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A new grey prediction model FGM(1, 1)

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ABSTRACT

The effectiveness of the first entry of the original series by GM(1, 1) is researched in this paper. The results show that the modelling values and forecasts are independent of the first entry of the original series. The grey prediction model presented in this paper is called first-entry GM(1, 1), abbreviated as FGM(1, 1), which is based on the existing GM(1, 1) but modelled with data including the first-entry's messages of the original series. A proof concerning this subject has been presented by other authors. However, the algorithm of their direct proof is too complicated. A more compact algorithm is presented in this paper to prove the first entry of the original series ineffective to the modelling values and forecasts by GM(1, 1). Then, an arbitrary number can be inserted in the front of the original series to extract the messages from its first entry. Only a few data (usually fewer than ten) are used for model building. This paper deals with the effectiveness of the first entry of the original series by GM(1, 1).

1. Introduction

Uncertainty can be divided into two categories: stochastic and fuzzy. Stochastic uncertainty can be analyzed by the probabilistic statistics, that is, by statistics. A large number of data are required by the mathematical statistics which is used to determine the statistical rule of the original data by a large sample. However, a typical rule may not exist within the data even with a large amount of data collected. An untypical process (for example, a non-normal distribution, a non-stationary process, etc.) is difficult to treat. Furthermore, a large amount of data such as the tensile strength of a material at different temperature, or the fatigue limits of a material with different chemical ingredients, are sometimes difficult to collect. Fuzzy uncertainty can be analyzed by the fuzzy mathematics or the grey system theory.

The grey system theory is fairly appropriate for prediction. The accumulated generating operation (abbreviated as AGO) [1,2] is the most important characteristic for the grey system theory and its purpose is to reduce the randomness of data. A non-negative smooth discrete function can be transformed into a sequence having the approximate exponential law which is the so-called grey exponential law. The 1-AGO data of the original series are used as the intermediate information for the grey prediction model building. Using the AGO technique efficiently reduces noise by converting ambiguous original time-series data to a monotonically increased series. The AGO technique is capable of identifying the systematic regularity quickly and easily. That is, the grey prediction just needs minimal data to construct a grey differential equation for prediction. The main feature of grey theory is its capability of using as few as four data items to forecast the future data [1]. The grey prediction has been widely used in engineering sciences, social sciences, agriculture, procreation, power consumption, management as well as other fields [3–7].

The main feature of grey theory is its capability of using as few as four data items to forecast the future data. For example, Chiang and Chen [1] apply GM(1, 1) to help estimate the thermal conductivity which is a nonlinear function of position and temperature. They apply GM(1, 1) by four entries to predict the temperature and decrease the number of measuring

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point accordingly. The grey prediction theory is methodology and it is necessary to constantly put forward new models or algorithm based on the theory to improve its performance, prediction accuracy especially. A proof concerning this subject has been presented by other authors [8]. However, their formulas just to expand almost each product of matrices patiently is too complicated and cannot see through the results without careful step by step calculation. For this purpose, this paper proposes a new prediction model, FGM(1, 1). A relatively ingenious and clear algorithm is presented in this paper that we add an arbitrary constant to the first entry of the original series to get a total of the new first entry and then prove the GM(1, 1) modelling values independent of the arbitrary constant, independent of the first entry accordingly, by the elementary column and row operation without complete expansion of the matrices. Then other authors can confidently apply FGM(1, 1) modelled by only three entries for prediction.

The effectiveness of the first entry of the original series by GM(1, 1) is researched in this paper. The results show that the modelling values (for the priori-sample period) and forecasts (for the post-sample period) are independent of the first entry of the original series. A proof concerning this subject have ever been presented by Lee et al. [8]. However, the algorithm of their direct proof is too complicated. A more compact algorithm is presented in this paper to prove the first entry of the original series ineffective to the modelling values and forecasts by GM(1, 1). Then, an arbitrary number can be inserted in front of the original series to extract the messages from its first entry. By so doing, even three history data can be modelled by FGM(1, 1) to forecast the future data.

2. The effectiveness of the first entry of the original series by GM(1, 1)

GM(1, 1) expresses a time series in the form of differential equation with the 1-AGO data of the original series being used as the intermediate information. The modelling values and forecasts by GM(1, 1) can be proved to be independent of the first entry of the original series as the followings:

2.1. The modelling method of the existing GM(1, 1)

Assume the original series (lifetime data, fatigue limit, or other series) to be

$$X^{(0)} = \left\{X^{(0)}(1), X^{(0)}(2), \ldots, X^{(0)}(r)\right\}, \tag{1}$$

and then the first-order accumulated generating operation (1-AGO) of $X^{(0)}$ is given by

$$X^{(1)}(k) = \sum_{i=1}^{k} X^{(0)}(i), \quad k = 1, 2, \ldots, r. \tag{2}$$

The grey generated model, based on the series

$$X^{(1)} = \left\{X^{(1)}(1), X^{(1)}(2), \ldots, X^{(1)}(r)\right\}, \tag{3}$$

is given by the differential equation

$$\frac{\mathrm{d}X^{(1)}(t)}{\mathrm{d}t} + aX^{(1)}(t) = v, \tag{4}$$

where the grey developmental coefficient a and grey control parameter v are the model parameters to be estimated. Eq. (4) is called the first order grey differential equation and denoted by GM(1, 1) [1,2] where the first 1 stands for the first-order derivative of 1-AGO data of X and the second 1 stands for only 1 series to have concern with the grey differential equation.

