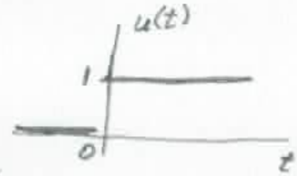
Piecewise Linear Functions and Their Laplace Transforms

Two simple but important functions for us are:

The unit-step function $u(t) = 1$ for $t > 0$
$u(t) = 0$ for $t < 0$
$u(0)$ can be any real number



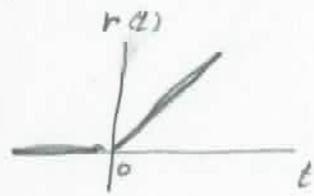
Its Laplace transform is

$$\mathcal{L}u(t) = \int_0^\infty u(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \frac{1}{s} \quad \text{if } \operatorname{Re} s > 0.$$

We then analytically extend $\frac{1}{s}$ over all of the complex plane to write

$\mathcal{L}u(t) = \frac{1}{s}$ ← This has a singularity at $s = 0$, called a "pole."

The unit-ramp function $r(t) = t\,u(t)$
$r(t) = t$ for $t > 0$
$r(t) = 0$ for $t < 0$



Its Laplace transform is

$$\mathcal{L}r(t) = \int_0^\infty t\,e^{-st}\,dt = \frac{1}{s^2} \quad \text{if } \operatorname{Re} s > 0.$$

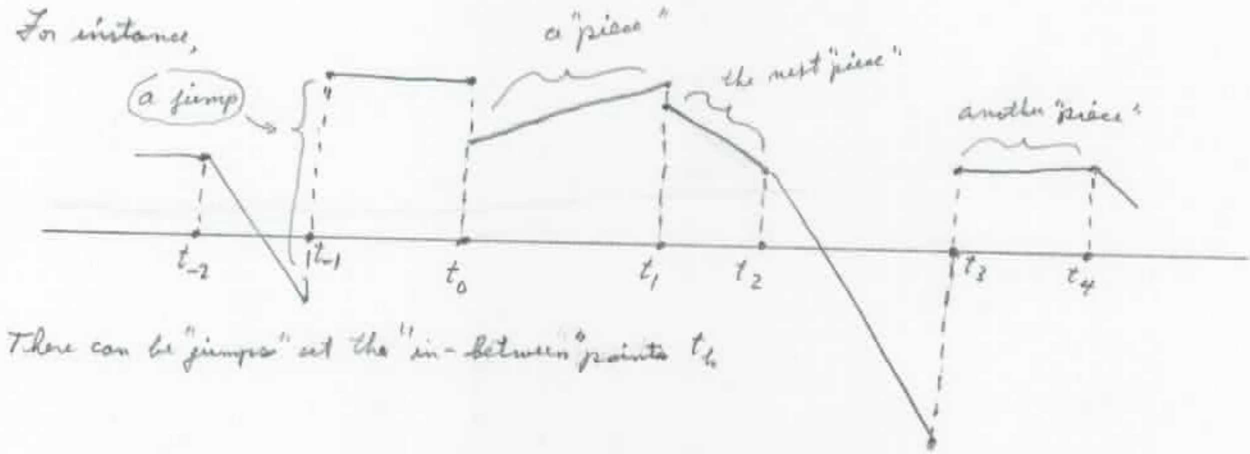
Through analytical continuation, we get

$\mathcal{L}r(t) = \frac{1}{s^2}$ for all s. There is a "double pole" at $s = 0$.

We can use these time functions (u(t) and r(t)) to decompose any piecewise-linear function as indicated on the next page.

Any piecewise-linear function can be obtained
by dividing up the abscissa (i.e., the horizontal axis)
into consecutive intervals and then assigning a straight line to each interval.

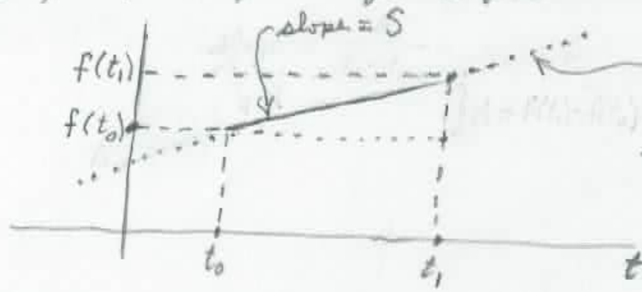
For instance,



There can be "jumps" at the "in-between" points t_k

We can get a mathematical formula for each "piece" by writing
the straight-line formula for that piece and then multiplying it
by the pulse function of unit height and of the same width as the piece.

