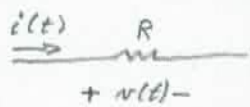
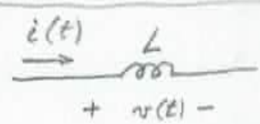
INTEGRODIFFERENTIAL EQUATIONS FOR CIRCUITS

We will now turn our attention to circuits wherein the sources are more general than constants for all time (DC) or sinusoids for all time (AC). Those sources will be general functions of time t that are Laplace transformable.
In this more general case, the voltages and currents are determined by integrodifferential equations written for $t > 0$.
Such equations are obtained from Kirchhoff's laws written in terms of the time-domain relations for electrical elements: The latter are:

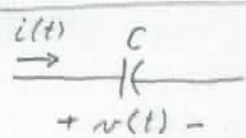
Resistors: 

$v(t) = R\, i(t)\,, \quad i(t) = \frac{1}{R} v(t)$

Inductors: 

$v(t) = L \frac{d}{dt} i(t)\,,$ Initial condition at $t = 0+$ is $i(0+)$

$$i(t) = \frac{1}{L} \int_0^t v(x)\,dx + i(0+)$$

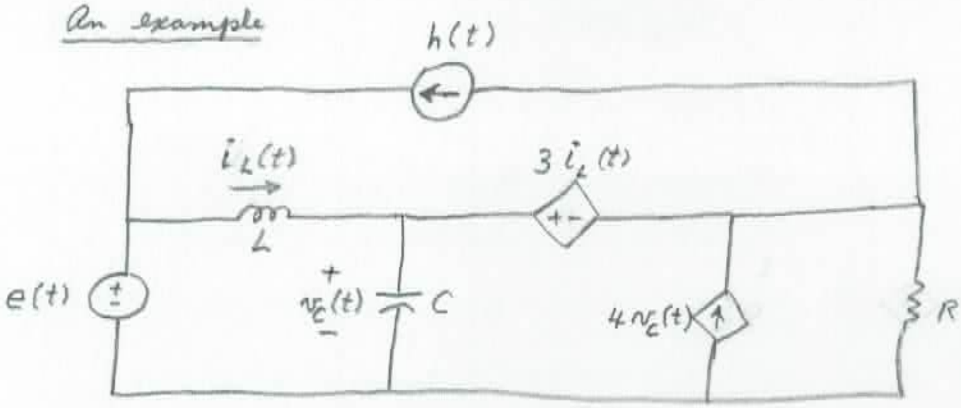
Capacitors: 

$i(t) = C \frac{d}{dt} v(t)\,,$ Initial condition at $t = 0+$ is $v(0+)$

$$v(t) = \frac{1}{C} \int_0^t i(x)\,dx + v(0+)$$

We will illustrate the writing of integrodifferential through an example.

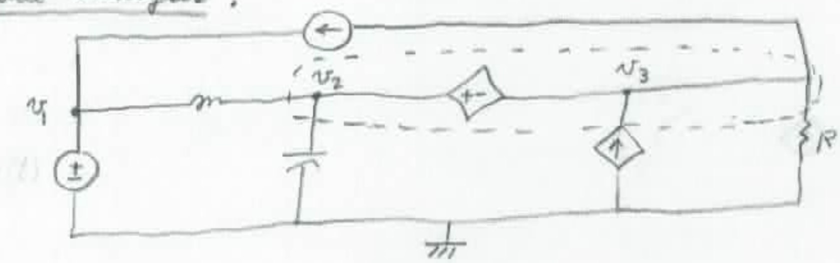
An example



All element values are given.
Also given are $i_L(0+)$ and $v_C(0+)$.

Write integrodifferential equation.

Nodal analysis :



v_1, v_2, v_3 are unknown node voltages that are functions of time t.

The node voltage $v_1(t)$ is equal to $e(t)$. (So we do not need a balloon around $e(t)$)

(1) $$v_1(t) = e(t)$$

We do need a balloon around the dependent voltage source.

Inside this balloon:

(2) $$v_2(t) - v_3(t) = 3\,i_L(t) = 3\left(\frac{1}{L}\int_0^t \left(v_1(x) - v_2(x)\right) dx + i_L(0+)\right)$$

KCL on this balloon:

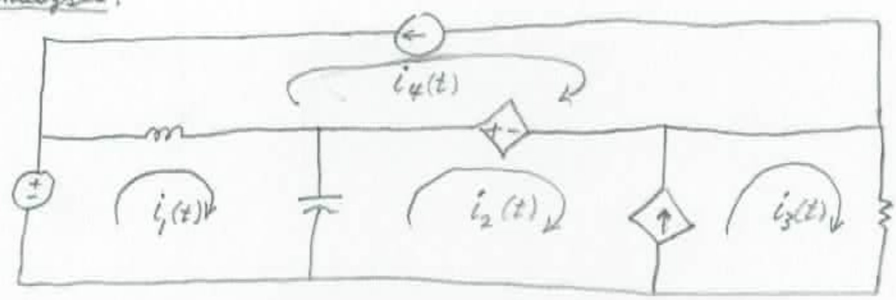
(3) $$\frac{1}{L}\int_0^t \left(v_2(x) - v_1(x)\right) dx - i_L(0+) + C\frac{d}{dt}v_2(t) - 4\,v_C(t) + \frac{v_3(t)}{R} + h(t) = 0$$

$$v_C(t) = v_2(t)$$

We solve these equations by using the Laplace transformation.