The grey derivative for the first-order grey differential equation with 1-AGO data as the intermediate information is conventionally represented as

$$\frac{\mathrm{d}X^{(1)}(t)}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{X^{(1)}(t+\Delta t) - X^{(1)}(t)}{\Delta t},$$

and

$$\frac{\mathrm{d}X^{(1)}(t)}{\mathrm{d}t} = \frac{\Delta X^{(1)}(t)}{\Delta t} = X^{(1)}(t+1) - X^{(1)}(t) = X^{(0)}(t+1)$$

when $\Delta t \to 1$ roughly [1]. The background value of $\frac{\mathrm{d}X^{(1)}(t)}{\mathrm{d}t}$, $X^{(1)}(t)$ is taken as the mean of $X^{(1)}(t)$ and $X^{(1)}(t+1)$. The solution to Eq. (4) with system parameters determined by least-squares method and initial condition $X^{(1)}(1) = X^{(0)}(1)$ is

$$\hat{X}^{(1)}(k) = \left[X^{(0)}(1) - \frac{v}{a}\right] \mathrm{e}^{-a(k-1)} + \frac{v}{a}, \quad k = 2, 3, \ldots, \tag{5}$$

where

$$[a, v]^{\mathrm{T}} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y_R, \tag{6}$$

$$B = \begin{bmatrix} -\frac{1}{2}(X^{(1)}(1) + X^{(1)}(2)), & 1.0 \\ -\frac{1}{2}(X^{(1)}(2) + X^{(1)}(3)), & 1.0 \\ \cdots & \\ -\frac{1}{2}(X^{(1)}(r-1) + X^{(1)}(r)), & 1.0 \end{bmatrix}, \tag{7}$$

and

$$Y_R = \left[X^{(0)}(2), X^{(0)}(3), \ldots, X^{(0)}(r)\right]^{\mathrm{T}}. \tag{8}$$

From Eq. (5), and by the first-order inverse accumulated generating operation (1-IAGO) of $\hat{X}^{(1)}$, the modelling value $\hat{X}^{(0)}$ can be derived to be

$$\hat{X}^{(0)}(1) = \hat{X}^{(1)}(1) = X^{(0)}(1), \tag{9a}$$

and

$$\hat{X}^{(0)}(k) = \hat{X}^{(1)}(k) - \hat{X}^{(1)}(k-1) = \left[X^{(0)}(1) - \frac{v}{a}\right](1 - \mathrm{e}^{a})\mathrm{e}^{-a(k-1)}, \quad k = 2, 3, \ldots \tag{9b}$$

Only four sets of data of the original series can be used for the GM(1, 1) model building [2], and therefore it is still significant to determine the system parameters a and v by the least squares method in Eqs. (6)–(8). The original series is not limited to be time dependent; it can be stress dependent, temperature dependent, etc.

2.2. *The first entry $X^{(0)}(1)$ ineffective by the existing GM(1, 1)*

In order to analyze the correlation between the first entry $X^{(0)}(1)$ in Eq. (1) and the grey developmental coefficient a together with the value $[X^{(0)}(1) - \frac{v}{a}]$ in Eq. (9), an arbitrary constant δ can be added to $X^{(0)}(1)$ to get a total of $X^{(0)}(1) + \delta$ and then each entry of the 1-AGO data will be added δ to get a total of $X^{(1)}(i) + \delta, i = 1, 2, \ldots, r$. Then according to the definition of 1-AGO by Eq. (2), the matrix B in Eq. (7) becomes D as

$$D = \begin{bmatrix} -\frac{1}{2}(X^{(1)}(1) + X^{(1)}(2)) - \delta, & 1 \\ -\frac{1}{2}(X^{(1)}(2) + X^{(1)}(3)) - \delta, & 1 \\ \cdots & \\ -\frac{1}{2}(X^{(1)}(r-1) + X^{(1)}(r)) - \delta, & 1 \end{bmatrix} = \begin{bmatrix} z_2 - \delta, & 1 \\ z_3 - \delta, & 1 \\ \cdots & \\ z_r - \delta, & 1 \end{bmatrix}, \tag{10}$$

where $z_i = -\frac{1}{2}(X^{(1)}(i-1) + X^{(1)}(i)),\ i = 2, 3, \ldots, r$.

