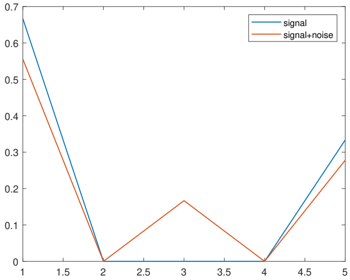

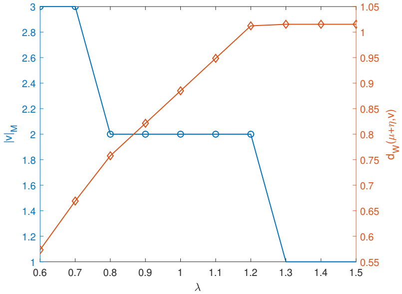

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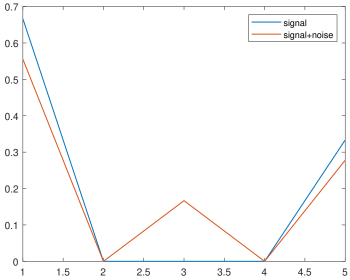

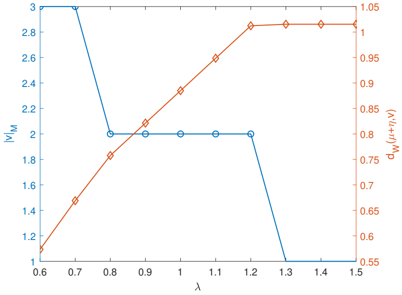

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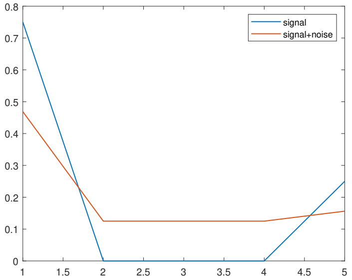

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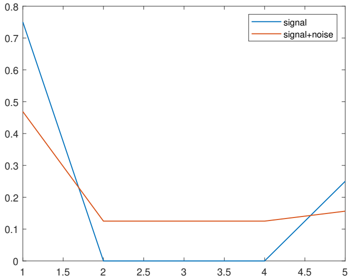

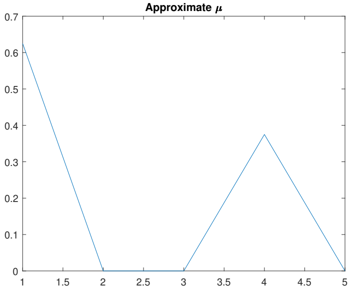

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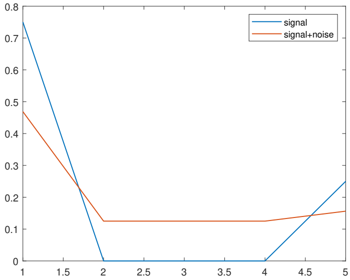

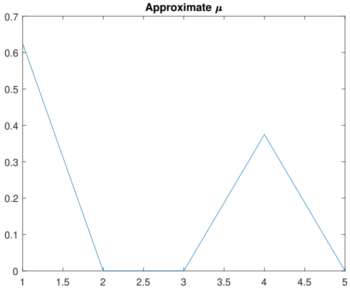

Approximate signal (left: lambda=[0.3, 0.7]) not equal to true signal for any lambda.

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Remark: When the signal-to-noise ratio (SNR) is too low, the reconstruction is underdetermined.

Application: Star Cluster Identification

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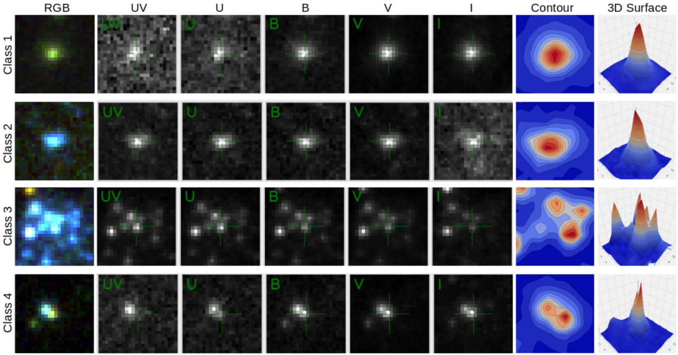

