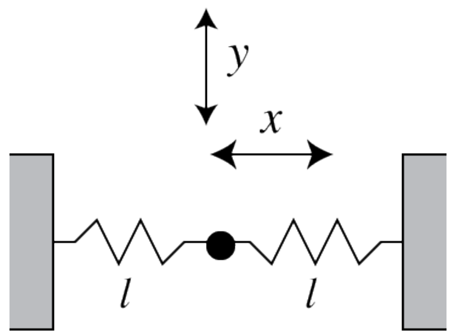
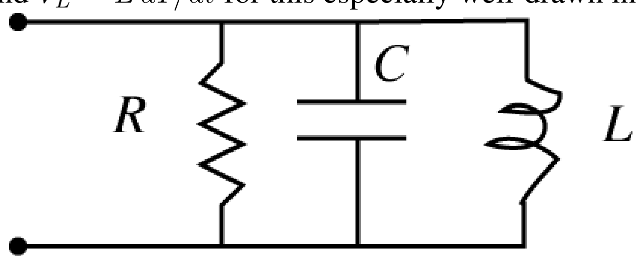
PHY 300, Spring 2006, Exam 1

Please show all work in your exam book. Calculators are allowed. You have been given an equation sheet. Do the easiest problem first, and the hardest last.

1. A mass m hangs on two equal springs supported between two facing walls a distance $2l$ apart. Both springs have the same spring constant k, and both are being stretched to a length l which is greater than their unstretched length l_0. First assume that the mass rests on a frictionless horizontal slider, and obtain an expression for the resonance frequency in the x direction. Next, assume that the mass stays at an equilibrium position in the horizontal direction, and calculate an approximate expression for the resonance frequency for small oscillations in the up-down direction.



2. The following electrical system is driven by a current $I_0 \cos \omega t$. Find the resonance frequency ω, the quality factor Q of the circuit, and the power absorbed at resonance. Note $V_R = IR$, $V_C = q/V$, and $V_L = L\,dI/dt$ for this especially well-drawn inductor.



3. A mechanical system has a maximum response at 500 Hz when damping is turned off, and 495 Hz when damping is turned on. The mass of the oscillating object is 0.2 kg. Calculate the following: the spring constant of the system, its quality factor, the full-width at half maximum (FWHM) of its damped resonance curve, and the time it takes for oscillations to decrease to 1/10 of an original value.

4. A wave of frequency 20 s^{-1} has a velocity of 40 m/s. How far apart are two points whose displacements are 45° apart in phase? At a given point, what is the phase difference between two displacements occurring at times separated by 0.01 sec?

5. Two identical pendulums ($\omega_0^2 = g/l$) are connected by a light coupling spring. Each pendulum has a length of 0.4 m. With the coupling spring connected, one pendulum is clamped and the period of the other is found to be 1.1 seconds. With neither pendulum clamped, what are the periods of the two normal modes?

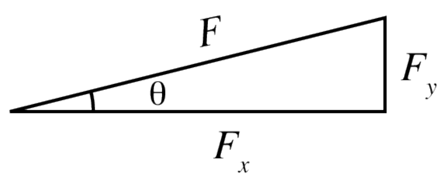
Solutions:

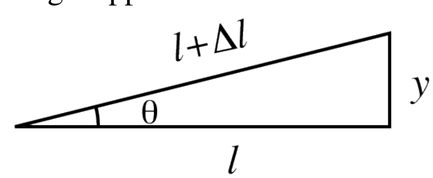
1. This is French problem 3-19. When you pull the mass to the right by a distance Δx, the force exerted to the left by the left hand spring is $F_{\text{left}} = -k[(l - l_0) + \Delta x]$, while the force exerted to the right by the right hand spring is $F_{\text{right}} = +k[(l - l_0) - \Delta x]$. The net force is then

$$F_{\text{net}} = F_{\text{left}} + F_{\text{right}} = -k[(l - l_0) + \Delta x] + k[(l - l_0) - \Delta x] = -2k\Delta x$$

so the effective spring constant in the horizontal direction is $k_x = 2k$, and the resonant frequency for horizontal motion is $\omega_x^2 = 2k/m$.