The system parameters of model (4) determined by least-squares method can be written as

$$\begin{bmatrix} a, & v \end{bmatrix}^{\mathrm{T}} = (D^{\mathrm{T}}D)^{-1}(D^{\mathrm{T}}Y_R). \tag{11}$$

The product theorem of determinant [9] can be applied to the analysis of the effectiveness of the first entry of the original series by GM(1, 1). Consider two matrices G and F of orders $p \times q$ and $q \times p$ $(p < q)$ respectively. Therefore then matrices G and F can be written as

$$G = \begin{bmatrix} g_{11}, & g_{12}, & \ldots, & g_{1q} \\ g_{21}, & g_{22}, & \ldots, & g_{2q} \\ & \cdots & & \\ g_{p1}, & g_{p2}, & \ldots, & g_{pq} \end{bmatrix}$$

and

$$F = \begin{bmatrix} f_{11}, & f_{12}, & \ldots, & f_{1p} \\ f_{21}, & f_{22}, & \ldots, & f_{2p} \\ & \cdots & & \\ f_{q1}, & f_{q2}, & \ldots, & f_{qp} \end{bmatrix}.$$

A partitioned matrix $\begin{bmatrix} G & 0 \\ -I & F \end{bmatrix}$ [9] can be formed from matrices G and F. If we premultiply the matrix $\begin{bmatrix} G & 0 \\ -I & F \end{bmatrix}$ by the elementary row operation matrix $\begin{bmatrix} I & G \\ 0 & I \end{bmatrix}$, we have

$$\begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \begin{bmatrix} G & 0 \\ -I & F \end{bmatrix} = \begin{bmatrix} 0 & GF \\ -I & F \end{bmatrix}. \tag{12}$$

The determinant of a matrix will not vary due to the elementary row operation [9], therefore

$$\begin{vmatrix} G & 0 \\ -I & F \end{vmatrix} = \begin{vmatrix} 0 & GF \\ -I & F \end{vmatrix} = |GF|\,. \tag{13a}$$

If matrices G and F are square matrices of the same order, we have

$$|GF| = |G|\,|F|\,. \tag{13b}$$

The formulas (13a) or (13b) are called the product theorem of determinant. If the adjoint matrix [9] of $(D^{\mathrm{T}}D)$ is denoted as $Adj(D^{\mathrm{T}}D)$, the multiplier $(D^{\mathrm{T}}D)^{-1}$ in Eq. (11) can be expressed by its adjoint matrix as $\frac{[Adj(D^{\mathrm{T}}D)]}{|D^{\mathrm{T}}D|}$, then Eq. (11) becomes

$$[a, v]^{\mathrm{T}} = \frac{[Adj(D^{\mathrm{T}}D)]}{\left|D^{\mathrm{T}}D\right|}(D^{\mathrm{T}}Y_R). \tag{14}$$

Applying the product theorem of determinant, $\left|D^{\mathrm{T}}D\right|$ in Eq. (14) will be independent of δ in Eq. (10). That is

$$\left|(D^{\mathrm{T}}D)\right| = \begin{vmatrix} D^{\mathrm{T}} & 0 \\ -I & D \end{vmatrix} = \begin{vmatrix} z_2-\delta, & z_3-\delta, & \ldots, & z_r-\delta, & 0, & 0 \\ 1, & 1, & \ldots, & 1, & 0, & 0 \\ -1, & 0, & \ldots, & 0, & z_2-\delta, & 1 \\ 0, & -1, & \ldots, & 0, & z_3-\delta, & 1 \\ & & \cdots & & \cdots & \\ 0, & 0, & \ldots, & -1, & z_r-\delta, & 1 \end{vmatrix}. \tag{15a}$$

If G is a matrix, then the elementary row operation $H_{ij}(c)G$ [9] is a matrix obtained by adding c times the jth row to the ith row of G, where $i \neq j$. Similarly, the elementary column operation $K_{ij}(c)G$ [9] is a matrix obtained by adding c times the jth column to the ith column of G, where $i \neq j$. Applying the elementary row operation $H_{12}(\delta)$ to Eq. (15a), we have

$$\left|(D^{\mathrm{T}}D)\right| = \begin{vmatrix} D^{\mathrm{T}} & 0 \\ -I & D \end{vmatrix} = \begin{vmatrix} z_2, & z_3, & \ldots, & z_r, & 0, & 0 \\ 1, & 1, & \ldots, & 1, & 0, & 0 \\ -1, & 0, & \ldots, & 0, & z_2-\delta, & 1 \\ 0, & -1, & \ldots, & 0, & z_3-\delta, & 1 \\ & & \cdots & & \cdots & \\ 0, & 0, & \ldots, & -1, & z_r-\delta, & 1 \end{vmatrix}, \tag{15b}$$

and then applying the elementary column operation $K_{r(r+1)}(\delta)$ to Eq. (15b),

$$\left|(D^{\mathrm{T}}D)\right| = \begin{vmatrix} D^{\mathrm{T}} & 0 \\ -I & D \end{vmatrix} = \begin{vmatrix} z_2, & z_3, & \ldots, & z_r, & 0, & 0 \\ 1, & 1, & \ldots, & 1, & 0, & 0 \\ -1, & 0, & \ldots, & 0, & z_2, & 1 \\ 0, & -1, & \ldots, & 0, & z_3, & 1 \\ & & \cdots & & \cdots & \\ 0, & 0, & \ldots, & -1, & z_r, & 1 \end{vmatrix} \tag{15c}$$

can be obtained. That is, from Eq. (13a),

$$\left|(D^{\mathrm{T}}D)\right| = \left|B^{\mathrm{T}}B\right|, \tag{16}$$

and then the grey developmental coefficient a and grey control parameter v in Eq. (11) can be written respectively as