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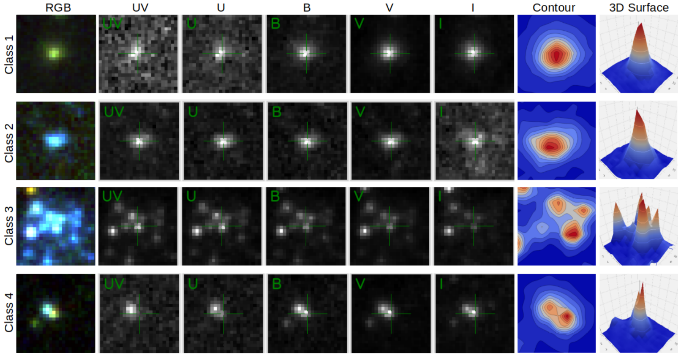

From [1], LEGUS classification scheme. Examples of candidates from the LEGUS images of NGC 1566 classified as Class 1 (symmetric star cluster; top),

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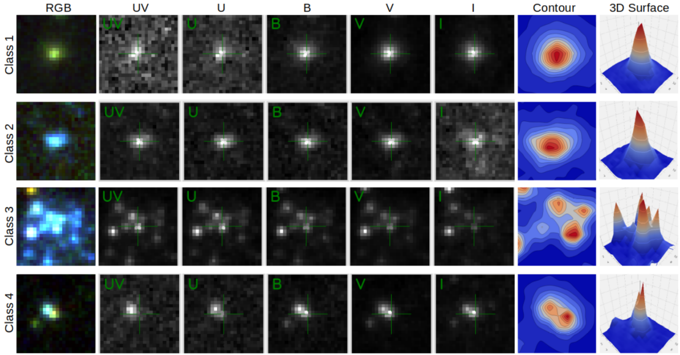

From [1], LEGUS classification scheme. Examples of candidates from the LEGUS images of NGC 1566 classified as Class 1 (symmetric star cluster; top), Class 2 (elongated star cluster; middle-top),

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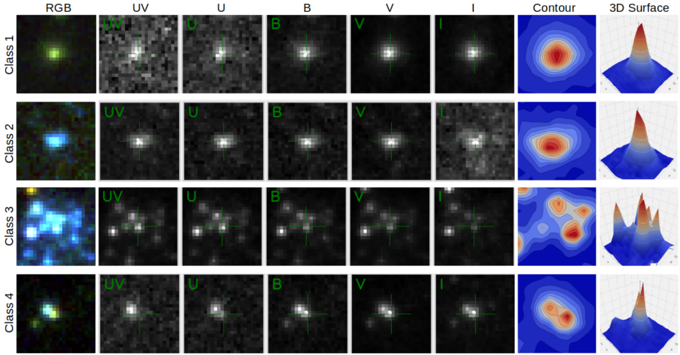

From [1], LEGUS classification scheme. Examples of candidates from the LEGUS images of NGC 1566 classified as Class 1 (symmetric star cluster; top), Class 2 (elongated star cluster; middle-top), Class 3 (compact, multi–peak association; middle-bottom), and

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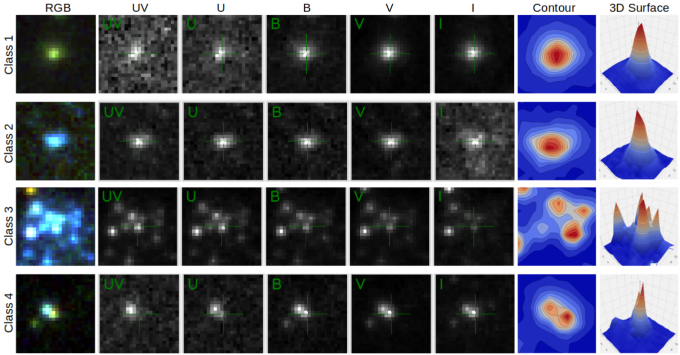

