Control problems of grey systems

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Received 9 September 1981
Revised 26 October 1981

The stability and stabilization of a grey system whose state matrix is triangular is studied. The displacement operator and established transfer developed by the author are the indispensable tool for the grey system.

Keywords: Linear system, Grey system, Unknown system, System analysis, System synthesis, Control theory.

1. The definition of the grey system

Definition 1. What is a grey system? According to the concept of the black box, a system containing knowns and unknowns is called a grey system. What is a grey matrix? A matrix with some of its properties (e.g. the number of columns or rows) known but some elements unknown is called a grey matrix.

2. Controllability

Definition 2. For a grey system

$$\dot{X} = AX + BU, \quad X, U \in R^n,$$
$$Y = CX, \quad Y \in R^n,$$

if (1) matrix B contains more than one known column b_i, (2) the product Ab_i is known, and (3) the rank of matrix M is n with

$$M = [b_i, Ab_i, \ldots, A^{n-1}b_i]$$

then we say that the grey system is controllable. Similarly, we can define the observability of a grey system.

3. The stability of a grey system

3.1. The symmetric system

The multimotor system with a common source and with the moments of inertia of the motors equal to each other is a symmetric system (see Fig. 1)

$$\dot{X} = \begin{bmatrix} \dfrac{-c_1^2}{J_1 r_1} & \dfrac{c_1 c_2 R_\Sigma}{J_1 r_1 r_2} \\ \dfrac{c_1 c_2 R_\Sigma}{J_2 r_1 r_2} & \dfrac{-c_2^2}{J_2 r_2} \end{bmatrix} X + BU, \quad J_1 = J_2.$$

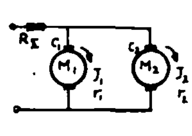

Fig. 1.

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Definition 3. A matrix whose elements $\otimes_{ij}$ and $\otimes_{ji}$, $i \neq j$ are one and the same unknown is called a grey symmetric matrix.

Theorem 1. *For a symmetric system whose state matrix A is*

$$A = \begin{bmatrix} a_{11} & \\ & \otimes \end{bmatrix}, \qquad a_{11} \text{ known}, \quad \otimes \text{ unknown},$$

the system having state matrix or principal submatrix $A^{(r)}$, $r = 1, 2, \ldots, n-1$, is unstable iff $a_{11} > 0$.

Proof. Theorem 1 is an extension of the separate theorem in algebra.

Let $A^{(r)}$ be a real symmetric matrix, that is,

$$A^{(r)} = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix}, \quad r = 1, 2, \ldots, n, \; a_{ij} = a_{ji}.$$

Let us arrange the eigenvalues according to their magnitude, that is,

$$\lambda_1^{(r)} \leqslant \lambda_2^{(r)} \leqslant \cdots \leqslant \lambda_r^{(r)}.$$

The separate theorem in algebra is:

$$\lambda_k^{(r+1)} \leqslant \lambda_k^{(r)} \leqslant \lambda_{k+1}^{(r+1)}.$$

For

$$r = 1, \qquad A^{(1)} = a_{11}, \quad \lambda_1^{(1)} = a_{11};$$

$$r = 2, \qquad A^{(2)} = \begin{bmatrix} a_{11} & \otimes \\ \otimes & \otimes \end{bmatrix}.$$

By using the separate theorem, we have

$$\lambda_1^{(2)} \leqslant \lambda_1^{(1)} \leqslant \lambda_2^{(2)}.$$

If $\lambda_1^{(1)} = a_{11} > 0$, then $\lambda_2^{(2)} > 0$. Certainly there is more than one positive eigenvalue in $A^{(n)} = A$.

3.2. The triangular system

Some practical systems are triangular, for example, the speed control system of a hydraulic turbine (see Fig. 2).

The state equation of this system is as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -e_n/T_a & 1/T_a & 0 & 0 \\ 0 & -h_1/T_w & h_2/T_w & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/T_d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & -1/T_a \\ h_3/T_y & 0 \\ 1/T_y & 0 \\ b_t/T_y & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix}.$$

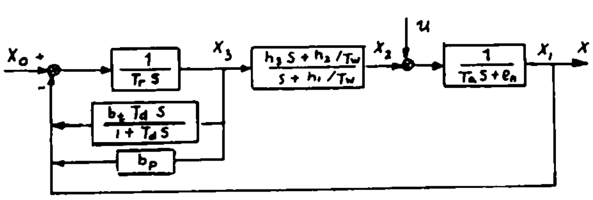

Fig. 2.

