Course information

- This is PHY 300: Waves and Optics. I'm Prof. Chris Jacobsen.
- We'll make heavy use of the course web site at `http://tinyurl.com/5sofbb` (which points to `xray1.physics.sunysb.edu/~jacobsen/phy300f2008`) where homework assignments and solutions will be posted, as will lecture notes, lab handouts and so on.
- Grades will be posted on `blackboard.stonybrook.edu`
- For more info, see `courseinfo.pdf` which is listed under lecture 1 on the course web site.

Creating these lecture notes

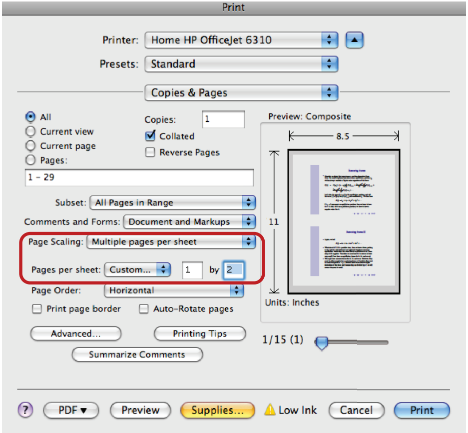
- These lecture notes were prepared using the Hannover style for the `beamer` package for LaTeX.
- TeX is a mathematical typesetting language created by Stanford mathematician Donald Knuth starting about 30 years ago.
- LaTeX is a simpler, lazy person's version of TeX developed originally by Leslie Lamport at SRI international. LaTeX is widely used by physicists and mathematicians, and has all sorts of great features for big, complicated documents.
- LaTeX is free, and available on nearly every computer type! See **TeX Live** at `http://www.tug.org/texlive`, or for Windows see `http://www.miktex.org`.
- These packages also include BIBTeX which is used for automatically formatting and numbering citations. The free, multiplatform program JabRef (`http://jabref.sourceforge.net`) provides a great way to manage BIBTeX citation databases.

Using these lecture notes

- I will make every effort to have lecture notes up on the course web page 24 hours in advance of each lecture.
- I will usually prepare far more slides than I think we'll get through, and then re-post the lecture to reflect where we actually stopped (and correct any typos).
- This means you don't have to write down every formula we discuss; instead, be an active listener and ask questions, and maybe make notes on particular points brought up in discussion.
- I will quite often use the chalkboard too! I hope you don't use the on-line notes as an excuse to not attend class, as I think the discussions are important for grasping the material.

If you want to print

These lecture notes use extra large fonts for easy viewing on a 1024×768 pixel projector, while printers typically produce the equivalent of about 2400×3000 pixels. Therefore if you want to print the lecture notes out, you might consider using the *Multiple pages per sheet* option in Adobe Acrobat.



Why study waves?

- Restoring forces are ubiquitous in nature, with displacements leading to oscillations. Often periodic: waves.
- This happens in optics (hence *PHY 300: Waves and Optics*), with liquid surfaces, gases, solids (earthquake waves, phonons)...
- It also happens with particles at very small scales: de Broglie wavelength $\lambda = h/p$ and the Schrödinger formulation of Quantum Mechanics.
- So it's very worthwhile to study wave motion!

Restoring forces

- Consider an object that experiences a position-dependent force $F(x)$. For small displacements about some equilibrium position x_0, we can always consider a Taylor series expansion of the force:

$$F(x) = F(x_0) + (x - x_0)\frac{d}{dx}F(x)|_{x=x_0} + \frac{(x-x_0)^2}{2!}\frac{d^2}{dx^2}F(x)|_{x=x_0} + \frac{(x-x_0)^3}{3!}\frac{d^3}{dx^3}F(x)|_{x=x_0} + \ldots$$

  Let's take the case of $x_0 = 0$ as the equilibrium position, and roll the terms $(1/n!)(d^n/dx^n)F(x)|_{x=0}$ into coefficients k_n. We then have

$$F(x) \simeq k_0 + k_1 x + k_2 x^2 + k_3 x^3 + \ldots$$

  If $x_0 = 0$ represents an equilibrium position, then we have to have $k_0 = 0$. Also, if it's an equilibrium position, we have to have a negative value for k_1.

