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General Solution for Linearized Systematic Error Propagation in Vehicle Odometry

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Abstract

Vehicle odometry is a nonlinear dynamical system in echelon form. Accordingly, a general solution can be written by solving the nonlinear equations in the correct order. Another implication of this structure is that a completely general solution to the linearized (perturbative) dynamics exists. The associated vector convolution integral is the general relationship between output error and both the input error and reference trajectory. Solutions for errors in individual coordinates are in the form of line integrals in state space. Response to initial conditions and translational scale errors, among others, is path independent and vanishes on all closed trajectories. Response to other errors is path dependent and can be reduced to expressions in error moments of the reference trajectory. These path dependent errors vanish on closed symmetric paths, among others. These theoretical results and the underlying error expressions have many uses in design, calibration, and evaluation of odometry systems.

1 Introduction

This paper addresses the problem of understanding the relationship between systematic error present in sensor indications in odometry, and the resultant error in computed vehicle pose. The word "understanding" has been chosen carefully because a numerical solution to the problem of "computing" resultant error is trivial. One simply corrupts the inputs as necessary, integrates both corrupted and uncorrupted systems over time, and measures the difference between corrupted and uncorrupted outputs.

1.1 Motivation

This work is motivated by a recurrent set of questions which arise when designing and constructing position estimation systems for mobile robots for which the answer always seems to require numerical solution. How good do the sensors need to be? What kind of localization error can be expected if we use this particular sensor? Why do some errors seem to cancel out on closed paths while others reverse when you drive backwards? What is the best way to calibrate the model of this sensor?

This paper provides one set of answers to these questions in the general case for systematic error sources. It also turns out that random error propagation yields to identical, but slightly more complicated analysis [1], and many conclusions here are common to both types of error sources.

1.2 Prior Work

The aerospace guidance community has enjoyed the benefits of a theoretical understanding of error propagation for at least five decades [4]. In inertial guidance, the governing differential equations and their solutions have long since been relegated to textbooks [10][11]. It is well known that, in the presence of gravity, most errors exhibit oscillation with the characteristic Schuler period of 84 minutes.

Likewise, the essentially geometric nature of satellite navigation system error relationships has been known since before the GPS satellites were in operation [5]. One of the earliest practical applications of the Kalman filter was in the reset of shipborne inertial systems [3].

Robotics has embraced these results, particularly the Kalman filter, several decades later [7][8]. Using the Kalman filter, the theoretical propagation of error, at least in numerical form, has been an essentially solved problem. However, the guidance community seems not to have provided us with the relevant analytical results for the land navigation systems which are typical of mobile robots - assemblies of wheel encoders, compasses, gyros, etc. Indeed, the problem of analytically computing the navigational error expected from even a given set of sensor errors on a given trajectory seems to be both a fundamental and an unsolved problem.

The AI community has justifiably placed significant emphasis on the question of how error and uncertainty should be represented [6]. However, robotics has expended far more effort addressing the question of how errors should be represented than it has expended understanding what they actually are and how they propagate.

Analytical analysis of error propagation in mobile robot odometry seems to have been largely ignored in the literature with a few exceptions. Early work in [12] concentrates on improving estimates for a single iteration of the estimation algorithm by incorporating knowledge of the geometry of the path followed between odometry updates. In [14], a method is presented which permits the calibration of systematic errors which are observable on rectangular closed trajectories by solving geometric relationships. In [13], a solution is obtained for non systematic error on constant curvature trajectories by solving a recurrence equation. This paper builds on this earlier work and presents the general solution for linearized systematic error propagation for any trajectory and any error model. The result is also extended elsewhere [1] in a natural way to apply to non systematic error.

1.3 Problem Description

One of the most important distinctions in position estimation is the distinction between triangulation and dead reckoning. These names are no longer as descriptive as they were when they were coined centuries ago [15] before mathematics developed past geometry. The essential difference from a mathematics perspective is whether the available observations project onto the states of interest, or onto their derivatives. Odometry is a form of dead reckoning and can be described, in the 2D case, by the following nonlinear system of differential equations:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \\ \omega(t) \end{bmatrix} \qquad \dot{\underline{x}}(t) = f(\underline{x}(t), \underline{u}(t)) \tag{1}$$

where the state vector $\underline{x}(t)$ and input vector $\underline{u}(t)$ are:

$$\underline{x}(t) = \begin{bmatrix} x(t) & y(t) & \theta(t) \end{bmatrix}^T \qquad \underline{u}(t) = \begin{bmatrix} V(t) & \omega(t) \end{bmatrix}^T$$

In order to make maximum use of available theory, we have used the technique of identifying the sensor inputs, normally denoted $\underline{z}(t)$ with the process inputs $\underline{u}(t)$. We have implicitly chosen the x axis as the heading datum. This situation is illustrated below:

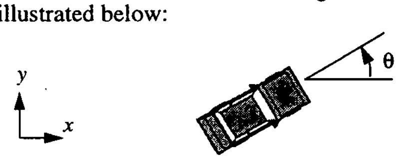


Figure 1: Coordinates for odometry.

