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Light polarization

- Some time ago we discussed the fact that Maxwell's equations give solutions for electric and magnetic fields of

$$E = E_0 e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \text{and} \qquad B = B_0 e^{-i(\vec{k}\cdot\vec{x}-\omega t)}$$

- We also discussed the Poynting vector $\vec{S} = \vec{E} \times \vec{B}$ which gives the direction of energy flow, and which indicates that the electric and magnetic fields are orthogonal to each other.

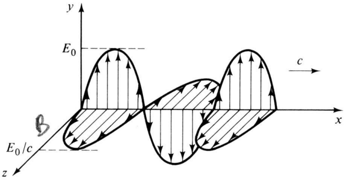

Figure 8.13

(and see Fowles Figs. 2.1 and 2.2)

- Once we've defined the direction of energy flow, we've specified the $\hat{x}$ direction. However, we might not have any reason to pick any *particular* orthogonal direction as being the $\hat{y}$ direction.

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Arbitrary polarizations

- So how do we deal with electric fields that oscillate in the $\hat{z}$ plane? By declaring them to represent a separate polarization.
- And what about electric fields that oscillate along some plane other than pure $\hat{y}$ or pure $\hat{z}$? We can treat *those* as a linear combination of the two orthogonal polarizations.
- How to deal with this? By treating any particular direction of electric field as being a combination of electric field oscillations in the $\hat{y}$ and $\hat{z}$ planes.
- This is all fine and dandy for one pure wave. If we have many waves mixing together, they may or may not have the same polarization. If part I_{pol} of the intensity is from polarized waves, and part I_{unpol} is not, then the degree of polarization P is

$$P = \frac{I_{\text{pol}}}{I_{\text{pol}} + I_{\text{unpol}}} \tag{1}$$

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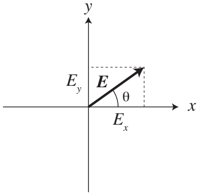
Mixing polarizations

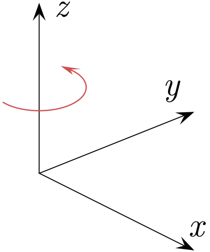
Let's think some more about waves with full linear polarization.

- We know from our study of refractive indices that electric field effects dominate, so we'll talk only about E for the wave.
- Let's also switch gears and talk about a wave $E_0 e^{-i(\vec{k}\cdot\vec{z}-\omega t-\varphi)}$ traveling in the $\hat{z}$ direction so we can use trigonometric relationships in $\hat{x}$ and $\hat{y}$ to talk about the polarization angle θ of a wave.
- We can then express the polarized wave in terms of a vector with (possibly) different starting phases φ in each direction:

$$\boldsymbol{E_0} = \hat{x}|E_{0x}|e^{i\varphi_x} + \hat{y}|E_{0y}|e^{i\varphi_y} \qquad (2)$$



Wave coming towards you



Right hand rule

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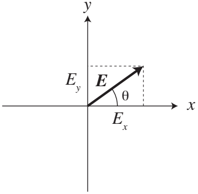
Polarization matrix

- A good way of expressing the net polarization $\boldsymbol{E_0} = \hat{x}|E_{0x}|e^{i\varphi_x} + \hat{y}|E_{0y}|e^{i\varphi_y}$ of Eq. 2 is in terms of a vector (see Fowles Sec. 2.5):

$$\boldsymbol{E_0} = \left[\begin{array}{c} |E_{0x}|e^{i\varphi_x} \\ |E_{0y}|e^{i\varphi_y} \end{array} \right] \Rightarrow \left[\begin{array}{c} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{array} \right] \qquad (3)$$

  where the latter form is a bit less explicit on requiring that all sign changes in E be built into φ but we'll work with that implicit assumption in what follows.



- An even better way to express wave polarization might be to use a normalized vector and pull the field magnitude out:

$$\boldsymbol{E_0} = E_0 \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array} \right] \qquad (4)$$

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Polarization matrices II

This is great! We can now represent lots of different polarizations by some rather simple matrices:

$$\text{Linear } \hat{x}\text{: } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ Linear } \hat{y}\text{: } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; \theta = 45^\circ\text{: } \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ General: } \begin{bmatrix} a \\ b \end{bmatrix}$$

and so on. But hey, all we required of our matrices is that they be normalized or $|a|^2 + |b|^2 = 1$. Well, this matrix is normalized:

$$\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}$$

What does it represent in terms of polarization? Well, it has the $\hat{y}$ field phase shifted by 90° relative to the $\hat{x}$ polarization, or $\varphi_y = \varphi_x + 90^\circ$ in Eqs. 2 and 3. That means when the $\hat{x}$ polarization is reaching zero, the $\hat{y}$ polarization is reaching a maximum, and vice versa. The polarization vector spins around in a circle!