Similarly, serial reservoirs and fishery are also triangular systems.

Definition 4. The matrix $A_l = (a_{ij})$, ($A_u = (a_{ij})$) is called a lower (upper) grey triangular matrix, where a_{ii} is known, $i = 1, \cdots n$, and

$$A_l: \quad a_{ij} = 0, \quad j > i; \quad a_{ij} = \otimes, \quad j < i,$$
$$A_u: \quad a_{ij} = 0, \quad j < i; \quad a_{ij} = \otimes, \quad j > i.$$

The matrix $A_c = (a_{ij})$ where a_{ii} is known, $i = 1, \cdots n$, $\otimes_{ij}$ is unknown, while

$$a_{ij} = 0, \quad i = 1, 3, 5, \cdots n-1, \quad j = i+1, i+2, \cdots n, \quad i < j,$$
$$a_{ij} = 0, \quad i = 3, 4, 5, \cdots n, \quad j = 2, 4, 6, \cdots n-1, \quad i > j,$$

is called a comb-shaped grey matrix.

Theorem 2. *A system whose state matrix A is a triangular or comb-shaped grey matrix is stable iff* $a_{ii} < 0$, $i = 1, \ldots, n$.

Proof. Omitted.

3.3. The common system

For the following system:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1^n \\ \vdots \\ y_n^n \end{bmatrix} = \begin{bmatrix} u_1^n \\ \vdots \\ u_n^n \end{bmatrix}, \quad Ay^n = u^n,$$

$$\begin{bmatrix} y_1^n \\ \vdots \\ y_n^n \end{bmatrix} = \begin{bmatrix} g_{11}^n & \cdots & g_{1n}^n \\ \vdots & & \vdots \\ g_{n1}^n & \cdots & g_{nn}^n \end{bmatrix} \begin{bmatrix} u_1^n \\ \vdots \\ u_n^n \end{bmatrix}, \quad y^n = A^{-1}u^n,$$

where A^{-1} is a transfer function matrix, we assume that the elements of A^{-1} can be obtained by experiment, and that there are more than one known elements in A^{-1}. If the information contained in g_{ij}^n's denominator is insufficient for calculating determinant $|A|$, then we may cancel the row related to u_j^n and the column related to y_i^n. Thus we have

$$y^{n-1} = \begin{bmatrix} g_{11}^{n-1} & \cdots & g_{1,n-1}^{n-1} \\ \vdots & & \vdots \\ g_{n-1,1}^{n-1} & \cdots & g_{n-1,n-1}^{n-1} \end{bmatrix} u^{n-1}$$

where g_{kl}^{n-1} is assumed to be known. Similarly, we may constrict the system until all of the knowns g_{ij}^n, $g_{kl}^{n-1}, \ldots, g_{pq}^1$ are obtained. Then the characteristic polynomial of this grey system is as follows:

$$|A| = \frac{1}{g_{ij}^n g_{kl}^{n-1} \cdots g_{pq}^1}. \tag{1}$$

Let us consider a third-order system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_1^3 \\ y_2^3 \\ y_3^3 \end{bmatrix} = \begin{bmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \end{bmatrix},$$

$$\begin{bmatrix} y_1^3 \\ y_2^3 \\ y_3^3 \end{bmatrix} = \underset{\text{transfer function}}{\begin{bmatrix} g_{11}^3 & g_{12}^3 & g_{13}^3 \\ g_{21}^3 & g_{22}^3 & g_{23}^3 \\ g_{31}^3 & g_{32}^3 & g_{33}^3 \end{bmatrix}} \begin{bmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \end{bmatrix} = \begin{bmatrix} |A_{11}| & |A_{21}| & |A_{31}| \\ |A_{12}| & |A_{22}| & |A_{32}| \\ |A_{13}| & |A_{23}| & |A_{33}| \end{bmatrix} \frac{1}{|A|} \begin{bmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \end{bmatrix}.$$