Now consider a vertical displacement only ($\Delta x = 0$). In this case, the magnitude of the force provided by the left hand spring is $F = k[(l - l_0) + \Delta l]$, where $(l - l_0)$ is the stretching that exists at $y = 0$ and $(l - l_0) + \Delta l$ is the stretching that exists at $y \neq 0$. The $\hat{y}$ component of that force is $F_y = -F \sin\theta$, or $-F\theta$ for a small angle approximation (see the diagram below left). Now consider the relationship between y and l, the length of the spring, as shown in the diagram at right below. Here we only care about the geometry of where the mass is located; this geometry is independent of how far the spring was stretched from its resting length l_0. In this case we have $\tan\theta = y/l$ or $\theta \simeq y/l$ for a small angle approximation.





We then have

$$F_y \simeq -F\theta = -k[(l - l_0) + \Delta l]\frac{y}{l} = -ky(1 - \frac{l_0}{l} + \frac{\Delta l}{l})$$

You can look at this result and point out that $\Delta l/l$ is likely to be negligible (that is, it's OK to do so and get full credit on the problem), but we'll go ahead and calculate Δl in this written-up solution. We have

$$\begin{aligned} l^2 + y^2 &= (l + \Delta l)^2 = l^2(1 + \frac{\Delta l}{l})^2 \simeq l^2(1 + 2\frac{\Delta l}{l}) = l^2 + 2l\,\Delta l \\ y^2 &\simeq 2l\,\Delta l \end{aligned}$$

or $\Delta l \simeq y^2/(2l)$. We can then write our expression for F_y as

$$F_y = -ky(1 - \frac{l_0}{l} + \frac{\Delta l}{l}) \simeq -ky(1 - \frac{l_0}{l} + \frac{y^2}{2l^2}) \simeq -k(1 - \frac{l_0}{l})y$$

We therefore have an effective spring constant in the $\hat{y}$ direction of $k_y = k(1 - l_0/l)$. That was for one spring; we have the same result for the other spring, so we double this result. We therefore arrive at a resonance frequency of $\omega_y^2 = (2k/m)(1 - l_0/l)$.

2. This is French problem 4-16. You've already see the solution at
`http://xray1.physics.sunysb.edu/~jacobsen/phy300s2006/hw3.pdf`
but we reproduce it here. The driving current equals the sum of current in the three circuit elements. That is,

$$I_0 \cos\omega t = q'_R + q'_C + q'_C.$$

The voltages across the three elements are equal. That is,

$$V_R = V_C = V_L \qquad \text{or} \qquad q'_R R = \frac{q_C}{C} = L q''_L.$$

The resistor is the dissipative element so let's solve for the current in it. We can therefore make substitutions

$$\begin{aligned} \frac{q_C}{C} &= R q'_R \qquad \Rightarrow \qquad q'_C = q''_R RC \\ L q''_L &= q'_R R \qquad \Rightarrow \qquad q'_L = q_R \frac{R}{L} \end{aligned}$$

Therefore

$$\begin{aligned} q'_R + q'_C + q'_L &= I_0 \cos\omega t \\ q'_R + RC q''_R + \frac{R}{L} q_R &= I_0 \cos\omega t \\ q''_R + \frac{1}{RC} q'_R + \frac{1}{LC} q_R &= \frac{I_0}{RC} \cos\omega t \\ \text{Compare: } x'' + \frac{b}{m} x' + \frac{k}{m} x &= \frac{F_0}{m} \cos\omega t \end{aligned}$$