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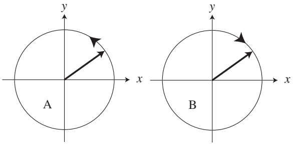
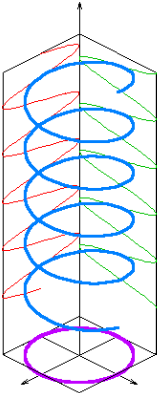
Circular polarization

We've found that we can talk about circular polarization, where the polarization vector spins around in a helix about the beam direction. This is described in terms of the handedness of the helix. Unfortunately, there is disagreement in the technical world on this! Let's say the wave is coming head-on at you:

$$1/\sqrt{2}\begin{bmatrix} 1 \\ i \end{bmatrix}$$



$$1/\sqrt{2}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$



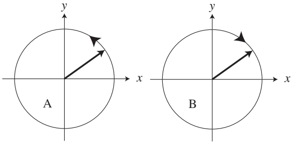
From Wikipedia

In figure A, the polarization vector is rotating counter-clockwise, and you'd use your right hand to describe the polarization change with your thumb pointing in the wave's direction. That must be right hand circular polarization, right?

Circular polarization II

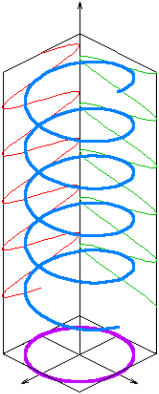
Again, you might think that figure A represents right-hand circular polarization, right?

$$1/\sqrt{2}\begin{bmatrix} 1 \\ i \end{bmatrix}$$



$$1/\sqrt{2}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Wrong! For whatever reason, the optics world has adopted the definition that, for waves traveling towards you, figure A is for left-handed circular polarization, and figure B is for right-handed circular polarization.



From Wikipedia: a RH circular polarized wave.

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Playing with polarization

- OK, so if we see an i in the lower array term, we know we have circular polarization. What about if i is in the top term?

$$\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\, i \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

  So this is really right-hand circular polarization with a prefactor that shifts the start in time.

- What about terms with different magnitudes?

$$\begin{bmatrix} 2 \\ i \end{bmatrix}$$

  This is a distorted circle with twice the width in the $\hat{x}$ direction as height in the $\hat{y}$ direction: that is, an ellipse (see Fowles Fig. 2.8c). Because the bottom term is positive, it represents left-hand elliptical polarization.

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Playing with polarization II

- What about terms that are neither pure real nor pure imaginary, such as this?

$$\begin{bmatrix} A \\ B + iC \end{bmatrix}$$

This is just a tilted ellipse, where the tilt angle α from the $\hat{x}$ axis is given by

$$\alpha = \arctan(C/B) \tag{5}$$

Again, the sign of the imaginary term in the denominator indicates whether it's a right- or left-handed, tilted, elliptical polarization vector.

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Modifying polarization

Let's consider a polarizing filter or analyzer, which is a device that only allows one polarization of light to be transmitted.

- In the microwave range, an array of wires will do this.
- Dr. William Bird Herapath found in 1852 that dogs treated with quinine had microscopic crystalline needles in their urine. He observed that overlapping perpendicular needles were black where they crossed, whereas parallel needles were clear.
- In 1928, Edwin Land (who dropped out of Harvard at age 17 in 1926) was inspired by Herapath's work to develop a synthetic sheet polarizer: iodized quinine salt crystals, ground for a month in a laboratory mill, suspended in a nitrocellulose lacquer solution, and subjected to a magnetic field. A plastic sheet is then dipped in the liquid suspension. Patent in 1929; eventual basis for the Polaroid corporation.

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Jones matrices

How do we handle this mathematically? Let's preserve the $\hat{x}$-polarized light only:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \cdot A + 0 \cdot B \\ 0 \cdot A + 0 \cdot B \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix} \tag{6}$$

This is called a Jones matrix for horizontal linear polarization. One can also write an array for a vertical linear polarizer, one at 45°, and so on; see the first three entries in Table 2.1 on p. 35 of Fowles. A linear polarizer with a transmission axis or TA at an angle θ from the horizontal axis can be described by

$$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \tag{7}$$

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Quarter-wave plate

A quarter-wave plate delays light of one polarization by 90° relative to the orthogonal polarization. Let's rotate $(-i)$ the phase of the vertical axis by 90°:

$$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 1 \\ 0 \cdot 1 + (-i) \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{8}$$

We've changed linearly polarized light along a 45° axis into right-hand circularly polarized light! Since this has a polarization vector that spins in the clockwise direction when looking head-on into the wave, the $\hat{y}$ polarization turns on before $\hat{x}$, so we say that the Fast Axis is in the vertical direction and the Slow Axis is in the horizontal direction. You'll explore this more in homework.