We know that $v_1(t) = e(t)$. After we determine $v_2(t)$ and $v_3(t)$, we can then get any other voltage or current in the circuit.

Mesh analysis:



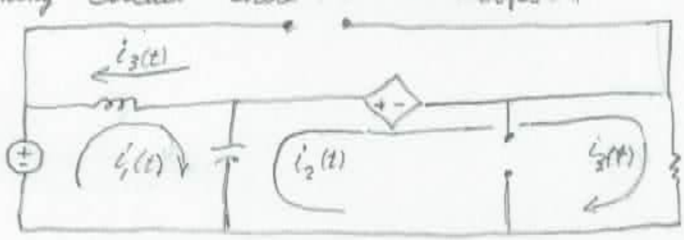
From the current sources we get two equations:

④ $$i_4(t) = -h(t)$$

⑤ $$i_3(t) - i_2(t) = 4\, v_c(t) = 4\left(\frac{1}{C}\int_0^t (i_1(x) - i_2(x))\,dx + v_c(0+)\right)$$

Now imagine that these two current sources are "erased". But, do not alter the currents going through the other elements.

In the "remaining circuit" there are two "loops":



KVL on these two loops give:

For i_1 loop: $$-e(t) + L\frac{d}{dt}\left(i_1(t) - i_3(t)\right) + \frac{1}{C}\int_0^t (i_1(x) - i_2(x))\,dx + v_c(0+) = 0 \quad ⑥$$

For i_2, i_3 loop: $$\frac{1}{C}\int_0^t (i_2(x) - i_1(x))\,dx - v_c(0+) + 3\left(i_1(t) - i_3(t)\right) + R\,i_3(t) = 0 \quad ⑦$$

By using the Laplace transformation, we can determine i_2, i_3, and i_4.
(i_1 is already determined by ④.)

After that is done, we can get any other voltage or current in the circuit.

We will now apply the Laplace transformation to these equations in order to observe how they might be solved.

Nodal analysis transforms

① transformed:

$$V_1(s) = E(s)$$

② transformed:

$$V_2(s) - V_3(s) = \frac{3}{L} \cdot \frac{V_1(s) - V_2(s)}{s} + 3\,\frac{i_L(0+)}{s}$$

③ transformed:

$$\frac{V_2(s) - V_1(s)}{Ls} - \frac{i_L(0+)}{s} + C\left(s V_2(s) - v_2(0+)\right) - 4 V_2(s) + \frac{V_3(s)}{R} + H(s) = 0$$

$$v_2(0+) = v_C(0+)$$

These equations may now be solved algebraically to get $V_2(s)$ and $V_3(s)$ as functions of s. ($V_1(s) = E(s)$ is already known.)
The next step is to determine the time functions $v_1(t)$, $v_2(t)$, and $v_3(t)$ corresponding to $V_1(s)$, $V_2(s)$, and $V_3(s)$ respectively. (We shall see how.)
Then any other voltage or current time function can be determined from $v_1(t)$, $v_2(t)$, and $v_3(t)$.

Mesh analysis

We now do a similar solving by applying the Laplace transformation to ④, ⑤, ⑥, and ⑦.

④ transformed: $I_4(s) = -H(s)$

⑤ transformed:

$$I_3(s) - I_2(s) = 4\,\frac{I_1(s) - I_2(s)}{Cs} + 4\,\frac{v_c(0+)}{s}$$

⑥ transformed:

$$-E(s) + Ls\left(I_1(s) - I_3(s)\right) - L\,i_2(0+) + \frac{I_1(s) - I_2(s)}{Cs} + \frac{v_c(0+)}{s} = 0$$

⑦ transformed:

$$\frac{I_2(s) - I_1(s)}{Cs} - \frac{v_c(0+)}{s} + 3\left(I_1(s) - I_3(s)\right) + R\,I_3(s) = 0$$

These equations can be solved <u>algebraically</u> to obtain $I_1(s)$, $I_2(s)$, and $I_3(s)$ as functions of s. ($I_4(s) = -H(s)$ is already known.)

Later on, we will discuss how to get $i_1(t)$, $i_2(t)$, $i_3(t)$, and $i_4(t)$ from these functions of s. Finally, any other time-dependent voltage or current (in the circuit) can be obtained from $i_1(t)$, $i_2(t)$, $i_3(t)$, and $i_4(t)$.