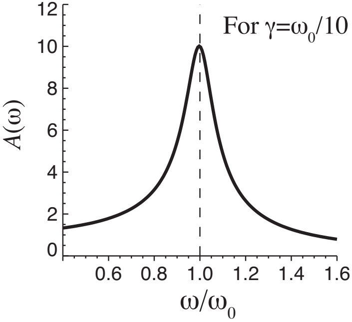
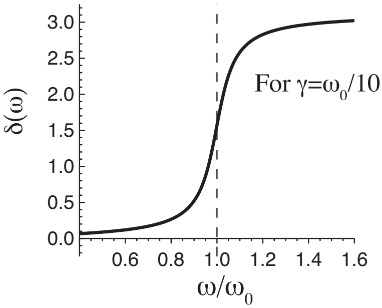
Review: DDHO

We discussed the damped, driven harmonic oscillator, with solutions

$$|A(\omega)| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \quad \text{and} \quad \tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

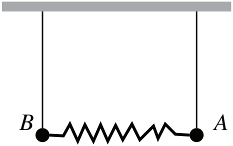




At resonance, the power dissipation is $P_{\text{resonance}} = F_0^2\omega_0 Q/(2k)$ and the FWHM of the resonance curve is ω_0/Q.

CO: coupled oscillators

We have discussed single, isolated oscillators. Now let's consider two oscillators that are coupled together:



Pendulum bob A has a position x_A, and bob B has a position x_B. We'll assume both pendulum bobs have the same mass and the same arm length, so that m and ω_0 is the same for each pendulum. Each pendulum bob has two separate restoring forces trying to bring it to the equilibrium position shown above: the gravitational restoring force which we showed earlier can be written as $m\omega_0^2 x$, and a spring force which depends on the separation $|x_A - x_B|$:

$$m\frac{d^2x_A}{dt^2} + m\omega_0^2 x_A + k(x_A - x_B) = 0 \quad (1)$$

$$m\frac{d^2x_B}{dt^2} + m\omega_0^2 x_B - k(x_A - x_B) = 0 \quad (2)$$

CO II

If we write the spring force resonant frequency as $\omega_c^2 = k/m$, we can rewrite Eq. 1:

$$\begin{aligned} m\frac{d^2x_A}{dt^2} + m\omega_0^2 x_A + k(x_A - x_B) &= 0 \\ \frac{d^2x_A}{dt^2} + (\omega_0^2 + \omega_c^2)x_A - \omega_c^2 x_B &= 0 \end{aligned} \tag{3}$$

and Eq. 2:

$$\begin{aligned} m\frac{d^2x_B}{dt^2} + m\omega_0^2 x_B - k(x_A - x_B) &= 0 \\ \frac{d^2x_B}{dt^2} + (\omega_0^2 + \omega_c^2)x_B - \omega_c^2 x_A &= 0 \end{aligned} \tag{4}$$

which together reproduce French Eq. 5-4. In other words, the two equations are coupled; Eqs. 3 and 4 each contain both x_A and x_B.

CO III

Again, we have Eqs. 3 and 4 of

$$\begin{aligned} \frac{d^2x_A}{dt^2} + (\omega_0^2 + \omega_c^2)x_A - \omega_c^2 x_B &= 0 \\ \frac{d^2x_B}{dt^2} + (\omega_0^2 + \omega_c^2)x_B - \omega_c^2 x_A &= 0 \end{aligned}$$

We can get two interesting other equations by addition, and also by subtraction, of these two equations:

$$\frac{d^2}{dt^2}(x_A + x_B) + \omega_0^2(x_A + x_B) = 0 \quad (5)$$

$$\frac{d^2}{dt^2}(x_A - x_B) + (\omega_0^2 + 2\omega_c^2)(x_A - x_B) = 0 \quad (6)$$

Can you think of a better way of expressing these two equations?

CO III

Again, we have Eqs. 5 and 6 of

$$\begin{aligned} \frac{d^2}{dt^2}(x_A + x_B) + \omega_0^2(x_A + x_B) &= 0 \\ \frac{d^2}{dt^2}(x_A - x_B) + (\omega_0^2 + 2\omega_c^2)(x_A - x_B) &= 0 \end{aligned}$$

Let's define $q_1 \equiv x_A + x_B$ for a *common mode* motion, so that Eq. 5 becomes

$$\frac{d^2 q_1}{dt^2} + \omega_0^2 q_1 = 0 \tag{7}$$

and define $q_2 \equiv x_A - x_B$ as well as $\omega' \equiv \sqrt{\omega_0^2 + 2\omega_c^2}$ for a *differential mode* motion:

$$\frac{d^2 q_2}{dt^2} + \omega'^2 q_2 = 0 \tag{8}$$

Thus we are now back to simple harmonic motion equations for the variables q_1 and q_2 with resonant frequencies ω_0 and ω', respectively.

