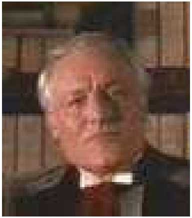
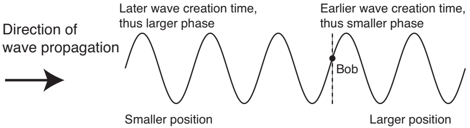
The wave equation

- We've talked about waves that oscillate with time: $e^{i\omega t}$ with $\omega = 2\pi/T$. That is, we go through a phase of 2π when the time T of one oscillation period has elapsed.
- Let's now also consider waves that oscillate in space. The wave also goes through a complete cycle over a wavelength λ, or a phase of 2π when $x = 0..\lambda$. Therefore just like we speak of an angular frequency $\omega = 2\pi/T$ and phase progression as ωt, we will speak of a wave number $k = 2\pi/\lambda$ and phase progression as kx.
- What about waves that move with deliberation, so that they are neither Lost in Time nor (what's your favorite TV show?) Lost in Space?



Wave directions

Imagine that you, Bob, are sitting in an artificial wave pool at a water park as the waves go by (look at the movie):



The "older" waves that passed you first are now to the right; the "younger" waves still coming towards you are on the left.

Waves generated at smaller (earlier) times t are on the right, and larger times t are on the left.

In our convention, waves generated earlier have smaller phase φ and are on the right, and while waves generated later have larger phase φ and are on the left.

Smaller position x is on the left, and larger position x is on the right.

We *must* say that the phase changes in position and time are opposite: either $(-kx + \omega t)$ or $(+kx - \omega t)$.

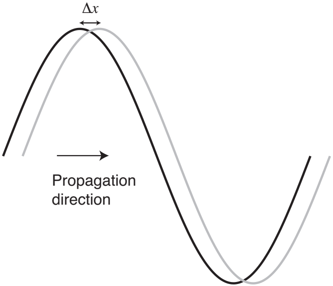
My preference (based on the water park example) for the arbitrary choice of overall sign is $(-kx + \omega t)$

Wave directions II

- For waves going in the $+x$ direction, we *must* say that phase evolves either as $(+kx - \omega t)$ or as $(-kx + \omega t)$, with $k = 2\pi/\lambda$.
- My personal preference is to pick $(-kx + \omega t)$, but different books make different choices!
- It's conventional to write the position first; we'll use $e^{-i(kx-\omega t)}$ in this course to represent forward $(+x)$ propagating waves.
- If we talk about not just a 1D coordinate system in x, but a 3D coordinate system, then we have $\vec{k} \cdot \vec{x}$ where $\vec{k}$ indicates the direction of wave propagation.
- If the wave is traveling in the $-x$ direction, we flip the sign of k, giving $e^{-i(-kx-\omega t)} = e^{i(kx+\omega t)}$ or $e^{+i(-kx-\omega t)} = e^{-i(kx+\omega t)}$.

Velocity of a traveling wave

Now let's ask how a traveling wave moves forward in the $\hat{x}$ direction. If we pick a point of stationary phase (i.e., a crest of a wave), we see that it advances to a new position as $\Delta x = v\Delta t$.



Therefore another relationship between position and time for the crest of a wave is

$$y(x,t) = A\exp[-i\frac{2\pi}{\lambda}(x - vt)]$$

Velocity II

Comparing

$$y(x,t) = A\exp[-i\frac{2\pi}{\lambda}(x - vt)] \qquad \text{with} \qquad y(x,t) = Ae^{-i(kx-\omega t)},$$

we find

$$k \equiv \frac{2\pi}{\lambda} \text{ as before} \tag{1}$$

$$\omega \equiv \frac{2\pi}{\lambda}v = \frac{2\pi}{\lambda}\frac{\lambda}{T} = \frac{2\pi}{T} \text{ as before} \tag{2}$$

In Eq. 2, we have made use of the fact that a wave travels a wavelength λ in a period T as a substitution for velocity v. We therefore find that the velocity v_{phase} of wave crests (the *phase velocity*) is

$$\omega = \frac{2\pi}{\lambda}v = kv \qquad \Rightarrow \qquad v_{\text{phase}} = \frac{\omega}{k} \tag{3}$$

