Addition to homework

- For Monday, Sep. 29, you already had an assignment of doing French problems 4-16, 17 and 5-6, 7, 9.
- Add three things to that assignment:
  - A) Show how to obtain Eq. 10 from Eq. 6 on today's lecture
  - B) Show how to obtain Eq. 12 from Eq. 6 on today's lecture
  - C) Using Eq. 10 from today's lecture, estimate the refractive index for air assuming that it is composed entirely of nitrogen (look up the density of air).
- This revised homework assignment is posted on the web page (as are the solutions to HW3).
- And looking ahead:
  - Monday, Sep. 29: lecture, plus homework due.
  - Wednesday, Oct. 1: no class (Rosh Hashana).
  - Monday, Oct. 6: Exam 1, in class. Exam will cover French chapters 1–5, plus the DDHO model of the refractive index (like the above problems).

Relating $\vec{E}$ and $\vec{B}$

Last time we said:

- For electrostatically neutral material, Gauss' law gives $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0$ which, for waves $\vec{E} = \vec{E}_0 e^{-i(kx-\omega t)}$ and $\vec{B} = \vec{B}_0 e^{-i(kx-\omega t)}$ traveling in the $\hat{x}$ direction, means that $(\vec{E}_0)_{\hat{x}} = (\vec{B}_0)_{\hat{x}} = 0$. Therefore for plane waves the $\vec{E}$ and $\vec{B}$ fields oscillate *transverse* to the direction of wave propagation.
- From Faraday's law of $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$, one obtains $-k(\vec{E}_0)_{\hat{z}} = \omega t(\vec{B}_0)_{\hat{y}}$ and $k(\vec{E}_0)_{\hat{y}} = \omega(\vec{B}_0)_{\hat{z}}$. Since multipication by i represents a rotation by 90° transverse to the $\hat{x}$ direction, this gives $\vec{B}_0 = (k/\omega)(\vec{i} \times \vec{E}_0)$ so $|B_0| = (k/\omega)|E_0| = (n/c)|E_0|$, and $\vec{E}$ and $\vec{B}$ are mutually perpendicular.
- See this figure from Griffiths' E&M book for linear polarization:

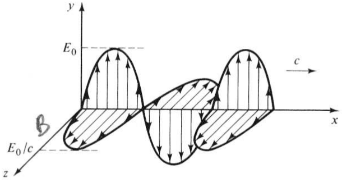


Figure 8.13

Expanding upon Gauss' law

- Consider $\vec{\nabla} \cdot \vec{E} = 0$ for no enclosed charge (electrostatically neutral medium) with $\vec{E} = \vec{E}_0 e^{-i(\vec{k}\cdot\vec{x}-\omega t)}$ and $\vec{k} = k_{\hat{x}}$:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right) \vec{E}_0 e^{-i(kx-\omega t)} \\ &= -ik\vec{E}_0 e^{-i(kx-\omega t)}\hat{x} + 0 + 0 = 0 \end{aligned} \tag{1}$$

  or $(\vec{E}_0)_{\hat{x}} = 0$.
- That is, Gauss' law does not allow us to have any $\hat{x}$ axis component to the electric field of an EM wave traveling in the $\hat{x}$ direction.
- Ditto with $\vec{B}$.
- Therefore for plane waves the $\vec{E}$ and $\vec{B}$ fields oscillate *transverse* to the direction of wave propagation.

Expanding upon Faraday's law

- Faraday's law says $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$.
- Consider the curl of $\vec{E} = \vec{E}_0 e^{-i(\vec{k}\cdot\vec{x}-\omega t)}$ with $\vec{k} = k_{\hat{x}}$:

$$\begin{aligned}
\vec{\nabla} \times \vec{E} &= (\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z})\,\hat{x} + (\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x})\,\hat{y} + (\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y})\,\hat{z} \\
&= \left[\Big(0-0\Big)\,\hat{x} + \Big(0-(-ik)E_{0,z}\Big)\,\hat{y} + \Big((-ik)E_{0,y}-0\Big)\,\hat{z}\right] \\
&\quad e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \\
&= \Big[ikE_{0,z}\,\hat{y} - ikE_{0,y}\,\hat{z}\Big]\,e^{-i(\vec{k}\cdot\vec{x}-\omega t)}. \qquad (2)
\end{aligned}$$