CO IV

Again, we had Eqs. 7 and 8 of

$$
\begin{aligned}
\frac{d^2 q_1}{dt^2} + \omega_0^2 q_1 &= 0 \qquad \text{with } q_1 \equiv x_A + x_B \\
\frac{d^2 q_2}{dt^2} + \omega'^2 q_2 &= 0 \qquad \text{with } q_2 \equiv x_A - x_B \text{ and } \omega' \equiv \sqrt{\omega_0^2 + 2\omega_c^2}
\end{aligned}
$$

The solution to each of these is of the form $q = Ae^{i\omega t + \delta}$. Going back to our original variables $x_A = (q_1 + q_2)/2$ and $x_B = (q_1 - q_2)/2$ and using C and D for the respective amplitudes of the q_1 and q_2 motions, we see that we can write the motion of each individual pendulum bob as

$$
\begin{aligned}
x_A &= \frac{1}{2} C e^{i\omega_0 t} + \frac{1}{2} D e^{i\omega' t + \delta} \qquad (9) \\
x_B &= \frac{1}{2} C e^{i\omega_0 t} - \frac{1}{2} D e^{i\omega' t + \delta} \qquad (10)
\end{aligned}
$$

We have chosen to assign a relative phase shift to the differential mode motion in ω' only, since we can always apply a common static phase shift to the combined motion.

CO: excitation by “tickling” A

Let’s now consider the case where we displace and hold the right-hand pendulum bob by an amplitude A_0 while the left-hand bob is at rest at its equilibrium position x_B. We thus have the following conditions:

$$t = 0 \qquad x_A = A_0 \qquad \frac{dx_A}{dt} = 0 \qquad x_B = 0 \qquad \frac{dx_B}{dt} = 0$$

Putting these conditions into Eq. 9 gives

$$x_A = \frac{1}{2}Ce^{i\omega_0 t} + \frac{1}{2}De^{i\omega' t+\delta} \qquad \Rightarrow \qquad x_A = A_0 = \frac{1}{2}C + \frac{1}{2}De^{i\delta} \quad (11)$$

and putting them into Eq. 10 gives

$$x_B = \frac{1}{2}Ce^{i\omega_0 t} - \frac{1}{2}De^{i\omega' t+\delta} \qquad \Rightarrow \qquad x_B = 0 = \frac{1}{2}C - \frac{1}{2}De^{i\delta} \quad (12)$$

Adding the two equations gives $C = A_0$ while subtraction gives $D = A_0$ if $\delta = 0$. Thus we see that the amplitudes will be the same for the common mode motion in q_1 as for the differential mode motion in q_2.

CO after “tickling” A II

Again, we’ve considered the starting condition of having each bob at rest but x_A displaced by A_0. We’ve found that both the common mode and differential mode oscillations have the same amplitudes, so Eqs. 11 and 12 give for the measureable, real parts of the complex quantities

$$x_A = \frac{1}{2}A_0(\cos\omega_0 t + \cos\omega' t) \tag{13}$$
$$x_B = \frac{1}{2}A_0(\cos\omega_0 t - \cos\omega' t) \tag{14}$$

which, by using the proper trigonometric identities, can be written as (see French Eq. 5-7)

$$x_A = A_0\cos\left(\frac{\omega' - \omega_0}{2}t\right)\cos\left(\frac{\omega' + \omega_0}{2}t\right) \tag{15}$$
$$x_B = A_0\sin\left(\frac{\omega' - \omega_0}{2}t\right)\sin\left(\frac{\omega' + \omega_0}{2}t\right) \tag{16}$$
$$= A_0\cos\left(\frac{\omega' - \omega_0}{2}t - \frac{\pi}{2}\right)\cos\left(\frac{\omega' + \omega_0}{2}t - \frac{\pi}{2}\right) \tag{17}$$

Thus we see that the motion looks like a beat frequency type phenomenon and that x_B lags x_A in phase by 90°.

