Matrix optics

To deal with more complicated optical systems, we developed matrices for each basic operation:

$$\text{Translation:} \qquad \mathcal{T} = \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \tag{1}$$

$$\text{Refraction:} \qquad \mathcal{R} = \begin{bmatrix} \dfrac{n}{n'} & \dfrac{1}{R}\left(\dfrac{n}{n'} - 1\right) \\ 0 & 1 \end{bmatrix} \tag{2}$$

$$\text{Reflection:} \qquad \mathcal{L} = \begin{bmatrix} 1 & \dfrac{2}{R} \\ 0 & 1 \end{bmatrix} \tag{3}$$

All these matrices have the property Det$\mathcal{M} = n/n'$; so should their products. Matrices can be used in sequences to track positions y and angles α:

$$\begin{bmatrix} \alpha_N \\ y_N \end{bmatrix} = \mathcal{M}_N \mathcal{M}_{N-1} \ldots \mathcal{M}_2 \mathcal{M}_1 \begin{bmatrix} \alpha_0 \\ y_0 \end{bmatrix} \tag{4}$$

so that we have to calculate matrix multiplications starting from the right and working our way left.

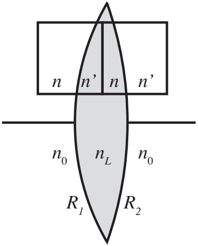
Matrix notation: inconsistencies

Some textbooks use $[y, \alpha]$ instead of $[\alpha, y]$. To handle this, you just swap rows and then columns, giving basic matrices of

$$\mathcal{T} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \qquad \mathcal{R} = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{R}\left(\dfrac{n}{n'} - 1\right) & \dfrac{n}{n'} \end{bmatrix} \qquad \mathcal{L} = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R} & 1 \end{bmatrix}$$

Matrix for a thin lens

What's the matrix for a thin lens of material with refractive index n_L in a medium with refractive index n_0?
First surface: $[n \to n_0, n' \to n_L, R \to R_1]$
Second surface $[n \to n_L, n' \to n_0, R \to R_2]$



Since $\mathcal{R} = \begin{bmatrix} \frac{n}{n'} & \frac{1}{R}\left(\frac{n}{n'} - 1\right) \\ 0 & 1 \end{bmatrix}$ the net effect is

$$\mathcal{R}_2\mathcal{R}_1 = \begin{bmatrix} \frac{n_L}{n_0} & \frac{1}{R_2}\left(\frac{n_L}{n_0} - 1\right) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{n_0}{n_L} & \frac{1}{R_1}\left(\frac{n_0}{n_L} - 1\right) \\ 0 & 1 \end{bmatrix} \tag{5}$$

Thin lens matrix II

Again, we had Eq. 5 of

$$\mathcal{R}_2\mathcal{R}_1 = \begin{bmatrix} \frac{n_L}{n_0} & \frac{1}{R_2}\left(\frac{n_L}{n_0} - 1\right) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{n_0}{n_L} & \frac{1}{R_1}\left(\frac{n_0}{n_L} - 1\right) \\ 0 & 1 \end{bmatrix}$$

The matrix multiplication result is

$$\begin{bmatrix} \frac{n_L}{n_0} \cdot \frac{n_0}{n_L} + \frac{1}{R_2}(\frac{n_L}{n_0} - 1) \cdot 0 & \frac{n_L}{n_0} \cdot \frac{1}{R_1}\left(\frac{n_0}{n_L} - 1\right) + \frac{1}{R_2}\left(\frac{n_L}{n_0} - 1\right) \cdot 1 \\ 0 \cdot \frac{n_0}{n_L} + 1 \cdot 0 & 0 \cdot \frac{1}{R_1}\left(\frac{n_0}{n_L} - 1\right) + 1 \cdot 1 \end{bmatrix}$$

which gives

$$\mathcal{R}_2\mathcal{R}_1 = \begin{bmatrix} 1 & \frac{1}{R_1}\left(1 - \frac{n_L}{n_0}\right) + \frac{1}{R_2}\left(\frac{n_L}{n_0} - 1\right) \\ 0 & 1 \end{bmatrix} \qquad (6)$$

Thin lens matrix III

We had Eq. 6 of

$$\mathcal{R}_2\mathcal{R}_1 = \begin{bmatrix} 1 & \frac{1}{R_1}\left(1-\frac{n_L}{n_0}\right)+\frac{1}{R_2}\left(\frac{n_L}{n_0}-1\right) \\ 0 & 1 \end{bmatrix}$$

We can simplify this:

$$\begin{aligned}\frac{1}{R_1}\left(1-\frac{n_L}{n_0}\right)+\frac{1}{R_2}\left(\frac{n_L}{n_0}-1\right) &= \frac{1}{R_1}\left(\frac{n_0-n_L}{n_0}\right)+\frac{1}{R_2}\left(\frac{n_L-n_0}{n_0}\right) \\ &= \left(\frac{n_L-n_0}{n_0}\right)\left(-\frac{1}{R_1}+\frac{1}{R_2}\right) = -\left(\frac{n_L-n_0}{n_0}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \quad (7)\end{aligned}$$

Look familiar?