- Consider the time derivative $-\partial\vec{B}/\partial t$ for $\vec{B} = \vec{B}_0 e^{-i(\vec{k}\cdot\vec{x}-\omega t)}$:

$$\frac{\partial}{\partial t}\vec{B} = - - i\omega\,(B_{0,x}\,\hat{x} + B_{0,y}\,\hat{y} + B_{0,z}\,\hat{z})\,e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (3)$$

- If we match up vector components from Eqs. 2 and 3, we have

$$\hat{x}:\ 0 = -i\omega B_{0,x} \qquad \hat{y}:\ ikE_{0,z} = -i\omega B_{0,y} \qquad \hat{z}:\ -ikE_{0,y} = -i\omega B_{0,z} \qquad (4)$$

which again indicates that there is no B field in the $\hat{x}$ direction.

Faraday's law II

- Again, from Eq. 4 we have

$$\hat{x}:\ 0 = -i\omega B_{0,x} \qquad \hat{y}:\ ikE_{0,z} = -i\omega B_{0,y} \qquad \hat{z}:\ -ikE_{0,y} = -i\omega B_{0,z}$$

- This tells us that there is no B field in the $\hat{x}$ direction, which we already knew.
- It tells us in the $\hat{y}$ direction that $B_{0,y} = -(k/\omega)E_{0,z} = -(n/c)E_{0,z}$
- It tells us in the $\hat{z}$ direction that $B_{0,z} = (k/\omega)E_{0,y} = (n/c)E_{0,y}$
- Again, we find out that $|B_0| = (k/\omega)|E_0| = (n/c)|E_0|$, and $\vec{E}$ and $\vec{B}$ are mutually perpendicular.

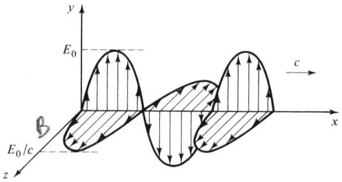


Figure 8.13

Refractive index as DDHO

- Recall that we used a damped, driven harmonic oscillator model with

$$\begin{aligned} F_{\text{total}} &= F_{\text{binding}} + F_{\text{damping}} + F_{\text{driving}} \\ m_e\ddot{x} &= -m_e\omega_j^2 x - m_e\gamma_j\dot{x} + q\,E_0 e^{i\omega t} \end{aligned}$$

- We found that this gave us an electric susceptibility of

$$\chi_e = \frac{n_a e^2}{m_e\epsilon_0}\sum_j \frac{f_j}{(\omega_j^2-\omega^2)+i\gamma_j\omega}. \tag{5}$$

- Because we found $n \simeq (1+\chi_e)^{1/2} \simeq (1+\chi_e/2)$ we obtained

$$\begin{aligned} n &= 1 + \frac{n_a e^2}{2m_e\epsilon_0}\sum_j \frac{f_j}{(\omega_j^2-\omega^2)+i\gamma_j\omega} \\ &= 1 - \frac{n_a e^2}{2m_e\epsilon_0}\sum_j \frac{f_j}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2}\left[(\omega^2-\omega_j^2)+i\gamma_j\omega\right] \end{aligned} \tag{6}$$

Refractive index as DDHO II

- We were able to write Eq. 6 in terms of separate real and imaginary parts:

$$\mathrm{Re}[n] = 1 - \frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{(\omega^2 - \omega_j^2)\, f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \tag{7}$$

$$\mathrm{Im}[n] = -\frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{\gamma_j \omega\, f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}. \tag{8}$$

ω=driving frequency
ω_j=the j^{th} oscillator's resonance frequency
f_j=the weight of the j^{th} oscillator, such that $\sum_j |f_j| \Rightarrow Z$
γ_j=the damping coefficient for the j^{th} oscillator.