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Aberrations

- From the Merriam-Webster online dictionary: **ab·er·ra·tion** (*n*):
  1. The fact or an instance of being aberrant (straying from the right or normal way) especially from a moral standard or normal state.
  2. Failure of a mirror, refracting surface, or lens to produce exact point-to-point correspondence between an object and its image.
  3. Unsoundness or disorder of the mind.

  plus other definitions... We will hope that neither the first nor third definitions apply to any of us, and proceed with the second.
- We will describe the aberration a of a wave in terms of the distance along a ray direction of the actual wave crest relative to the ideal wave crest.

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Ray aberrations

Start by considering the effect of a wavefront aberration a on where a ray lands:

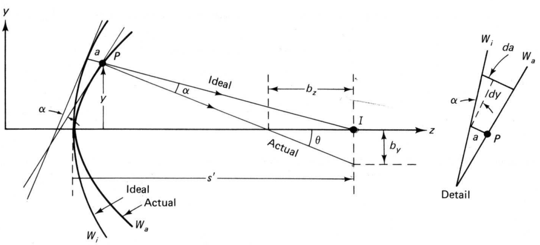

Figure 5-2 Construction used to relate the ray aberrations b_y and b_z to the wave aberration a. The detail shows how to relate a change da in wave aberration to a change dy in the aperture dimension.

- Following a ray from the surface normal of a wavefront, the angular error of the ray is $\alpha = da/dy$ for changes in the aberration a as a function of height from the optical axis y.
- The lateral ray aberration is $b_y = \alpha s' = s'\, da/dy$.
- The longitudinal ray aberration b_z is well approximated by $b_z = b_y / \tan\theta \simeq b_y(s'/y)$.

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Spherical aberration

Consider refraction from a single spherical interface:

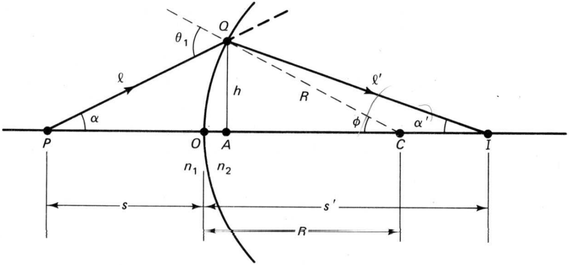

Figure 5-3 Refraction of a ray at a spherical surface.

The imaging condition is given by Fermat's principle of least time; all valid rays have the same (least) time. Normally this gives $n_1\ell + n_2\ell' = n_1 s + n_2 s'$. We'll consider the deviations from this situation to be described by an error in the optical path $a(Q)$:

$$a(Q) = (n_1\ell + n_2\ell') - (n_1 s + n_2 s'). \tag{9}$$

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Spherical aberration II

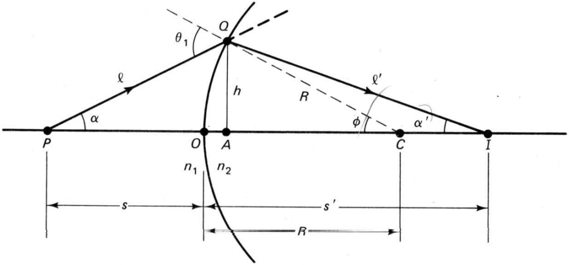

Figure 5-3 Refraction of a ray at a spherical surface.

Consider the distances ℓ and ℓ' using the law of cosines $C^2 = A^2 + B^2 - 2AB\cos\theta$:

$$\ell^2 = R^2 + (R+s)^2 - 2R(R+s)\cos\varphi \quad (10)$$

$$\begin{aligned} \ell'^2 &= R^2 + (s'-R)^2 - 2R(s'-R)\cos(180^\circ - \varphi) \\ &= R^2 + (s'-R)^2 + 2R(s'-R)\cos\varphi \end{aligned} \quad (11)$$

Note that $\sin\varphi = h/R$; we'll use that next.