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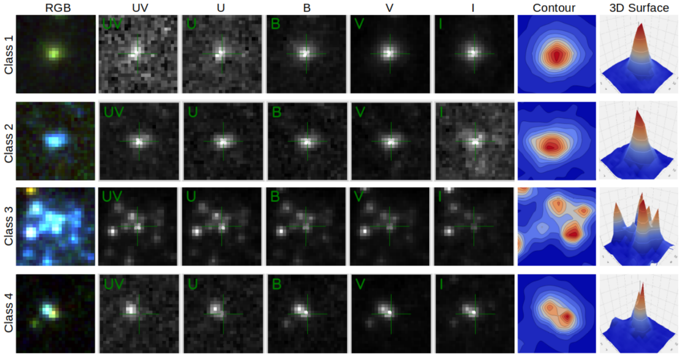

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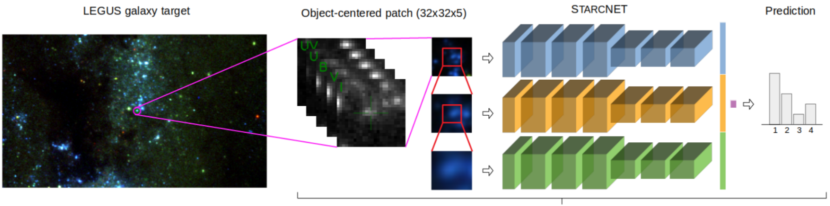
Star Cluster Identification

Star Cluster Identification

Convolutional Neural Network:

Star Cluster Identification

Convolutional Neural Network:

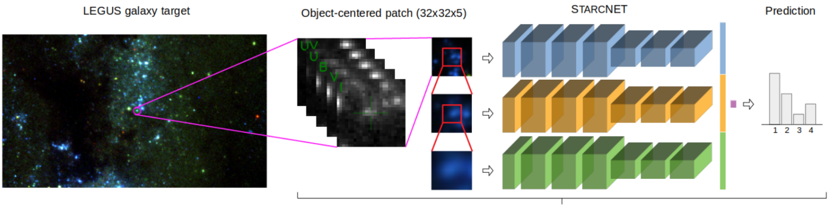


(a) Objects detected with SExtractor on HST observations

(b) Overview of the pipeline

Star Cluster Identification

Convolutional Neural Network:



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From [1], the StarcNet pipeline.

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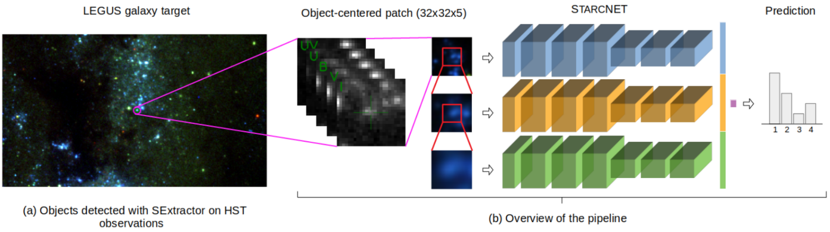
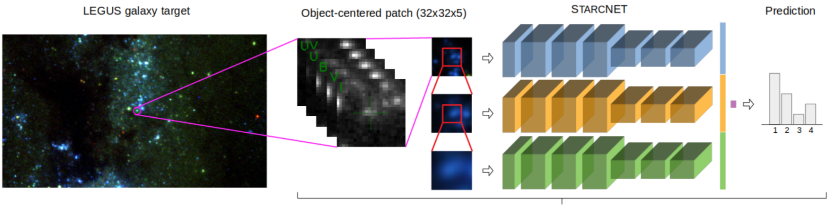


From [1], the StarcNet pipeline. Graphic sketch of the machine learning pipeline used in this work to classify cluster candidates in the LEGUS images.