Restoring forces II

- Again, we had

$$F(x) \simeq k_0 + k_1 x + k_2 x^2 + k_3 x^3 + \dots$$

  with $k_0 = 0$ and k_1 a negative number.
- What about k_2? If it's a positive term, then we have a force pushing to the right for both leftward and rightward displacements from equilibrium, which does not sound like an equilibrium at all. Same story if it's negative. Therefore we must have k_2 be zero or at least very small if we have an equilibrium (same for k_4, k_6, and so on).
- We might have non-zero terms for k_3, k_5, and so on. However, they must be either negative for a restoring force, or if positive they must be small compared to $-k_1 x$. Because these terms involve higher derivatives of the force, and because they are divided by $n!$, we will assume they can be small.

Simple restoring forces

- OK, we've determined that if a particle has an equilibrium position $x_0 = 0$ then the force it experiences can be approximated as

$$F(x) \simeq -kx$$

at least for small displacements.

- Newton tells us that this is mass times acceleration:

$$m\frac{d^2x}{dt^2} = -kx \tag{1}$$

A good trial to this differential equation is $x = A\sin(\omega t + \varphi_0)$. Let's try it out:

$$\begin{aligned} x &= A\sin(\omega t + \varphi_0) && (2)\\ x' = \frac{dx}{dt} &= \omega A\cos(\omega t + \varphi_0) \\ x'' = \frac{d^2x}{dt^2} &= -\omega^2 A\sin(\omega t + \varphi_0) && (3) \end{aligned}$$

Simple restoring forces II

- Insert the Eq. 3 result of $d^2x/dt^2 = -\omega^2 A\sin(\omega t + \varphi_0)$ into Eq. 1 of $m(d^2x/dt^2) = -kx$ to obtain

$$-m\omega^2 A\sin(\omega t + \varphi_0) = -kA\sin(\omega t + \varphi_0)$$

from which we find

$$\begin{aligned} m\omega^2 &= k \\ \omega &= \sqrt{\frac{k}{m}} \end{aligned} \tag{4}$$

- We could have also chosen a cosine solution, but since $\cos(x - \pi/2) = \sin(x)$ the only difference would be in the value of the starting phase φ_0.

Harmonic oscillator

- We have found that a simple restoring force with equilibrium position $x_0 = 0$ is well described by simple harmonic motion (referred to by French as SHM, but you should not confuse this with "single Hispanic male" in the personals section of the want ads):

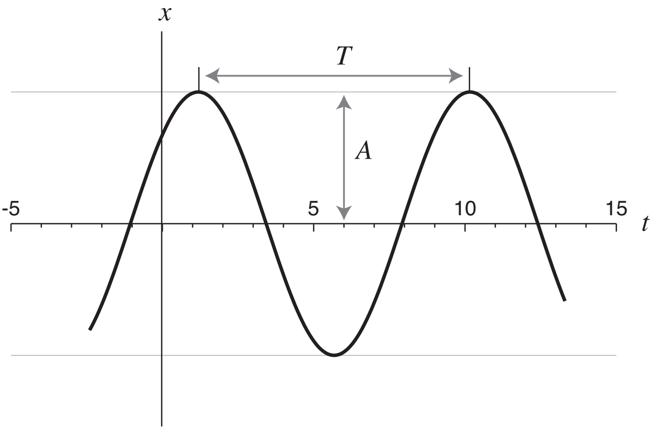
$$x = A\sin(\omega t + \varphi_0) \qquad \text{with} \qquad \omega = \sqrt{\frac{k}{m}}$$

- How to interpret the coefficient ω? Well, if we displace the particle it returns to the same displacement position when $\omega t - \omega t_0 = 2\pi$. This defines a period of oscillation T to be

$$T = \frac{2\pi}{\omega} \tag{5}$$

giving $\omega = 2\pi/T$ as angular frequency (expressed in radians per second). We often prefer to talk about a frequency in cycles per second or Hertz, or $f = \omega/2\pi = 1/T$.