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Spherical aberration III

We know the exact result $\sin\varphi = h/R \equiv x$. We want to find an expression for $\cos\varphi$. Let's first get an approximate expression for $\varphi(x)$ in the approximation $x \ll 1$:

$$\begin{aligned} x &= \sin\varphi = \varphi - \frac{\varphi^3}{3!} + \ldots \simeq \varphi\left(1 - \varphi^2/6\right) \\ \frac{x}{\left(1 - \varphi^2/6\right)} &\simeq \varphi \\ x\left(1 + \varphi^2/6\right) &\simeq \varphi \end{aligned}$$

If $x \ll 1$, then to first order we can say $x \simeq \varphi$. If we say *that*, then we can also write the above as

$$\varphi \simeq x(1 + x^2/6) = \frac{h}{R}\left(1 + \frac{h^2}{6r^2}\right)$$

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Spherical aberration IV

Again, we have $\varphi \simeq (h/R)[1 + h^2/(6R^2)]$, and we want to find $\cos\varphi$:

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \ldots \simeq 1 - \varphi^2/2 + \varphi^4/24$$

We can do this in Maple using `x:=h/R;` and `phi:=x*(1+x^2/6);` and `taylor(cos(phi),h=0,5);` to obtain

$$\cos\varphi \simeq 1 - \frac{h^2}{2R^2} - \frac{h^4}{8R^4} + O(h^6) \tag{12}$$

We will want to use this in determining the lengths of Eqs. 13 and 15:

$$\begin{aligned} \ell^2 &= R^2 + (R+s)^2 - 2R(R+s)\cos\varphi \\ \ell'^2 &= R^2 + (s'-R)^2 + 2R(s'-R)\cos\varphi \end{aligned}$$

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Determining ℓ

From Eqs. 13 and 12 we have

$$
\begin{aligned}
\ell^2 &= R^2 + (R+s)^2 - 2R(R+s)\left(1 - \frac{h^2}{2R^2} - \frac{h^4}{8R^4}\right) \\
&= R^2 + s^2 + 2sR + R^2 - 2sR - 2R^2 + 2R(R+s)\frac{h^2}{2R^2} \\
&\qquad +2R(R+s)\frac{h^4}{8R^4} \\
&= s^2 + 2R(R+s)\frac{h^2}{2R^2} + 2R(R+s)\frac{h^4}{8R^4}
\end{aligned}
$$

giving

$$
\ell = s\left\{1 + \left[h^2\frac{(R+s)}{s^2R} + h^4\frac{(R+s)}{4s^2R^3}\right]\right\}^{1/2} \qquad (13)
$$

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Determining ℓ II

Start again from Eq. 13:

$$\begin{aligned}
\ell &= s\left\{1+\left[h^2\frac{(R+s)}{s^2R}+h^4\frac{(R+s)}{4s^2R^3}\right]\right\}^{1/2} \\
&\simeq s\left\{1+\frac{1}{2}\left[h^2\frac{(R+s)}{s^2R}+h^4\frac{(R+s)}{4s^2R^3}\right]\right. \\
&\qquad \left.-\frac{1}{8}\left[h^2\frac{(R+s)}{s^2R}+h^4\frac{(R+s)}{4s^2R^3}\right]^2\right\} \\
&\simeq s\left\{1+h^2\frac{(R+s)}{2s^2R}+h^4\left[\frac{(R+s)}{8s^2R^3}-\frac{(R+s)^2}{8s^4R^2}\right]+O(h^6)\right\} \qquad (14)
\end{aligned}$$

where we have used the second order binomial expansion $(1+x)^{1/2} \simeq 1+x/2-x^2/8$ for $x \ll 1$ or $h \ll s$ and $h^2 \ll sR$.

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Determining ℓ'

Let's do the same for ℓ'. From Eqs. 15 and 12 we have

$$\begin{aligned}
\ell'^2 &= R^2 + (s'-R)^2 + 2R(s'-R)\left(1 - \frac{h^2}{2R^2} - \frac{h^4}{8R^4}\right) \\
&= R^2 + s'^2 - 2s'R + R^2 + 2s'R - 2R^2 + 2R(R-s')\frac{h^2}{2R^2} \\
&\quad +2R(R-s')\frac{h^4}{8R^4} \\
&= s'^2 + 2R(R-s')\frac{h^2}{2R^2} + 2R(R-s')\frac{h^4}{8R^4}
\end{aligned}$$

giving

$$\ell' = s'\left\{1 + \left[h^2\frac{(R-s')}{s'^2R} + h^4\frac{(R-s')}{4s'^2R^3}\right]\right\}^{1/2} \tag{15}$$

