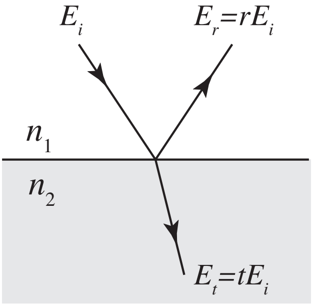
Today's lecture

- We outlined the steps of deriving the Fresnel equations, and found how they tell us about reflection and transmission at refractive interfaces.
- There's more to say about the Fresnel equations, but we also need to discuss things relevant to the Fabry-Perot interferometer for tomorrow's lab.
- We'll then jump back to finishing off the Fresnel equations.

Rays across an interface

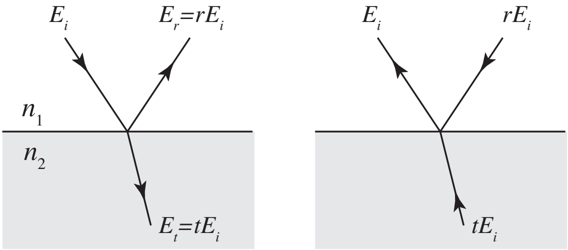
We've talked about how light rays can interact with a refractive interface. We go from an incident wave with field E_i to a reflected wave with field $E_r = rE_i$ and a transmitted wave with field $E_t = tE_i$.



The derivation of the Fresnel expressions for r and t involved involved vector components, and basic E&M rules like $\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}$ and $E_{1\|} = E_{2\|}$. Now Snell's law works perfectly well in reverse. Can we reverse the rays at right?

Rays across an interface II

Can't we just reverse the direction of all the rays?

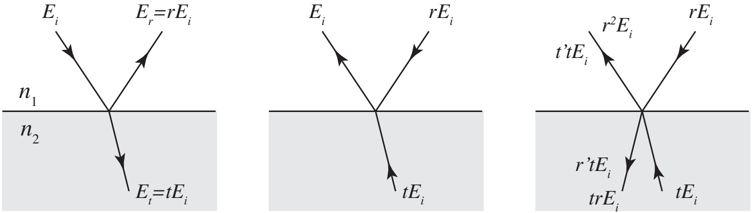


In the reversed diagram at right:

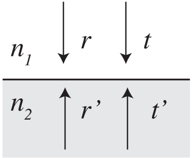
- Where's the transmitted ray from rE_i?
- Where's the reflected ray from tE_i?
- Do things in fact not work in reverse? Is it like the observation of Søren Kierkegaard (1813–1855) that "Life can only be understood backwards, but it must be lived forward"?

Rays across an interface III

Well, maybe we can fix reversibility! To do so, we need to have the transmitted ray from rE_I cancel out the reflected ray from tE_i. Here's the diagram with that ray included:



In producing that diagram for $n_2 > n_1$, we have adopted the following convention:



The Stokes equations

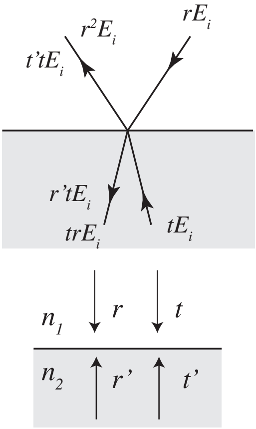
- This is in fact a reasonable way to think of things! (Due to George Gabriel Stokes, FRS, 1819–1903). For a relative refractive index of $n < 1$ (that is, rays going from n_2 to n_1) we have a π phase shift upon reflection or $r = -r'$.