For our purposes, it will occasionally be useful to add an observer equation which permits the input to be overdetermined:

$$\underline{z}(t) = h(\underline{x}(t), \underline{u}(t)) \tag{2}$$

This device could be used to model, for example, a situation where encoders are provided on 4 wheels even though two are enough to determine position and heading.

Many alternative formulations of odometry are possible but the above formulation has two key properties. First, it is memoryless because the zero input response is zero. Second, it is in echelon form because any given equation depends only on the states below it in the order listed. As a result of the second property, the solution is immediate and well-known:

$$\begin{aligned} \theta(t) &= \theta(0) + \int_0^t \omega(t)dt \\ x(t) &= x(0) + \int_0^t V(t)\cos\theta(t)dt \\ y(t) &= y(0) + \int_0^t V(t)\sin\theta(t)dt \end{aligned} \tag{3}$$

Closed-form solutions to integrals of general functions do not exist. The best that can be achieved is to eliminate the original self reference of the state derivative to the state itself and write an explicit integral for the trajectory resulting from the input. This is as closed-form as a general differential equation solution can be.

This paper addresses the following problem. Let the inputs to the system be corrupted by systematic additive errors as follows:

$$V'(t) = V(t) + \delta V(t) \qquad \omega'(t) = \omega(t) + \delta\omega(t)$$

Using these input errors and the system dynamics, determine the associated errors in the computed vehicle pose:

$$\begin{aligned} x'(t) &= x(t) + \delta x(t) \\ y'(t) &= y(t) + \delta y(t) \\ \theta'(t) &= \theta(t) + \delta\theta(t) \end{aligned}$$

2 Linearized Error Dynamics

A derivative distinction between dead reckoning and triangulation is that errors in triangulation are felt when they occur whereas errors in dead reckoning are felt forever thereafter. This is equivalent to noting that errors in dead reckoning exhibit dynamics. For example, consider a single discrete error in the input angular velocity:

$$\delta\omega(t) = \Delta\theta\delta(t-\tau)$$

This single error rotates the entire subsequent computed trajectory about the point $[x(\tau), y(\tau)]$. If $\underline{r}(t)$ denotes the position vector in the plane, then the magnitude of the error at any subsequent time is:

$$\delta\underline{r}(t) = \sqrt{\delta x^2(t) + \delta y^2(t)} = |\underline{r}(t) - \underline{r}(\tau)|\Delta\theta$$

This expression is linear in distance from the point where the error occurred, so it is unbounded and vanishes upon return to $\underline{r}(\tau)$. This simple illustration foreshadows much of the sequel.

In principle, the direct solution for error dynamics is to substitute the perturbed inputs into the general solution and attempt to integrate it and cancel the reference trajectory. An attempt to do so above quickly leads to the need to integrate expressions like:

$$\int_0^t \delta V(t)\{\cos\theta(t)\sin\delta\theta(t)\}dt$$

With errors hidden inside trig functions inside integrals, this is the limit of simplification achievable without making strong assumptions on the form of the errors. While closed-form solutions exist for nonlinear propagation of constant errors, for example, we will pursue linearized solutions here for two reasons. First, it then becomes possible to generate general solutions. Second, linear results can be readily compared with the stochastic error propagation results anticipated in future work.

2.1 Perturbative Dynamics

Perturbative techniques linearize nonlinear dynamical systems in order to study their first order behavior. As long as errors are small, the perturbative dynamics are a good approximation to the exact behavior, and for our purposes, will be far more illuminating. Equation (1) is linearized as follows:

$$\delta\dot{\underline{x}}(t) = F\{\underline{x}(t), \underline{u}(t)\}\delta\underline{x}(t) + G\{\underline{x}(t), \underline{u}(t)\}\delta\underline{u}(t) \tag{4}$$

where the Jacobians may depend on the state and the input, and are evaluated on some reference trajectory:

$$F(t) = \frac{\partial}{\partial \underline{x}} \underline{f} \bigg|_{\underline{x}} \qquad G(t) = \frac{\partial}{\partial \underline{u}} \underline{f} \bigg|_{\underline{x}}$$

Although equation (4) may still be nonlinear in the state and the input, it is linear in the perturbations.

2.2 Solution for Commutative Dynamics

Our linearized differential equation is of the form of a time varying linear system:

$$\dot{\underline{x}}(t) = F(t)\underline{x}(t) + G(t)\underline{u}(t)$$

While the transition matrix $\Phi(t, \tau)$ (which is tantamount to a solution to such systems) is known to exist, there is no guarantee that it will be easy to find. However, consider the particular matrix exponential:

$$\Psi(t, \tau) = \exp\left(\int_{\tau}^{t} F(\zeta)d\zeta\right) \tag{5}$$

where the matrix exponential is defined as usual by the infinite matrix power series:

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

When this commutes [2] with the system dynamics matrix:

$$\Psi(t, \tau)F(t) = F(t)\Psi(t, \tau)$$

it is the transition matrix which solves the associated time-varying linear system and the total solution is of the form of the vector convolution integral:

$$\underline{x}(t) = \Phi(t, t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)\underline{u}(\tau)d\tau \tag{6}$$

This property of commutative dynamics is the key for solving odometry error propagation in closed form.