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Determining ℓ' II

Carry on from Eq. 15:

$$\begin{aligned}
\ell' &= s'\left\{1+\left[h^2\frac{(R-s')}{s'^2R}+h^4\frac{(R-s')}{4s'^2R^3}\right]\right\}^{1/2} \\
&\simeq s'\left\{1+\frac{1}{2}\left[h^2\frac{(R-s')}{s^2R}+h^4\frac{(R-s')}{4s^2R^3}\right]\right. \\
&\qquad \left. -\frac{1}{8}\left[h^2\frac{(R-s')}{s^2R}+h^4\frac{(R-s')}{4s^2R^3}\right]^2\right\} && (16) \\
&\simeq s'\left\{1+h^2\frac{(R-s')}{2s'^2R}+h^4\left[\frac{(R-s')}{8s'^2R^3}-\frac{(R-s')^2}{8s'^4R^2}\right]+O(h^6)\right\} && (17)
\end{aligned}$$

where we have again used the second order binomial expansion $(1+x)^{1/2}\simeq 1+x/2-x^2/8$ for $x\ll 1$ or $h\ll s'$ and $h^2\ll s'R$.

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Getting to the aberration term

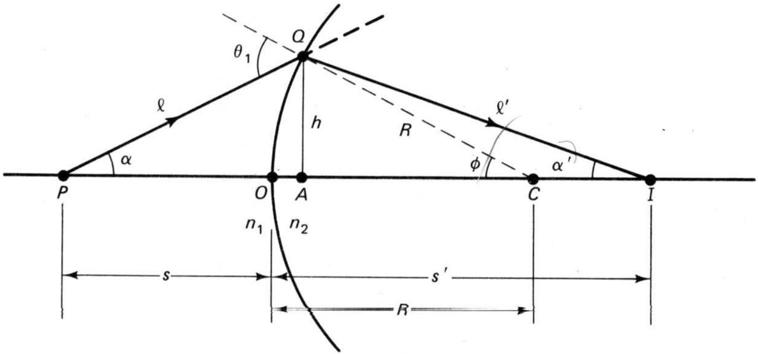

Figure 5-3 Refraction of a ray at a spherical surface.

Now that we have tractable expressions for ℓ and ℓ', let's remind ourselves of what we're doing! We want to find the aberration function $a(Q)$ as given by Eq. 9 of

$$a(Q) = (n_1\ell + n_2\ell') - (n_1 s + n_2 s')$$

Determining $a(Q)$

We want to calculate $a(Q)$ from Eq. 9:

$$\begin{aligned}
a(Q) &= (n_1\ell + n_2\ell') - (n_1 s + n_2 s') \\
&= n_1 s\left\{1 + h^2\frac{(R+s)}{2s^2R} + h^4\left[\frac{(R+s)}{8s^2R^3} - \frac{(R+s)^2}{8s^4R^2}\right]\right\} - n_1 s \\
&\quad + n_2 s'\left\{1 + h^2\frac{(R-s')}{2s'^2R} + h^4\left[\frac{(R-s')}{8s'^2R^3} - \frac{(R-s')^2}{8s'^4R^2}\right]\right\} - n_2 s' \\
&= \frac{h^2}{2}\left[n_1\frac{(R+s)}{sR} + n_2\frac{(R-s')}{s'R}\right] \qquad (18) \\
&\quad + \frac{h^4}{8}\left[n_1\frac{(R+s)}{sR^3} - n_1\frac{(R+s)^2}{s^3R^2} + n_2\frac{(R-s')}{s'R^3} - n_2\frac{(R-s')^2}{s'^3R^2}\right]
\end{aligned}$$

Expanding the $h^2[\,]$ term in Eq. 18 gives

$$\frac{n_1}{s} + \frac{n_1}{R} + \frac{n_2}{s'} - \frac{n_2}{R} = \left(\frac{n_1}{s} + \frac{n_2}{s'}\right) - \left(\frac{n_2 - n_1}{R}\right) \qquad (19)$$

This should ring a bell!

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The h^2 term in $a(Q)$

Look again at the h^2 term of Eq. 19:

$$\frac{h^2}{2}\left[\left(\frac{n_1}{s}+\frac{n_2}{s'}\right)-\left(\frac{n_2-n_1}{R}\right)\right]$$

Recall that the imaging condition for a single refractive surface (which is what we've calculated the aberration function $a(Q)$ for) is given by

$$\frac{n_1}{s}+\frac{n_2}{s'}=\frac{n_2-n_1}{R}$$

so the h^2 term in $a(Q)$ vanishes when you're in focus!