From velocity to wave equation

Again, our forward traveling wave goes like

$$y(x,t) = Ae^{-ik(x-vt)}$$

Let's take partial derivatives in space and time:

$$\begin{aligned}
\frac{\partial y}{\partial t} = \frac{\partial}{\partial t} Ae^{i(kvt-kx)} &= ikv\,Ae^{i(kvt-kx)} = ikv\,y \text{ or } ky = \frac{1}{i}\frac{1}{v}\frac{\partial y}{\partial t} \\
\frac{\partial y}{\partial x} = \frac{\partial}{\partial x} Ae^{-i(kx-vt)} &= -ik\,Ae^{-i(kx-vt)} = -ik\,y \text{ or } ky = -\frac{1}{i}\frac{\partial y}{\partial x} \\
\text{Combined: } ky = -\frac{1}{i}\frac{\partial y}{\partial x} &= \frac{1}{i}\frac{1}{v}\frac{\partial y}{\partial t} \\
\text{or} \frac{\partial y}{\partial x} &= -\frac{1}{v}\frac{\partial y}{\partial t}
\end{aligned}$$

If we were to do the same for a backward traveling wave of $y(x,t) = Ae^{-ik(-x-vt)}$, we would find

$$\frac{\partial y}{\partial x} = +\frac{1}{v}\frac{\partial y}{\partial t}$$

A universal wave equation

Again, we had

$$\begin{aligned} \text{Forward wave: } \frac{\partial y}{\partial x} &= -\frac{1}{v}\frac{\partial y}{\partial t} \\ \text{Backward wave: } \frac{\partial y}{\partial x} &= +\frac{1}{v}\frac{\partial y}{\partial t} \end{aligned} \tag{4}$$

so we don't yet have a single rule. However, we *do* get something universal if we take the second derivatives, in which case we find

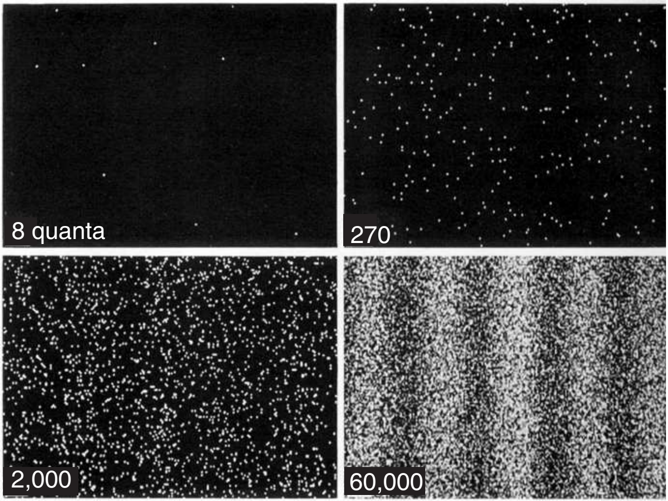
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{5}$$

This condition will be a requirement for a traveling wave to obey.

From waves to optics

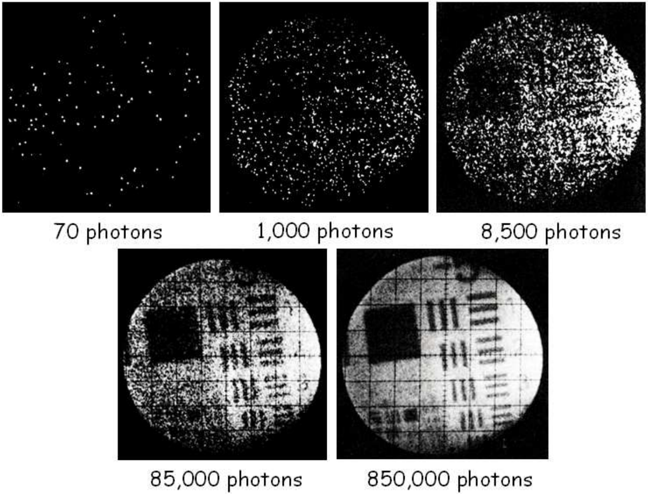
- We have discussed in detail many properties of waves of many sorts. We are now going to link that in to the study of optics. To do so, we'll have to race through some material that you'll explore further in the E&M sequence PHY 301/302.
- And we'll ignore photons for the time being, and treat light strictly as a classical electromagnetic wave.
- If you took PHY 251, you know that from Planck and Einstein we have a picture of light as coming in discrete energy packets called photons which have an energy $E = h\nu$ and momentum $p = Ec$
  - Planck's constant: $h = 6.626 \times 10^{-34}$ J·s
  - Speed of light: $c = 2.99 \times 10^{8}$ m/s
  - Handy-dandy: $hc = 1240$ eV· nm, for use in $E = hc/\lambda$
- And you'll know that eventually the arrival pattern of photons becomes that predicted by the intensity pattern of a wave.

Photons arriving one-by-one



Electrons, really, but you get the idea. . . From A. Tonomura, *Electron Holography* (Springer-Verlag, 1993), p. 14.