2.3 Influence Vectors

The only unknown in equation (6) is the transition matrix. For convenience, we can define the (potentially nonsquare) input transition matrix as:

$$\bar{\Phi}(t, \tau) = \Phi(t, \tau)G(\tau)$$

As we shall see, this matrix is the defining matrix for each form of odometry because it captures both the effects of system dynamics and the state observer. Let $\bar{\Phi}_i$ denote the ith column of the input transition matrix. Notice that for a given element u_i of $\underline{u}$, its contribution to the integrand is:

$$d\underline{x}(t, \tau) = \bar{\underline{\Phi}}_i u_i d\tau$$

The vectors $\bar{\underline{\Phi}}_i$ define the projection of each element of the input (measurement) error vector onto the entire output (state) error vector. Indeed, equation (6) can be rewritten:

$$\underline{x}(t) = \Phi(t, t_0)\underline{x}(t_0) + \int_{t_0}^{t}\left(\sum_i \bar{\underline{\Phi}}_i u_i\right)d\tau \tag{7}$$

Note that the order of integration and summation can be reversed when convenient.

2.4 Transition Matrix

It was noted earlier that our odometry system is in echelon form. Essentially, this means that the system Jacobian $F(t)$ is strictly upper triangular:

$$F = \{f_{ij} | f_{ij} = 0 \text{ when } (j \le i)\}$$

and since $\Psi(t, \tau)$ is composed entirely of definite integrals of $F(t)$, it is also strictly upper triangular. It can be shown that the nth power (and hence all subsequent powers) of an $n \times n$ strictly upper triangular matrix vanishes. This means that the matrix exponential can be easily written by summing the first few nonzero terms so we will be able to write closed-form expressions for the transition matrix.

2.5 Linearized Observer Formulation

When an observer (equation (2)) is used to observe the input indirectly, it can be linearized as follows:

$$\delta\underline{z}(t) = H(t)\delta\underline{x}(t) + M(t)\delta\underline{u}(t) \tag{8}$$

Where the observer Jacobians are:

$$H(t) = \frac{\partial}{\partial \underline{x}} \underline{h} \bigg|_{\underline{x}} \qquad M(t) = \frac{\partial}{\partial \underline{u}} \underline{h} \bigg|_{\underline{x}} \tag{9}$$

If $\delta\underline{u}(t)$ is not known directly but $\delta\underline{z}(t)$ is, we can solve the above equation for $\delta\underline{u}(t)$ by first writing:

$$M(t)\delta\underline{u}(t) = \delta\underline{z}(t) - H(t)\delta\underline{x}(t)$$

In the event that the measurements determine or overdetermine the inputs, the left pseudoinverse applies:

$$M^{LI} = [M^T(t)M(t)]^{-1}M^T(t)$$

and we can write:

$$\delta\underline{u}(t) = M^{LI}[\delta\underline{z}(t) - H(t)\delta\underline{x}(t)]$$

This is the input which minimizes the residual:

$$\delta\underline{z}(t) - [H(t)\delta\underline{x}(t) + M(t)\delta\underline{u}(t)]$$

Substituting this back into the state perturbation equation we have:

$$\delta\dot{\underline{x}}(t) = F(t)\delta\underline{x}(t) + G(t)\{M^{LI}[\delta\underline{z}(t) - H(t)\delta\underline{x}(t)]\}$$

Which reduces to:

$$\delta\dot{\underline{x}}(t) = \{F(t) - G(t)M^{LI}H(t)\}\delta\underline{x}(t) + G(t)M^{LI}\delta\underline{z}(t) \tag{10}$$

This is of the same form as the original perturbation equation with modified matrices and the measurements acting as the input:

$$\delta\dot{\underline{x}}(t) = \bar{F}(t)\delta\underline{x}(t) + \bar{G}(t)\delta\underline{z}(t)$$

3 Application to Odometry

This section will derive the error propagation equations for a few common forms of odometry.

3.1 Direct Heading Odometry

The term direct heading odometry will be used to refer to the case where a direct measurement of heading is available rather than its derivative. For example, a compass could be used to measure heading directly and a transmission encoder could be used to measure the linear velocity of the center of an axle of the vehicle.