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The h^4 terms in $a(Q)$

We have shown that the h^2 term vanishes in $a(Q)$ when in focus. The remaining term (with a slight rearrangement) in Eq. 18 is

$$a(Q) = \frac{h^4}{8}\left[n_1\frac{(R+s)}{sR^3} + n_2\frac{(R-s')}{s'R^3} - n_1\frac{(R+s)^2}{s^3R^2} - n_2\frac{(R-s')^2}{s'^3R^2}\right]. \tag{20}$$

But we already showed when dealing with Eq. 19 that the following goes to zero when we are at the thin lens imaging point:

$$\left[n_1\frac{(R+s)}{sR} + n_2\frac{(R-s')}{s'R}\right] \qquad \Rightarrow \qquad 0$$

In Eq. 20, this is multiplied by $1/R^2$ but even so it's still zero! Therefore our only remaining term in the in-focus aberration function is

$$a(Q) = -\frac{h^4}{8}\left[n_1\frac{(R+s)^2}{s^3R^2} + n_2\frac{(R-s')^2}{s'^3R^2}\right] \tag{21}$$

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h^4 in $a(Q)$ continued

Continuing with Eq. 21, we have

$$\begin{aligned}
a(Q) &= -\frac{h^4}{8}\left[n_1\frac{(R+s)^2}{s^3R^2}+n_2\frac{(R-s')^2}{s'^3R^2}\right] \\
&= -\frac{h^4}{8}\left[\frac{n_1R^2}{s^3R^2}+2\frac{n_1sR}{s^3R^2}+\frac{n_1s^2}{s^3R^2}+\frac{n_2R^2}{s'^3R^2}-2\frac{n_2s'R}{s'^3R^2}+\frac{n_2s'^2}{s'^3R^2}\right] \\
&= -\frac{h^4}{8}\left[\frac{n_1}{s^3}+2\frac{n_1}{s^2R}+\frac{n_1}{sR^2}+\frac{n_2}{s'^3}-2\frac{n_2}{s'^2R}+\frac{n_2}{s'R^2}\right] \\
&= -\frac{h^4}{8}\left[\frac{n_1}{s}\left(\frac{1}{s^2}+2\frac{1}{sR}+\frac{1}{R^2}\right)+\frac{n_2}{s'}\left(\frac{1}{s'^3}-2\frac{1}{s'R}+\frac{1}{R^2}\right)\right] \\
&= -\frac{h^4}{8}\left[\frac{n_1}{s}\left(\frac{1}{s}+\frac{1}{R}\right)^2+\frac{n_2}{s'}\left(\frac{1}{s'}-\frac{1}{R}\right)^2\right] \qquad (22)
\end{aligned}$$

Whew! At the end of the day, we have found that when a single refractive surface is used to image a point to a point, the imaging is not perfect: aberrations increase as the fourth power (h^4) of radius, even when you're at the paraxial focal point. You have a homework problem that will help you explore this further.

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What spherical aberration looks like

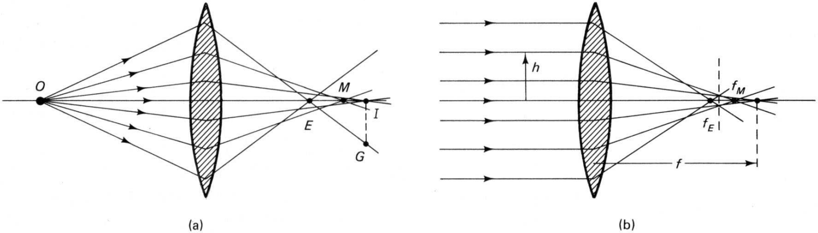

Figure 5-6 Spherical aberration of a lens, producing in (a) different image distances and in (b) different focal lengths, depending on the lens aperture.

From Pedrotti and Pedrotti.

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- You've seen from Eq. 22 that the h^4 term from a single refractive interface is multiplied by

$$\frac{n_1}{s}\left(\frac{1}{s}+\frac{1}{R}\right)^2+\frac{n_2}{s'}\left(\frac{1}{s'}-\frac{1}{R}\right)^2$$

For a second surface, we will have $s_2 = -s_1'$ with a thin lens, and two radii R_1 and R_2.

- It turns out that one can then play with different values of R_1 and R_2 to minimize spherical aberration for different object and image distances. Lenses can have different choices of R_1 and R_2 but the same overall focal length $1/f = (n-1)(1/R_1 - 1/R_2)$. Besides the focal length, one can characterize lenses in terms of their Coddington shape factor $\sigma \equiv (R_2 + R_1)/(R_2 - R_1)$.

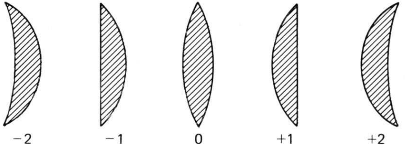

Figure 5-7 "Bending" of a single lens into various versions having the same focal length. The Coddington shape factor below each version serves to classify them.

