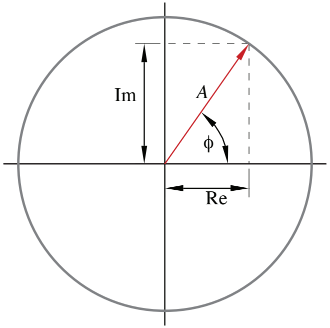
Complex algebra

- Recall that we decided that there are advantages to using complex algebra to represent wave motion:

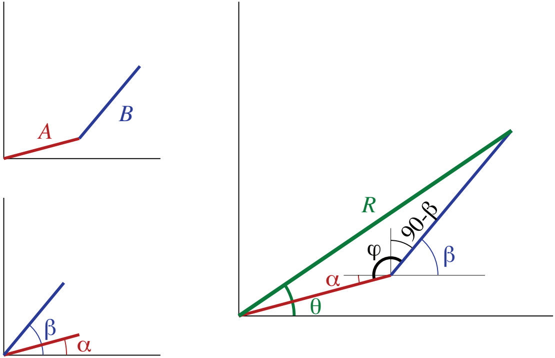
$$A\cos(\omega t + \varphi_0) = \mathrm{Re}[Ae^{i\omega t + \varphi_0}] \tag{1}$$

- A is magnitude (some say amplitude), while $\omega t + \varphi_0$ is the phase (φ_0 is the starting phase, and ω is the angular frequency $\omega = 2\pi f$). Together they make a complex amplitude.
- Phase lets us keep track of whether we're at the max, zero, or min of the wavefield.
Look at the movie.



Addition

Consider the addition of two vectors $Ae^{i\omega t+\alpha}$ and $Be^{i\omega t+\beta}$, viewed at a moment in time:



If the two waves have the exact same frequency ω, both terms have a common factor $e^{i\omega t}$ which we can pull out, and thus deal with the static case. We found in the last lecture that the length is given by

$$R^2 = A^2 + B^2 - 2AB\cos(\varphi) = A^2 + B^2 + 2AB\cos(\alpha - \beta) \qquad (2)$$

Addition: many waves at same frequency

- With many waves (indexed by i) at the same frequency ω but differing magnitudes A_i, we found we could express the sum as

$$R^2 = \sum_{i=1}^{N} A_i^2 + 2 \sum_{j>i}^{N} \sum_{i=1}^{N} A_i A_j \cos(\theta_i - \theta_j). \tag{3}$$

- Let's first consider the case where the phases are distributed evenly from $-\pi$ to $+\pi$, so that $(\theta_i - \theta_j)$ takes on all values around the circle with even probability. Since $\cos\theta$ gives the part of a vector that projects along the $+\hat{x}$ axis in Cartesian coordinates, the addition of vectors all pointing out at random directions from the center will tend to give a net vector of zero, thus wiping out the second part of Eq. 3.

Sum of incoherent waves

- Look again at the result of Eq. 3 of

$$R^2 = \sum_{i=1}^{N} A_i^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} A_i A_j \cos(\theta_i - \theta_j)$$

  in the case of adding up completely random phases so that the second term goes away.
- We've just learned something important! If we add up a bunch of incoherent waves which all have the same amplitude A_i but random phase, we have for *incoherent superposition* a resultant length of

$$|R| = \sqrt{\sum_{i=1}^{N} A_i^2} = \sqrt{N \cdot A_i^2} = \sqrt{N}|A| \tag{4}$$

- This principle comes up many times in physics. Consider the walk of a drunken sailor: each step is of a known distance, but in a completely random direction. We can't say that the sailor will go due east, or southwest; but we can say that the net distance traveled for N steps is $\sqrt{N}$ times the distance of one step. Einstein used this to get Avogadro's number from observations of Brownian motion.

Sum of coherent waves

Now consider the case where all waves have the same amplitude and the same phase. Return again to the second term of Eq. 3:

$$R^2 = \sum_{i=1}^{N} A_i^2 + 2 \sum_{j>i}^{N} \sum_{i=1}^{N} A_i A_j \cos(\theta_i - \theta_j)$$

In this case, the angle distances $(\theta_i - \theta_j)$ will all be zero! We then have

$$R^2 = \sum_{i=1}^{N} A_i^2 + 2 \sum_{j>i}^{N} \sum_{i=1}^{N} A_i A_j \qquad (5)$$

as the resultant length for the coherent superposition case.