The heading and error in heading are respectively equal at all times to the heading measurement and its error, so there is no need for a state to track this. Considering the heading to be an input, the state equations are:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \end{bmatrix}$$

This system is clearly memoryless since the states do not appear on the right hand side. Perturbing this, we have:

$$\frac{d}{dt}\begin{bmatrix} \delta x(t) \\ \delta y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \delta x(t) \\ \delta y(t) \end{bmatrix} + \begin{bmatrix} \cos\theta(t) & -V(t)\sin\theta(t) \\ \sin\theta(t) & V(t)\cos\theta(t) \end{bmatrix}\begin{bmatrix} \delta V(t) \\ \delta\theta(t) \end{bmatrix}$$

The vanishing system Jacobian clearly satisfies equation (5). The transition matrix is trivial:

$$\Phi(t,\tau) = \exp\left(\int_{\tau}^{t} F(\zeta)d\zeta\right) = \exp[0] = I$$

Substituting into the general solution in equation (6) gives:

$$\delta \underline{x}(t) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \delta V(\tau)\cos\theta(\tau) \\ \delta V(\tau)\sin\theta(\tau) \end{bmatrix} d\tau + \int_{path} \begin{bmatrix} -\delta\theta(\underline{x})dy \\ \delta\theta(\underline{x})dx \end{bmatrix} \quad (14)$$

This is the general (linearized) solution for the propagation of systematic error in 2D direct heading odometry for any trajectory and any error model.

3.2 Integrated Heading Odometry

In integrated heading odometry, an angular velocity indication is available and a heading state is necessary which is then integrated to get the heading. For example, a gyro could be used to measure heading rate and a transmission encoder, groundspeed radar, or fifth wheel encoder could be used to measure the linear velocity of the center of an axle of the vehicle. This is the case given in equation (1) repeated here for reference:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \\ \omega(t) \end{bmatrix}$$

Let us also define notation for curvature:

$$\omega(t) = \kappa(t)V(t)$$

Perturbing it we have:

$$\begin{bmatrix} \delta \dot{x}(t) \\ \delta \dot{y}(t) \\ \delta \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -V(t)\sin\theta(t) \\ 0 & 0 & V(t)\cos\theta(t) \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \delta x(t) \\ \delta y(t) \\ \delta\theta(t) \end{bmatrix} + \begin{bmatrix} \cos\theta(t) & 0 \\ \sin\theta(t) & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \delta V(t) \\ \delta\omega(t) \end{bmatrix}$$

3.2.1 Relevant Properties of Dynamical Systems

Certain important properties can be illustrated on these two equations. Note that the original three nonlinear equations can be divided by velocity (without creating a singularity) to convert the independent variable from time to distance ($\omega/V = \kappa$), but this may or may not be true of the perturbed system depending on the form of the input errors. When it is true, the errors are motion dependent and they stop accumulating when all motion stops.

Also, note that if the errors are odd functions of the inputs, then switching the signs of the inputs will switch the signs of the derivatives. Such systems are reversible because accumulated error is exactly erased by moving backward over the original trajectory.

Lastly, note that the original nonlinear system is linear in velocity under the substitution $\omega = \kappa V$. Hence, the position computed for a scaled speed $V' = (1+\alpha)V$, is scaled in exactly the same way. As a result, differential distance and velocity scale errors are distinguished in that they preserve the shape of the trajectory, changing only its size.

3.2.2 General Error Propagation

Let us define the following expressions for the coordinates of the endpoint from the perspective of the point $[x((\tau), y(\tau))]$:

$$\Delta x(t,\tau) = [x(t) - x(\tau)] \qquad \Delta y(t,\tau) = [y(t) - y(\tau)]$$

Next, the integrated system Jacobian is:

$$R(t,\tau) = \int_{\tau}^{t} F(\zeta)d\zeta = \begin{bmatrix} 0 & 0 & -\int_{\tau}^{t} V(\xi)\sin\theta(\xi)d\xi \\ 0 & 0 & \int_{\tau}^{t} V(\xi)\cos\theta(\xi)d\xi \\ 0 & 0 & 0 \end{bmatrix}$$

which reduces to:

$$R(t,\tau) = \begin{bmatrix} 0 & 0 & -\Delta y(t,\tau) \\ 0 & 0 & \Delta x(t,\tau) \\ 0 & 0 & 0 \end{bmatrix}$$

Since $R^2(t,\tau) = 0$, we have the transition matrix as:

$$\Phi(t,\tau) = \exp[R(t,\tau)] = I + R(t,\tau) = \begin{bmatrix} 1 & 0 & -\Delta y(t,\tau) \\ 0 & 1 & \Delta x(t,\tau) \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting into the general solution in equation (6) gives:

$$\delta \underline{x}(t) = \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix}\delta \underline{x}(0) + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \delta V(\tau) \\ \delta\omega(\tau) \end{bmatrix} d\tau \quad (15)$$

which is convenient to discuss when error sources are separated into influence vector terms as follows:

$$\delta \underline{x}(t) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \\ 0 \end{bmatrix} + \begin{bmatrix} -y(t)\delta\theta(0) \\ x(t)\delta\theta(0) \\ \delta\theta(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \cos\theta(\tau) \\ \sin\theta(\tau) \\ 0 \end{bmatrix}\delta V(\tau)d\tau + \int_0^t \begin{bmatrix} -\Delta y(t,\tau) \\ \Delta x(t,\tau) \\ 1 \end{bmatrix}\delta\omega(\tau)d\tau \quad (16)$$