- Since a wave propagates in real space as e^{-inkx}, this gives separately an absorption and a phase shift:

$$e^{-inkx} = e^{-i\,\mathrm{Re}[n]\,kx}\, e^{-i\,(-i|\mathrm{Im}[n]|)\,kx} = e^{-ikx} e^{-i(\mathrm{Re}[n]-1)kx}\, e^{-|\mathrm{Im}[n]|\,kx}$$

- The Kramers-Kronig relations let us calculate $\mathrm{Re}[n(\omega)]$ from $\mathrm{Im}[n(\omega)]$ and vice-versa.

Plasma frequency

- In examining Eq. 6 of

$$n = 1 - \frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \left[(\omega^2 - \omega_j^2) + i \gamma_j \omega \right]$$

and ignoring the driving term ω and damping terms γ_j, we can solve for a dominant resonant frequency ω_j which is referred to as the plasma frequency ω_p:

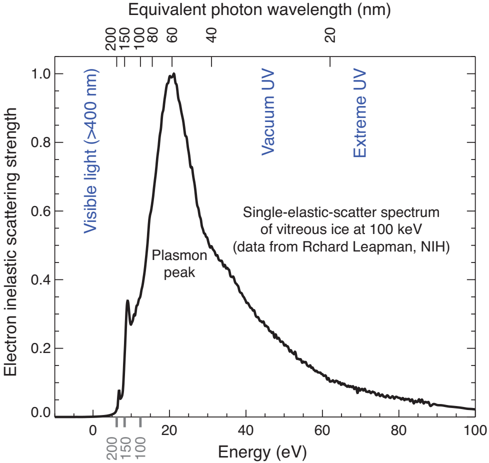
$$\omega_p^2 = \frac{n_a e^2}{m_e \epsilon_0} \sum_j |f_j| = \frac{n_a e^2 Z}{m_e \epsilon_0} \tag{9}$$

where we have recognized that the term $\sum_j f_j$ is just the sum of all the oscillators, which is the total number of electrons Z.

- The *plasma frequency* is the frequency at which we have collective oscillations of the electrons in a solid. For most solids, the plasma frequency corresponds to an energy of $\hbar \omega_p \simeq$ 10–20 eV.

EELS and the plasma frequency

100 keV electrons onto a thin film. Measure what fraction of electrons lose what amount of energy in single inelastic scattering events.



Visible light: the low frequency limit

- For most materials, the plasma frequency corresponds to light in the $\lambda = 100$–200 nm wavelength range. This is why glass is opaque to UV light.
- So visible light represents a low frequency limit! A Taylor series expansion of with $\omega \ll \omega_j$ then gives

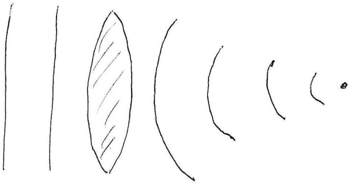
$$\mathrm{Re}[n] \simeq 1 + \left(\frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{f_j}{\omega_j^2} \right) + \omega^2 \left(\frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{f_j}{\omega_j^4} \right) \tag{10}$$

which agrees with a well-known empirical parameterization of the refractive index in glass: Cauchy's equation of

$$\mathrm{Re}[n] \simeq 1 + A(1 + B/\lambda^2) \tag{11}$$

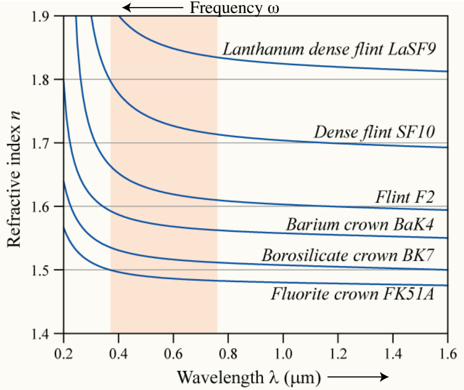
What does the refractive index do?

- If we ignore attenuation, plane waves propagate as $\exp[-iknz] = \exp[-ikz]\exp[-ik(n-1)z]$.
- Thus there is a phase delay (wave slows down) of $k(n-1)z = 2\pi\frac{(n-1)}{\lambda}z$.
- If $n = 1.5$, it means that light gets delayed by half a wave per wavelength. This can be used to make focusing lenses!