Star Cluster Identification

Convolutional Neural Network:



(a) Objects detected with SExtractor on HST observations

(b) Overview of the pipeline

From [1], the StarcNet pipeline. Graphic sketch of the machine learning pipeline used in this work to classify cluster candidates in the LEGUS images. (Left): The Hubble Space Telescope images as processed by the LEGUS project through a custom pipeline to generate automatic catalogs of cluster candidates, which are part of the public LEGUS catalogs release (Calzetti et al. 2015; Adamo et al. 2017); we apply StarcNet to the LEGUS catalogs and images.

Star Cluster Identification

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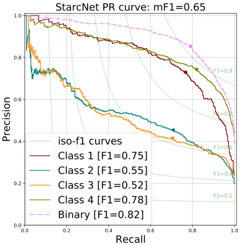
Convolutional Neural Network:

Star Cluster Identification

Convolutional Neural Network:

Human Classification \ StarcNet Prediction	Class 1	Class 2	Class 3	Class 4
1	0.78 (410)	0.14 (75)	0.01 (6)	0.07 (37)
2	0.17 (107)	0.55 (335)	0.12 (73)	0.16 (97)
3	0.01 (3)	0.15 (92)	0.45 (279)	0.39 (243)
4	0.03 (45)	0.06 (80)	0.09 (115)	0.82 (1098)

Human \ StarcNet Prediction	C	NC
C	0.81 (927)	0.19 (213)
NC	0.11 (220)	0.89 (1735)



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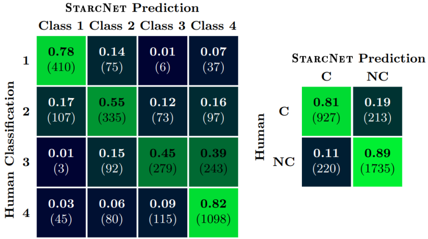

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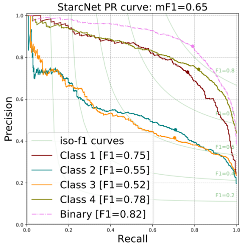

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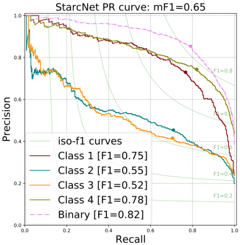

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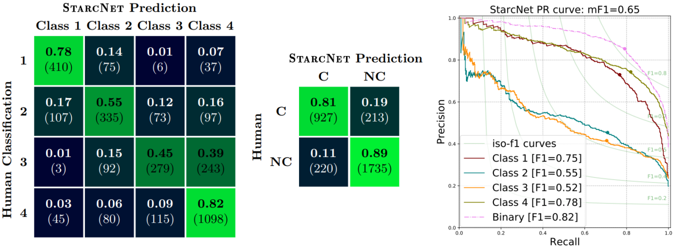

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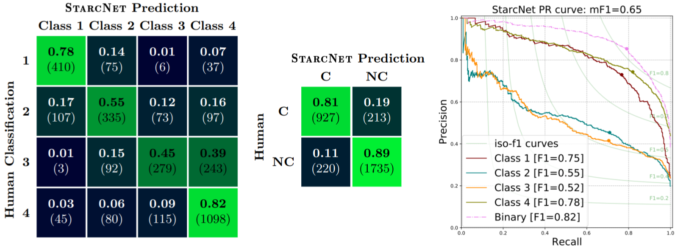

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Star Cluster Identification

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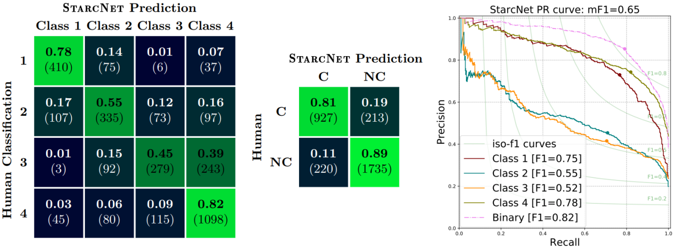

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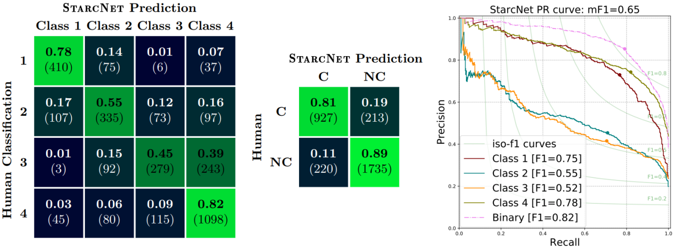
Star Cluster Identification