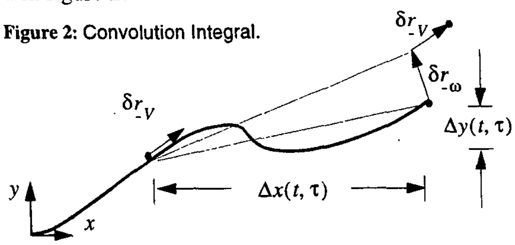
This is the general (linearized) solution for the propagation of systematic error in 2D integrated heading odometry for any trajectory and any error model.

A few points are worth noting. Overall, the result consists of the sum of the response to initial conditions and the response to the inputs. The former are path independent terms and the latter may or may not be depending on whether they can be integrated in closed form. When they cannot, the errors are path dependent integrals (functionals) evaluated on the reference trajectory.

3.3 Intuitive Interpretation

Now that the solution is written out, it is clear that it could have been written by inspection. The initial conditions affect the endpoint error in a predictable manner and the remaining terms amount to an addition of the effects felt at the endpoint at time t of the linear and angular errors occurring at each time τ between the start and end as illustrated in figure 2:

Figure 2: Convolution Integral.



The matrix relating input errors occurring at time τ to their later effect at time t is:

$$\begin{bmatrix} \delta x(t) \\ \delta y(t) \\ \delta\theta(t) \end{bmatrix} = \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta V(\tau) \\ \delta\omega(\tau) \end{bmatrix} d\tau$$

This is exactly what equation (16) is integrating. Linearization amounts to treating all errors as if they were independent in the sense that the endpoint is not changed to reflect the result of previous errors as the integral proceeds forward through time.

3.4 Differential Heading Odometry

Differential heading odometry is a special case of integrated heading odometry where angular velocity is derived from the differential indications of wheel linear velocities and the wheel tread W. Let there be a left wheel and a right wheel on either side of the vehicle reference point as shown below:

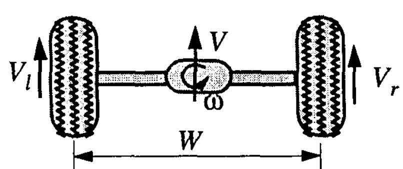


Figure 3: Differential Heading Odometry.

While it is possible in this simple case to solve for the equivalent integrated heading input $\underline{u}(t)$ and use the previous result, we will formulate this form of odometry with an observer to illustrate the more general case where the measurements $\underline{z}(t)$ may depend nonlinearly on both the state and the input, and may overdetermine the input. Let the measurement vector be the velocities of the two wheels:

$$\underline{z}(t) = \begin{bmatrix} V_r(t) & V_l(t) \end{bmatrix}^T$$

The relationship between these and the equivalent integrated heading inputs is:

$$\underline{z}(t) = M\underline{u}(t) \qquad \begin{bmatrix} V_r(t) \\ V_l(t) \end{bmatrix} = \begin{bmatrix} 1 & W/2 \\ 1 & -W/2 \end{bmatrix} \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix}$$

This is a particularly simple version of the more general form of the observer in equation (8). The inverse relationship is immediate:

$$\underline{u}(t) = M^{-1}\underline{z}(t) \qquad \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/W & -1/W \end{bmatrix} \begin{bmatrix} V_r(t) \\ V_l(t) \end{bmatrix}$$

The observer Jacobians are:

$$H(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M(t) = \begin{bmatrix} 1 & W/2 \\ 1 & -W/2 \end{bmatrix}$$

The left pseudoinverse reduces to the inverse in this case of a square M matrix:

$$M(t)^{LI} = M(t)^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/W & -1/W \end{bmatrix}$$

From equation (10), the perturbation dynamics are:

$$\delta\dot{\underline{x}}(t) = F(t)\delta\underline{x}(t) + G(t)M^{LI}\delta\underline{z}(t)$$

This is now identical to the integrated heading case. Substituting $\delta\underline{u}(t) = M^{LI}\delta\underline{z}(t)$ we have:

$$\delta\underline{x}(t) = \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix} \delta\underline{x}(0) + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} \delta V_r(t) \\ \delta V_l(t) \end{bmatrix} d\tau \quad (18)$$

This is the general (linearized) solution for the propagation of systematic error in 2D differential heading odometry for any trajectory and any error model.

4 First Moments of Error

The main results are equations (14), (16), and (18). Notice that, in all cases, the input response is a line integral evaluated on the reference trajectory. This is explicitly so in the case of the last term in direct heading. It becomes clearer in the other cases by considering the following generalization. Consider a term of the form:

$$\delta x(t) = \int_0^t f[x(\tau)]\delta V(\tau)d\tau$$

If the input error is motion dependent, which is to say proportional to a power of velocity (or some other position variable derivative):

$$\delta V(\tau) = \alpha(\tau)V^n(\tau)$$

then the integral becomes:

$$\delta x(s) = \int_0^s f[x(s)]\alpha(s)V^{n-1}(s)ds$$

which is a line integral where s is arc length (or some other position variable) along the trajectory.