Thin lens matrix IV

Again, after some algebraic manipulation we had

$$\mathcal{R}_2\mathcal{R}_1 = \begin{bmatrix} 1 & -\left(\frac{n_L - n_0}{n_0}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\ 0 & 1 \end{bmatrix}$$

The careful observer should note that the upper right matrix element is simply the lensmaker's equation! (See e.g., Fowles 10.7 or Eq. 17 on l8.pdf). Our thin lens matrix F becomes

$$\mathcal{F} = \mathcal{R}_2\mathcal{R}_1 = \begin{bmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{bmatrix} \tag{8}$$

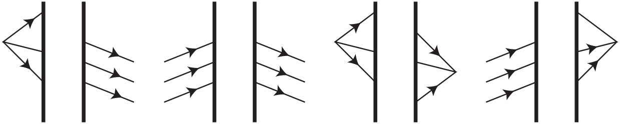
Matrix elements: A

We've found that we can use matrix optics to make some calculations in a neat and clever way. Can we milk matrices for more insight?

The changes to rays of $\alpha_1 = A\alpha_0 + By_0$ and $y_1 = C\alpha_0 + Dy_0$ are in the matrix

$$\begin{bmatrix} \alpha_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \alpha_0 \\ y_0 \end{bmatrix}$$

When $A = 0$, we have $\alpha_1 = By_0$ with no dependence on α_0. That is, rays with any angle leaving a point y_0 in the input plane will all have an angle By_0 in the output plane. We will call this input position the first focal plane of the optical system.



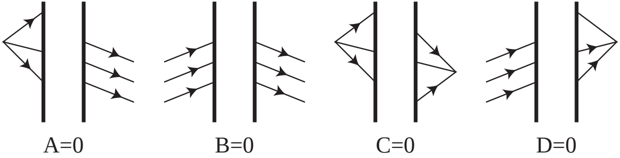
A=0 B=0 C=0 D=0

Matrix elements: B

The changes to rays of $\alpha_1 = A\alpha_0 + By_0$ and $y_1 = C\alpha_0 + Dy_0$ are in the matrix

$$\begin{bmatrix} \alpha_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \alpha_0 \\ y_0 \end{bmatrix}$$

When $B = 0$, we have $\alpha_1 = A\alpha_0$ no matter what the value of y_0 is. This is the case for many telescope designs, and $A = \alpha_1/\alpha_0$ gives the angular magnification.

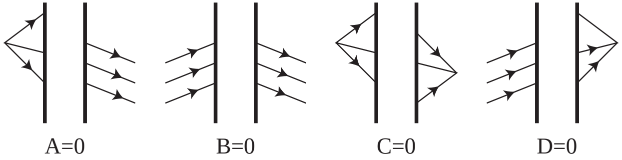


Matrix elements: C

The changes to rays of $\alpha_1 = A\alpha_0 + By_0$ and $y_1 = C\alpha_0 + Dy_0$ are in the matrix

$$\begin{bmatrix} \alpha_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \alpha_0 \\ y_0 \end{bmatrix}$$

When $C = 0$, we have $y_1 = Dy_0$ with no dependence on α_0. That is, we image a point to a point, and the transverse magnification is $D = y_1/y_0$.

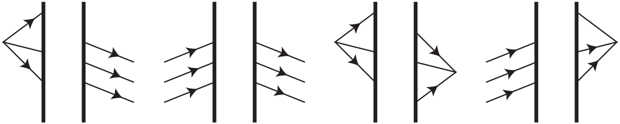


Matrix elements: D

The changes to rays of $\alpha_1 = A\alpha_0 + By_0$ and $y_1 = C\alpha_0 + Dy_0$ are in the matrix

$$\begin{bmatrix} \alpha_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \alpha_0 \\ y_0 \end{bmatrix}$$

When $D = 0$, we have $y_1 = C\alpha_0$ with no dependence on y_0. That is, rays from any position but entering at the angle α_0 will be focused to a common point. We will call this output position the second focal plane of the optical system.