Coherent sum II

Again, our coherent superposition result of Eq. 5 is

$$R^2 = \sum_{i=1}^{N} A_i^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} A_i A_j.$$

Let's reverse the process used to expand $(\sum_i A_i \sin\theta_i)^2$ in the first lecture:

$$\begin{aligned}
(\sum_{i=1}^{N} A_i)^2 &= (A_1 + A_2 + A_3 + \ldots)^2 \\
&= A_1^2 + A_2^2 + A_3^2 \; + 2A_1A_2 + 2A_1A_3 + 2A_2A_3 \; + \ldots \\
&= \sum_{i=1}^{N} A_i^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} A_i A_j
\end{aligned} \tag{6}$$

Therefore we see that the coherent superposition sum of Eq. 5 is just

$$R^2 = (\sum_{i=1}^{N} A_i)^2 = (NA)^2 = N^2A^2 \tag{7}$$

where we have assumed that all amplitudes A_i are the same.

Coherent sum III

Again, we have now expressed the coherent sum result in the form of Eq. 7:

$$R^2 = (\sum_i^N A_i)^2 = (NA)^2 = N^2 A^2$$

Let's consider the length of the resultant vector $|R|$:

$$|R| = \sqrt{R^2} = \sqrt{N^2 A^2} = N|A|. \tag{8}$$

This is considerably different than the incoherent superposition result (Eq. 4) of $|R| = \sqrt{N}|A|$.

Sums: different frequencies

Now let's consider the addition of two waves with equal amplitude but different frequency: $Ae^{i\omega_1 t} + Ae^{i\omega_2 t}$. In fact, we could just as well consider wave motion along a direction x as well as along a time t, so that we have $Ae^{-i(k_1 x - \omega_1 t)} + Ae^{-i(k_2 x - \omega_2 t)}$. But let's make our life easier and say $\alpha \equiv -k_1 x + \omega_1 t$ and $\beta \equiv -k_2 x + \omega_2 t$. We then have

$$Ae^{i\alpha} + Ae^{i\beta} = A[\cos\alpha + \cos\beta] + iA[\sin\alpha + \sin\beta]. \tag{9}$$

Now let's dig deep into our toolbox of trig identities:

$$\begin{aligned} \cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} && (10) \\ \sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} && (11) \end{aligned}$$

With these identities, we can write Eq. 9 of $Ae^{i\alpha} + Ae^{i\beta}$ as

$$= 2A\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2iA\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \tag{12}$$

Different frequencies II

Again, Eq. 12 is

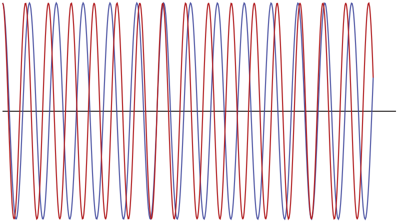
$$\begin{aligned} Ae^{i\alpha} + Ae^{i\beta} &= 2A\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2iA\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ &= 2A\left[\cos\frac{\alpha+\beta}{2} + i\sin\frac{\alpha+\beta}{2}\right]\cos\frac{\alpha-\beta}{2} \end{aligned} \tag{13}$$

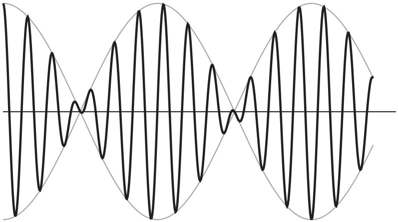
Now let's write $\bar{\theta} \equiv (\alpha+\beta)/2$ as the average angle, and $d\theta \equiv (\alpha-\beta)/2$ as the typical difference from the average. We then see that the result of mixing waves at the two frequencies is

$$Ae^{i\alpha} + Ae^{i\beta} = 2Ae^{i\bar{\theta}}\cos(d\theta). \tag{14}$$

That is, the resulting wave motion can be described as a wave with an average frequency $\bar{\omega} = (\omega_1+\omega_2)/2$ which is modulated by a cosine envelope of much lower frequency $d\omega = (\omega_1-\omega_2)/2$. This "beat" envelope is illustrated on the following page.

Beat envelope





If we were to mix two audible frequencies, the beat would be perceived at twice the frequency. Can you explain why?