Such integrals are the sources of path dependent error in odometry. These path functionals are equivalent to the moments of mechanics evaluated on curves whose "mass" at a given location is the error magnitude suffered at that location. Two types of moments have appeared.

4.1 Spatial Moments

The spatial moments are analogous to the first moment of inertia and apply to angular velocity errors. For a general error $\delta u(\tau)$, the first spatial error moments are:

$$
\begin{aligned}
U_x(t) &= \int_0^t [x(t) - x(\tau)]\delta u(\tau) d\tau \\
U_x(s) &= \int_0^s [x(s) - x(\zeta)]\delta u(\zeta) d\zeta \\
U_x(\theta) &= \int_0^\theta [x(\theta) - x(\xi)]\delta u(\xi) d\xi
\end{aligned}
\tag{19}
$$

In order, these will be called the first spatial duration, excursion, and rotation moments of error. Equivalent moments for the y coordinate are immediate.

4.2 Fourier Moments

The Fourier moments apply to linear velocity errors. For a general error $\delta u(\tau)$, the first Fourier error moments are:

$$
\begin{aligned}
U_c(t) &= \int_0^t \cos\theta(\tau)\delta u(\tau) d\tau \\
U_c(s) &= \int_0^s \cos\theta(\zeta)\delta u(\zeta) d\zeta \\
U_c(\theta) &= \int_0^\theta \cos\theta(\xi)\delta u(\xi) d\xi
\end{aligned}
\tag{20}
$$

In order, these will be called the first Fourier cosine duration, excursion, and rotation moments of error. Equivalent moments for the sine function are immediate.

4.3 Relationships of Trajectory Moments to Influence Vectors

In general, the influence vector projects each differential error source onto the associated differential state error. Due to linearization, the total error in the state can be separated into the components caused by each error source. For example, equation 16 can be written as:

$$
\delta \underline{x}(t) = \Phi(t, 0)\delta \underline{x}(0) + \int_0^t \underline{\tilde{\Phi}}_1 \delta V(\tau) d\tau + \int_0^t \underline{\tilde{\Phi}}_2 \delta\omega(\tau) d\tau
$$

Each of the last two terms is an error moment vector - the integral of the product of the error source and its influence vector. In the particular case where the error u_i is constant (or can be made so by a change of variable), the total effect of each error source can be evaluated solely as a function of the reference trajectory. For example, the above becomes:

$$
\delta \underline{x}(t) = \Phi(t, 0)\delta \underline{x}(0) + \delta V \int_0^t \underline{\tilde{\Phi}}_0 d\tau + \delta\omega \int_0^t \underline{\tilde{\Phi}}_1 d\tau
$$

These integrals, or path functionals, can be called trajectory moments. They are loosely analogous to the Laplace Transform in that they convert the differential equations of dynamical system estimation into algebraic ones - once the trajectory is specified.

They are intrinsic geometric properties of the path which give the output error when multiplied by the constant input error magnitude. A table of such moments provides essentially immediate answers for position error on known reference trajectories.

5 Errors and Trajectories

The error results available are functions of the reference trajectory, as is characteristic of perturbations of nonlinear systems. They are also functions of the error models chosen. Specific trajectories and error models will be assumed here in order to get specific results.

5.1 Error Models

For direct heading, we will assume a speed encoder scale error (due, for example to incorrect wheel radius) and a compass error due to a magnetic field produced by the vehicle. For integrated heading, the same speed encoder scale error and a constant gyro bias will be assumed. For differential heading, two different encoder scale errors will be used. These assumptions are summarized in the following table.

Table 1: Error Sources

Odometry Class	Error Sources
Direct Heading	$\delta V = \alpha V$ where $(\alpha \ll 1)$ $\delta\theta = A\cos\theta + B\sin\theta$
Integrated Heading	$\delta V = \alpha V$ where $(\alpha \ll 1)$ $\delta\omega = b$
Differential Heading	$\delta V_r = \alpha_r V_r$ where $(\alpha_r \ll 1)$ $\delta V_l = \alpha_l V_l$ where $(\alpha_l \ll 1)$

Under these assumptions, all error moments in the general solutions become constants multiplied by trajectory moments. The solution for direct heading becomes:

$$
\delta \underline{x}(s) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \end{bmatrix} + \alpha \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \int_0^s \begin{bmatrix} -As\theta c\theta \\ Bs\theta c\theta \end{bmatrix} ds + \int_0^s \begin{bmatrix} -Bs^2\theta \\ Ac^2\theta \end{bmatrix} ds
$$