Determining coefficients



- Besides determining the period, we can measure the amplitude A from measuring the maximum displacement x from $x = 0$. The total range of motion is from $-A$ to $+A$.

Determining coefficients II

- How to determine φ_0? Well, if we know $x_{t=0}$, we can say

$$\begin{aligned} x_{t=0} &= A\sin(\omega\cdot 0+\varphi_0) \\ \frac{x_{t=0}}{A} &= \sin(\varphi_0) \\ \varphi_0 &= \sin^{-1}(\frac{x_{t=0}}{A}) \end{aligned}$$

Alternatively we can find a time $t_{x=0}$ where the position is zero:

$$\begin{aligned} 0 &= A\sin(\omega t_{x=0}+\varphi_0) \\ n\pi &= \omega t_{x=0}+\varphi_0 \\ \varphi_0 &= n\pi-\omega t_{x=0} \end{aligned}$$

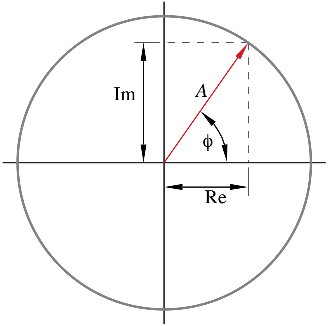
If we also know the velocity dx/dt, we will want to use this as well in determining φ_0 (the net effect will be to change to $\varphi_0=(2n+n_0)\pi-\omega t_{x=0}$ where n is an integer and n_0 is 0 or 1).

Complex algebra

- Our equation for periodic motion is $x = A\sin(\omega t + \varphi_0)$. This is not an entirely satisfactory way of writing things. Consider the case when $\omega t + \varphi_0 = 2n\pi$ such that $x \to 0$.
- A good way to do this is to use complex notation:

$$x = \mathrm{Re}\Big[A\exp(i\omega t + \varphi_0)\Big] = \mathrm{Re}[Ae^{i\omega t + \varphi_0}]$$

- A is magnitude (some say amplitude), while φ is phase. Together they make a complex amplitude.
- Phase lets us keep track of whether we're at the max, zero, or min of the wavefield.
  Look at the movie.



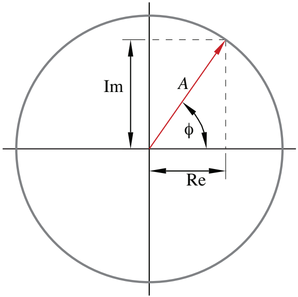
An eye on i

- Again, we can describe wave displacement as

$$x = \mathrm{Re}\left[A\exp(i\omega t + \varphi_0)\right] = \mathrm{Re}[Ae^{i\omega t+\varphi_0}]$$

where A again gives the amplitude, ω gives the angular frequency, and φ_0 gives the starting phase.

- French uses j for $\sqrt{-1}$ and says that this is what most physics and engineering books use (bottom of p. 11). In my experience, engineers tend to use j but physicists tend to use i. I'll use i for $\sqrt{-1}$.



Euler's expression

How can we relate $Ae^{i\theta}$ to sines and cosines? Let's do a Taylor expansion on $\sin\theta$ about $\theta = 0$:

$$\begin{aligned}
\sin\theta &\simeq \sin\theta|_{\theta=0} + (\theta-0)\frac{d}{d\theta}\sin\theta|_{\theta=0} + \frac{(\theta-0)^2}{2!}\frac{d^2}{d\theta^2}\sin\theta|_{\theta=0} \\
&\quad + \frac{(\theta-0)^3}{3!}\frac{d^3}{d\theta^3}\sin\theta|_{\theta=0} + \dots \\
&\simeq 0 + \theta(\cos 0) + \frac{\theta^2}{2}(-\sin 0) + \frac{\theta^3}{3!}(-\cos 0) + \dots \\
&\simeq \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots
\end{aligned}$$