Conditions for arbitrary ω?

Let's step back from what we know and consider motion at some arbitrary frequency ω according to

$$\begin{aligned} x_A &= Ce^{i\omega t} \\ x_B &= C'e^{i\omega t} \end{aligned}$$

Let's insert these assumed solutions into Eqs. 3 and 4:

$$\begin{aligned} \frac{d^2x_A}{dt^2} + (\omega_0^2 + \omega_c^2)x_A - \omega_c^2 x_B &= 0 \\ \frac{d^2x_B}{dt^2} + (\omega_0^2 + \omega_c^2)x_B - \omega_c^2 x_A &= 0 \end{aligned}$$

This gives

$$(-\omega^2 + \omega_0^2 + \omega_c^2)C - \omega_c^2 C' = 0 \quad (18)$$

$$-\omega_c^2 C + (-\omega^2 + \omega_0^2 + \omega_c^2)C' = 0 \quad (19)$$

Coupling I

The result of Eq. 18 of

$$(-\omega^2 + \omega_0^2 + \omega_c^2)C - \omega_c^2 C' = 0$$

gives

$$\frac{C}{C'} = \frac{\omega_c^2}{-\omega^2 + \omega_0^2 + \omega_c^2} \tag{20}$$

while the result of Eq. 19 of

$$-\omega_c^2 C + (-\omega^2 + \omega_0^2 + \omega_c^2)C' = 0$$

gives

$$\frac{C}{C'} = \frac{-\omega^2 + \omega_0^2 + \omega_c^2}{\omega_c^2} \tag{21}$$

Thus we must say

$$\frac{\omega_c^2}{-\omega^2 + \omega_0^2 + \omega_c^2} = \frac{-\omega^2 + \omega_0^2 + \omega_c^2}{\omega_c^2} \tag{22}$$

Coupling II

Again, we had Eq. 22 of

$$\frac{\omega_c^2}{-\omega^2+\omega_0^2+\omega_c^2} = \frac{-\omega^2+\omega_0^2+\omega_c^2}{\omega_c^2}$$

which gives

$$\begin{aligned} (-\omega^2+\omega_0^2+\omega_c^2)^2 &= (\omega_c^2)^2 \\ -\omega^2+\omega_0^2+\omega_c^2 &= \pm\omega_c^2 \\ \omega^2 &= \omega_0^2+\omega_c^2\pm\omega_c^2 \end{aligned}$$

so we must have two solutions for ω:

$$\begin{aligned} \omega'^2 &= \omega_0^2+2\omega_c^2 \\ \omega''^2 &= \omega_0^2 \end{aligned}$$

What have we learned from this? Nothing, and everything.

Coupling III

Our solutions of

$$\begin{aligned} \omega'^2 &= \omega_0^2 + 2\omega_c^2 \\ \omega''^2 &= \omega_0^2 \end{aligned}$$

are of the same form as what we found for a *particular* problem of displacing A and letting things go. We have now seen that these are *general* solutions. Let's go back to Eq. 20 of

$$\frac{C}{C'} = \frac{\omega_c^2}{-\omega^2 + \omega_0^2 + \omega_c^2}$$

and insert $\omega^2 = \omega'^2 = \omega_0^2 + 2\omega_c^2$:

$$\frac{C}{C'} = \frac{\omega_c^2}{-\omega_0^2 - 2\omega_c^2 + \omega_0^2 + \omega_c^2} = \frac{\omega_c^2}{-\omega_c^2} = -1$$

Coupling IV

Let's also try $\omega^2 = \omega''^2 = \omega_0^2$ in Eq. 20:

$$\frac{C}{C'} = \frac{\omega_c^2}{-\omega^2 + \omega_0^2 + \omega_c^2} = \frac{\omega_c^2}{-\omega_0^2 + \omega_0^2 + \omega_c^2} = \frac{\omega_c^2}{\omega_c^2} = +1$$

Again, we have either $C/C' = +1$ or -1. Our general equations of motion must always have the same magnitude since $|C| = |C'|$, though the differential motion at ω' can have a $+1$ or -1 sign associated with it. We can in general write solutions for coupled oscillators of

$$x_A = Ce^{i\omega_0 t} \qquad \text{and} \qquad x_B = Ce^{i\omega_0 t} \tag{23}$$

for common mode motion, or

$$x_A = De^{i\omega' t} \qquad \text{and} \qquad x_B = -De^{i\omega' t} \tag{24}$$

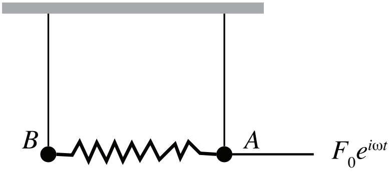
Because our original differential equations of motion were all linear, we can also have sums and differences of these two solutions.