The solution for integrated heading becomes:

$$
\delta \underline{x}(t) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \\ 0 \end{bmatrix} + \begin{bmatrix} -y(t)\delta\theta(0) \\ x(t)\delta\theta(0) \\ \delta\theta(0) \end{bmatrix} + \alpha \begin{bmatrix} x(t) \\ y(t) \\ 0 \end{bmatrix} + b \int_0^t \begin{bmatrix} -\Delta y(t, \tau) \\ \Delta x(t, \tau) \\ 1 \end{bmatrix} d\tau
$$

and similarly, the differential heading solution is:

$$
\delta \underline{x}(t) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \\ 0 \end{bmatrix} + \begin{bmatrix} -y(t)\delta\theta(0) \\ x(t)\delta\theta(0) \\ \delta\theta(0) \end{bmatrix} + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t, \tau) \\ s\theta & \Delta x(t, \tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & \frac{\beta W}{4} \\ \frac{\beta}{W} & \alpha \end{bmatrix} \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} d\tau
$$

where:

$$
\alpha = (\alpha_r + \alpha_l)/2 \qquad \beta = (\alpha_r - \alpha_l)
$$

5.2 Trajectories

A linear trajectory, starting at the origin, parallel to the x axis is defined by the following inputs:

$$
\omega(t) = 0 \qquad V(t) = arbitrary
$$

and the associated solution to equation (1):

$$
x(t) = s(t) \qquad y(t) = 0 \qquad \theta(t) = 0 \tag{21}
$$

A constant curvature (arc) trajectory, starting at the origin, initially parallel to the x axis is defined by the following inputs:

$$\omega(t) = \kappa(t)V(t) = V(t)/R \qquad V(t) = arbitrary$$

and the associated solution to equation (1):

$$\theta(s) = \kappa s \quad x(s) = R\sin(\kappa s) \quad y(s) = R[1-\cos(\kappa s)]$$

We also define for later $T = 1/\omega$.

6 Solutions to Special Cases

Using the above assumed errors and trajectories, linearized systematic error propagation is completely determined.

6.1 Straight Trajectory

The solution for direct heading becomes:

$$\delta \underline{x}(s) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \end{bmatrix} + \begin{bmatrix} \alpha s \\ A s \end{bmatrix}$$

Both translational errors are linear in distance - but for different reasons. The x error is due to the encoder scale error while the y error is due to the constant compass error at zero heading.

The solution for integrated heading becomes:

$$\delta \underline{x}(t) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s(t)\delta\theta(0) \\ \delta\theta(0) \end{bmatrix} + \alpha \begin{bmatrix} s(t) \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ Vt^2/2 \\ t \end{bmatrix}$$

Constant velocity was assumed in getting the term quadratic in time. Alongtrack error is linear in distance while heading error is linear in time. Crosstrack error includes a term linear in distance and another term which is quadratic in time or distance for constant velocity.

Similarly, the differential heading solution is:

$$\delta \underline{x}(s) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \\ 0 \end{bmatrix} + \begin{bmatrix} -y(s)\delta\theta(0) \\ x(s)\delta\theta(0) \\ \delta\theta(0) \end{bmatrix}$$

$$+ \begin{bmatrix} \alpha x(s) \\ 0 \\ (\beta/W)s \end{bmatrix} + \frac{\beta}{W}\begin{bmatrix} 0 \\ s^2/2 \\ 0 \end{bmatrix} + \frac{\beta W}{4}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

All terms are motion dependent. As in the integrated heading case, alongtrack error is linear in distance. However, in this case the heading error is also linear in distance (rather than time). The crosstrack error is quadratic in distance. Note that if s reverses, the entire error vector reverses, because all terms depend explicitly on s.

6.2 Constant Curvature Trajectory

The solution for direct heading becomes:

$$\delta \underline{x}(s) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \end{bmatrix} + \alpha \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}$$

$$+ R\begin{bmatrix} -A/4 \\ B/4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -Bs \\ As \end{bmatrix} + \frac{R}{4}\begin{bmatrix} A\cos 2\kappa s + B\sin 2\kappa s \\ -B\cos 2\kappa s + A\sin 2\kappa s \end{bmatrix}$$

In addition to linear terms relating to position coordinates and distance travelled, there are pure oscillation terms that depend only on the distance travelled along the arc but which cycle twice per orbit of a complete circle.

The solution for integrated heading becomes:

$$\delta \underline{x}(t) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \\ 0 \end{bmatrix} + \begin{bmatrix} -y(t)\delta\theta(0) \\ x(t)\delta\theta(0) \\ \delta\theta(0) \end{bmatrix} + \begin{bmatrix} (\alpha - bT)x(t) \\ btx(t) \\ 0 \end{bmatrix} + \begin{bmatrix} bt[R-y(t)] \\ (\alpha - bT)y(t) \\ bt \end{bmatrix}$$

Constant velocity was assumed in computing the first two elements of the last term. The heading error is linear in time whereas the position errors are entirely oscillatory but of increasing amplitude as time increases. Note that $R - y(t)$ oscillates around zero. Note that the linear increase in amplitude is a first order approximation to the true nonlinear behavior of a beat frequency. Eventually, the amplitude decreases again in the exact solution.