$$a = \frac{A}{\left|D^{\mathrm{T}}D\right|} \tag{17}$$

and

$$v = \frac{V}{\left|D^{\mathrm{T}}D\right|} \tag{18}$$

where A and V are equal to the corresponding determinants of the matrices obtained by replacing the first and second rows of $D^{\mathrm{T}}D$ by $(D^{\mathrm{T}}Y_R)^{\mathrm{T}}$ (or $Y_R^{\mathrm{T}}D$). Also, expressing $\left|(D^{\mathrm{T}}D)\right|$ as the partitioned matrix (15) by the product theorem of determinant, A is equal to determinant of the matrix obtained by replacing the first $(r-1)$ entries of the first row of $D^{\mathrm{T}}D$ by Y_R^{T} and V is equal to determinant of the matrix obtained by replacing the first $(r-1)$ entries of the second row of $D^{\mathrm{T}}D$ by Y_R^{T}. There then follows

$$a=\frac{1}{\left|D^{\mathrm{T}}D\right|}\begin{vmatrix} X^{(0)}(2), & X^{(0)}(3), & \ldots, & X^{(0)}(r), & 0, & 0\\ 1, & 1, & \ldots, & 1, & 0, & 0\\ -1, & 0, & \ldots, & 0, & b_2-\delta, & 1\\ 0, & -1, & \ldots, & 0, & b_3-\delta, & 1\\ & & \cdots & & \cdots & \\ 0, & 0, & \ldots, & -1, & b_r-\delta, & 1 \end{vmatrix} \tag{19}$$

and

$$v=\frac{1}{\left|D^{\mathrm{T}}D\right|}\begin{vmatrix} b_2-\delta, & b_3-\delta, & \ldots, & b_r-\delta, & 0, & 0\\ X^{(0)}(2), & X^{(0)}(3), & \ldots, & X^{(0)}(r), & 0, & 0\\ -1, & 0, & \ldots, & 0, & b_2-\delta, & 1\\ 0, & -1, & \ldots, & 0, & b_3-\delta, & 1\\ & & \cdots & & \cdots & \\ 0, & 0, & \ldots, & -1, & b_r-\delta, & 1 \end{vmatrix}. \tag{20}$$

Applying elementary column operation $K_{r(r+1)}(\delta)$ to Eqs. (19) and (20) respectively, we have

$$a=\frac{1}{\left|D^{\mathrm{T}}D\right|}\begin{vmatrix} X^{(0)}(2), & X^{(0)}(3), & \ldots, & X^{(0)}(r), & 0, & 0\\ 1, & 1, & \ldots, & 1, & 0, & 0\\ -1, & 0, & \ldots, & 0, & b_2, & 1\\ 0, & -1, & \ldots, & 0, & b_3, & 1\\ & & \cdots & & \cdots & \\ 0, & 0, & \ldots, & -1, & b_r, & 1 \end{vmatrix} \tag{21}$$

and

$$v=\frac{1}{\left|D^{\mathrm{T}}D\right|}\begin{vmatrix} b_2-\delta, & b_3-\delta, & \ldots, & b_r-\delta, & 0, & 0\\ X^{(0)}(2), & X^{(0)}(3), & \ldots, & X^{(0)}(r), & 0, & 0\\ -1, & 0, & \ldots, & 0, & b_2, & 1\\ 0, & -1, & \ldots, & 0, & b_3, & 1\\ & & \cdots & & \cdots & \\ 0, & 0, & \ldots, & -1, & b_r, & 1 \end{vmatrix}. \tag{22}$$

Therefore, the grey developmental coefficient a is independent of δ in Eq. (10), but the grey control parameter v has concern with δ in Eq. (10). By Eqs. (21) and (22), $\frac{\Delta v}{a}$ can be obtained as

$$\frac{\Delta v}{a}=\begin{vmatrix} -\delta, & -\delta, & \ldots, & -\delta, & 0, & 0\\ X^{(0)}(2), & X^{(0)}(3), & \ldots, & X^{(0)}(r), & 0, & 0\\ -1, & 0, & \ldots, & 0, & b_2, & 1\\ 0, & -1, & \ldots, & 0, & b_3, & 1\\ & & \cdots & & \cdots & \\ 0, & 0, & \ldots, & -1, & b_r, & 1 \end{vmatrix}\Bigg/\begin{vmatrix} X^{(0)}(2), & X^{(0)}(3), & \ldots, & X^{(0)}(r), & 0, & 0\\ 1, & 1, & \ldots, & 1, & 0, & 0\\ -1, & 0, & 0, & 0 & b_2, & 1\\ 0, & -1 & 0, & 0 & b_3, & 1\\ & & \cdots & & \cdots & \\ 0, & 0, & 0, & -1, & b_r, & 1 \end{vmatrix}=\delta \tag{23}$$

where $\frac{\Delta v}{a}$ represents the variation of $\frac{v}{a}$ due to the first entry of the original series by GM(1, 1) varying from $X^{(0)}(1)$ to $X^{(0)}(1)+\delta$. Then, $X^{(0)}(1)-\frac{v}{a}$ in Eq. (9a) can be obtained as

$$[X^{(0)}(1)+\delta]-\left(\frac{v}{a}+\frac{\Delta v}{a}\right)=[X^{(0)}(1)+\delta]-\left(\frac{v}{a}+\delta\right)=X^{(0)}(1)-\frac{v}{a}. \tag{24}$$

Therefore, the modelling values and forecasts by GM(1, 1) in Eq. (9a) are independent of the arbitrary constant δ and also independent of the first entry of the original series accordingly.