Coupled, forced I

Let's now consider coupled oscillators with a driving force applied to oscillator A:

$$m\frac{d^2x_A}{dt^2} = -m\omega_0^2 x_A - k(x_A - x_B) + F_0 e^{i\omega t} \quad (25)$$

$$m\frac{d^2x_B}{dt^2} = -m\omega_0^2 x_B + k(x_A - x_B) \quad (26)$$



Dividing through by m, using $\omega_c^2 = k/m$, and rearranging gives

$$\frac{d^2x_A}{dt^2} + (\omega_0^2 + \omega_c^2)x_A - \omega_c^2 x_B = \frac{F_0}{m}e^{i\omega t} \quad (27)$$

$$\frac{d^2x_B}{dt^2} + (\omega_0^2 + \omega_c^2)x_B - \omega_c^2 x_A = 0. \quad (28)$$

Coupled, forced II

Again, we had Eqs. 27 and 28 of

$$\begin{aligned} \frac{d^2x_A}{dt^2} + (\omega_0^2+\omega_c^2)x_A - \omega_c^2 x_B &= \frac{F_0}{m}e^{i\omega t} \\ \frac{d^2x_B}{dt^2} + (\omega_0^2+\omega_c^2)x_B - \omega_c^2 x_A &= 0. \end{aligned}$$

Let's first add these two equations:

$$\frac{d^2x_A}{dt^2} + \frac{d^2x_B}{dt^2} + (\omega_0^2+\omega_c^2)(x_A+x_B) - \omega_c^2(x_A+x_B) = \frac{F_0}{m}e^{i\omega t} \qquad (29)$$

And let's subtract them:

$$\frac{d^2x_A}{dt^2} - \frac{d^2x_B}{dt^2} + (\omega_0^2+\omega_c^2)(x_A-x_B) + \omega_c^2(x_A-x_B) = \frac{F_0}{m}e^{i\omega t} \qquad (30)$$

Coupled, forced III

Again, we had Eq. 29 of

$$\frac{d^2x_A}{dt^2} + \frac{d^2x_B}{dt^2} + (\omega_0^2 + \omega_c^2)(x_A + x_B) - \omega_c^2(x_A + x_B) = \frac{F_0}{m}e^{i\omega t}$$

If we again use $q_1 \equiv x_A + x_B$ we obtain

$$\frac{d^2q_1}{dt^2} + \omega_0^2 q_1 = \frac{F_0}{m}e^{i\omega t}. \tag{31}$$

And we had Eq. 30 of

$$\frac{d^2x_A}{dt^2} - \frac{d^2x_B}{dt^2} + (\omega_0^2 + \omega_c^2)(x_A - x_B) + \omega_c^2(x_A - x_B) = \frac{F_0}{m}e^{i\omega t}$$

If we again use $q_2 \equiv x_A - x_B$ and $\omega'^2 \equiv \omega_0^2 + 2\omega_c^2$, we obtain

$$\frac{d^2q_2}{dt^2} + \omega'^2 q_2 = \frac{F_0}{m}e^{i\omega t} \tag{32}$$

Coupled, forced IV

Again, we have from Eqs. 31 and 32

$$\frac{d^2q_1}{dt^2} + \omega_0^2 q_1 = \frac{F_0}{m} e^{i\omega t} \qquad \text{and} \qquad \frac{d^2q_2}{dt^2} + \omega'^2 q_2 = \frac{F_0}{m} e^{i\omega t}$$