To first order, the y error vanishes upon return to the origin whereas the x coordinate of the perturbed trajectory lags or leads the unperturbed trajectory by an additional distance of $Rbt = 2\pi R(b/\omega)$ for each orbit of the unperturbed trajectory.

The solution for differential heading becomes:

$$\delta \underline{x}(s) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \\ 0 \end{bmatrix} + \begin{bmatrix} -y(s)\delta\theta(0) \\ x(s)\delta\theta(0) \\ \delta\theta(0) \end{bmatrix} + \begin{bmatrix} (\alpha' - b'R)x(s) \\ b'sx(s) \\ b's \end{bmatrix} + \begin{bmatrix} b's(R-y(s)) \\ (\alpha' - b'R)y(s) \\ 0 \end{bmatrix}$$

Where we have defined:

$$\alpha' = \left(\alpha + \frac{\beta W}{4R}\right) \qquad b' = \left(\frac{\beta}{W} + \frac{\alpha}{R}\right)$$

and then:

$$\frac{\beta W}{4R} - \frac{\beta R}{W} = \left(\alpha + \frac{\beta W}{4R}\right) - \left(\frac{\beta R}{W} + \alpha\right) = \alpha' - b'R$$

The structure is analogous to the integrated heading case. All terms are motion dependent. The heading error is linear in distance whereas the position errors are entirely oscillatory but of increasing amplitude as distance increases. Note that $R - y(s)$ oscillates around zero. the x coordinate of the perturbed trajectory lags or leads the unperturbed trajectory by an additional distance of $R(\beta s + \alpha\theta R)$ for each orbit of the unperturbed trajectory.

7 Simulation

The results of the last section were verified by comparing the linearized solutions above with an exact nonlinear numerical solution. For example, a simulation was conducted for the integrated heading case, for a 1% transmission encoder error and 3 degree/hour gyro bias. On 5 orbits of a 4m radius at 0.25 m/s speed, and 0.5 secs timestep, the difference between the linearized and exact solutions remains under 1/2 mm.

Figure 4 illustrates an arbitrary trajectory chosen to exhibit no particular symmetry.

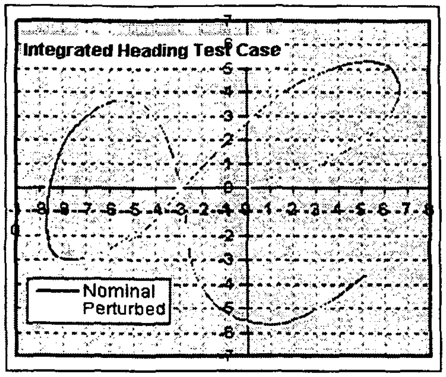


Figure 4: Test Trajectory

The same encoder and gyro errors mentioned above are applied to this test run. The perturbed (white) and unperturbed (black) trajectories can barely be distinguished at full scale in the figure. Comparison of the exact and linearized solutions is presented in figure 5.

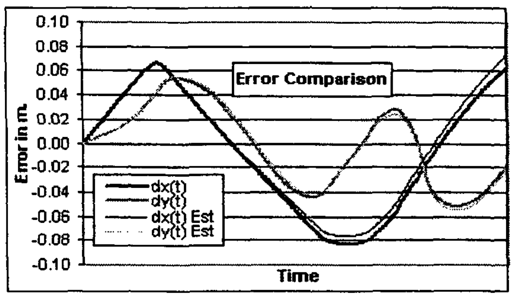


Figure 5: Error Comparison

Clearly, the linearized solution is an excellent approximation for errors of this magnitude. Throughout the test, the difference between exact and linearized solutions never exceeds 1 cm.

8 Conclusions

The commonly used heuristic that systematic odometry error is linear in distance turns out to be correct only for small excursions. Errors which affect angle are integrated once to get angular error and then a second time to get the associated position error. As a result, all cases studied generate a term quadratic in distance or time depending on the variable of integration.

A common technique for assessing accuracy is to verify that the computed trajectory associated with a closed reference trajectory also closes. As we have shown, some systematic error terms are path independent and therefore must vanish on closed trajectories, so the path closure test cannot observe such systematic sensor errors. However, the results of this paper provide the means to extract error magnitudes of any type from any trajectory by measuring total error externally and solving the equations produced by the error moments for the error magnitudes.

In addition to their pedagogic value, the results of this paper can be used in design to determine acceptable levels of systematic sensor error for a given target system performance. They can be used in development to accentuate response to individual error sources for on-line or off-line calibration or evaluative purposes. They can be used in operation to plan trajectories in order to minimize exposure to specific error sources. The non systematic error version of the present results also may have application to small footprint embedded optimal estimators since the linear variance equation reduces in many cases to computation of second order versions of the trajectory moments mentioned here.

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