A similar expansion of $\cos\theta$ about $\theta = 0$ gives

$$\cos\theta \simeq 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots$$

Euler's expression II

Now let's consider $\cos\theta + i\sin\theta$ using these series expansions and $i^2 = -1$:

$$\begin{aligned} \cos\theta + i\sin\theta &\simeq 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots \\ &\simeq 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots \end{aligned} \tag{6}$$

Do a Taylor expansion on e^x about $x = 0$, recognizing that $de^x/dx = e^x$ and $e^0 = 1$:

$$\begin{aligned} e^x &\simeq e^x|_{x=0} + (x-0)\frac{d}{dx}e^x|_{x=0} + \frac{(x-0)^2}{2!}\frac{d^2}{dx^2}e^x|_{x=0} \\ &\quad + \frac{(x-0)^3}{3!}\frac{d^3}{dx^3}e^x|_{x=0} + \frac{(x-0)^4}{4!}\frac{d^4}{dx^4}e^x|_{x=0}\ldots \\ &\simeq e^0 + (x-0)\cdot e^0 + \frac{(x-0)^2}{2!}e^0 + \frac{(x-0)^3}{3!}e^0 + \frac{(x-0)^4}{4!}e^0 + \ldots \\ &\simeq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \end{aligned} \tag{7}$$

Euler's expression III

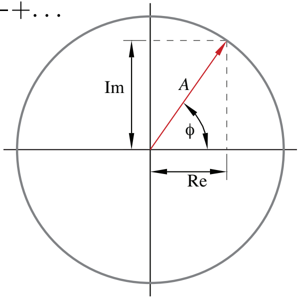
Again, we had from Eq. 6 the series expansion

$$\cos\theta + i\sin\theta \simeq 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots$$

and from Eq. 7 the series expansion

$$e^x \simeq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

These two series expansions are identical if we set $x \equiv i\theta$, so we have shown Euler's result that $e^{i\theta} = \cos\theta + i\sin\theta$.
Now you can see that if we treat the measurable position as $A\cos\theta$, it's just the real part of the complex exponential $\mathrm{Re}[Ae^{i\theta}]$.



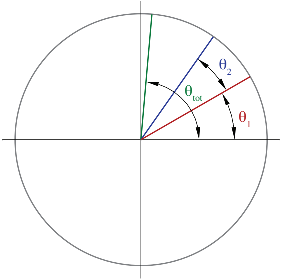
$$A\cos\varphi = \mathrm{Re}[Ae^{i\varphi}]$$

Rotations in complex space

- Consider the multiplication of $e^{i\theta_1}$ with $e^{i\theta_2}$:

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

That is, we simply add the two rotations together.



- What about a rotation of $\pi/2$?

$$e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = 0 + i \cdot 1 = i$$

That is, multiplication by i is the same as rotation by 90° counter-clockwise. Multiplication by $i^2 = -1$ equals a rotation by 180°, and so on.

Re-runs of the movie

Here again is the movie of simple harmonic motion $\cos(\omega t + \varphi_0)$. It can be expressed as the real part of a complex exponential $\mathrm{Re}\left[e^{i\omega t + \varphi_0}\right]$ which is steadily rotating around in the complex plane.

Other tricks

We can use all the machinery developed for exponential functions in other ways. Here's a derivative:

$$\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}\,d\theta = e^{i\theta+\pi/2}\,d\theta$$

which involves rotation by 90° and multiplication by $d\theta$ to get the tangent. Compare with differentiating sine and cosine:

$$\begin{aligned}\frac{d}{d\theta}(\cos\theta + i\sin\theta) &= (-\sin\theta + i\cos\theta)\,d\theta = (i\cos\theta + i\cdot i\sin\theta)\,d\theta \\ &= i(\cos\theta + i\sin\theta)\,d\theta\end{aligned}$$

so we have again shown Euler's relationship to hold true. Here's how we do powers of exponentials:

$$(e^{i\theta})^{\varphi} = e^{i\theta\varphi}$$

It's a piece of cake as a complex number $(e^{i\theta})^{\varphi}$, right? Try doing $(\cos\theta + i\sin\theta)^{\varphi}$ for some simple non-integer value of φ!