2.3. The modelling method of FGM(1, 1) presented in this paper

An arbitrary number can be inserted in front of the original series to extract the messages from its first entry. The modelling procedure of FGM(1, 1) is carried out in detail as follows.

Assume the original series (lifetime data, fatigue limit, or other series) to be

$$X^{(0)}=\left\{X^{(0)}(0),X^{(0)}(1),\ldots,X^{(0)}(r)\right\}, \tag{25}$$

where the first entry $X^{(0)}(0)$ is an arbitrary number, and the first-order accumulated generating operation of $X^{(0)}$ is given by

$$X^{(1)}(k) = \sum_{i=0}^{k} X^{(0)}(i), \quad k = 0, 1, \dots, r. \tag{26}$$

The grey generated model, based on the series

$$X^{(1)} = \left\{X^{(1)}(0), X^{(1)}(1), \dots, X^{(1)}(r)\right\}, \tag{27}$$

is given by the differential equation

$$\frac{\mathrm{d}X^{(1)}(t)}{\mathrm{d}t} + aX^{(1)}(t) = v, \tag{28}$$

where the grey developmental coefficient a and grey control parameter v are the model parameters to be estimated. Eq. (28) is called the first order grey differential equation and denoted by FGM(1, 1) where the first 1 stands for the first-order derivative of 1-AGO data of X and the second 1 stands for only 1 series to have concern with the grey differential equation.

The grey derivative for the first-order grey differential equation with 1-AGO data as the intermediate information is conventionally represented as

$$\frac{\mathrm{d}X^{(1)}(t)}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{X^{(1)}(t+\Delta t) - X^{(1)}(t)}{\Delta t},$$

and

$$\frac{\mathrm{d}X^{(1)}(t)}{\mathrm{d}t} = \frac{\Delta X^{(1)}(t)}{\Delta t} = X^{(1)}(t+1) - X^{(1)}(t) = X^{(0)}(t+1)$$

when $\Delta t \to 1$ [1]. The background value of $\frac{\mathrm{d}X^{(1)}(t)}{\mathrm{d}t}$, $X^{(1)}(t)$ is taken as the mean of $X^{(1)}(t)$ and $X^{(1)}(t+1)$. The solution to Eq. (28) with system parameters determined by least-squares method and initial condition $X^{(1)}(0) = X^{(0)}(0)$ is

$$\hat{X}^{(1)}(k) = \left[X^{(0)}(1) - \frac{v}{a}\right] \mathrm{e}^{-a(k-1)} + \frac{v}{a}, \quad k = 1, 2, \dots, \tag{29}$$

where

$$[a, v]^{\mathrm{T}} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y_R, \tag{30}$$

$$B = \begin{bmatrix} -\frac{1}{2}(X^{(1)}(0) + X^{(1)}(1)), & 1.0 \\ -\frac{1}{2}(X^{(1)}(1) + X^{(1)}(2)), & 1.0 \\ \dots & \\ -\frac{1}{2}(X^{(1)}(r-1) + X^{(1)}(r)), & 1.0 \end{bmatrix}, \tag{31}$$

and

$$Y_R = \left[X^{(0)}(1), X^{(0)}(2), \dots, X^{(0)}(r)\right]^{\mathrm{T}}. \tag{32}$$

From Eq. (29), and by the first-order inverse accumulated generating operation (1-IAGO) of $\hat{X}^{(1)}$, the modelling value $\hat{X}^{(0)}$ can be derived to be

$$\hat{X}^{(0)}(0) = X^{(0)}(0), \tag{33a}$$

and

$$\hat{X}^{(0)}(k) = \hat{X}^{(1)}(k) - \hat{X}^{(1)}(k-1) = \left[X^{(0)}(1) - \frac{v}{a}\right](1 - \mathrm{e}^{a})\mathrm{e}^{-a(k-1)}, \quad k = 1, 2, \dots \tag{33b}$$

By so doing, even three history data can be modelled by FGM(1, 1) to forecast the future data.

3. Example

There are two series, one monotonically increasing and the other monotonically decreasing, to demonstrate the performance by FGM(1, 1) better than that by GM(1, 1).

3.1. The failure time prediction of products

There is a kind of product, whose failure time is Weibull distribution [10–12]. 30 samples of such product are randomly placed on test until 15 samples fail. The failure times are tabulated in Table 1 in sequence [12]. We can determine shape

Table 1
Lifetimes $t^{(0)}$ obtained by actual test, $\hat{t}^{(0)}$ predicted by GM(1, 1), and $m\hat{t}^{(0)}$ predicted by FGM(1, 1).

i	$t^{(0)}(i)$	$\hat{t}^{(0)}(i)$	$m\hat{t}^{(0)}(i)$
0	7	7	0.0
1	9.4	9.4	9.9725
2	12.5	12.1584	11.7379
3	14.0	14.0920	13.8159
4	15.9	16.3332	16.2617
5	19.3	18.9309	19.1406
6	24.1	21.9416	22.5291
7	25.8	25.4312	26.5174
8	28.7	29.4758	31.2118
9	39.6	34.1636	36.7373
10	42.2	39.5969	43.2410
11	58.3	45.8944	50.8960
12	77.5	53.1934	59.9061
13	89.6	61.6533	70.5114
14	98.0	71.4586	82.9941
15	106.4	82.8233	97.6867

parameter m and characteristic life η of the product by good linear unbiased estimator (GLUE) as following [11,12]:

We can determine $\hat{\sigma}$ and $\hat{\mu}$ by GLUE since $r = 15 < 0.9 \times 30$. There then follow

$$\hat{\sigma} = \frac{(r-1)\ln X^{(0)}(r) - \sum_{i=1}^{r-1} \ln X^{(0)}(i)}{nK_{r,n}} = \frac{65.3409 - 47.8389}{16.4445} = 1.0643, \tag{34}$$

and

$$\hat{\mu} = \ln X^{(0)}(r) - \hat{\sigma} E(Z_r) = 4.6672 - 1.0643 \times (-0.4253) = 5.1199, \tag{35}$$

where $nK_{r,n} = 16.4445$ and $E(Z_r) = -0.4253$ are obtained from “Reliability test table” [11]. Therefore, the estimates of m and η there follow

$$\hat{m} = \frac{g_{r,n}}{\hat{\sigma}} = \frac{0.9382}{1.0643} = 0.8815, \tag{36}$$

and

$$\hat{\eta} = \mathrm{e}^{\hat{\mu}} = 167.31 \text{ h}, \tag{37}$$

where $g_{r,n} = g_{15,30} = 0.9382$ is also obtained from “Reliability test table”.

If the test is terminated for several samples (for example, 5 instead of 15 failed samples), then the rest could be predicted by GM(1, 1) and FGM(1, 1). The first five lifetime data tabulated in Table 1 by actual test are

$$t^{(0)} = \{t^{(0)}(1), t^{(0)}(2), \ldots, t^{(0)}(5)\} = \{9.4, 12.5, 14.0, 15.9, 19.3\} \tag{38a}$$

or

$$t^{(0)} = \{t^{(0)}(0), t^{(0)}(1), \ldots, t^{(0)}(5)\} = \{0.0, 9.4, 12.5, 14.0, 15.9, 19.3\} \tag{38b}$$

where the entry 0.0 is an arbitrarily assigned number. We will predict the last 10 data respectively by the existing GM(1, 1) and the FGM(1, 1) presented in this paper, and then calculate the shape parameter m and characteristic life η of the product by GLUE.

3.1.1. *The failure time prediction by the existing GM(1, 1)*

B and Y_R in Eqs. (7) and (8) can be obtained by the first five lifetime data (38a) as

$$B = \begin{bmatrix} -15.65, & -28.90, & -43.85, & -61.45 \\ 1.0, & 1.0, & 1.0, & 1.0 \end{bmatrix}^{\mathrm{T}},$$

and

$$Y_R = [12.5, 14.0, 15.9, 19.3]^{\mathrm{T}},$$

and the system parameters of model (4) determined by least-squares method,

$$\begin{bmatrix} a \\ v \end{bmatrix} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y_R = \begin{bmatrix} -0.1476 \\ 9.8959 \end{bmatrix},$$

can be evaluated by Eq. (6).

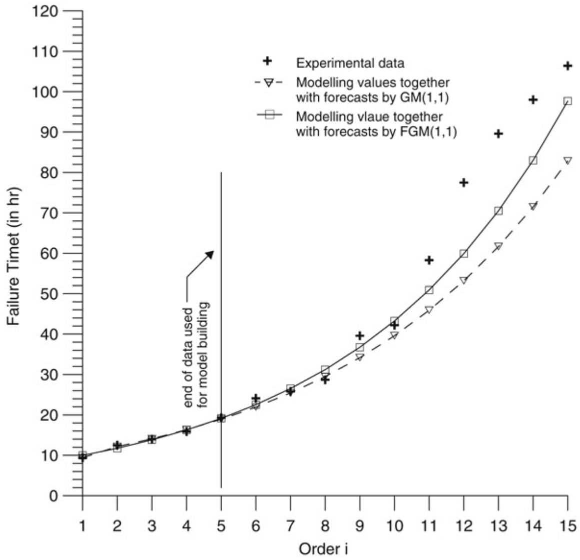

Fig. 1. The experimental data, and modelling values together with forecasts of lifetime data by GM(1, 1) and FGM(1, 1) respectively.

Table 2
Shape parameter $\hat{m}$ and characteristic life $\hat{\eta}$ based on the actual data, the first five actual data combined respectively with the other predicted values by GM(1, 1) and by FGM(1, 1).