A=0 B=0 C=0 D=0

Cardinal points

With thin lenses, we had some nice, simple ray tracing rules that we found to be helpful:

1. Rays through the lens center are undeviated.
2. Rays parallel to optical axis on the input side go through the focal point on the output side of the lens.
3. Rays through the focal point on the input side emerge parallel to the optical axis on the output side.

We wish to find some equivalents for thick lenses. The equivalents are:

Nodal points: rays that come in towards N_1 emerge at the same angle from N_2.

Principal planes: Rays that come in parallel to the optical axis appear to bend at the principal plane H_2 and emerge through the output focal point f_2 (and vice versa for going through f_1 to H_1 and then parallel to optic axis).

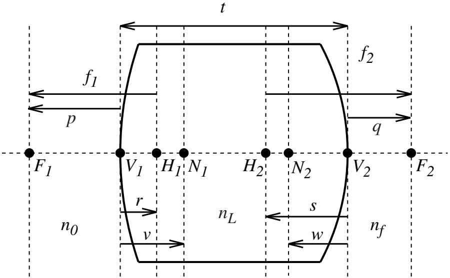
Cardinal points II

Again, our cardinal points are:

Nodal points: rays that come in towards N_1 emerge at the same angle from N_2.

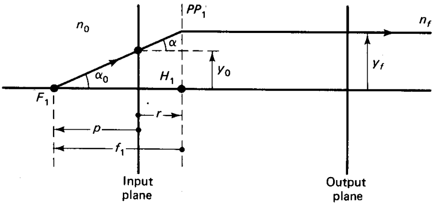
Principal planes: Rays that come in parallel to the optical axis appear to bend at the principal plane H_2 and emerge through the output focal point f_2 (and vice versa for going through f_1 to H_1 and then parallel to optic axis).

We draw these assuming that all arrows that point backwards involve negative numbers.



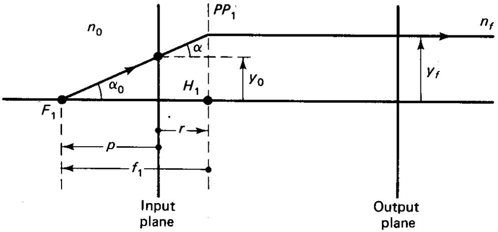
Principal planes

Rays that come in parallel to the optical axis appear to bend at the principal plane H_2 and emerge through the output focal point f_2 (and vice versa for going through f_1 to H_1 and then parallel to optic axis). Consider a ray through the first focal point to the first principal plane:



Because $\alpha_{H_1} = 0$, if we consider the ray from the first vertex V_1 (the input surface) to the first principal plane H_1, we have $\alpha_{H_1} = 0 = A\alpha_0 + By_0$ or $\alpha_0 = -(B/A)y_0$. At the same time, we see that $\alpha_0 = y_0/(-p)$ in the small angle approximation, so we have $y_0/(-p) = -(B/A)y_0$ giving $p = A/B$. We also have $\alpha_0 = y_f/(-f_1)$.

Principal planes II

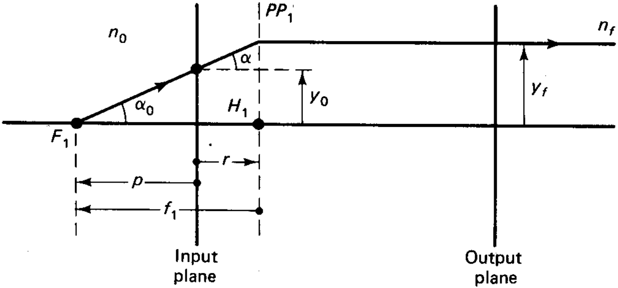


Again, we have $p = A/B$ and $\alpha_0 = y_f/(-f_1)$, leading to

$$\begin{aligned} f_1 &= \frac{-y_f}{\alpha_0} = \frac{-(C\alpha_0 + Dy_0)}{\alpha - 0} = \frac{-(C\alpha_0 - D(A/B)\alpha_0)}{\alpha_0} \\ &= -C + \frac{DA}{B} = \frac{AD - BC}{B} = \frac{\mathrm{Det}\mathcal{M}}{B} = \frac{n_0}{n_f}\frac{1}{B}. \end{aligned} \tag{9}$$

We used here the result for determinants from the last lecture (slide 31 of l9.pdf).