- For the lower-left ray to indeed have a net field of zero, we need to have

$$\begin{aligned} trE_i + r'tE_i &= 0 \\ \text{or} \qquad t(r+r') &= 0 \\ \text{or} \qquad r &= -r' \quad \text{(yes!)} \quad (1) \end{aligned}$$

and the upper-left ray must equal E_i:

$$\begin{aligned} r^2E_i + t'tE_i &= E_i \\ \text{or} \qquad r^2 + t't &= 1 \qquad (2) \end{aligned}$$

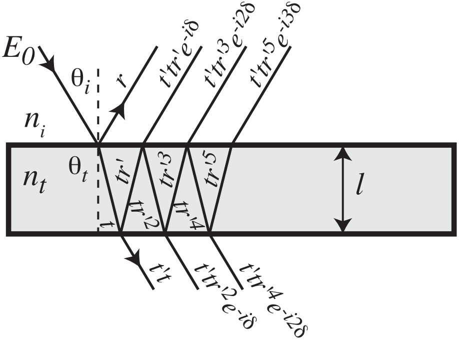


Relating rays

Again, we found from Eq. 1 that we expect $r = -r'$ or a π phase shift upon internal reflection (as confirmed by the Fresnel equations), and we found from Eq. 2 the condition

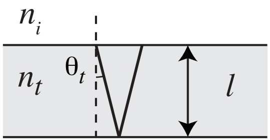
$$r^2 + t't = 1 \qquad \Rightarrow \qquad t't = 1 - r^2 \tag{3}$$

Let's now use the formalism we've developed to consider multiple bounces from a slab of glass of thickness ℓ, assuming that we can calculate the phase difference δ between successive reflected beams:



What's δ?

The phase difference between successive reflected beams can be found from the optical path length between the successive beams:



The distance Δ that the ray travels going down and back is $2\ell/\cos\theta_t$, so the net phase delay δ can be found from

$$\delta = \frac{2\pi}{\lambda} n_t \Delta = \frac{2\pi}{\lambda} n_t \frac{2\ell}{\cos\theta_t} = 4\pi \frac{\ell}{\lambda} \frac{n_t}{\sqrt{1 - \sin^2\theta_t}} = 4\pi \frac{\ell}{\lambda} \frac{n_t}{\sqrt{1 - \frac{n_i^2}{n_t^2}\sin^2\theta_i}} \tag{4}$$

This is a phase *lag* for the next emerging beam, or a phase shift of $e^{-i\delta}$.

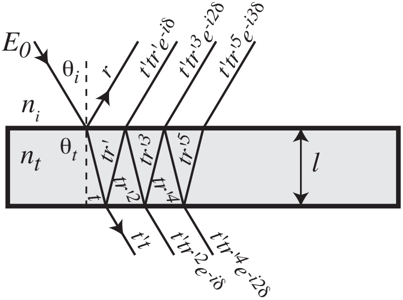
The net reflected wave

The net reflected wave is

$$\begin{aligned} E_r &= E_0[r + t'tr'e^{-i\delta} + t'tr'^3e^{-i2\delta} + t'tr'^5e^{-i3\delta} + \dots] \\ &= E_0[r + t't\sum_{n=2}^{\infty} r'^{(2n-3)}e^{-i(n-1)\delta}] \qquad (5) \end{aligned}$$

as can be seen with some examples:

n	$(2n-3)$	$(n-1)$
2	$(2\cdot 2-3)=1$	(2-1)=1
3	$(2\cdot 3-3)=3$	(3-1)=2
4	$(2\cdot 4-3)=5$	(4-1)=3



Net reflected wave II

Again, we found that the net reflected wave is given by Eq. 5 of

$$E_r E_0[r + t' t \sum_{n=2}^{\infty} r'^{(2n-3)} e^{-i(n-1)\delta}]$$

Now pull one $r' e^{-i\delta}$ out:

$$\begin{aligned}
E_r &= E_0 \left[r + t' t r' e^{-i\delta} \sum_{n=2}^{\infty} r'^{(2n-4)} e^{-i(n-2)\delta} \right] \\
&= E_0 \left[r + t' t r' e^{-i\delta} \sum_{n=2}^{\infty} (r'^2)^{(n-2)} e^{-i(n-2)\delta} \right] \\
&= E_0 \left[r + t' t r' e^{-i\delta} \sum_{n=2}^{\infty} (r'^2 e^{-i\delta})^{(n-2)} \right] \\
&= E_0 \left[r + t' t r' e^{-i\delta} \sum_{n=0}^{\infty} (r'^2 e^{-i\delta})^{n} \right] \qquad (6)
\end{aligned}$$