By the actual data $t^{(0)}(i)$, $i = 1, 2, \ldots, 15$	$\hat{m}$	0.8815
	$\hat{\eta}$	167.31 h
	Error of $\hat{\eta}$	7
By the first five actual data, $t^{(0)}(i)$, $i = 1, 2, \ldots, 5$, combined with the predicted data by GM(1, 1)	$\hat{m}$	0.9894
	$\hat{\eta}$	123.96 h
	Error of $\hat{\eta}$	−25.91%
By the first five actual data, $t^{(0)}(i)$, $i = 1, 2, \ldots, 5$, combined with the predicted data by FGM(1, 1)	$\hat{m}$	0.9016
	$\hat{\eta}$	152.07 h
	Error of $\hat{\eta}$	−9.11%

From Eqs. (9a) and (9b), there follows

$$\hat{t}^{(0)}(1) = X^{(0)}(1) = 9.4,$$

and

$$\begin{aligned}\hat{t}^{(0)}(i) &= \left(9.4 - \frac{9.8959}{-0.1476}\right)(1 - \mathrm{e}^{-0.1476})\mathrm{e}^{0.1476(i-1)} \\ &= 10.4901 \times \mathrm{e}^{0.1476(i-1)}, \quad (i = 2, 3, \ldots).\end{aligned}$$

The modelling values and predicted failure time $\hat{t}^{(0)}$ by GM(1, 1) are tabulated in Table 1 and as shown in Fig. 1, and the estimating shape parameter $\hat{m}$ and characteristic life $\hat{\eta}$ based on the first five actual data and the predicted values by GM(1, 1) are tabulated in Table 2.

3.1.2. The failure time prediction by the FGM(1, 1) presented in this paper

B and Y_R in Eqs. (31) and (32) can be obtained by the first six lifetime data (38b) (including the arbitrarily assigned number 0.0) as

$$B = \begin{bmatrix} -4.70, & -15.65, & -28.90, & -43.85, & -61.45 \\ 1.0, & 1.0, & 1.0, & 1.0, & 1.0 \end{bmatrix}^{\mathrm{T}},$$

and

$$Y_R = [9.4, 12.5, 14.0, 15.9, 19.3]^{\mathrm{T}},$$

and the system parameters of model (28) determined by least-squares method,

$$\begin{bmatrix} a \\ v \end{bmatrix} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y_R = \begin{bmatrix} -0.1630 \\ 9.1818 \end{bmatrix},$$

can be evaluated by Eq. (30).

Table 3
The tensile strength $s^{(0)}$ obtained by actual test, $\hat{s}^{(0)}$ predicted by GM(1, 1) and $m\hat{s}^{(0)}$ predicted by FGM(1, 1) of Samuel [13]: For temperature 400–1300 °F.

i (Temperature °F)	Tensile strength by actual test $s^{(0)}(i)$ (MPa)	Modelling values and forecasts/error (%) of tensile strength by GM(1, 1) $\hat{s}^{(0)}(i)$ (MPa)	Modelling values and forecasts/error (%) of tensile strength by FGM(1, 1) $m\hat{s}^{(0)}(i)$ (MPa)
0	7	7	0.0
1 (400)	1931	1931/0.0	1933.34/0.12
2 (500)	1724	1720.23/−0.22	1713.41/−0.61
3 (600)	1517	1519.19/0.14	1518.49/0.10
4 (700)	1345	1341.65/−0.25	1345.74/0.06
5 (800)	1207	1184.86/−1.83	1192.65/−1.19
6 (900)	1069	1046.39/−2.12	1056.97/−1.13
7 (1000)	952	924.10/−2.93	936.73/−1.60
8 (1100)	848	816.10/−3.76	830.17/−2.10
9 (1200)	745	720.73/−3.26	735.73/−1.24
10 (1300)	669	636.50/−4.86	652.03/−2.54

From Eq. (33b), there follows

$$\begin{aligned} m\hat{t}^{(0)}(i) &= \left(0.0 - \frac{9.1818}{-0.1630}\right)(1 - e^{-0.1630})e^{0.1630(i-1)} \\ &= 8.4726 \times e^{0.1630(i-1)}, \quad (i = 1, 2, \ldots). \end{aligned}$$

The modelling values and predicted failure time $m\hat{t}^{(0)}$ by FGM(1, 1) are tabulated in Table 1 and as shown in Fig. 1, and the estimating shape parameter $\hat{m}$ and characteristic life $\hat{\eta}$ based on the first five actual data and the predicted values by FGM(1, 1) are tabulated in Table 2.

3.2. *The tensile strength prediction of materials*

The hardness and strength of materials will always decrease monotonically with increasing temperature. The prediction of the tensile strength corresponding to higher temperature of a material can be made by GM(1, 1) and FGM(1, 1). Consider the series in Table 124 of Samuel [13], the experimental data of the tensile strength of heat-treated steel C1144 (coarse grained) from 400 °F through 1300 °F. There are ten observations tabulated in Table 3. The material is annealed at 1450 °F, normalized at 1650 °F and quenched in water from 1550 °F. In general, it takes about 2 min to measure each tensile strength datum; insulation is needed for a higher temperature measurement (usually over 700 °F or 800 °F). If only the first four tensile strength data corresponding to 400–700 ^{0}F are actually tested, then the rest could be predicted by GM(1, 1) and FGM(1, 1). The first five lifetime data tabulated in Table 3 by actual test are

$$s^{(0)} = \{s^{(0)}(1), s^{(0)}(2), \ldots, s^{(0)}(4)\} = \{1931, 1724, 1517, 1345\} \tag{39a}$$

or

$$s^{(0)} = \{s^{(0)}(0), s^{(0)}(1), \ldots, s^{(0)}(4)\} = \{0.0, 1931, 1724, 1517, 1345\} \tag{39b}$$

where the entry 0.0 is an arbitrarily assigned number. We will predict the last 6 data by the existing GM(1, 1) and the FGM(1, 1) presented in this paper.