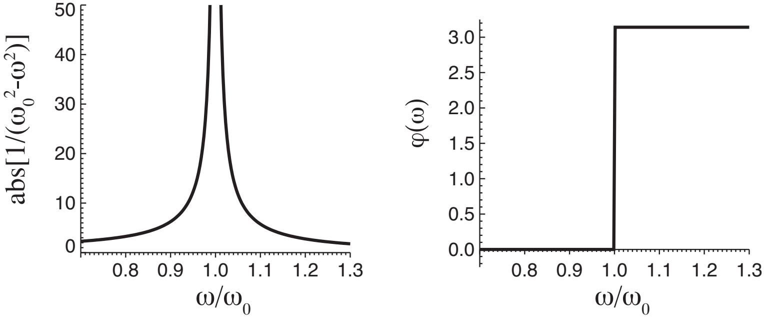
To quote Yogi Berra, “it’s like déjà vu all over again.” We’ve solved differential equations of this form when we considered the driven harmonic oscillator! We found solutions that look like $q = Ce^{i\omega t}$, which when inserted into the differential equations above give

$$\begin{aligned} (-\omega^2 + \omega_0^2) Ce^{i\omega t} &= \frac{F_0}{m} e^{i\omega t} \\ -\omega^2 + \omega_0^2 &= \frac{F_0}{mC} \\ C &= \frac{F_0/m}{\omega_0^2 - \omega^2} \end{aligned} \tag{33}$$

Coupled, forced V

Again, our solutions to Eqs. 31 and 32 look like $q = Ce^{i\omega t}$ with C given by Eq. 33:

$$C = \frac{F_0/m}{\omega_0^2 - \omega^2} \qquad \text{or } |C| \text{ and } \varphi$$



Coupled, forced VI

These were solutions for the individual modes of motion: common mode $q_1 \equiv x_A + x_B$ at frequency ω_0, and differential mode $q_2 = x_A - x_B$ at frequency $\omega'^2 = \omega_0^2 + 2\omega_c^2$. So let's look at the net motion of A:

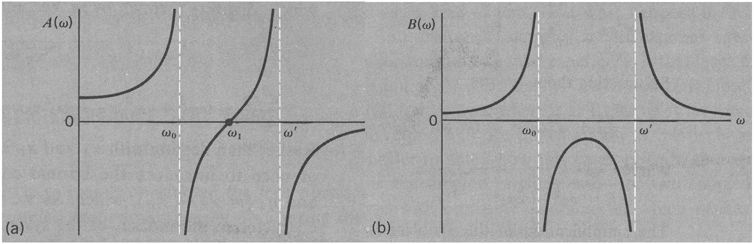
$$x_A = Ae^{i\omega t} = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}(Ce^{i\omega t} + De^{i\omega t}) \quad (34)$$

$$x_B = Be^{i\omega t} = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}(Ce^{i\omega t} - De^{i\omega t}) \quad (35)$$

so we find

$$\begin{aligned} A(\omega) &= \frac{F_0/m}{2}\left(\frac{1}{\omega_0^2 - \omega^2} + \frac{1}{\omega'^2 - \omega^2}\right) \\ &= \frac{F_0}{2m}\left(\frac{(\omega'^2 - \omega^2) + (\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)(\omega'^2 - \omega^2)}\right) \\ &= \frac{F_0}{2m}\left(\frac{\omega_0^2 + \omega'^2 - 2\omega^2}{(\omega_0^2 - \omega^2)(\omega'^2 - \omega^2)}\right) \quad (36) \\ B(\omega) &= \frac{F_0}{2m}\left(\frac{2\omega_c^2}{(\omega_0^2 - \omega^2)(\omega'^2 - \omega^2)}\right) \quad (37) \end{aligned}$$

Combined motion

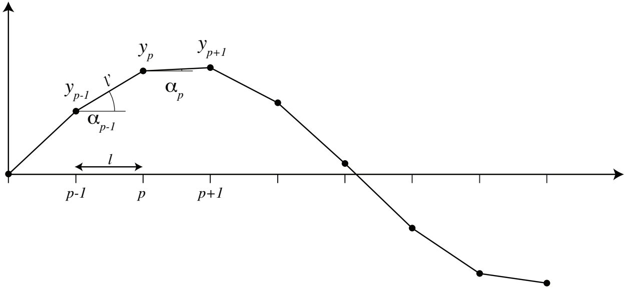


French Fig. 5.8, showing the amplitudes of the oscillators $A(\omega)$ and $B(\omega)$.

Coupled oscillators: from 2 to oodles

Let's move from two coupled pendulums to N coupled oscillators.

- Now each oscillator experiences a force from a neighbor on each side. We'll assume that the coupling between oscillators is dominant, and talk about them being on a string with tension T.
- We'll index each oscillator with an integer p for position.
- Each position will be a distance ℓ apart along the axis, or ℓ' along the string.