Simple harmonic motion: review

Recall that for any restoring force, the lowest order Taylor series expansion gives $F = -kx$. At equilibrium we have a static system, and thus no change in velocity, and thus no net force, or

$$ma = m\frac{d^2x}{dt^2} = -kx. \tag{15}$$

As we saw, one allowable solution is

$$x(t) = A\cos(\omega t + \varphi_0) \tag{16}$$

in which case we were able to show that

$$\omega \equiv \sqrt{\frac{k}{m}}. \tag{17}$$

Complex exponential solution

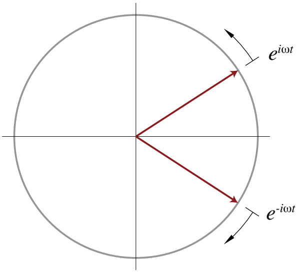
Let's now try a solution to Eq. 15 of $m(d^2x/dt^2) = -kx$ involving a complex exponential form of $x = Ae^{i\omega t}$ and see what we get:

$$\begin{aligned} m(i\omega)(i\omega)Ae^{i\omega t} &= -kAe^{i\omega t} \\ -m\omega^2 &= -k \\ \omega &= \sqrt{\frac{k}{m}} \end{aligned}$$

so we see that we have the same result for the frequency of motion. However, we could just as well try a solution of $x = Ae^{-i\omega t}$; this also satisfies the differential equation and again gives $\omega = \sqrt{k/m}$. Does this present a problem? What do we do with these two solutions?

SHM: two solutions

- Again, we found that there were two equally valid solutions to motion with a $F = -kx$ restoring force: $A_1 e^{+i\omega t}$ and $A_2 e^{-i\omega t}$. They correspond to rotation in opposite directions in the complex plane.
- They both give the same value of $\mathrm{Re}[Ae^{\pm i\omega t}]$.



SHM: two solutions II

The two solutions $A_1 e^{+i\omega t}$ and $A_2 e^{-i\omega t}$ should have the same maximum amplitude if they have the same energy, or $A_1 = A_2 = A$. Let's examine a sum of these two solutions with complementary starting phases φ:

$$\begin{aligned} x &= Ae^{i\omega t+\varphi} + Ae^{-i\omega t-\varphi} \\ &= A\Big[\cos(\omega t+\varphi) + i\sin(\omega t+\varphi)\Big] \\ &\quad +A\Big[\cos(-\omega t-\varphi) + i\sin(-\omega t-\varphi)\Big] \\ &= A\Big[\cos(\omega t+\varphi) + i\sin(\omega t+\varphi)\Big] \\ &\quad +A\Big[\cos(\omega t+\varphi) - i\sin(\omega t+\varphi)\Big] \\ &= 2A\cos(\omega t+\varphi) \end{aligned}$$

The take-home message: in most cases we can make an arbitrary choice on the sign of the exponent in the complex exponential, since the measureable part is real only. We'll usually consider the phase to increase with time, and thus use $e^{+i\omega t}$.

SHM: elastic deformation

- We've seen that SHM crops up with linear restoring forces $F = -kx$. What physical situations give rise to SHM?
- Consider the deformation of materials. As long as we stay within the elastic yield of a material, we can say

$$\frac{\text{stress}}{\text{strain}} = \frac{\Delta P/A}{\Delta \ell/\ell_0} = Y \tag{18}$$

where Y is a constant of proportionality called the *Young's modulus*, ΔP in this case refers to a tensile force applied to an object with cross-sectional area A, and $\Delta \ell$ is the plastic deformation (stretching) length relative to a certain tension-free length ℓ_0.
- Let $\Delta F = -\Delta P$ be the force exerted by the object under tension to fight against the force that's trying to stretch it out, so as to have dynamic equilibrium, and go over to infintessimal derivatives. We then can write Eq. 18 as

$$dF = -\frac{AY}{\ell_0}\, d\ell \tag{19}$$

Elastic deformation II

Again, we had from Eq. 19 the expression

$$dF = -\frac{AY}{\ell_0}\,d\ell$$

which upon integration gives

$$F = -\frac{AY}{\ell_0}x \tag{20}$$

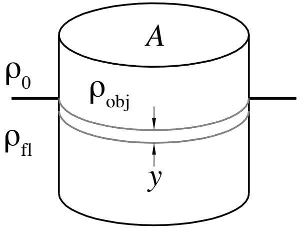
if we write the position of the stretched end as x instead of ℓ in the convention $x = 0$ when $F = 0$. That is, we have a linear restoring force with $k = AY/\ell_0$ and harmonic motion applies. Think of compression waves travelling in rock during earthquakes.