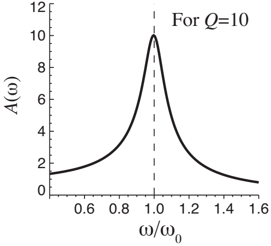


Visible light dispersion

Again, we expected strong absorption below $\lambda = 200$ nm or 0.2 μm, and $n \simeq 1 + A(1 + B/\lambda^2)$:



Frequency ω $\longrightarrow$



$\longleftarrow$ Wavelength λ

http://en.wikipedia.org/wiki/Image:Dispersion-curve.png

On to high frequencies: X rays!

- We had from Eq. 6 the result

$$n = 1 - \frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \left[(\omega^2 - \omega_j^2) + i \gamma_j \omega \right]$$

- At high frequencies $\omega \gg \omega_j$ and low damping $\gamma \to 0$, this reduces to

$$n \simeq 1 - \frac{n_a e^2}{2 m_e \epsilon_0 \omega^2} \sum_j f_j = 1 - \alpha \lambda^2 (f_1 + i f_2) = 1 - \delta - i\beta \quad (12)$$

- At a particular driving frequency, we're using the complex quantity $(f_1 + i f_2)$ for the net effect of all oscillators. We're also using

$$\alpha = n_a r_e / (2\pi)$$

where $r_e = e^2 / m_e c^2 = 2.818 \times 10^{-15}$ m is the classical radius of the electron (the radius at which the Coulomb energy equals $m_e c^2$).

How to write the x-ray refractive index

- It's completely arbitrary to say that forward wave propagation is $\exp[-ikz]$ instead of $\exp[+ikz]$. Unfortunately, different books use different conventions...
- With forward wave propagation as $\exp[-ikz]$, we have $n = 1 - \delta - i\beta = 1 - \alpha\lambda^2(f_1 + if_2)$.
- With forward wave propagation as $\exp[+ikz]$, we have $n = 1 - \delta + i\beta = 1 + \alpha\lambda^2(f_1 - if_2)$.
- In any case, X rays in media have their phase advanced and their amplitude attenuated.
- Soft x-ray literature ($E \lesssim 5$ keV): $(f_1 + if_2)$ is common.
- Hard x-ray literature ($E \gtrsim 5$ keV): $(f_0 + f' + if'')$ is common, where $f_0 \simeq Z$.

But X rays have $n < 1$!

- $n \simeq 1 - \delta - i\beta$ so $(n-1)$ is small and negative.
- Crest of waves: *phase* velocity is $v_p = \frac{\omega}{k} \simeq c(1 + \alpha f_1 \lambda^2)$, which is faster than the speed of light!
- But main body of wave moves more slowly: *group* velocity is $v_g = \frac{d\omega}{dk} \simeq c(1 - \alpha f_1 \lambda^2)$, as we'll see later...
- X rays undergo dispersion, and thus are attenuated. Think of breakers crashing at the beach...

86 A. Einstein, [Nr. 9/12.

Lassen sich Brechungsexponenten der Körper für Röntgenstrahlen experimentell ermitteln?

Von A. Einstein.

(Eingegangen am 21. März 1918.)

Vor einigen Tagen erhielt ich von Herrn Prof. A. Köhler (Wiesbaden) eine kurze Arbeit[1]), in welcher eine auffallende Erscheinung bei Röntgenaufnahmen geschildert ist, die sich bisher nicht hat deuten lassen. Die reproduzierten Aufnahmen — zumeist menschliche Gliedmaßen darstellend — zeigen an der Kontur einen hellen Saum von etwa 1 mm Breite, in welchem die Platte heller bestrahlt zu sein scheint als in der (nicht beschatteten) Umgebung des Röntgenbildes.