Multiplication

Multiply complex numbers with lengths other than 1:

$$Ae^{i\theta_1} \cdot Be^{i\theta_2} = ABe^{i(\theta_1+\theta_2)}$$

We can again show this with the trig version:

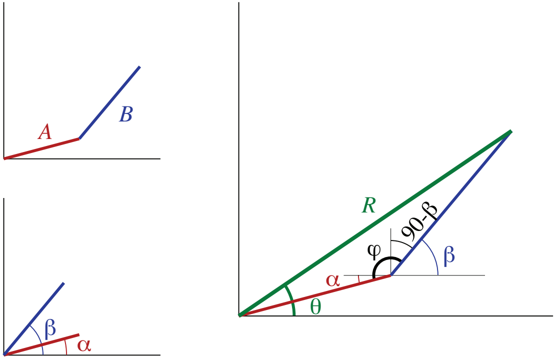
$$\begin{aligned}
(A\cos\theta_1 + iA\sin\theta_1) \\ \cdot(B\cos\theta_2 + iB\sin\theta_2) &= AB\cos\theta_1\cos\theta_2 - AB\sin\theta_1\sin\theta_2 \\
&\quad +iAB\sin\theta_1\cos\theta_2 + iAB\cos\theta_1\sin\theta_2 \\
&= AB\Big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\Big] \\
&= ABe^{i(\theta_1+\theta_2)}
\end{aligned}$$

where we have made use of the trig identities

$$\begin{aligned}
\sin(\theta_1 \pm \theta_2) &= \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2 \\
\cos(\theta_1 \pm \theta_2) &= \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2
\end{aligned}$$

Addition

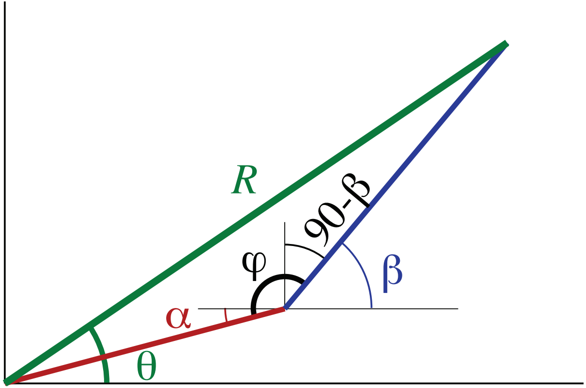
Consider the addition of two vectors $Ae^{i\omega t+\alpha}$ and $Be^{i\omega t+\beta}$, viewed at a moment in time:



If the two waves have the exact same frequency ω, both terms have a common factor $e^{i\omega t}$ which we can pull out, and thus deal with the static case. How do we calculate the length of the resultant vector? From the law of cosines, we have

$$R^2 = A^2 + B^2 - 2AB\cos\varphi \tag{8}$$

Addition II



We can express $\cos\varphi$ as

$$\cos\varphi = \cos\Big(\alpha + 90^\circ + (90^\circ - \beta)\Big) = \cos\Big(180^\circ + (\alpha - \beta)\Big) \tag{9}$$

Let's use the trig identity $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$:

$$\cos(180^\circ + \theta) = \cos 180^\circ \cos\theta - \sin 180^\circ \sin\theta = -\cos\theta \tag{10}$$

Addition III

Repeating the results of Eqs. 9 and 10, we have

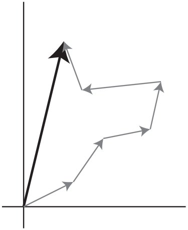
$$\cos\varphi = \cos\Big(180^\circ + (\alpha - \beta)\Big) = -\cos(\alpha - \beta) \tag{11}$$

We can then substitute this into Eq. 8 to find that the length of the resultant vector is

$$R^2 = A^2 + B^2 - 2AB\cos(\varphi) = A^2 + B^2 + 2AB\cos(\alpha - \beta) \tag{12}$$

Let's extend this result to consider adding up lots of waves. The net phase angle θ is given by

$$\tan\theta = \frac{\sum_i A_i \sin\theta_i}{\sum_i A_I \cos\theta_i} \tag{13}$$