SHM: bobbing buoys

Consider a buoyant object at the boundary between two fluids (one might by air). The buoyant force is equal to the displaced mass times gravitational acceleration. If the object is pressed down by a distance $-y$, then we will have displaced a mass of the fluid below of $Ay \cdot \rho_{\text{fl}}$ and replaced it by a mass of the fluid above of $Ay \cdot \rho_0$, giving a net force of

$$F = -Ay(\rho_{\text{fl}} - \rho_0)g = -[g(\rho_{\text{fl}} - \rho_0)A]y \tag{21}$$

Again, we have a linear restoring force $k = g(\rho_{\text{fl}} - \rho_0)A$ so simple harmonic motion applies.

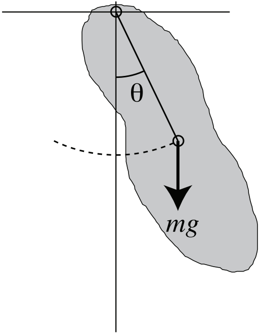


The pendulum

Consider a general pendulum. When rotated by an angle θ, it experiences a torque $\tau = -rF \sin \varphi$ where $F = mg$. That is, the net torque is $\tau = -rmg \sin \theta$. Now the equivalent for $F = m(d^2x/dt^2)$ for torque is $\tau = I(d^2\theta/dt^2)$ where I is the moment of inertia, so we have

$$I\frac{d^2\theta}{dt^2} = -rmg \sin \theta \qquad (22)$$

as the equation of motion for the pendulum.



Pendulum: small angles

Again, we had Eq. 22 of motion for the general pendulum of

$$I\frac{d^2\theta}{dt^2} = -rmg\sin\theta$$

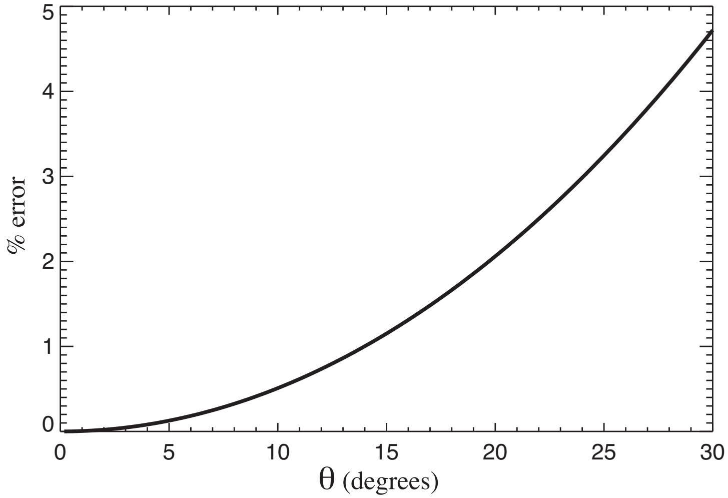
This is not an easy differential equation to solve directly. However, if we could make the substitution $\sin\theta \to \theta$ then all would be right with the world. When can we do that? Consider the Taylor series of $\sin\theta$:

$$\begin{aligned} \sin\theta &\simeq \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots \\ &\simeq \theta\left[1 - \frac{\theta^2}{3!} + \frac{\theta^4}{4!} + \ldots\right] \end{aligned}$$

This is true when the fractional contribution of the second term is small, or $\theta^2/3! \ll 1$ or $\theta \ll \sqrt{6}$. This ought to be pretty easy to satisfy, and indeed it is as we show on the next slide.

$\sin\theta \simeq \theta$

Here's the error of the approximation $\sin\theta \simeq \theta$, expressed as $100(\theta - \sin\theta)/\sin\theta$:



Pendulum

Return once more to Eq. 22 of motion for the general pendulum:

$$I\frac{d^2\theta}{dt^2} = -rmg\sin\theta$$

If we say $\sin\theta \simeq \theta$, this again becomes a simple linear restoring force equation:

$$I\frac{d^2\theta}{dt^2} = -rmg\theta \tag{23}$$

so again we're going to have simple harmonic motion with

$$\omega = \sqrt{\frac{k}{m}} \rightarrow \sqrt{\frac{rmg}{I}} \tag{24}$$

If the object is a mass m at the end of a massless rod or string of length r, then the moment of inertia is $I = (1/2)mr^2$ and the frequency of motion becomes

$$\omega = \sqrt{\frac{rmg}{I}} = \sqrt{\frac{rmg}{(1/2)mr^2}} = \sqrt{\frac{2g}{r}} \tag{25}$$