Oodles I

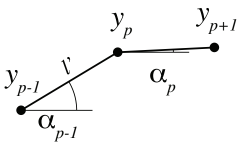
Angles between points:

$$\alpha_{p-1} = \tan^{-1}\left(\frac{y_p - y_{p-1}}{\ell}\right) \simeq \frac{y_p - y_{p-1}}{\ell} \tag{38}$$

What's ℓ'? we can write

$$\ell' = \frac{\ell}{\cos\alpha} \simeq \frac{\ell}{1-\alpha^2/2} \simeq \ell(1+\alpha^2/2) \tag{39}$$

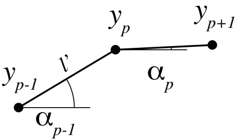
So $\ell' - \ell = \ell\alpha^2/2$. We'll ignore α^2 effects throughout. If $\ell' \simeq \ell$, then the tension T on each point is the same.



Oodles II

- Since the tension is independent of α (in the limit $\alpha^2 \ll 1$), we say T is a constant.
- Net force in x on p:

$$\begin{aligned} F_x &= -T\cos\alpha_{p-1} + T\cos\alpha_p \\ &\simeq T(-1 + \frac{\alpha_{p-1}^2}{2} + 1 - \frac{\alpha_p^2}{2}) \\ &\simeq \frac{T}{2}(\alpha_{p-1}^2 - \alpha_p^2) \qquad (40) \end{aligned}$$

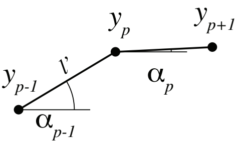


- Since this depends on α^2, we will ignore this force; each point stays at a constant position x_p.

Oodles III

Consider now the net force in y:

$$\begin{aligned} F_y &= -T\sin\alpha_{p-1} + T\sin\alpha_p \\ &\simeq T(\alpha_p - \alpha_{p-1}) \\ &\simeq \frac{T}{\ell}(y_{p+1} - y_p - y_p + y_{p-1}) \\ &\simeq \frac{T}{\ell}(-2y_p + y_{p+1} + y_{p-1}) \quad (41) \end{aligned}$$



where we have made use of the result of Eq. 38 of

$$\alpha_{p-1} \simeq \frac{y_p - y_{p-1}}{\ell}$$

Oodles IV

We have found that we can ignore F_x, so that x_p is a constant. In the y direction, the net force of Eq. 41 produces acceleration:

$$\begin{aligned} m\frac{d^2y_p}{dt^2} &= \frac{T}{\ell}(-2y_p + y_{p+1} + y_{p-1}) \\ \frac{m\ell}{T}\frac{d^2y_p}{dt^2} + 2y_p - (y_{p+1} + y_{p-1}) &= 0 \end{aligned} \tag{42}$$

We have a puzzle here: we have a differential equation in y_p, but it is also coupled to neighboring positions y_{p-1} and y_{p+1}. Still, this looks enough like a harmonic oscillator that we will assume that we are looking for solutions of the form $y_p = A_p e^{i\omega t}$, in which case Eq. 42 becomes

$$-\omega^2\frac{m\ell}{T}A_p e^{i\omega t} + 2A_p e^{i\omega t} + (A_{p+1} + A_{p-1})e^{i\omega t} = 0 \tag{43}$$

We see that ω^2 has the same dimensions as $T/(m\ell)$, so we will make the definition

$$\omega_0^2 \equiv \frac{T}{m\ell} \tag{44}$$

Oodles V

With the definition of Eq. 44 of $\omega_0^2 \equiv T/(m\ell)$, we can rewrite Eq. 42 as

$$\frac{d^2y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0 \tag{45}$$

If we return to our assumption of $y_p = A_p e^{i\omega t}$, this becomes

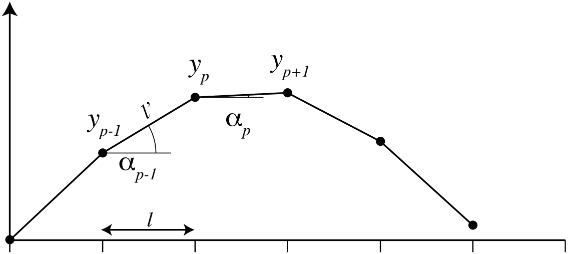
$$\begin{aligned} -\omega^2 A_p e^{i\omega t} + 2\omega_0^2 A_p e^{i\omega t} - \omega_0^2(A_{p+1} + A_{p-1})e^{i\omega t} &= 0 \\ \text{or} \qquad (-\omega^2 + 2\omega_0^2)A_p - \omega_0^2(A_{p+1} + A_{p-1}) &= 0 \\ \text{or} \qquad (-\omega^2 + 2\omega_0^2)A_p &= \omega_0^2(A_{p+1} + A_{p-1}) \end{aligned} \tag{46}$$