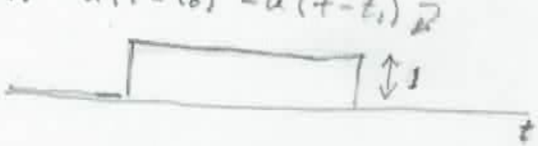
For example, to get a formula for the piece between t_0 and t_1 write:



A formula for this straight line is:

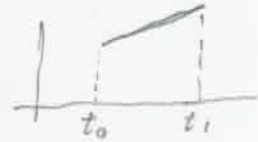
$$f(t_0) + \underbrace{\frac{f(t_1) - f(t_0)}{t_1 - t_0}}_{S \,=\, \text{slope of this line}} (t - t_0)$$

The pulse needed is: $u(t-t_0) - u(t-t_1)$



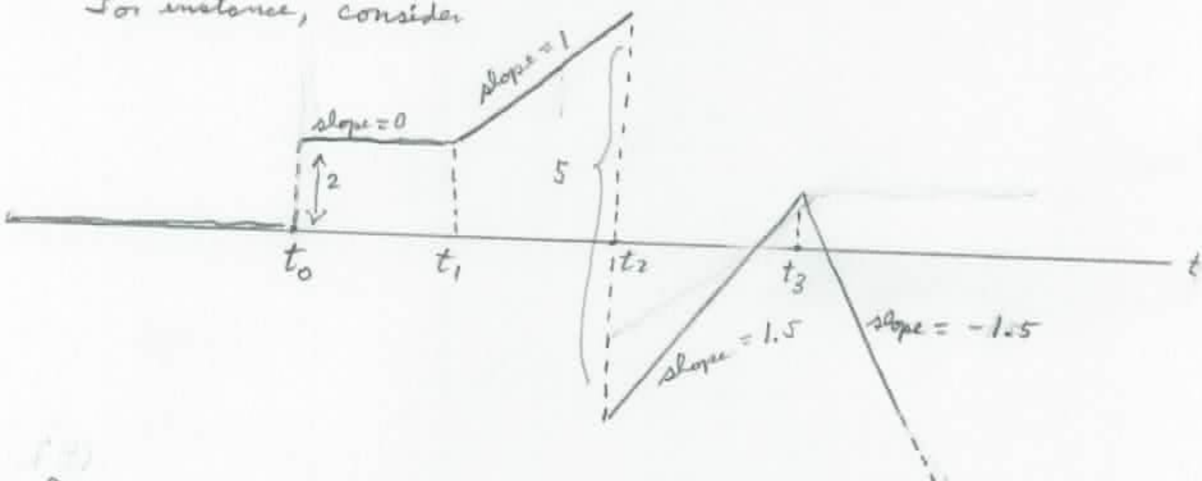
So, the formula for this piece is

$$\Big(f(t_0) + (t-t_0)S\Big)\Big(u(t-t_0) - u(t-t_1)\Big)$$



Another way to get a formula for a piecewise-linear function can be used when the function is equal to 0 for an interval like $-\infty < t < t_0$. In this case, just add in the increments in the straight-line formulas as "t" is traced from left to right.

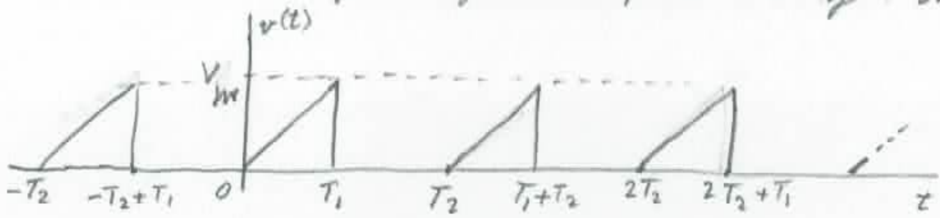
For instance, consider



$$f(t) = 2u(t-t_0) + r(t-t_1) - 5u(t-t_2) + .5\,r(t-t_2) - 3r(t-t_3).$$

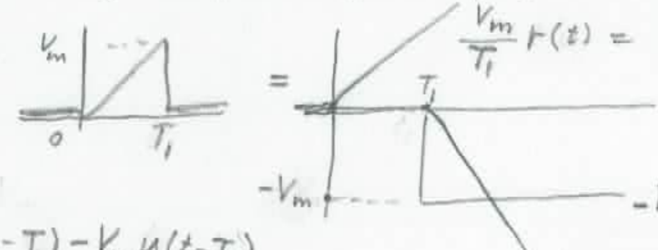
Either method will give a correct formula for the piecewise-linear function even though the two formulas may look different. (They are actually the same.)

Example: Here's a formula for the "repeated triangles" on page AvP2 :



(period $= T_2$)

Using the method on PLin 3, we get the following formula for the "first" triangle



$\frac{V_m}{T_1} r(t) = \frac{V_m}{T_1}(t-T_1)u(t-T_1)$

$-V_m u(t-T_1)$

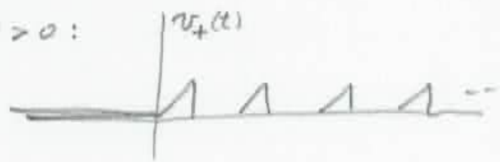
$-\frac{V_m}{T_1} r(t-T_1)$

$$p(t) = \frac{V_m}{T_1} r(t) - \frac{V_m}{T_1} r(t-T_1) - V_m u(t-T_1)$$

So, for the repeated triangles we have:

$$v(t) = \sum_{k=-\infty}^{\infty} p(t-kT_2)$$

If the triangles exist only for $t>0$:



we get

$$v_+(t) = \sum_{k=0}^{\infty} p(t-kT_2)$$

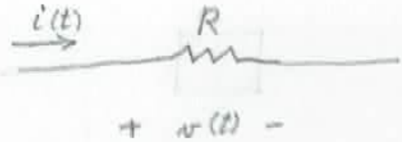
Applying the Laplace transform $\mathcal{L}$, we get

$$\mathcal{L}p(t) = P(s) = \frac{V_m}{T_1}\cdot\frac{1}{s^2} - \frac{V_m}{T_1}\cdot\frac{e^{-sT_1}}{s^2} - V_m\frac{e^{-sT_1}}{s}$$

$$\mathcal{L}v_+(t) = V_+(s) = \sum_{k=0}^{\infty} P(s)e^{-skT_2} = P(s)\frac{1}{1-e^{-skT_2}} \quad \text{where } \operatorname{Re} s > 0$$

$$\boxed{\begin{array}{l} \text{If } |x|<1, \ \frac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{k=0}^{\infty} x^k \\ \text{Also, } |e^{-skT_2}| < 1 \text{ when } \operatorname{Re} s > 0 \text{ and } k = 0, 1, 2, \ldots \end{array}}$$

Average Power for Other Wave Forms

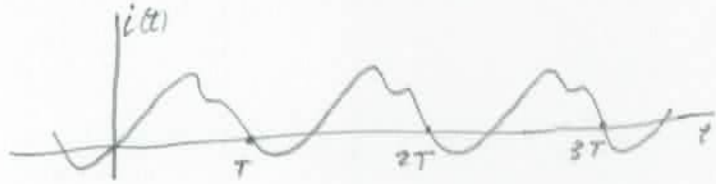


$v(t) = R\,i(t)$

$p(t)$ = instantaneous power dissipated in R

$$p(t) = v(t)\,i(t) = R\,(i(t))^2 = \frac{1}{R}(v(t))^2.$$

If $i(t)$ is a periodic wave of period T:

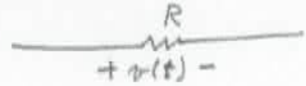


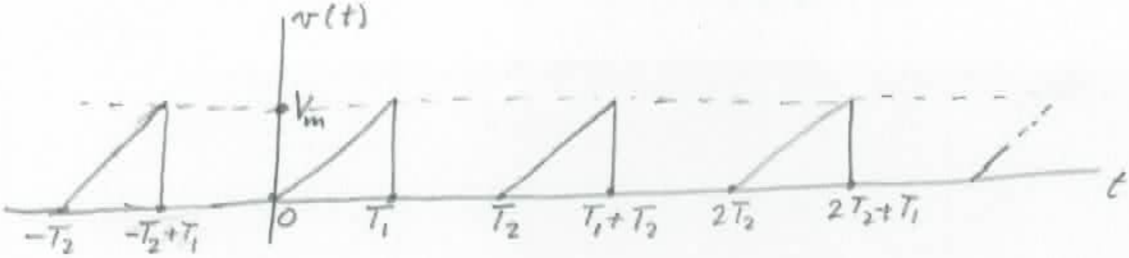
then $$P_{av} = \text{average power} = \frac{1}{T}\int_0^T R\,(i(t))^2\,dt = \frac{1}{T}\int_0^T \frac{1}{R}(v(t))^2\,dt$$

also, $$I_{eff} = \sqrt{\frac{1}{T}\int_0^T (i(t))^2\,dt}$$ where now R is taken to be equal to 1. $(R=1)$

$$V_{eff} = \sqrt{\frac{1}{T}\int_0^T (v(t))^2\,dt}$$

Example





$$v(t) = \frac{V_m}{T_1} t \quad \text{for} \quad 0 < t < T_1$$

$$v(t) = 0 \quad \text{for} \quad T_1 < t < T_2$$

$$P_{av} = \frac{1}{T_2}\left(\int_0^{T_1} \frac{1}{R}\left(\frac{V_m}{T_1} t\right)^2 dt + \int_{T_1}^{T_2} 0 \, dt\right)$$

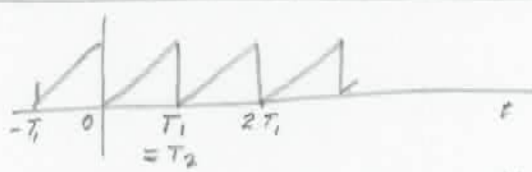
$$= \frac{V_m^2}{T_2 R T_1^2} \int_0^{T_1} t^2 \, dt = \frac{V_m^2}{T_2 R T_1^2} \cdot \frac{T_1^3}{3}$$

$$= \frac{T_1 V_m^2}{T_2 R 3}.$$

also, $V_{eff} = \sqrt{P_{av}}$ where $R = 1$ now

$$= V_m \sqrt{\frac{T_1}{T_2 3}}$$

If $T_1 = T_2$,



$$P_{av} = \frac{V_m^2}{3R}$$

← This is the sweep signal in an oscilloscope or TV set

$$V_{eff} = \frac{V_m}{\sqrt{3}}$$