This time it's really photons



Position-sensing photomultiplier, ITT Electro-Optical Products; Fig. 1.1 of Hecht, Optics (4th ed., Addison-Wesley, 2002)

E&M I: Gauss' laws

We now interrupt this program for a short review of electricity and magnetism.
Gauss' law for electrostatics tells how we can calculate an electric field from a charge distribution ρ:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{6}$$

where $\epsilon_0 = 8.85 \times 10^{-12}$ C/(N·m^2), the *electric permittivity of free space*, is the constant in Coulomb's law for electrostatic force:

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$$

The divergence $\vec{\nabla}\cdot$ is written in Cartesian coordinates as

$$\vec{\nabla}\cdot = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \tag{7}$$

More on Gauss

You might know Gauss's law in its integral form, but this can be manipulated to give Eq. 6

$$\int_{\text{surface}} \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} q_{\text{enclosed}} = \frac{1}{\epsilon_0} \int_{\text{volume}} \rho \, d\tau \tag{8}$$

$$\text{Divergence theorem: } \int_{\text{volume}} (\vec{\nabla} \cdot \vec{E}) d\tau = \int_{\text{volume}} \left(\frac{1}{\epsilon_0} \rho \right) d\tau \tag{9}$$

$$\text{Result: } \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \text{ (Eq. 6)}$$

For magnetics, we have no units of magnetic charge (monopoles), and Gauss' law tells us

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{10}$$

E&M II: Ampére's law

Ampére's law tells how currents J and changing electric fields can produce magnetic fields:

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}) \tag{11}$$

with $\mu_0 = 4\pi \times 10^{-7}$ N/A^2. You've probably learned Ampére's law as

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \vec{I}_{\text{enclosed}},$$

which can be manipulated to give the first part of Eq. 11, but the extra term $\partial E / \partial t$ was found by Maxwell to be necessary as you'll learn in your E&M course. Here $\vec{\nabla}\times$ is the curl of a vector, which in Cartesian coordinates is

$$\vec{\nabla} \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$$

Faraday's law

Finally, Faraday's law tells how changing magnetic fields can induce voltages:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{12}$$

Maxwell's silver hammer

The above equations for Gauss's electrostatic and magnetostatic laws, and Ampére's law for currents and changing E fields, and Faraday's law for changing magnetic fields, were put together by James Clerk Maxwell over the period 1855–1873. Four equations describe all the fundamental laws of E&M: a triumph of unification!

$$\begin{aligned}
\text{Faraday:}\quad \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}\\
\text{Ampère:}\quad \vec{\nabla}\times\vec{B} &= \mu_0(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t})\\
\text{Gauss (electric):}\quad \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\epsilon}\\
\text{Gauss (magnetic):}\quad \vec{\nabla}\cdot\vec{B} &= 0.
\end{aligned}$$

James Clerk Maxwell (1831-1879)

Using Ohm's law $\vec{J}=\sigma\vec{E}$, Ampère's law becomes

$$\vec{\nabla}\times\vec{B} = \sigma\mu_0\vec{E}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}.$$

Maxwell's equations in free space

Maxwell's equations in free space (no charge, no current) are

$$
\begin{aligned}
\text{Faraday:} \quad & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\text{Ampère:} \quad & \vec{\nabla} \times \vec{B} &= \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\
\text{Gauss (electric):} \quad & \vec{\nabla} \cdot \vec{E} &= 0 \\
\text{Gauss (magnetic):} \quad & \vec{\nabla} \cdot \vec{B} &= 0
\end{aligned}
$$

Taking the curl of Gauss' law, and using some of the tools of multivariable calculus, gives

$$
\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \\
\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} &= \frac{\partial}{\partial t} \left(-\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\
\nabla^2 \vec{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad (13)
\end{aligned}
$$

Maxwell II

Taking the curl of Ampére's law gives

$$
\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) &= \vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) \\
\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \\
\nabla^2 B &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \qquad (14)
\end{aligned}
$$

Maxwell III

What hath we wrought? We have from Eqs. 13 and 14 the following:

$$
\begin{aligned}
\nabla^2 \vec{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\
\nabla^2 B &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}
\end{aligned}
$$

Hmm. . . A relationship between second derivative in position and second derivative in time? Sounds like our wave equation of Eq. 5:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

To make this so, all we have to do is to associate the velocity with these electromagnetic properties:

$$
\begin{aligned}
\frac{1}{v^2} &= \mu_0 \epsilon_0 \\
v \Rightarrow c &\equiv \frac{1}{\sqrt{\mu_0 \epsilon_0}}
\end{aligned}
$$

and of course you know that $c = 2.99 \times 10^8$ m/s or 30 cm per nanosecond. A*ha*!!!