Convolutional Neural Network:



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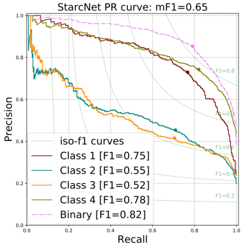

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Optimal Transportation Star Cluster Identification

Our method:

Optimal Transportation Star Cluster Identification

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Algorithm 6: Optimal Transportation Star Cluster Identification

Input:

X : images $N \times m \times m$

δ : noise level

Our method:

Algorithm 7: Optimal Transportation Star Cluster Identification

Input:

X : images $N \times m \times m$

δ : noise level

Output:

C : star cluster classification $N \times 1$

Optimal Transportation Star Cluster Identification

Our method:

Algorithm 8: Optimal Transportation Star Cluster Identification

Input:

X : images $N \times m \times m$

δ : noise level

Output:

C : star cluster classification $N \times 1$

for $n = 1, 2, ..., N$ **do**

$V_n = \arg\min_{v' : d_W(X_n, v') < \delta} H(v')$

Optimal Transportation Star Cluster Identification

Our method:

Algorithm 9: Optimal Transportation Star Cluster Identification

Input:

X : images $N \times m \times m$

δ : noise level

Output:

C : star cluster classification $N \times 1$

for $n = 1, 2, ..., N$ **do**

  $V_n = \arg\min_{v' : d_W(X_n, v') < \delta} H(v')$

  if $rank(H_0(V_n^{-1}([0.9, 1]))) == 1$ **then**

Optimal Transportation Star Cluster Identification

Our method:

Algorithm 10: Optimal Transportation Star Cluster Identification

Input:
X : images $N \times m \times m$
δ : noise level
Output:
C : star cluster classification $N \times 1$
for $n = 1, 2, ..., N$ **do**
  $V_n = \arg\min_{v' : d_W(X_n, v') < \delta} H(v')$
  if $rank(H_0(V_n^{-1}([0.9, 1]))) == 1$ **then**
    $C_n = 1$
  else
    $C_n = 0$
  end
end

Optimal Transportation Star Cluster Identification

Evaluating the expression $\arg\min_{v':d_W(x,v')<\delta} H(v')$

Optimal Transportation Star Cluster Identification

Evaluating the expression $\arg\min_{v':d_W(x,v')<\delta} H(v')$
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Normalize vectors to stay in probability space.

Optimal Transportation Star Cluster Identification

Evaluating the expression $\arg\min_{v' : d_W(x, v') < \delta} H(v')$
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Optimal Transportation Star Cluster Identification

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Optimal Transportation Star Cluster Identification

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Optimal Transportation Star Cluster Identification

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2. Random search:

Optimal Transportation Star Cluster Identification

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3. Gradient Descent:

Optimal Transportation Star Cluster Identification

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   1. Converges quickly to local minimum (usually)

Optimal Transportation Star Cluster Identification

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2. Random search:
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3. Gradient Descent:
   1. Converges quickly to local minimum (usually)
   2. Misses global minimum and can even diverge

Results:
Preliminary results on class 2 cluster detection:

Optimal Transportation Star Cluster Identification

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$precision = \frac{tp}{tp+fp} = 1$
$recall = \frac{tp}{tp+fn} = 2/5$
$F_1 = 2\frac{precision \cdot recall}{precision+recall} = \frac{4}{7} = 57\%$

Optimal Transportation Star Cluster Identification

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Future Work:

Optimal Transportation Star Cluster Identification

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Future Work:

1. Investigate transporting data across spectrum, not just spacially.

Optimal Transportation Star Cluster Identification

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Future Work:

1. Investigate transporting data across spectrum, not just spacially.
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3. Speedup on GPU.

Bibliography

G. Pérez, M. Messa, D. Calzetti, S. Maji, D. E. Jung, A. Adamo, and M. Sirressi.
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