Ich möchte die Fachgenossen auf diese Erscheinung hinweisen und beifügen, daß die Erscheinung wahrscheinlich auf Totalreflexion beruht. Nach der klassischen Dispersionstheorie müssen wir erwarten, daß der Brechungsexponent n für Röntgenstrahlen nahe an 1 liegt, aber im allgemeinen doch von 1 verschieden ist. n wird kleiner bzw. größer als 1 sein, je nachdem der Einfluß derjenigen Elektronen auf die Dispersion überwiegt, deren Eigenfrequenz kleiner oder größer ist als die Frequenz der Röntgenstrahlen. Die Schwierigkeit einer Bestimmung von n liegt darin, daß $(n-1)$ sehr klein ist (etwa 10^{-6}). Es ist aber leicht einzusehen, daß bei nahezu streifender Inzidenz der Röntgenstrahlen im Falle $n < 1$ eine nachweisbare Totalreflexion auftreten muß.

How X rays propagate

- Wave propagates as

$$\begin{aligned} \exp[-iknz] &= \exp[-ikz]\exp[-ik(n-1)z] \\ &= \exp[-ikz]\ \exp[+ik\delta z]\exp[-k\beta z] \end{aligned}$$

- Phase is *advanced* by $\exp[+ik\delta z]$
- Wave is attenuated by $\exp[-k\beta z]$

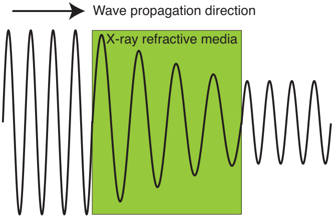
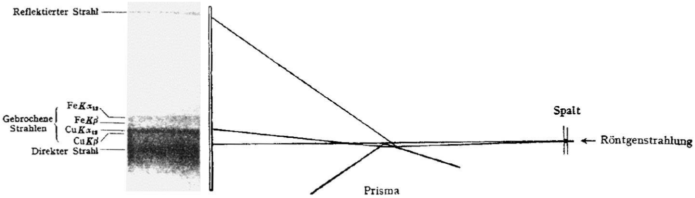


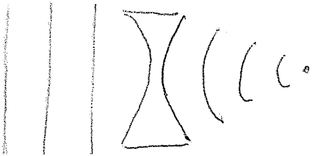
Figure inspired by Benjamin Hornberger

Refraction is opposite!

- Because phase is *advanced* in materials, prisms work the opposite way from visible light. Can that be so? Yes! Larsson, Siegbahn, and Waller, *Naturwiss.* (1924):



- Lenses work the opposite way, too: concave lenses focus, rather than defocus!



Kramers-Kronig for X rays

- For X rays we have $n = 1 - \alpha\lambda^2(f_1 + if_2)$
- The Kramers-Kronig relationship then becomes (Henke, 1981)

$$f_1(E) = Z + \frac{2}{E}\int_0^\infty \frac{\epsilon^2 f_2(E)}{E^2 - \epsilon^2}d\epsilon - \Delta f_r,$$

  where Δf_r is a small relativistic correction term that can usually be ignored.
- Since x-ray intensity is attenuated in a material thickness t by

$$\exp[-2k\beta t] = \exp[-4\pi\alpha\lambda f_2 t] = \exp[-\mu t]$$

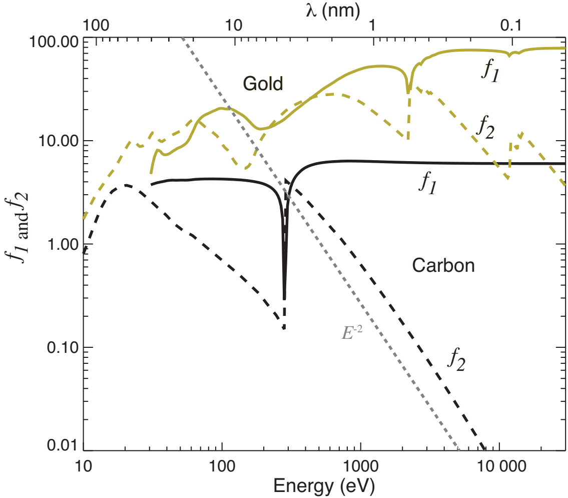
  we can determine f_2 over "all" x-ray energies from absorption measurements.
- We can then use this complete set of $f_2(E)$ to determine the phase-shifting part of the refractive index $f_1(E)$ using Kramers-Kronig!