Speed of light

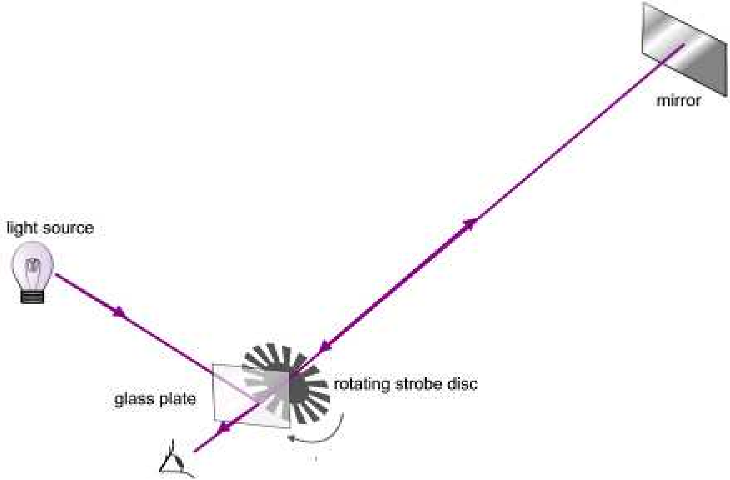
- Inferred to be finite from astronomical measurements by Romer (1676)
- Huygens soon estimated 2.2×10^8 m/sec.
- Refined by Bradley (1728) to 2.98×10^8 m/sec.
- First terrestrial measurement by Fizeau (1849): 3.13×10^8 m/sec.

Maxwell on the similarity of $1/\sqrt{\mu_0 \epsilon_0}$ to the above, as quoted in Griffiths' *Introduction to Electrodynamics* (Pearson, 1998):

> *This velocity is so nearly that of light, that it seems we have strong reasons to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws.*

Armand Hippolyte Louis Fizeau, 1819–1896

Fizeau's experiment



As drawn on Wikipedia by Theresa Knott.

Waves in electromagnetic media

- With enough light around, we've seen that light travels at a speed identical to that expected for electromagnetic waves!
- Can we explain other characteristics of light using electromagnetic waves?
- To find out, we need to consider not just electromagnetic waves in a vacuum, but EM waves in gases, glass, liquids. . .

Dielectric media

- In a dielectric medium, an applied electric field pulls electron orbitals a bit away from nuclei.
- For typical materials, there's a linear relationship between applied electric field E and the bulk polarization P of

$$\vec{P} = \epsilon_0 \chi_e \vec{E}. \tag{15}$$

- The same basic relationship can be characterized by several parameters:
  electric susceptability χ_e
  electric permittivity $\epsilon = \epsilon_0(1 + \chi_e)$
  dielectric constant $K = \epsilon/\epsilon_0 = (1 + \chi_e)$
- Typical dielectric constants for nonconductors within region of linearity:
  Gasses: $K \simeq$1.0001–1.0006
  Solids: $K \simeq$2–8
  Liquids: $K \simeq$20–90

Magnetic media

In non-ferromagnetic materials, the relationships between the magnetism M, the applied magnetic field B, and the auxiliary field H are:

$$\vec{M} = \chi_m \vec{H} \tag{16}$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1 + \chi_m)\vec{H} \tag{17}$$

Again, the same basic relationship can be described with several parameters:

$$\text{magnetic susceptibility} \equiv \chi_m \tag{18}$$

$$\text{permeability} \equiv \mu = \mu_0(1 + \chi_m) \tag{19}$$

$$\text{permeability of free space} \equiv \mu_0 = 4\pi \times 10^{-7}\ \mathrm{N \cdot C^2/s^2} \tag{20}$$

Diamagnetic materials: $\chi_m < 0$ (typically $\chi_m \approx -10^{-5}$)
Paramagnetic materials: $\chi_m > 0$ (typically $\chi_m \approx 10^{-5}$ except for lanthanides where $\chi_m \approx 10^{-2}$

Maxwell's equations with media

Earlier we had Eqs. 13 and 14 for Maxwell's equations in free space:

$$\begin{aligned} \nabla^2 \vec{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\ \nabla^2 B &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \end{aligned}$$

With nonconducting, electrostatically neutral, linear media, these are modified as follows:

$$\nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{21}$$

$$\nabla^2 \vec{B} = \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \tag{22}$$

Evidently the speed of wavefront propagation is modified as followed:

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \qquad \Rightarrow \qquad v_{\text{phase}} = \frac{1}{\sqrt{\mu \epsilon}} \tag{23}$$