If g_{21}^3 is known, then we have

$$g_{21}^3 = \frac{|A_{12}|}{|A|}.$$

If the message in g_{21}^3 for calculating $|A|$ is insufficient, in other words, if the part of the message in $|A|$ is absorbed by $|A_{12}|$ then $|A|$ can be expressed as follows:

$$|A| = |A_{12}|A^*, \qquad g_{21}^3 = \frac{|A_{12}|}{|A|} = \frac{|A_{12}|}{|A_{12}|A^*} = \frac{1}{A^*},$$

where g_{21}^3 and A^* are known. To calculate $|A|$, we have to find $|A_{12}|$, A_{12} being expressed as:

$$A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}.$$

How to obtain the submatrix A_{12}? After canceling the row related to u_1^3 and the column related to y_2^3 in the third-order system, the second-order system A_{12} is obtained. For the second-order system, we have

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \begin{bmatrix} y_1^2 \\ y_2^2 \end{bmatrix} = \begin{bmatrix} u_1^2 \\ u_2^2 \end{bmatrix}, \qquad \begin{bmatrix} y_1^2 \\ y_2^2 \end{bmatrix} = \begin{bmatrix} g_{11}^2 & g_{12}^2 \\ g_{21}^2 & g_{22}^2 \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_2^2 \end{bmatrix} = \begin{bmatrix} a_{33} & -a_{23} \\ -a_{11} & a_{21} \end{bmatrix} \frac{1}{|A_{12}|} \begin{bmatrix} u_1^2 \\ u_2^2 \end{bmatrix}.$$

If g_{12}^2 is known in the second-order system, we have

$$g_{12}^2 = \frac{-a_{23}}{|A_{12}|}, \qquad |A_{12}| = \frac{-a_{23}}{g_{12}^2}.$$

If the element a_{23} is known, then

$$|A| = |A_{12}| \frac{1}{g_{21}^3} = \frac{-a_{23}}{g_{12}^2 g_{21}^3}.$$

How to obtain a_{23}? Here, we assume that the message in g_{12}^2 is insufficient, then we may cancel the row related to u_2^2 and the column related to y_1^2 in the second-order system to obtain a_{23}.

4. The control of a grey system

For a white (known) system the control problem means a map which is from state space to control space. For a grey system, however, there are two maps: the first is from grey state space to white state space; the second is from white state space to white control space. Now we want to propose a transfer for the first map, this transfer (map) is called an established transfer (map) (see Fig. 3).

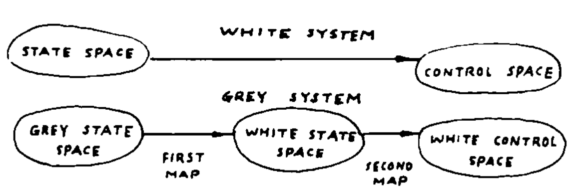

Fig. 3.

4.1. The displacement operator

Let us define an operator ρ_{ij} as follows:

$$\rho_{ij} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & a_{ij} & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad a_{ij} = 1, \qquad \rho_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where only $a_{ij} = 1$ is nonzero. ρ_{ij} is called a displacement operator.

4.2. Extension

Matrix T_b is always the linear combination of displacement operators. For instance,

$$T_b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = b_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + b_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \sum_{i=1}^{2}\sum_{j=1}^{2} b_{ij}\rho_{ij}.$$

4.3. Transfer

The following product of matrices

$$M = \rho_{ij} A \rho_{kl}$$

implies that we extract the element a_{jk} from A and set it in matrix M as an element m_{il}. Here, only m_{il} in matrix M is nonzero. For instance,

$$M = \rho_{41} A \rho_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \underset{A \text{ is grey}}{\begin{bmatrix} \otimes & a_{12} & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \end{bmatrix}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \underset{M \text{ is white}}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12} & 0 \end{bmatrix}}$$

where A is grey but M is white.

4.4. The established transfer

If matrices T_a, A and T_b are grey, while

$$M = T_a A T_b$$

is white, then we say this product is an established transfer.