So $m \Leftrightarrow C$, $F_0 \Leftrightarrow I_0/R$, $b \Leftrightarrow 1/R$, and $k \Leftrightarrow 1/L$. Then $\gamma = b/m \Rightarrow 1/RC$, and

$$Q = \frac{\omega_0}{\gamma} = R\sqrt{\frac{C}{L}} \qquad \text{and} \qquad P_{\text{max}} = \frac{F_0^2}{2b} = \frac{I_0^2}{R^2}\frac{R}{2} = \frac{I_0^2}{2R}$$

3. We have $\omega_0 = 2\pi \cdot 500\ \text{s}^{-1}$, and $\omega = 2\pi \cdot 495\ \text{s}^{-1}$. The spring constant is $k = m\omega_0^2 = (0.2)(2\pi \cdot 500)^2 = 1.97 \times 10^6$ N/m. To find the quality factor $Q = \omega_0/\gamma$, we first find γ:

$$\begin{aligned} \omega^2 &= \omega_0^2 - \frac{\gamma^2}{4} \\ \gamma &= 2\sqrt{\omega_0^2 - \omega^2} \\ Q &= \frac{\omega_0}{\gamma} = \frac{\omega_0}{2\sqrt{\omega_0^2 - \omega^2}} = \frac{1}{2\sqrt{1 - \omega^2/\omega_0^2}} = \frac{1}{2\sqrt{1 - 495^2/500^2}} = 3.54 \end{aligned}$$

The full-width at half-max or FWHM of the resonance curve is found from $2\Delta\omega = \omega_0/Q = 2\pi \cdot 500/3.54 = 887.5$ radians/sec or 141 Hz. For damping, we have

$$\begin{aligned} \frac{E_0}{10} &= E_0 e^{-\gamma t_{10}} \\ \ln 10 &= \gamma t_{10} \\ t_{10} &= \frac{\ln 10}{\gamma} = \frac{\ln 10}{4\pi\sqrt{500^2 - 495^2}} = 2.6 \text{ milliseconds} \end{aligned}$$

4. This is French problem 7-4 with the numbers changed to protect the innocent. We have $\nu = 20$ s^{-1} and $v_p = 40$ m/s. Then $v_p = \omega/k = (2\pi\nu)/(2\pi/\lambda)$ so $v_p = \lambda\nu$ or $\lambda = v_p/nu = 40/20 = 2$ m. Then from $(45/360) = x/\lambda$ we have $x = \lambda\cdot(45/360) = 2\cdot(45/360) = 0.25$ m as the distance between phases that are 45° apart. Two points separated by 0.01 seconds are separated by a phase angle θ found from $[0.01/(1/20)] = \theta/360$ or $\theta = 360 \cdot 0.01 \cdot 20 = 72°$.

5. French problem 5-2 (first part, at least, and with a slightly different number for T_1). Each individual pendulum has a resonance frequency of $\omega_0 = \sqrt{g/l} = \sqrt{9.8/0.4} = 4.95$ radians/sec. When the two pendula are connected by a spring but one pendulum is clamped, we have

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -m\omega_0^2 x - kx = k_1 x \\ \text{so} \qquad k_1 &= m\omega_0^2 + k \\ \text{or} \qquad m\omega_c^2 = k &= k_1 - m\omega_0^2 = m\omega_1^2 - m\omega_0^2 \\ \omega_c &= \sqrt{\omega_1^2 - \omega_0^2} = \sqrt{(\frac{2\pi}{T_1})^2 - \omega_0^2} = \sqrt{(\frac{2\pi}{1.1})^2 - 4.950^2} = 2.85 \text{ radians/sec} \end{aligned}$$

The two normal mode frequencies are then $\omega_0 = 4.950$ radians/sec, and

$$\omega' = \sqrt{\omega_0^2 + 2\omega_c^2} = \sqrt{4.950^2 + 2 \cdot 2.85^2} = 6.38 \text{ radians/sec}$$

or periods $t = 2\pi/\omega$ of 1.27 and 0.98 seconds, respectively.