The Henke data

- At the University of Hawaii, Burt Henke worked with a series of undergraduates to make exceedingly careful measurements of x-ray absorption of all elements, over a wide energy range, using x-ray tubes. (He did this while losing his eyesight over his career, too!) We owe him a lot!
- At the end of his career, he moved to the Center for X-ray Optics at Lawrence Berkeley Lab, where Erik Gullikson (a former student of Henke) continues to refine the work.
- The Henke data: f_2 from 10 eV to 30,000 eV over 501 energies, and f_1 from 50 eV to 30,000 eV.
  - Original version: Henke *et al.*, *Atomic Data and Nuclear Data Tables* **27**, 1–144 (1982).
  - Latest version: Henke, Gullikson, and Davis, *Atomic Data and Nuclear Data Tables* **54**, 181–342 (1993) with subsequent updates. See `http://henke.lbl.gov/optical_constants/`

Behavior of $(f_1 + if_2)$

Notice: $f_1 \to Z$ at high energies, and $f_2 \propto E^{-2}$ with edge jumps



Phase and group velocities

- Now let's consider the case of two different wave frequencies and propagation velocities but with the same amplitude. That is, consider $\psi_1 = Ae^{-i(k_1 x - \omega_1 t)}$ and $\psi_2 = Ae^{-i(k_2 x - \omega_2 t)}$.
- As in lecture 2, let $\alpha \equiv -k_1 x + \omega_1 t$ and $\beta \equiv -k_2 x + \omega_2 t$. Our two-wave sum ψ is then

$$\psi = Ae^{i\alpha} + Ae^{i\beta} = A[\cos\alpha + \cos\beta] + iA[\sin\alpha + \sin\beta]. \qquad (13)$$

- As before, we dig deep into our toolbox of trig identities:

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad (14)$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad (15)$$

- With these identities, we can write Eq. 13 as

$$\psi = 2A\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2iA\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad (16)$$

Phase and group velocities II

Again, Eq. 16 is

$$\begin{aligned}\psi &= Ae^{i\alpha} + Ae^{i\beta} \\ &= 2A\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2iA\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ &= 2A\left[\cos\frac{\alpha+\beta}{2} + i\sin\frac{\alpha+\beta}{2}\right]\cos\frac{\alpha-\beta}{2} \end{aligned} \tag{17}$$

Now let's write $\bar{\theta} \equiv (\alpha+\beta)/2$ as the average angle, and $\Delta\theta \equiv (\alpha-\beta)/2$ as the typical difference from the average. We then see that the result of mixing waves at the two frequencies is

$$\psi = Ae^{i\alpha} + Ae^{i\beta} = 2Ae^{i\bar{\theta}}\cos(\Delta\theta) = 2Ae^{i\bar{\theta}}\,\mathrm{Re}[e^{i\Delta\theta}] \tag{18}$$

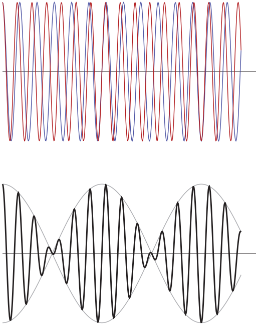
where $\bar{\theta} = -[(k_1x-\omega_1t)+(k_2x-\omega_2t)]/2 = -(\bar{k}x-\bar{\omega}t)$
and $\Delta\theta = -[(k_1x-\omega_1t)-(k_2x-\omega_2t)]/2 = -(\Delta k\,x-\Delta\omega\,t)$

Phase and group velocities III

Again, we have shown that

$$\psi = Ae^{-i(k_1 x - \omega_1 t)} + Ae^{-i(k_2 x - \omega_2 t)} = 2Ae^{-i(\bar{k}x - \bar{\omega}t)}\,\mathrm{Re}[e^{-i(\Delta k\, x - \Delta\omega\, t)}]. \tag{19}$$

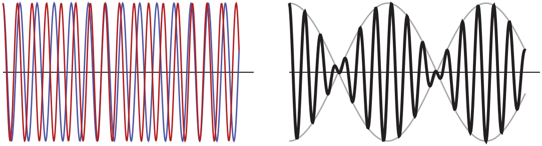
where $\bar{\theta} = -[(k_1 x - \omega_1 t) + (k_2 x - \omega_2 t)]/2 = -(\bar{k}x - \bar{\omega}t)$
and $\Delta\theta = -[(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)]/2 = -(\Delta k\, x - \Delta\omega\, t)$.