Theorem 3. *For a grey triangular matrix A,*

$$A = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ \otimes & & a_{nn} \end{bmatrix},$$

if the transfer matrices T_a and T_b are

$$T_a = \sum_{i=1}^{\delta-1} \otimes_{ii}\rho_{ii} + q_{\delta\delta}\rho_{\delta\delta} = \begin{bmatrix} \otimes & & & & & & \\ & \ddots & & & 0 & & \\ & & \otimes & & & & \\ & & & q_{\delta\delta} & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}, \tag{2}$$

$$T_b = \sum_{k=1}^{n} \sum_{j=\delta+1}^{n} \otimes_{jk}\rho_{jk} + b_{\delta\delta}\rho_{\delta\delta} = \left[\begin{array}{ccc|c} 0 & & 0 & 0 \\ & \ddots & & \\ & & 0 & \\ 0 & & b_{\delta\delta} & \\ \hline & \otimes & & \end{array}\right], \tag{3}$$

then the transferred matrix

$$M = T_a A T_b = \operatorname{diag}(0,\ldots,0, q_{\delta\delta}a_{\delta\delta}b_{\delta\delta}, 0,\ldots,0)$$

is white.

Proof. Omitted.

5. The stabilization of a grey system

For the system

$$\dot{X} = AX + BU, \quad X, U \in R^n,$$
$$Y = CX, \quad Y \in R^n,$$

A is a triangular grey matrix, whose diagonal elements are known. By using the output feedback

$$U = -K^{\mathrm{T}}Y = -K^{\mathrm{T}}CX$$

we obtain the closed-loop polynomial as follows:

$$\det(\lambda I - A + BK^{\mathrm{T}}C).$$

Now, the task confronting us is to obtain the gain K^{T}. For a normal system, this problem is easy to solve. For a grey system, however, it is difficult to solve because the characteristic polynomial is abnormal owing to the existence of a lot of unknowns. This problem can be solved in two steps. First, we transfer the grey closed-loop characteristic polynomial into a normal one. Second, from the normal polynomial, we can obtain the gain K^{T}.

But how to transfer the grey characteristic polynomial into a normal one? In general, matrices B and C are given. However, for a grey problem, we consider that B and C can be designed. By using Theorem 3 that is, if B and C are given as follows:

$$B = \sum_{k=1}^{n} \sum_{j=\delta+1}^{n} \otimes_{jk}\rho_{jk} + b_{\delta\delta}\rho_{\delta\delta} = \left[\begin{array}{ccc|c} 0 & & 0 & 0 \\ & \ddots & & \\ & & 0 & \\ 0 & & b_{\delta\delta} & \\ \hline & \otimes & & \end{array}\right], \tag{4}$$

$$C=\sum_{j=1}^{\delta-1}\otimes_{jj}\rho_{jj}+c_{\delta\delta}\rho_{\delta\delta}=\begin{bmatrix}\otimes & & & & & \\ & \ddots & & 0 & & \\ & & \otimes & & \cdot & \\ & & & c_{\delta\delta} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & 0 & & & & & 0\end{bmatrix}. \tag{5}$$

According to the matrix operation,

$$\begin{aligned}\det(\lambda I-A+BK^{\mathrm{T}}C)&=\det\left[(\lambda I-A)\left(I_n+BK^{\mathrm{T}}C(\lambda I-A)^{-1}\right)\right]\\&=\det(\lambda I-A)\det\left[I_n+BK^{\mathrm{T}}C(\lambda I-A)^{-1}\right].\end{aligned}\tag{6}$$

Let K^{T} be

$$\underset{n\times n}{K^{\mathrm{T}}}=\underset{n\times 1}{q}\ \underset{1\times n}{k},$$

then

$$\begin{aligned}\det(\lambda I-A+BK^{\mathrm{T}}C)&=\det(\lambda I-A)\det\left[I_n+BqkC(\lambda I-A)^{-1}\right]\\&=\det(\lambda I-A)\left[I+kC(\lambda I-A)^{-1}Bq\right]\\&=\det(\lambda I-A)+kC\left[\operatorname{adj}(\lambda I-A)\right]Bq.\end{aligned}$$

As matrix A is triangular, $\det(\lambda I-A)$ is white. Letting C and B have the form of Equations (4) and (5) to be designed, according to Theorem 3, we have

$$C\left[\operatorname{adj}(\lambda I-A)\right]B=\text{white matrix}$$

because this is an established transfer and the diagonal elements in $\operatorname{adj}(\lambda I-A)$ are known. Hence (6) becomes normal.

Acknowledgement

The author wishes to express his gratitude to Prof. Y.C. Ho for encouragment to submit this paper. Thanks are also due to Prof. J. Song for his support of this work since 1979 and to Prof. T.J. Tarn for stimulating discussions.