Finite geometric series

We have a series expression for E_r. Consider a finite geometrical series:

$$S^m = \sum_{n=0}^{m} x^n = 1 + x + \ldots + x^m. \quad (7)$$

We can then say

$$\begin{aligned} S^m(1+x) &= S^m + xS^m \\ &= S^m + (x + x^2 + \ldots + x^{m+1}) \\ &= S^m + S^{m+1} - 1 \end{aligned}$$

Rearranging gives

$$\begin{aligned} S^m(1+x-1) &= S^{m+1} - 1 \\ xS^m &= S^{m+1} - 1 \quad (8) \end{aligned}$$

Continuing from Eq. 8:

$$\begin{aligned} xS^m &= S^m + x^{m+1} - 1 \\ 1 - x^{m+1} &= S^m(1-x) \\ S^m &= \frac{1 - x^{m+1}}{1-x}. \quad (9) \end{aligned}$$

In the case where $x < 1$ and $m \to \infty$, we can ignore x^{m+1} and we have

$$\begin{aligned} S^m &= \sum_{n=0}^{m} x^n \\ &= 1 + x + \ldots + x^m \\ &\simeq \frac{1}{1-x}. \quad (10) \end{aligned}$$

Net reflected wave III

We can use the series result from Eq. 10 of $1 + x + \ldots + x^m \simeq 1/(1-x)$ to look again at our expression for the net reflected field of Eq. 6:

$$E_r = E_0 \left[r + t'tr'e^{-i\delta} \sum_{n=0}^{\infty} (r'^2 e^{-i\delta})^n \right] = E_0 \left[r + \frac{t'tr'e^{-i\delta}}{1 - r'^2 e^{-i\delta}} \right]. \quad (11)$$

Using the results of Eq. 1 of $r' = -r$ and Eq. 3 of $t't = 1 - r^2$, we can write E_r as

$$\begin{aligned} E_r &= E_0 \left[r + \frac{-(1-r^2)re^{-i\delta}}{1 - r^2 e^{-i\delta}} \right] \\ &= E_0 \left[\frac{r - r^3 e^{-i\delta} - re^{-i\delta} + r^3 e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right] \\ &= E_0 \left[r \frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right]. \quad (12) \end{aligned}$$

Net reflected wave V

The reflected irradiance is $\propto E_r \cdot E_r^*$ or

$$\begin{aligned} I_r &= I_0 \left[r \frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}} \right] \left[r \frac{1 - e^{+i\delta}}{1 - r^2 e^{+i\delta}} \right] \\ &= I_0 r^2 \frac{1 - e^{-i\delta} - e^{+i\delta} + e^{-i\delta + i\delta}}{1 - r^2 e^{-i\delta} - r^2 e^{+i\delta} + r^4 e^{-i\delta + i\delta}} \\ &= I_0 r^2 \frac{2 - \left(e^{-i\delta} + e^{+i\delta}\right)}{1 + r^4 - r^2 \left(e^{-i\delta} + e^{+i\delta}\right)}. \end{aligned} \tag{13}$$

Now $e^{i\theta} = \cos\theta + i \sin\theta$ and $e^{-i\theta} = \cos\theta - i \sin\theta$, so $\left(e^{i\theta} + e^{-i\theta}\right) = 2\cos\theta$. This gives

$$\begin{aligned} I_r &= I_0 r^2 \frac{2 - 2\cos\delta}{1 + r^4 - 2r^2 \cos\delta} \\ &= I_0 \frac{(2r^2)(1 - \cos\delta)}{1 + r^4 - 2r^2 \cos\delta} \end{aligned} \tag{14}$$

Reflectance

Again, the reflectance of Eq. 14 is

$$I_r = I_0 \frac{(2r^2)(1 - \cos\delta)}{1 + r^4 - 2r^2 \cos\delta}$$

When $\delta = 2m\pi$ with m any integer, $\cos\delta = 1$ so the reflectance is zero. When $\delta = 2(m+1)\pi$, $\cos\delta = -1$ and the reflectance becomes

$$I_r = I_0 \frac{4r^2}{1 + 2r^2 + r^4} = I_0 \frac{4r^2}{(1 + r^2)^2} \tag{15}$$