The combined result (at bottom) looks like two different waves propagating!

Phase and group velocities IV

Again, we have $\psi = 2Ae^{-i(\bar{k}x - \bar{\omega}t)}\,\mathrm{Re}[e^{-i(\Delta k\,x - \Delta\omega\,t)}]$



- Because $k = 2\pi/\lambda$ giving $\lambda = 2\pi/k$, and $\omega = 2\pi f = 2\pi/T$ giving $T = 2\pi/\omega$, the velocity of a wave (which travels a distance λ in a time T) is given by

$$v = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k} \tag{20}$$

- We therefore have two velocities going on here:

$$v_{\text{wave crest}} = \frac{\bar{\omega}}{\bar{k}} \qquad v_{\text{beat envelope}} = \frac{\Delta\omega}{\Delta k}$$

Phase and group velocities V

Again, we had

$$v_{\text{wave crest}} = \frac{\bar{\omega}}{\bar{k}} \qquad v_{\text{beat envelope}} = \frac{\Delta\omega}{\Delta k}$$

In the limit of infinitessimal changes in wavelength and frequency, this leads to two different frequencies!

- The *phase velocity* v_p is the speed of wave crests:

$$v_p = \frac{\omega}{k} \tag{21}$$

- The *group velocity* v_g is the speed at which the main wave body travels (*i.e.*, the speed at which most energy is carried):

$$v_g = \frac{d\omega}{dk} \tag{22}$$

- *Dispersion* arises in situations where the differences between v_p and v_g are noticeable!

Calculating v_p

- On Eq. 4 of l6.pdf, we found that electromagnetic waves in linear media obey the relationship $k^2 = \mu\epsilon\omega^2$, giving

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} \tag{23}$$

with $n \equiv \sqrt{\mu\epsilon}/\sqrt{\mu_0\epsilon_0}$.

- We therefore see that the phase velocity is trivial to calculate if we know n:

$$v_p = \frac{\omega}{k} = \frac{c}{n} \tag{24}$$

Calculating v_g

- To calculate the group velocity, let's start with Eq. 23 of $\omega/k = c/n$, rewrite it as $ck = \omega n$, and differentiate:

$$\begin{aligned}
d[\omega\, n(\omega)] &= d[ck] \\
d\omega \cdot n(\omega) + \omega \cdot d[n(\omega)] &= c\, dk \\
d\omega \cdot n(\omega) + d\omega \cdot \omega \cdot \frac{d[n(\omega)]}{d\omega} &= c\, dk \\
d\omega \left[n(\omega) + \omega \frac{dn(\omega)}{d\omega} \right] &= c\, dk \\
\frac{d\omega}{dk} &= \frac{c}{n(\omega) + \omega \dfrac{dn(\omega)}{d\omega}}
\end{aligned}$$

- We therefore see that the group velocity v_g can be calculated from

$$v_g = \frac{d\omega}{dk} = \frac{c}{n(\omega) + \omega \dfrac{dn(\omega)}{d\omega}} \tag{25}$$

v_p and v_g for visible light

- For visible light, we found the Cauchy form of Eqs. 10 and 11

$$\begin{aligned}\mathrm{Re}[n] &\simeq 1+\left(\frac{n_a e^2}{2m_e\epsilon_0}\sum_j \frac{f_j}{\omega_j^2}\right)+\omega^2\left(\frac{n_a e^2}{2m_e\epsilon_0}\sum_j \frac{f_j}{\omega_j^4}\right)\\ &\simeq 1+A'+B'\omega^2 \simeq 1+A(1+B/\lambda^2).\end{aligned}$$

- The phase velocity is

$$v_p=\frac{c}{n}=\frac{c}{1+A'+B'\omega^2}\simeq c\left(1-A'-B'\omega^2\right) \tag{26}$$

where in the last step we've used the binomial expansion $(1+x)^n \simeq 1+nx$ for $x \ll 1$.