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Other aberrations

Spherical aberration is just one of many types of aberration a lens can have. These aberrations are characterized in terms of the off-axis distance h of the object, off-axis distance r at the lens' plane (beware: we used h for the off-axis distance at the lens' plane in our spherical aberration derivation!), and azimuthal angle θ. The main monochromatic or Siedel aberrations are then

$$a(Q) = {}_0C_{40}r^4 + {}_1C_{31}r^3\cos\theta + {}_2C_{22}h'^2r^2\cos^2\theta + {}_2C_{20}h'^2r^2 + {}_3C_{11}h'^3r\cos\theta$$

where the coefficients are:

- Spherical aberration ${}_0C_{40}$
- Coma ${}_1C_{31}$: comet-like "tails" of light from off-axis image points
- Astigmatism ${}_2C_{22}$: different focal lengths in orthogonal directions
- Field curvature ${}_2C_{20}$:
- Distortion ${}_3C_{11}$: "barrel" or "pincushion" effect when a square object is imaged

Lens designers make their living by finding complicated combinations of lenses to minimize these distortions for various optical systems.

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Aberrations: illustrations

Aberrations in optical systems form a dense, complex subject. But the book *Optics* by Eugene Hecht has some nice illustrations.

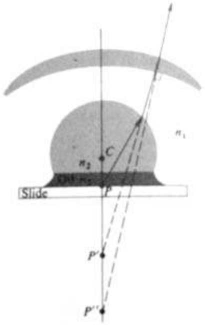

Figure 6.18 An oil-immersion microscope objective.

Aberrations illustrated II

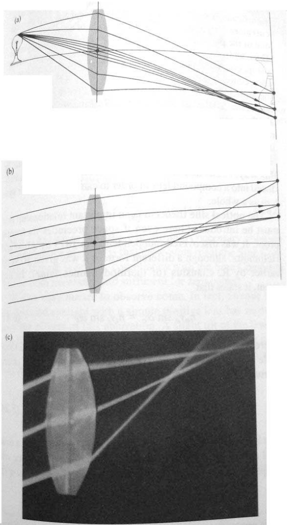

Figure 6.21 (a) Negative coma. (b) and (c) Positive coma.
(Photo by E.H.)

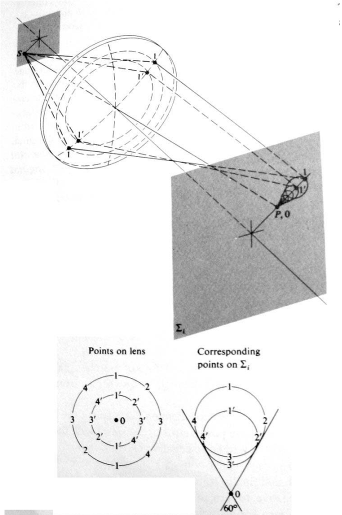

Figure 6.22 The geometrical coma image of a monochromatic point source. The central region of the lens forms a point image at the vertex of the cone.

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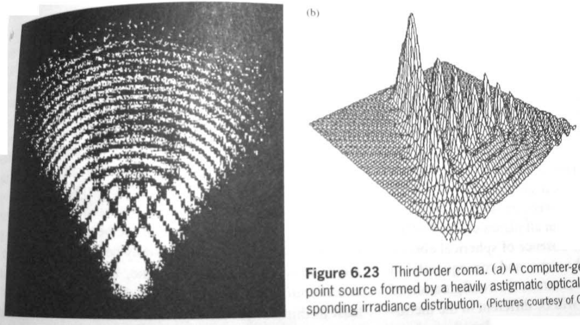

Figure 6.23 Third-order coma. (*a*) A computer-generated diagram of the image of a point source formed by a heavily astigmatic optical system. (*b*) A plot of the corresponding irradiance distribution. (Pictures courtesy of OPAL Group, St. Petersburg, Russia.)

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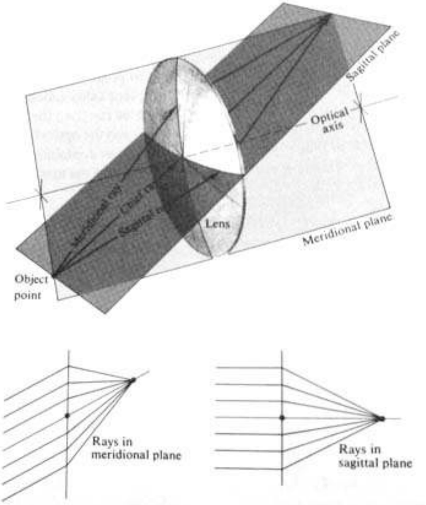

Figure 6.26 The sagittal and meridional planes.