Transmittance

One can do an analogous calculation on the net transmitted beam to find

$$I_t = I_0 \frac{(1-r^2)^2}{1 + r^4 - 2r^2 \cos\delta} \tag{16}$$

or, using $\cos\delta = 1 - 2\sin^2(\delta/2)$,

$$I_t = I_0 \frac{(1-r^2)^2}{(1-r^2)^2 + 4r^2 \sin^2\dfrac{\delta}{2}} = I_0 \frac{1}{1 + \dfrac{4r^2}{(1-r^2)^2}\sin^2\dfrac{\delta}{2}}. \tag{17}$$

It's common to define a coefficient of finesse F as

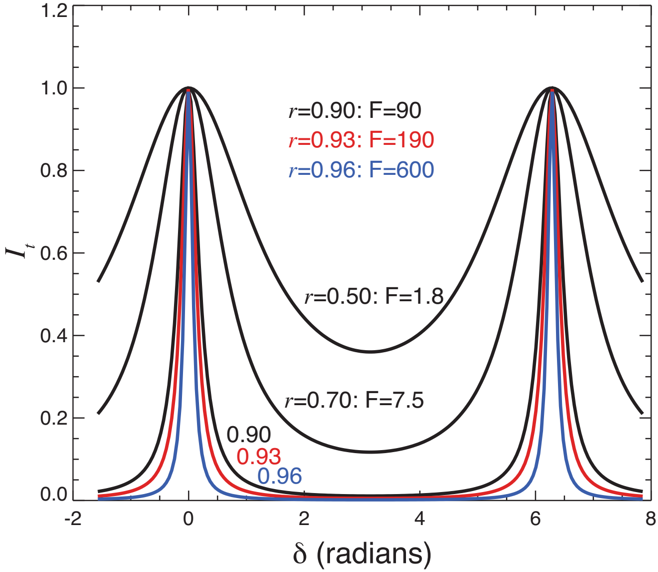
$$F \equiv \frac{4r^2}{(1-r^2)^2} \tag{18}$$

which along with Eq. 17 gives something we'll use for Fabry-Perot interferometers:

$$\frac{I_t}{I_0} = \frac{1}{1 + F\sin^2\dfrac{\delta}{2}}. \tag{19}$$

Transmission versus r, F

Here's a plot of $1/[1 + F\sin^2(\delta/2)]$:



FWHM of transmission

- At what phase deviation δ_{HWHM} do we reach half the maximum? (HWHM=half width at half maximum).

$$\frac{1}{2} = \frac{1}{1 + F\sin^2(\delta_{\mathrm{HWHM}}/2)} \qquad \Rightarrow 2 = 1 + F\sin^2(\delta_{\mathrm{HWHM}}/2)$$

$$\Rightarrow \sin(\frac{\delta_{\mathrm{HWHM}}}{2}) = \frac{1}{\sqrt{F}} \simeq \frac{\delta_{\mathrm{HWHM}}}{2}$$

$$\text{giving} \;\; \delta_{\mathrm{FWHM}} = 2\delta_{\mathrm{HWHM}} = \frac{4}{\sqrt{F}} \tag{20}$$

where we've used the small angle approximation $\sin\theta \simeq \theta$.

- Therefore if the full width at half max of fringes is only a fraction x of a period, we can say $x = \delta_{\mathrm{FWHM}}/(2\pi)$ and thus estimate F from Eq. 20.
- If we say $r = 1 - \epsilon$, we can rearrange Eq. 18 to find ϵ:

$$F = \frac{4(1-\epsilon)^2}{\epsilon^4} \qquad \Rightarrow \epsilon \simeq (\frac{4}{F})^{1/4} \tag{21}$$

The Fresnel equations

By considering the properties of EM fields across interfaces for two orthogonal orientations, we arrived at the Fresnel equations (see Pedrotti and Pedrotti Eqs. 20-23 – 20-26, or Fowles Eqs. 2-56 – 2-59):

$$\text{TE: } r_\perp = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \tag{22}$$

$$\text{TM: } r_\parallel = \frac{n^2\cos\theta - \sqrt{n^2 - \sin^2\theta}}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \tag{23}$$

$$\text{TE: } t_\perp = \frac{2\cos\theta}{\cos\theta + \sqrt{n^2 - \sin^2\theta}} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i + \theta_t)} \tag{24}$$

$$\text{TM: } t_\parallel = \frac{2n\cos\theta}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} \tag{25}$$

In these expressions, $n \equiv n_t/n_i$ and $\theta \equiv \theta_i$. The case of $n_i > n_t$ or $n < 1$ is that of *internal reflection*, while the case of $n_t > n_i$ or $n > 1$ is that of *external reflection*.