This gives us the relationship

$$\frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} \tag{47}$$

Oodles VI

Let's think again about the relationship between amplitudes of successive points:



Let's test a reasonable guess for the relationship between amplitudes of successive points: $A_p = C \sin(p\theta)$. That is, θ is an increment of amplitude from one point to another for a standing wave solution, as we'll see later. If we consider a standing wave with fixed ends such that $A_{p=0} = 0$ and $A_{p=N+1} = 0$, we can say that $(N+1)\theta = n\pi$ with $(n = 1, 2, 3, \ldots)$ and thus write the amplitude A_p as

$$A_p = C_n \sin\left(\frac{pn\pi}{N+1}\right) \qquad \text{with } n = 1, 2, 3, \ldots \tag{48}$$

Oodles VII

With the assumption $A_p = C\sin(p\theta)$, we have

$$\begin{aligned}
A_{p-1} + A_{p+1} &= C\Big[\sin\big((p-1)\theta\big) + \sin\big((p+1)\theta\big)\Big] \\
&= C\Big[\sin(p\theta)\cos(-\theta) + \cos(p\theta)\sin(-\theta) \\
&\qquad + \sin(p\theta)\cos(\theta) + \cos(p\theta)\sin(\theta)\Big] \\
&= C\Big[\sin(p\theta)\cos(\theta) - \cos(p\theta)\sin(\theta) \\
&\qquad + \sin(p\theta)\cos(\theta) + \cos(p\theta)\sin(\theta)\Big] \\
&= 2C\sin(p\theta)\cos(\theta) \\
&= 2A_p\cos\theta \qquad (49)
\end{aligned}$$

where we have used the trig identity for $\sin(\alpha+\beta)$, and $\cos(-\theta) = \cos(\theta)$, and $\sin(-\theta) = -\sin(\theta)$ to reproduce French Eq. 5-21.

Oodles VIII

Again, from Eq. 49 we have $A_{p-1} + A_{p+1} = 2A_p \cos\theta$. If we now use the result $(N+1)\theta = n\pi$ arrived at before Eq. 48, and Eq. 47, we have

$$\frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} = 2\cos\left(\frac{n\pi}{N+1}\right) \tag{50}$$

Let's define $\beta \equiv n\pi/(N+1)$ and solve for the variable frequency ω:

$$\begin{aligned}
-\omega^2 + 2\omega_0^2 &= 2\omega_0^2 \cos\beta \\
\omega^2 &= 2\omega_0^2(1 - \cos\beta) \\
&= 4\omega_0^2 \frac{1 - \cos\beta}{2} \\
&= 4\omega_0^2 \sin^2(\frac{\beta}{2}) \\
\omega &= 2\omega_0 \sin(\frac{\beta}{2}) = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) \qquad (51)
\end{aligned}$$

were we have made use of the trig identity $\sin^2\beta/2 = (1/2)(1 - \cos\beta)$ in arriving at the final result.

Oodles IX

What hath we wrought? We have from Eq. 48

$$y_p = A_p e^{i\omega t} = C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega t}$$

and from Eq. 51 the result

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) = 2\omega_0 \sin\left(\frac{n}{N+1}\,\frac{\pi}{2}\right) \qquad \text{with } n = 1, 2, 3, \ldots$$

so we should really write the position of the p^{th} particle as $y_{pn}(t)$. Now let us consider the frequencies allowed by Eq. 51. If we increase n from 1 up to $N+1$ (the number of oscillating points, because p goes from 0 to $N+1$), we will have unique values of ω_n. However, when n goes to $N+2$, we have

$$\frac{N+2}{N+1} = \frac{N+1}{N+1} + \frac{1}{N+1}$$

but since $\sin(\pi/2 + \beta) = \sin(\pi/2 - \beta)$, we'll have the same result for Eq. 51 for $n = (N+1) + 1$ as for $n = (N+1) - 1$. That is, we have only $n = 1, 2, \ldots, N+1$ unique frequencies.