Harmonic motion is ubiquitous

- What we have seen is that simple harmonic motion is ubiquitous in nature. Fortunately it's an easy type of motion to solve, and we can describe it very nicely using complex wave notation $Ae^{i\omega t}$.
- Real life is a bit more complicated! Motion tends to be damped, so that it does not go on forever.
- Damping is generally proportional to velocity, and in a direction opposing the motion. Consider drag forces in fluids: in the turbulent flow regime $F_d \propto v^2$ while in the laminar flow regime it is $F_d \propto v$.
- Of course we can always do a Taylor series expansion on any damping force, and assume that it is linear for weak damping.
- As a result we will want to go on and consider damped wave motion equations of the form

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \tag{26}$$

Damped harmonic motion

Again we want to consider Eq. 26 for damped harmonic wave motion:

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

Using $\omega_0 \equiv \sqrt{k/m}$ and $\gamma \equiv b/m$, we can rewrite this as

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{27}$$

Let's assume a complex exponential solution of the form $x = Ae^{i(pt+\alpha)}$. Putting this into Eq. 27, we have

$$\left((ip)^2 + (ip)\gamma + \omega_0^2\right)Ae^{i(pt+\alpha)} = 0 = \left(-p^2 + ip\gamma + \omega_0^2\right)Ae^{i(pt+\alpha)} \tag{28}$$

which must be true for any time t. Therefore we require

$$-p^2 + ip\gamma + \omega_0^2 = 0 \tag{29}$$

if our trial solution of $x = Ae^{i(pt+\alpha)}$ is to work.

Damped harmonic motion II

Again, from Eq. 29 we require

$$-p^2 + ip\gamma + \omega_0^2 = 0$$

from our trail solution $x = Ae^{i(pt+\alpha)}$. Now for this to be satisfied, the real part and the imaginary part must both be zero. We can meet this condition if we assume that $p = n + is$ with n and s both real. In this case

$$p^2 = n^2 - s^2 + i2ns$$

and Eq. 29 becomes

$$-n^2 + s^2 - s\gamma + \omega_0^2 + i(-2ns + n\gamma) = 0 \tag{30}$$

which means we must have

$$\begin{aligned} -n^2 + s^2 - s\gamma + \omega_0^2 &= 0 && (31) \\ -2ns + n\gamma &= 0 \quad \rightarrow \quad s = \frac{\gamma}{2} && (32) \end{aligned}$$

Damped harmonic motion III

Substituting the Eq. 32 result of $s = \gamma/2$ into the Eq. 31 result of $-n^2 + s^2 - s\gamma + \omega_0^2 = 0$ gives

$$n^2 = \omega_0^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{4m^2}$$

where we have used the substitutions coming just before Eq. 27. We see that n is real (as required when we assumed $p = n + is$) only when $\gamma \leq 2\omega_0$ so that limits the validity of our solution. We can then put this back into our trial solution for the differential equation with $p = n + is$:

$$\begin{aligned} x &= Ae^{i(pt+\alpha)} \rightarrow Ae^{int-st+i\alpha} \\ &= Ae^{-st}e^{i(nt+\alpha)} = Ae^{-(\gamma/2)t}e^{i(nt+\alpha)} \end{aligned}$$

That's the notation used in French; it's perhaps clearer to write this as

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} \tag{33}$$

$$x = Ae^{-(\gamma/2)t}e^{i(\omega t+\alpha)} \tag{34}$$

Damped harmonic motion IV

Again we have found from Eq. 34 that the motion goes as

$$x = Ae^{-(\gamma/2)t}e^{i(\omega t+\alpha)}$$

where ω is given from Eq.33 as

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \omega_0^2 - \frac{b^2}{4m^2}$$

In the limit of weak damping, we can approximate the damped resonant frequency as

$$\begin{aligned} \omega &= \left[\omega_0^2(1 - \frac{\gamma^2}{4\omega_0^2})\right]^{1/2} \\ &\simeq \omega_0(1 - \frac{\gamma^2}{8\omega_0^2}) = \omega_0(1 - \frac{1}{8Q^2}) \end{aligned} \tag{35}$$

where we have used the binomial expansion of $(1+x)^n \simeq 1 + nx$ for $x \ll 1$ which is just the leading term of a Taylor series expansion, and defined what is called a quality factor $Q \equiv \omega_0/\gamma$.