Brewster's angle, and R and T

Brewster's angle:

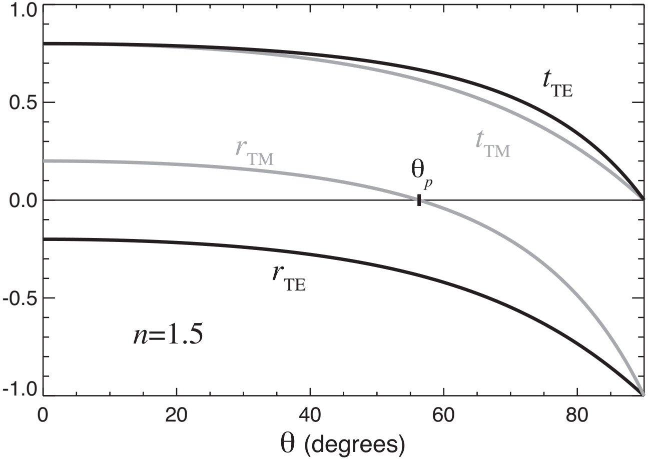
- We found that when $\theta_i + \theta_t = \pi/2$, the reflection of TM waves is killed.
- This is at Brewster's angle as given by $\tan\theta_p = n$.
- For $n = 1.3$ we have $\theta_p = 52°$ while for $n = 1.5$ we have $\theta_p = 56°$.
- Polarization upon reflection. Polaroid sunglasses.

Reflection R and transmission T:

$$R = r^2 \qquad T = n\frac{\cos\theta_t}{\cos\theta_i}t^2$$

Coefficients for $n = 1.5$

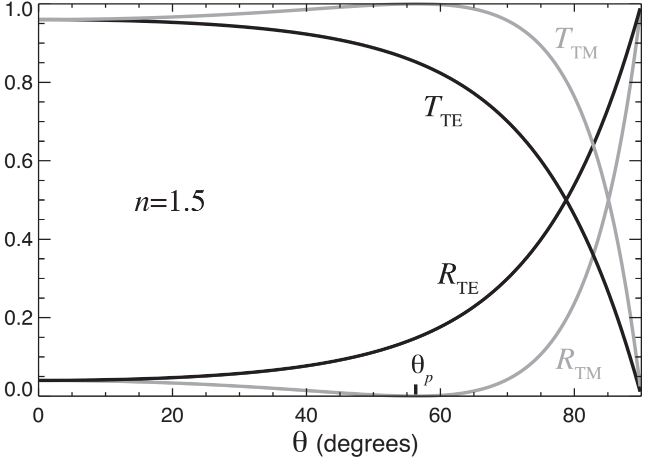
We plotted $r_{TE} = r_\perp$, $r_{TM} = r_\parallel$, $t_{TE} = t_\perp$, and $t_{TM} = t_\parallel$ for $n = 1.5$:



The negative values of r describe conditions where the phase is inverted by 180° upon reflection.

Reflectivity and transmittance for $n = 1.5$

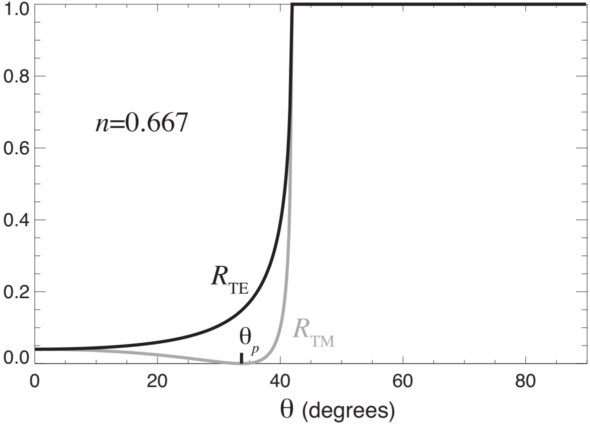
Using $R = r^2$ and $T \simeq n(\cos\theta_t/\cos\theta_i)t^2$, we plotted the reflectivity and transmittance for $n = 1.5$:



Notice Brewster's angle $\theta_p = \arctan(n)$

Reflectivity: $n = 1/1.5$

Now let's deal with $n < 1$, and consider what happens beyond the critical angle. Here's a plot of reflectivity R for $n = 1/1.50$:



Beyond the critical angle

When we go beyond the critical angle, the reflection coefficients become complex. That is, when $n < 1$ we have

$$\sqrt{n^2 - \sin^2\theta} \rightarrow i\sqrt{\sin^2\theta - n^2}.$$

In this case we write the reflection coefficients of Eqs. 22 and 23 as

$$\text{TE: } r_\perp = \frac{\cos\theta - i\sqrt{\sin^2\theta - n^2}}{\cos\theta + i\sqrt{\sin^2\theta - n^2}} \tag{26}$$

$$\text{TM: } r_\parallel = \frac{n^2\cos\theta - i\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta + i\sqrt{\sin^2\theta - n^2}}. \tag{27}$$

These both have the same form of

$$\frac{a - ib}{a + ib} \tag{28}$$

which we will exploit on the next slide.

Beyond critical II

Again, we had in Eq. 28 expressions of the form $(a - ib)/(a + ib)$. Both the numerator and the denominator have the same magnitude $M = \sqrt{a^2 + b^2}$. This suggests a graphical interpretation of the result, as shown at right. We can therefore say that

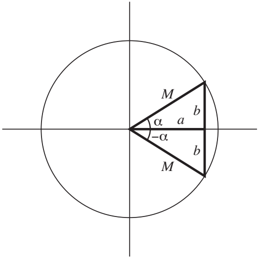
$$\tan\alpha = \frac{b}{a}$$

and express the reflection coefficient as

$$r = |r|e^{i\varphi_r} = \frac{a - ib}{a + ib} = \frac{e^{i(-\alpha)}}{e^{i(+\alpha)}} = e^{i(-2\alpha)}$$

from which we obtain

$$\varphi_r = -2\alpha = -2\arctan\left(\frac{b}{a}\right) \qquad (29)$$



Beyond critical III

Let us now use the result of Eq. 29 of $\varphi_r = -2\arctan(b/a)$ to find the phase of the reflection coefficients of Eqs. 26 and 27:

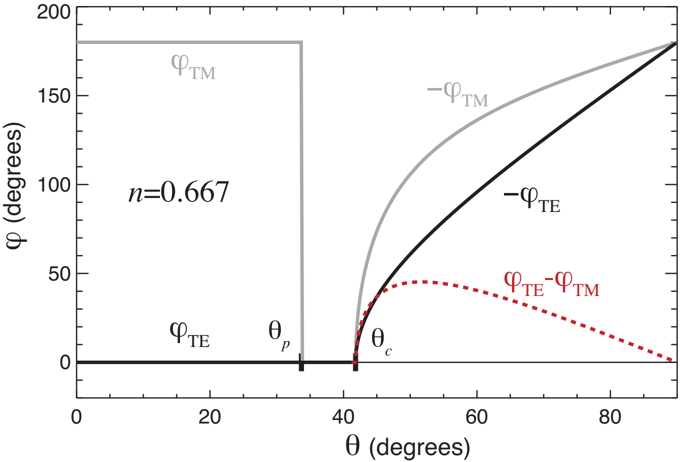
$$\varphi_{\mathrm{TE}} = -2\arctan\left(\frac{\sqrt{\sin^2\theta - n^2}}{\cos\theta}\right) \tag{30}$$

$$\varphi_{\mathrm{TM}} = -2\arctan\left(\frac{\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta}\right) \tag{31}$$

These phases, along with the phases associated with the sign of r in the expressions of Eqs. 22 and 23 for $n < 1$, are plotted on the next page.

