Left off equation sheet: $c = 2.99 \times 10^8$ m/s

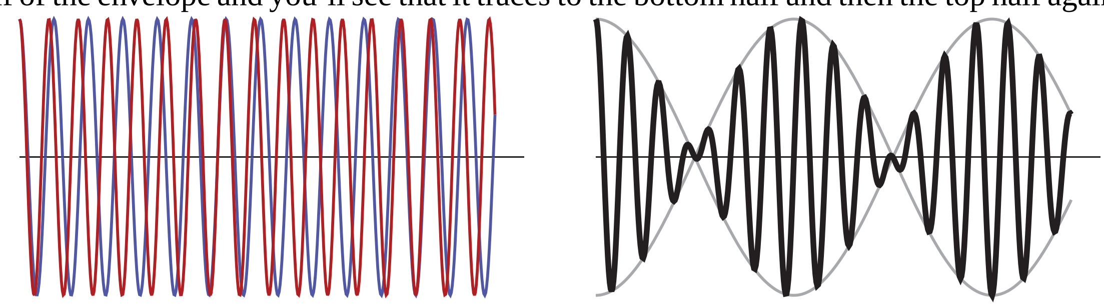
1. A mechanical system has a maximum response at 1 kHz when damping is turned off, and the resonant frequency shifts by 0.2 Hz when damping is turned on. The mass of the oscillating object is 0.01 kg. Calculate the following: the spring constant of the system, its quality factor, the full width at half maximum (FWHM) of its damped resonance curve, and the time it takes for the damped oscillations to decrease to 1/10$^{\text{th}}$ of their original value.
Answer: The spring constant is $k = m\omega_0^2 = (0.01)[2\pi \cdot (1 \times 10^3)]^2 = 3.95 \times 10^5$ N/m. To find the quality factor $Q = \omega_0/\gamma$, we first find γ:

$$\begin{aligned} \omega^2 &= \omega_0^2 - \frac{\gamma^2}{4} \\ \gamma &= 2\sqrt{\omega_0^2 - \omega^2} \\ Q &= \frac{\omega_0}{\gamma} = \frac{\omega_0}{2\sqrt{\omega_0^2 - \omega^2}} = \frac{1}{2\sqrt{1 - \omega^2/\omega_0^2}} = \frac{1}{2\sqrt{1 - 999.8^2/1000^2}} = 25.0 \end{aligned}$$

The full-width at half-max or FWHM of the damped resonance curve is found from $2\Delta\omega = \omega_0/Q$ giving $2\Delta\omega = 2\pi(1000)/(25.0) = 251$ radians/sec or 40 Hz. For damping, we have

$$\begin{aligned} \frac{E_0}{10} &= E_0 e^{-\gamma t_{10}} \\ \ln 10 &= \gamma t_{10} \\ t_{10} &= \frac{\ln 10}{\gamma} = \frac{\ln 10}{2\sqrt{(2\pi)^2(1000^2 - 999.8^2)}} = 9.2 \text{ milliseconds} \end{aligned}$$

2. A mandolin player is trying to tune her two E strings to be at the same pitch. She hears the combined effect as a tone at 658 Hz and an intensity fluctuation at 2 Hz. What are the frequencies at which the two strings vibrate? By what percentage should the tension on one string be adjusted to bring it into tune?
Answer: The intensity fluctuation at 2 Hz means the cosine beat envelope is at 1 Hz; pick the top half of the envelope and you'll see that it traces to the bottom half and then the top half again:



Therefore we have $\bar{f} = 658$ and $\Delta f = 1$, so the two frequencies are $f_1 = 657.5$ and $f_2 = 658.5$ Hz. The increase in tension $\Delta\tau$ needed to bring the lower string up to the pitch of the higher frequency string is found from $2\pi f_1 = \sqrt{\tau/\mu}$ (where τ is the tension and μ is the mass per length):

$$2\pi(f_1 + \Delta f) = \sqrt{\frac{\tau + \Delta\tau}{\mu}}$$

$$\begin{aligned}
f_1 + \Delta f &= \sqrt{\frac{\tau+\Delta\tau}{4\pi^2\mu}} = \sqrt{\frac{\tau}{4\pi^2\mu}(1+\frac{\Delta\tau}{\tau})} \simeq \sqrt{\frac{\tau}{4\pi^2\mu}}(1+\frac{\Delta\tau}{2\tau}) \\
f_1 + \Delta f &= f_1(1+\frac{\Delta\tau}{2\tau}) \\
1+\frac{\Delta f}{f_1} &= 1+\frac{\Delta\tau}{2\tau} \\
\frac{\Delta\tau}{\tau} &= 2\frac{\Delta f}{f_1} = 2\frac{1}{657.5} = 0.00304
\end{aligned}$$

so the tension on string 1 should be increased by 0.3%.