- The group velocity can be found using $d[1+A'+B'\omega^2]=2B'\omega$:

$$\begin{aligned}v_g &= \frac{c}{n(\omega)+\omega\dfrac{dn(\omega)}{d\omega}}=\frac{c}{(1+A'+B'\omega^2)+\omega(2B'\omega)}\\ &= \frac{c}{1+A'+3B'\omega^2}\simeq c(1-A'-3B'\omega^2)\end{aligned} \tag{27}$$

v_p and v_g for visible light II

- Again, for visible light with $n \simeq 1 + A' + B'\omega^2$ we had velocities from Eqs. 26 and 27 of

$$v_p \simeq c(1 - A' - B'\omega^2) \qquad \text{and} \qquad v_g \simeq c(1 - A' - 3B'\omega^2)$$

- The phase and group velocities are both less than the speed of light, making us happy.
- The phase and group velocities are slightly different, so *dispersion* effects must be considered in some cases. (An example we'll soon see involves *chromatic aberrations*).

v_p and v_g with X rays

- From Eq. 12 we had $n \simeq 1 - \alpha\lambda^2(f_1 + if_2)$. Let's simplify with $\tilde{f} = (f_1 + if_2)$, and $\alpha' = \alpha(2\pi c)^2$, so that we have

$$n \simeq 1 - \alpha'\tilde{f}\omega^{-2}$$

- The phase velocity is

$$v_p = \frac{c}{n} = \frac{c}{1 - \alpha'\tilde{f}\omega^{-2}} \simeq c(1 + \alpha'\tilde{f}\omega^{-2}) \tag{28}$$

- The group velocity can be found using $dn/d\omega = d[1 - \alpha'\tilde{f}\omega^{-2}]/d\omega = +2\alpha'\tilde{f}\omega^{-3}$ so that we have

$$\begin{aligned} v_g &= \frac{c}{n(\omega) + \omega\frac{dn(\omega)}{d\omega}} = \frac{c}{(1 - \alpha'\tilde{f}\omega^{-2}) + \omega(+2\alpha'\tilde{f}\omega^{-3})} \\ &= \frac{c}{1 + \alpha'\tilde{f}\omega^{-2}} \\ &\simeq c(1 - \alpha'\tilde{f}\omega^{-2}) \end{aligned} \tag{29}$$

v_p and v_g with X rays II

- Again, for X rays with $n \simeq 1 - \alpha' \tilde{f} \omega^{-2}$ we had velocities from Eqs. 28 and 29 of

$$v_p \simeq c(1 + \alpha' \tilde{f} \omega^{-2}) \qquad \text{and} \qquad v_g \simeq c(1 - \alpha' \tilde{f} \omega^{-2})$$

- While the phase velocity is faster than the speed of light, the group velocity is slower. Therefore wave crests outrun wave bodies, leading to breakers on the beach. . . Oops, I mean attenuation of x-ray beams in media.

Deep ocean waves

Wikipedia has some good info:
http://en.wikipedia.org/wiki/Ocean_surface_wave
http://en.wikipedia.org/wiki/Gravity_wave

- The phase velocity is approximately $v_p \simeq \sqrt{g/k}$, and the group velocity is approximately $v_g = v_p/2$.
- This dispersion is why sometimes you can get rogue waves (many wavelengths superimposing) and sometimes relatively calm spots.

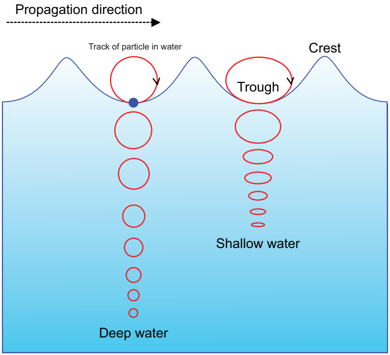


Figure courtesy Wikipedia

The Great Wave off Kanagawa

1832 woodcut by Hokusai; one of *36 Views of Mount Fuji*