Beyond critical IV

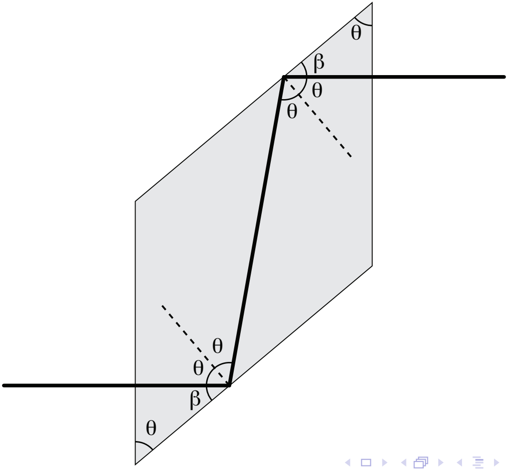
Here are the phases upon reflection for the case of $n < 1$:



Notice how the phase difference between TE and TM modes is about 45° at an internal incidence angle of about 50°?

The Fresnel Rhomb

The Fresnel rhomb is a very simple optical device for producing circularly polarized light. Two internal reflections are each at an incidence angle of $\theta \simeq 50^\circ$ in glass with $n \simeq 1/1.5$, such that $\varphi_{\mathrm{TM}} - \varphi_{\mathrm{TE}} = 45^\circ$ leading (after two bounces) to a net phase shift of the TM mode of 45+45=90° relative to the TE mode.



At the critical angle?

Think back to when we derived the Fresnel equations. We had $E_t = E_{0t} e^{-i(\vec{k}_t \cdot \vec{r} - \omega t)}$, with

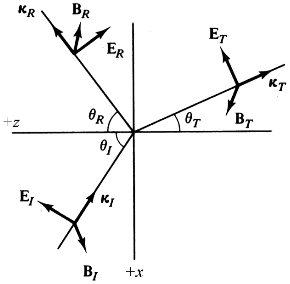
$$\vec{k}_t \cdot \vec{r} = k_t(-\sin\theta_t, 0, -\cos\theta_t, 0) \cdot (x, y, z) = -k_t \left[-\hat{x}(x \sin\theta_t) - \hat{z}(z \cos\theta_t)\right] \qquad (32)$$

at the interface ($z = 0$). We can express $\cos\theta_t$ as

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{\sin^2\theta_i}{n^2}} \qquad (33)$$

using Snell's law and the relative refractive index $n = n_t/n_i$. If we go to an angle $\theta_i > \theta_c$ beyond the critical angle $\sin^2\theta_c = n^2$, we are better off writing Eq. 33 as

$$\cos\theta_t = i\sqrt{\frac{\sin^2\theta}{n^2} - 1} \qquad (34)$$



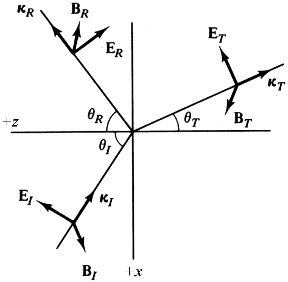
At critical II

We can now rewrite Eq. 32 of $\vec{k}_t \cdot \vec{r} = -k_t \left[-\hat{x} \cdot (x \sin\theta_t) - \hat{z} \cdot (z \cos\theta_t) \right]$ using the result of Eq. 34 of $\cos\theta_t = i\sqrt{\sin^2\theta/n^2 - 1}$ to obtain

$$\vec{k}_t \cdot \vec{r} = \left[\hat{x} \cdot (k_t x \frac{\sin\theta}{n}) + \hat{z} \cdot (i k_t z \sqrt{\frac{\sin^2\theta}{n^2} - 1}) \right] \tag{35}$$

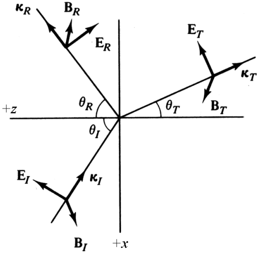
for wave propagation in the $\hat{x}$ and $\hat{z}$ directions for the case when $\theta > \theta_c$.