Oodles: summary

We have determined that we have only $n = 1, 2, \ldots, N+1$ unique frequencies in the result of Eq. 48 of

$$y_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}$$

In fact, when $n = N+1$, the amplitude is $C_{N+1} \sin(p\pi)$, and since p is an integer the amplitude for $n = N+1$ is zero. (Well, duh; this was built into our assumption that $(N+1)\theta = n\pi$ when arriving at Eq. 48). So really we have only $n = 1, 2, \ldots, N$ unique frequencies with non-zero amplitude. Also, just as we found that there are only N unique frequencies, the same argument applied to the ampltudes again shows that there are only N unique results. We've learned something important: **N oscillators between fixed points have N allowed modes of oscillation** and we should write Eq. 48 as

$$y_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t} \qquad \text{with } n = 1, 2, \ldots, N \tag{52}$$

Standing waves

Again, we have Eq. 52 of

$$y_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t} \text{ with } n = 1, 2, \ldots, N \text{ and } p = 0, 1, \ldots, N+1$$

Let's consider the $n = 1$ case:

$$y_{p1} = C_1 \sin\left(\frac{p}{N+1}\pi\right) e^{i\omega_1 t}$$

Each point p oscillates at the frequency ω_1 with an amplitude of C_1 times $\sin\theta$ with θ going from 0 to π. This is a standing wave with maximum amplitude in the center (French Fig. 5-13).
The $n = 2$ case looks like

$$y_{p2} = C_2 \sin\left(\frac{p}{N+1}2\pi\right) e^{i\omega_2 t}$$

which goes like $\sin\theta$ with $\theta = 0 \to 2\pi$. We have a node in the middle (French Fig. 5-14). Get the pattern?

Strings to springs

Let's now consider $p = 1, 2, \ldots, N$ masses m coupled by springs with spring constant $k = m\omega_0^2$ (points $p = 0$ and $p = N + 1$ will be fixed points at either end of the system). Let y_p represent the displacement of each point from its equilibrium position. The force that a point p feels is given by the relative spring force it feels from each side:

$$\begin{aligned} m\frac{d^2 y_p}{dt^2} &= k(y_{p+1} - y_p) - k(y_p - y_{p-1}) \\ \frac{d^2 y_p}{dt^2} &= \frac{k}{m}(-2y_p + y_{p+1} + y_{p-1}) \\ \frac{d^2 y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) &= 0 \end{aligned}$$

This is exactly the same mathematical form as we had in Eq. 45! In that case we had $\omega_0^2 = T/(m\ell)$, while now we have $\omega_0^2 = k/m$; and we interpreted y_p as the vertical displacement of a string stretched horizontally rathe than the displacement from a longitudinal equilibrium position, but everything we've done above also applies to a series of N masses coupled by springs.

String frequencies

Let's consider the case of very large values of N. For standing waves on a string, the total length of the string is $L = (N+1)\ell$ and its total mass is $M = Nm$ with a mass per unit length of $\mu \equiv m/l$. Now our spectrum of allowed frequencies was given by Eq. 51 as

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

which in the limit $n \ll N$ becomes

$$\begin{aligned}
\omega_n &\simeq 2\sqrt{\frac{T}{m\ell}}\frac{n\pi}{2(N+1)} \\
&\simeq \sqrt{\frac{T}{m/\ell}}\frac{n\pi}{\ell(N+1)} = \sqrt{\frac{T}{\mu}}\frac{n\pi}{L}. \qquad (53)
\end{aligned}$$

We see that heavier strings have lower frequencies for the same length and tension, which tells us about things like how to build guitars and pianos.

Frequency cutoff

The highest mode is with $n = N$, which from Eq. 51 gives in the limit $N \gg 1$

$$\omega_n = 2\omega_0 \sin\left(\frac{N\pi}{2(N+1)}\right) \simeq 2\omega_0 \sin\left(\frac{\pi}{2}\right) \simeq 2\omega_0$$

What does the motion look like at this frequency? We had from Eq. 52 the result of

$$\begin{aligned} y_{pn}(t) &= C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t} \qquad \Rightarrow \qquad y_{pN} = C_N \sin\left(\frac{pN\pi}{N+1}\right) \\ &= C_N \sin\left(p\pi - \frac{p\pi}{N+1}\right) \end{aligned}$$

so the position of each successive point p has an opposite sign.