3.2.1. *The tensile strength prediction by the existing GM(1, 1)*

B and Y_R in Eqs. (7) and (8) can be obtained by the first four tensile strength (39a) as

$$B = \begin{bmatrix} -2793.0, & -4413.5, & -5844.5 \\ 1.0, & 1.0, & 1.0 \end{bmatrix}^{\mathrm{T}},$$

and

$$Y_R = [1724.0, 1517.0, 1345.0]^{\mathrm{T}},$$

and the system parameters of model (4) determined by least-squares method,

$$\begin{bmatrix} a \\ v \end{bmatrix} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y_R = \begin{bmatrix} 0.1243 \\ 2069.3212 \end{bmatrix},$$

can be evaluated by Eq. (6).

From Eqs. (9a) and (9b), there follows

$$\hat{s}^{(0)}(1) = X^{(0)}(1) = 1931.0,$$

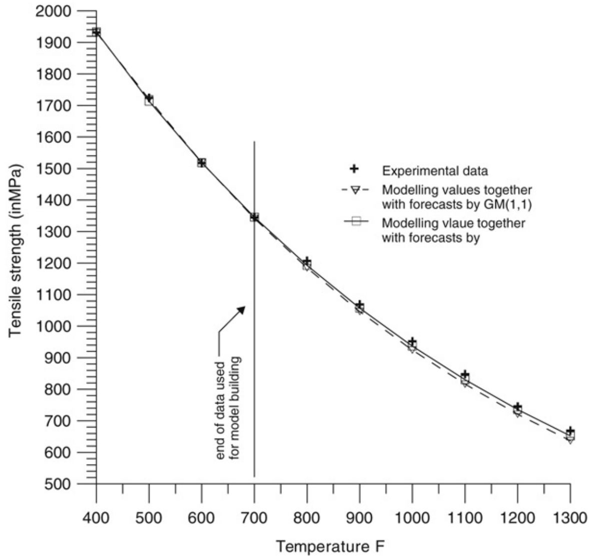

Fig. 2. Forecasts and the modelling values respectively by GM(1, 1) and FGM(1, 1) respectively and the experimental data of the tensile strength of Samuel [13]: for temperature 400–1300 °F.

and

$$\begin{aligned}\hat{s}^{(0)}(i) &= \left(1931.0 - \frac{2069.3212}{0.1243}\right)(1 - e^{0.1243})e^{-0.1243(i-1)} \\ &= 1947.8719 \times e^{-0.1243(i-1)}, \quad (i = 2, 3, \ldots).\end{aligned}$$

The modelling values and predicted failure time $\hat{s}^{(0)}$ by GM(1, 1) are tabulated in Table 3 and as shown in Fig. 1.

3.2.2. The tensile strength prediction by the FGM(1, 1) presented in this paper

B and Y_R in Eqs. (31) and (32) can be obtained by the first six lifetime data (39b) (including the arbitrarily assigned number 0.0) as

$$B = \begin{bmatrix} -965.5, & -2793.0, & -4413.5, & -5844.5 \\ 1.0, & 1.0, & 1.0, & 1.0 \end{bmatrix}^{\mathrm{T}},$$

and

$$Y_R = [1931.0, 1724.0, 1517.0, 1345.0]^{\mathrm{T}},$$

and the system parameters of model (28) determined by least-squares method,

$$\begin{bmatrix} a \\ v \end{bmatrix} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y_R = \begin{bmatrix} 0.1208 \\ 2052.4368 \end{bmatrix},$$

can be evaluated by Eq. (30).

From Eq. (33b), there follows

$$\begin{aligned}m\hat{s}^{(0)}(i) &= \left(0.0 - \frac{2052.4368}{0.1208}\right)(1 - e^{0.1208})e^{-0.1208(i-1)} \\ &= 2181.5148 \times e^{-0.1208(i-1)}, \quad (i = 1, 2, \ldots).\end{aligned}$$

The modelling values and tensile strength $m\hat{s}^{(0)}$ by FGM(1, 1) are tabulated in Table 3 and as shown in Fig. 2.

4. Conclusions

The modelling values and forecasts by GM(1, 1) have been proved to be independent of the first entry of the original series in this paper. An arbitrary number can be inserted in front of the original series to extract the messages from its first entry. Only a few data (usually fewer than ten) are used for model building and the research in this paper is significant in dealing with the effectiveness of the first entry of the original series by GM(1, 1).

The incompletion of information is the primary characteristic of the grey system, and it is necessary to whiten a system by way of inserting more messages in the system. The predicted characteristic life of the product by the FGM(1, 1) presented in this paper is more accurate than that by the existing GM(1, 1) as shown in Table 2. The results with the same modelling data also show that FGM(1, 1) can extract the messages from the data more sufficiently than the existing GM(1, 1).

Besides, a proof concerning this subject has been presented by Lee et al. [8]. However, the algorithm of their direct proof is too complicated and not easy to read. A more compact algorithm is presented in this paper to prove the first entry of the original series ineffective to the modelling values and forecasts by GM(1, 1).

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