Addition IV

For adding up lots of waves at the same frequency ω, the net length is

$$R^2 = \left(\sum_i A_i \sin\theta_i\right)^2 + \left(\sum_i A_i \cos\theta_i\right)^2 \tag{14}$$

which at first glance doesn't appear to be very illuminating. However, let's imagine expanding the first three terms of these squares:

$$\begin{aligned}
\left(\sum_{i=1}^{3} A_i \sin\theta_i\right)^2 &= (A_1 \sin\theta_1 + A_2 \sin\theta_2 + A_3 \sin\theta_3) \\
&\quad \cdot (A_1 \sin\theta_1 + A_2 \sin\theta_2 + A_3 \sin\theta_3) \\
&= A_1^2 \sin^2\theta_1 + A_2^2 \sin^2\theta_2 + A_3^2 \sin^2\theta_3 \\
&\quad + 2A_1A_2 \sin\theta_1 \sin\theta_2 + 2A_1A_3 \sin\theta_1 \sin\theta_3 \\
&\quad + 2A_2A_3 \sin\theta_2 \sin\theta_3
\end{aligned} \tag{15}$$

Addition V

Again, we had in Eq. 16 the result

$$\begin{aligned}\left(\sum_{i=1}^{3} A_i \sin\theta_i\right)^2 &= (A_1\sin\theta_1 + A_2\sin\theta_2 + A_3\sin\theta_3)\\ &\quad\cdot(A_1\sin\theta_1 + A_2\sin\theta_2 + A_3\sin\theta_3)\\ &= A_1^2\sin^2\theta_1 + A_2^2\sin^2\theta_2 + A_3^2\sin^2\theta_3\\ &\quad +2A_1A_2\sin\theta_1\sin\theta_2 + 2A_1A_3\sin\theta_1\sin\theta_3\\ &\quad +2A_2A_3\sin\theta_2\sin\theta_3\end{aligned}$$

Examination of this result shows that we can generalize the sine sum to

$$\left(\sum_{i=1}^{N} A_i \sin\theta_i\right)^2 = \sum_{i=1}^{N} A_i^2\sin^2\theta_i + 2\sum_{j>i}^{N}\sum_{i=1}^{N} A_iA_j\sin\theta_i\sin\theta_j \tag{16}$$

Addition VI

Now let's return to Eq. 14 of $R^2 = \left(\sum_i A_i \sin\theta_i\right)^2 + \left(\sum_i A_i \cos\theta_i\right)^2$ and make use of the result of Eq. 16 for $\left(\sum_{i=1}^N A_i \sin\theta_i\right)^2$ and the equivalent cosine sum:

$$\begin{aligned} R^2 &= \sum_{i=1}^N A_i^2 \sin^2\theta_i + 2\sum_{j>i}^N \sum_{i=1}^N A_i A_j \sin\theta_i \sin\theta_j \\ &\quad + \sum_{i=1}^N A_i^2 \cos^2\theta_i + 2\sum_{j>i}^N \sum_{i=1}^N A_i A_j \cos\theta_i \cos\theta_j \\ &= \sum_{i=1}^N A_i^2 (\sin^2\theta_i + \cos^2\theta_i) \\ &\quad +2\sum_{j>i}^N \sum_{i=1}^N A_i A_j (\sin\theta_i \sin\theta_j + \cos\theta_i \cos\theta_j) \end{aligned} \tag{17}$$

Addition VII

- Where were we? Oh yes, at Eq. 17:

$$
\begin{aligned}
R^2 &= \sum_{i=1}^{N} A_i^2(\sin^2\theta_i + \cos^2\theta_i) \\
&\quad +2\sum_{j>i}^{N}\sum_{i=1}^{N} A_iA_j(\sin\theta_i\sin\theta_j + \cos\theta_i\cos\theta_j) \\
&= \sum_{i=1}^{N} A_i^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} A_iA_j\cos(\theta_i-\theta_j) \qquad (18)
\end{aligned}
$$

where we have again made use of the trig identity $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$.