Principal planes III

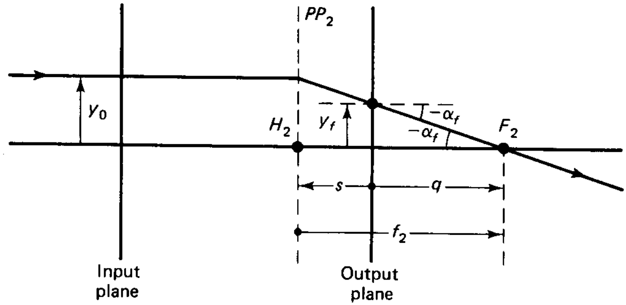


We can solve for r based on our knowledge of p and f_1:

$$r = p - f_1 = \frac{A}{B} - \frac{n_0}{n_f}\frac{1}{B} = \frac{1}{B}\left(A - \frac{n_0}{n_f}\right). \tag{10}$$

Second principal plane

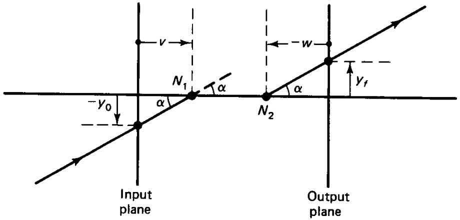
A similar calculation can be carried out for the second principal plane:



We will simply give the corresponding result in our summary.

Nodal points

Rays that come in towards the nodal point N_1 emerge at the same angle from the nodal point N_2:



We see $\alpha = -y_0/v$. Because rays through the nodal points have $\alpha_0 = \alpha_f = \alpha$, we can write

$$\alpha_f = A\alpha_0 + By_0 \qquad \alpha = A\alpha + By_0$$

$$\alpha = \frac{B}{1-A} y_0 = \frac{-y_0}{v} \qquad v = \frac{A-1}{B}.$$

A calculation involving the matrix determinant, like what was used in Eq. 9, leads to $w = (n_0/n_f - D)/B$.

Letting Maple do the dirty work

- OK, we've laid out the basic calculations to determine the positions of principal planes and nodal points. We still have some work to do to come up with more complete expressions!
- Being lazy physicists, let's let Maple do the dirty work. The notation here is from Maple 8; you can download a Maple 12 worksheet from the course web page as thin_lens_matrix.mw.
- Let's start by considering the matrix solution for a simple thin lens:

```
> m1:=array([[(n/nl),((1/R1)*(n/nl-1))],[0,1]]);
```

$$m1 := \begin{bmatrix} \dfrac{n}{nl} & \dfrac{\dfrac{n}{nl} - 1}{R1} \\ 0 & 1 \end{bmatrix}$$

Now that we've seen the Maple input and output, the other two inputs are

```
> m2:=array([[1,0],[0,1]]);
> m3:=array([[(nl/n),((1/R2)*(nl/n-1))],[0,1]]);
```

Maple's thin lens

Having input the basic matrices, we can then get the thin lens matrix from

```
> mlens:=simplify(evalm(m3&*(m2&*m1)));
```

which gives

$$mlens \ := \begin{bmatrix} 1 & -\dfrac{(n - nl)\,(-R2 + R1)}{n\,R1\,R2} \\ 0 & 1 \end{bmatrix}$$

Let's rearrange Maple's result for the upper-right matrix element:

$$\begin{aligned} -\frac{(n-n_L)(-R_2+R_1)}{n\,R_1\,R_2} &= -\left(\frac{n_L-n}{n}\right)\left(\frac{R_2-R_1}{R_1\,R_2}\right) \\ &= -\left(\frac{n_L-n}{n}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = -\frac{1}{f} \end{aligned}$$

which again gives a net matrix for the lens of $\begin{bmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{bmatrix}$

Thick lens surrounded by n_0 and n_f

Let's make Maple work a bit harder by doing a thick lens with different refractive media front and back. Again, I show here entries in Maple 8 notation, and make available a Maple 12 (partial) worksheet at thick_lens_matrix.mw:

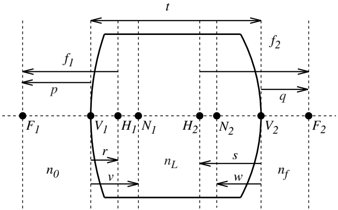
```
m1:=array([[(n0/nl),(n0-nl)/(nl*R1)],[0,1]]);
m2:=array([[1,0],[t,1]]);
m3:=array([[(nl/nf),(nl-nf)/(nf*R2)],[0,1]]);
mlens:=simplify(evalm(m3&*(m2&*m1)));
```