We see that the wave propagates in the $\hat{x}$ direction as $e^{-i[k_t(\sin\theta_t/n)x - \omega t]}$ which is normal wave propagation. What about in the $\hat{z}$ direction?



At critical III

In the $\hat{z}$ direction, the wave propagates as $e^{-i(k_{t,z}z-\omega t)} = e^{-ik_{t,z}z}e^{i\omega t}$. Ignoring the normal time dependence $e^{i\omega t}$, and considering a propagation direction out of the medium of $+\zeta = -z$, we can use Eq. 35 to write the propagation of the wave out of the refractive medium in the $-\hat{z}$ direction as



$$\begin{aligned} \exp[-ik_{t,z}z] &= \exp\left[-i \cdot ik_t\sqrt{\frac{\sin^2\theta}{n^2} - 1} \cdot z\right] \\ &= \exp\left[+k_t\sqrt{\frac{\sin^2\theta}{n^2} - 1} \cdot (-\zeta)\right] \\ &= \exp\left[-k_t\sqrt{\frac{\sin^2\theta}{n^2} - 1} \cdot \zeta\right] \qquad (36) \end{aligned}$$

At critical IV

Again, we found from Eq. 36 that the wave propagates according to

$$\exp\left[-k_t\sqrt{\frac{\sin^2\theta}{n^2}-1}\cdot\zeta\right]$$

Therefore we see that the electric field penetrates out of the refractive material as $\exp[-\zeta/\alpha]$ characterized by a $1/e$ distance α of

$$\alpha \equiv \frac{1}{k_t\sqrt{\sin^2\theta/n^2-1}} = \frac{\lambda}{2\pi\sqrt{\sin^2\theta/n^2-1}} \tag{37}$$

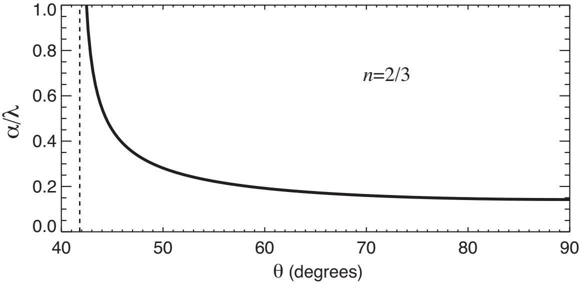
so we wouid like to understand the behavior of

$$\frac{\alpha}{\lambda} = \frac{1}{2\pi\sqrt{\sin^2\theta/n^2-1}} \tag{38}$$

as a function of incident angle θ.

Evanescent waves

The "leakage" of electric field out of the refractive bounday after total internal reflection (TIR) is referred to in terms of *evanescent waves*. The electric field decreases with distance out from the boundary according to $\exp[-\zeta/\alpha]$ with a characteristic distance α expressed in terms of a fraction of a wavelength (Eq. 38) as $\alpha/\lambda = 1/[2\pi\sqrt{\sin^2\theta/n^2 - 1}]$. The dependence of α/λ on incident angle θ is



To a good approximation, $\alpha \simeq 0.2\lambda$ over a broad range of TIR angles.

Evanescent waves II

The fact that evanescent waves tunnel a distance of about 0.2λ or about 100 nm for visible light has some very interesting consequences including:

- Fiber optics work (in a simplistic view) by keeping light confined within the angle of total internal reflection inside the fiber medium. You can "spy" on the fiber if you can bring a sensor (like another fiber with a grating to couple the signal in) within $\sim 0.2\lambda$ or about 300 nm for the most common communication wavelength of $\lambda = 1.5\ \mu$m. This is used for deliberate signal coupling, and possibly for espionage. (During the 1970s, US submarines tapped into undersea Soviet copper communication cables in Operation Ivy Bells: see the book *Blind Man's Bluff* by Sontag and Drew).
- If you pump light at an excitation wavelength into the side of a glass coverslip, you can excite fluorescence in molecules that drift within a distance of $\sim 0.2\lambda$ (or < 100 nm at UV wavelengths) of the coverslip. This technique, known as TIRF, can be useful for studies of molecular diffusion, molecular binding on a coverslip with a biologically active surface coating, and so on.