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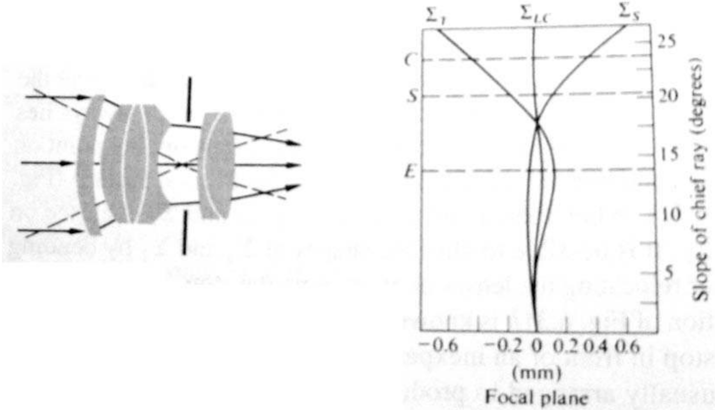


Figure 6.32 A typical Sonnar. The markings *C*, *S*, and *E* denote the limits of the 35-mm film format (field stop), that is, corners, sides, and edges. The Sonnar family lies between the double Gauss and the triplet.

Aberrations illustrated VI

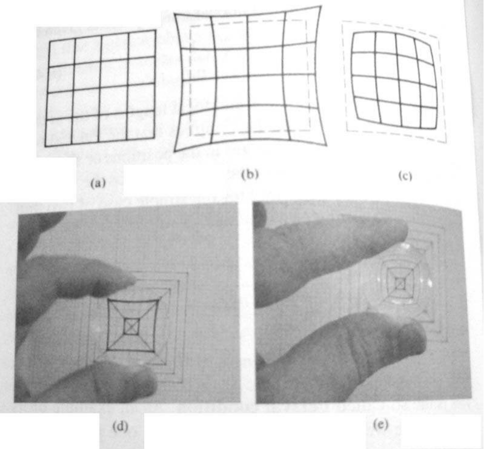
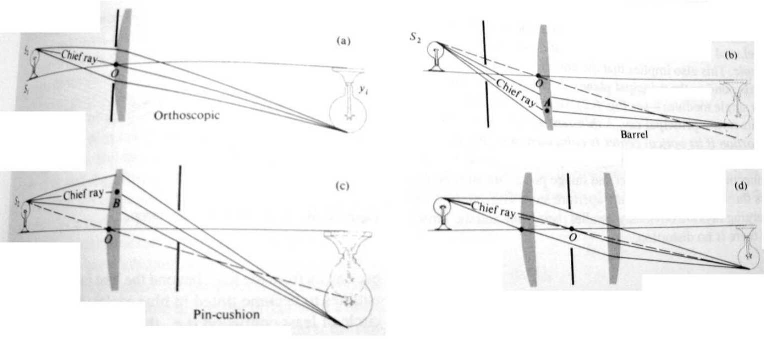


Figure 6.33 (*a*) Undistorted object. (*b*) When the magnification on the optical axis is less than the off-axis magnification, pincushion distortion results. (*c*) When it is greater on axis than off, barrel distortion results. (*d*) Pincushion distortion in a single thin lens. (*e*) Barrel distortion in a single thin lens. (Photos by E.H.)

Aberrations illustrated VII



igure 6.34 The effect of stop location on distortion.

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GRIN: gradient refractive index

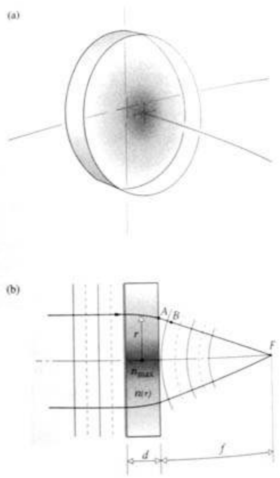

Figure 6.42 A disk of transparent glass whose index of refraction decreases radially out from the central axis. (b) The geometry corresponding to the focusing of parallel rays by a GRIN lens.

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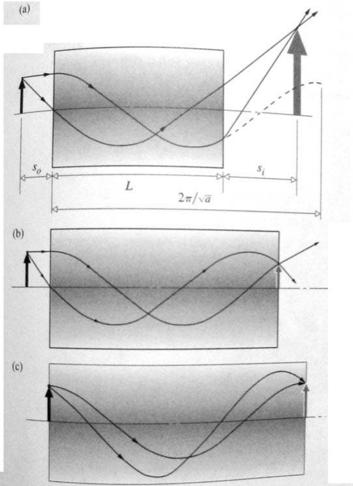

Figure 6.43 (a) A radial-GRIN rod producing a real, magnified, erect image. (b) Here the image is formed on the face of the rod. (c) This is a convenient setup for use in a copy machine.