This gives $mlens :=$

$$\begin{aligned} & \left[\left[-\frac{(-nl\,R2 - t\,nl + t\,nf)\,n0}{R2\,nf\,nl}, \quad -\frac{-nl\,R2\,n0 + nl^2\,R2 - t\,nl\,n0 + t\,nl^2 + t\,nf\,n0 - t\,nf\,nl - R1\,nl^2 + R1\,nl\,nf}{R2\,nf\,R1\,nl}\right],\right. \\ & \left.\left[\frac{t\,n0}{nl}, \frac{t\,n0 - t\,nl + R1\,nl}{R1\,nl}\right]\right] \\ = \quad & \left[[A, B], [C, D]\right] \end{aligned}$$

Getting our p's and q's from Maple

In fact, the lens with thickness t and input and output refractive indices of n_0 and n_f respectively is just the thing we're looking for in our general set of cardinal points:



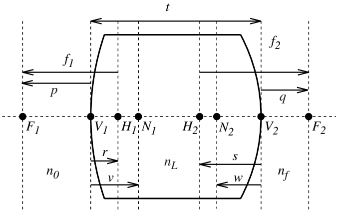
We can then use Maple to obtain solutions for our cardinal points from the result we had on the previous slide for *mlens*. The Maple input is:

$p = A/B$: `p:=simplify(combine(mlens[1,1]/mlens[1,2]));`

$q = -A/C$: `q:=simplify(combine(-mlens[2,2]/mlens[1,2]));`

$r = (D - n_0/n_f)/C$: `r:=simplify(combine((mlens[1,1]-(n0/nf))/mlens[1,2]));`

More p's and q's



$s = (1 - A)/C$: `s:=simplify(combine((1-mlens[2,2])/mlens[1,2]));`

$v = (D - 1)/C$: `v:=simplify(combine((mlens[1,1]-1)/mlens[1,2]));`

$w = (n_0/n_f - A)/C$: `w:=simplify(combine(((n0/nf)-mlens[2,2])/mlens[2,1]));`

$1/f_1 = C/(n_0/n_f)$: `inverse_f1:=simplify(combine(1/(p-r)));`

$1/f_2 = -C$: `inverse_f2:=simplify(combine(1/(q-s)));`

Summary of cardinal points

With a little more work, one can find

$$
\begin{aligned}
\frac{1}{f_1} &= \frac{n_L - n_f}{n_0 R_2} - \frac{n_L - n_0}{n_0 R_1} - \frac{(n_L - n_0)(n_L - n_f)}{n_0\, n_L} \frac{t}{R_1\, R_2} = \frac{B}{n_0/n_f} \\
f_2 &= \frac{-n_f}{n_0} f_1 = \frac{-1}{B} \\
p &= r + f_1 = \frac{D}{C} \\
q &= 1 \\
r &= \frac{n_L - n_f}{n_L R_2} f_1 t = \frac{A - n_0/n_f}{B} \\
s &= -\frac{n_L - n_0}{n_L R_1} f_2 t = \frac{1 - D}{B} \\
v &= \left(1 - \frac{n_f}{n_0} + \frac{n_L - n_f}{n_L R_2} t\right) f_1 = \frac{A - 1}{B} \\
w &= \left(1 - \frac{n_0}{n_f} - \frac{n_L - n_0}{n_L R_1} t\right) f_2 = \frac{n_0/n_f - D}{B}
\end{aligned}
$$

Further fun facts

Note that if $n_0 = n_f$, we can make some simplifying statements:

- $r = v$ and $s = w$, so principal and nodal points coincide.
- $f_1 = -f_2$, so first and second focal lengths are equal in magnitude.
- $r - s = v - w$, so the separation of the principal points is the same as the separation of nodal points.

Most importantly, with the above expressions for focal lengths measured from the principal planes, it can be shown that the equivalent of $1/s_o + 1/s_i = 1//f$ is

$$-\frac{f_1}{s_o} + \frac{f_2}{s_i} = 1 \qquad \text{with magnification} \qquad m = -\frac{n_0 s_i}{n_f s_o}$$

So what's all this good for?

- The treatment we have come up with describes thick lenses, and even combinations of several thick lenses, in terms of net properties of the optical system.
- This is done either by finding the A, B, C, D coefficients for the net array, or by calculating the distances $p, q, \ldots$ based on the equations we have derived.
- We've done all this for small angles. In general, good lens design seeks to satisfy this condition because it tends to minimize aberrations. So in fact the basic matrix methods described above are pretty useful.
- If you need to worry about large angles, then you can always do numerical ray tracing on a computer. There are lots of commercial packages that do that!
- Even so, knowing about principal planes and nodal points is important for building even simple optical systems.