3. Two vectors are to be added, one with a length of 1.0 and the other with a length of 0.1. What's the percentage change in the length of the result when the shorter vector is at first $0°$ and then $5°$ from the longer vector? When it's at $90°$ and then $95°$ from the longer vector? *This problem will become immediately practical when we talk about Zernike phase contrast.*
Answer: In the first case, the two net vector components in $\hat{x}$ and $\hat{y}$ are

$$\begin{aligned}
\hat{x}: &\quad \{1+0.1\cos(0°), 1+0.1\cos(5°)\} = \{1.1, 1.099\,62\} \\
\hat{y}: &\quad \{0+0.1\sin(0°), 0+0.1\sin(5°)\} = \{0, 0.008\,72\} \\
r = \sqrt{x^2+y^2} &= \{\sqrt{1.1^2+0^2}, \sqrt{1.099\,62^2+0.008\,72^2}\} = \{1.100\,00, 1.099\,65\}
\end{aligned}$$

with a percentage difference of $100\cdot(1.1-1.099\,65)/1.1 = 0.032\%$. In the second case, the two net vector components are

$$\begin{aligned}
\hat{x}: &\quad \{1+0.1\cos(90°), 1+0.1\cos(95°)\} = \{1, 0.991\,28\} \\
\hat{y}: &\quad \{0+0.1\sin(90°), 0+0.1\sin(95°)\} = \{0.1, 0.099\,62\} \\
r = \sqrt{x^2+y^2} &= \{\sqrt{1^2+0.1^2}, \sqrt{0.991\,28^2+0.099\,62^2}\} = \{1.004\,99, 0.996\,27\}
\end{aligned}$$

with a percentage difference of $100\cdot(1.004\,99-0.996\,27)/(1.004\,99) = 0.87\%$. Rotating the smaller vector by $90°$ means that small changes in its angle give relatively larger changes in the net intensity.

4. Two identical pendula ($\omega_0^2 = g/l$) are connected by a light coupling spring. Each pendulum has a length of 0.5 m. With the coupling spring connected, one pendulum is clamped and the period of the other is then found to be 1.1 seconds. With neither pendulum clamped, what are the periods of the two normal modes? Justify your answer by showing the relevant differential equation.
Answer: French problem 5-2 (first part, at least, and with a slightly different number for T_1). Each individual pendulum has a resonance frequency of $\omega_0 = \sqrt{g/l} = \sqrt{9.8/0.5} = 4.427$ radians/sec. The spring by itself has a constant k leading to a spring-only, coupled resonant frequency of $\omega_c = \sqrt{k/m}$. When the two pendula are connected by a spring but one pendulum is clamped, we can find the resulting clamped resonant frequency ω_{clamped} as follows:

$$m\frac{d^2x}{dt^2} = -m\omega_0^2 x - kx = k_{\text{clamped}} x$$

$$\begin{aligned} \text{so} \qquad k_{\text{clamped}} &= m\omega_0^2 + k \\ \text{or} \qquad m\omega_c^2 = k &= k_{\text{clamped}} - m\omega_0^2 = m\omega_{\text{clamped}}^2 - m\omega_0^2 \\ \omega_c &= \sqrt{\omega_{\text{clamped}}^2 - \omega_0^2} = \sqrt{(\frac{2\pi}{T_{\text{clamped}}})^2 - \omega_0^2} = \sqrt{(\frac{2\pi}{1.1})^2 - 4.427^2} = 3.609 \text{ radians/sec} \end{aligned}$$

Therefore we now have established the two fundamental frequencies going into the combined motion, so that the common mode frequency is $\omega_0 = 4.427$ radians/second, while the differential mode frequency ω' is given by

$$\omega' = \sqrt{\omega_0^2 + 2\omega_c^2} = \sqrt{4.427^2 + 2 \cdot 3.609^2} = 6.756 \text{ radians/sec}$$

or periods $t = 2\pi/\omega$ of 1.42 and 0.93 seconds, respectively.

5. A certain clear plastic has a dielectric constant of $K = \epsilon/\epsilon_0 = 1.5$. At what speed do light wavefronts travel in this material?
Answer: Because the refractive index goes like $n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$ and because magnetic effects are negligible in most media, we have $n \simeq \sqrt{\epsilon/\epsilon_0} = \sqrt{K} = \sqrt{1.5} = 1.22$. Wavefronts travel at the phase velocity $v_p = c/n = 2.99 \times 10^8/1.22 = 2.45 \times 10^8$ m/sec.