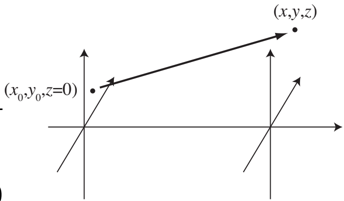
Adding up point sources

- Recall that we start with a wavefield at an input plane $(x_0, y_0, z = 0)$. We treat it as a bunch of Huygens point sources, each with magnitude and phase modulated by $\tilde{g}(x_0, y_0)$.
- To get the field at a downstream position (x, y, z), add up the contribution from the input plane sources:

$$\psi(x,y,z) = \psi_0 \frac{\lambda}{A} \int_{x_0} \int_{y_0} \tilde{g}(x_0, y_0) \frac{\exp[-ikr]}{r} \cos\theta. \qquad (1)$$

- The radius r is given by

$$\begin{aligned} r &= \sqrt{z^2 + (x - x_0)^2 + (y - y_0)^2} \\ &= z\sqrt{1 + \frac{(x - x_0)^2}{z^2} + \frac{(y - y_0)^2}{z^2}} \end{aligned} \qquad (2)$$



Fresnel approximation

We then did a Taylor series expansion of r from the form in Eq. 2 to obtain

$$r \simeq z \left[1 + \frac{(x-x_0)^2}{2z^2} + \frac{(y-y_0)^2}{2z^2} - \frac{(x-x_0)^4}{8z^4} - \frac{(y-y_0)^4}{8z^4} + \ldots \right] \quad (3)$$

The *Fresnel approximation* involved discarding terms like $(x-x_0)^4/(8z^4)$ in phase, or saying

$$z^3 \gg \rho^4/(2\lambda) \qquad \text{(Fresnel approximation)}$$

with $\rho = \sqrt{(x-x_0)^2 + (y-y_0)^2}$. We also said that $1/r \Rightarrow 1/z$ for the magnitude term.
This led to the Fresnel-Kirchoff diffraction integral:

$$\begin{aligned} \psi(x,y,z) &= \psi_0 \frac{\lambda}{z}\frac{1}{A} \exp\left[-i\frac{2\pi z}{\lambda}\right] \exp\left[-i\pi\frac{x^2+y^2}{\lambda z}\right] \\ &\qquad \int_{x_0}\int_{y_0} \tilde{g}(x_0,y_0) \exp\left[-i\pi\frac{x_0^2+y_0^2}{\lambda z}\right] \exp\left[i2\pi\frac{xx_0+yy_0}{\lambda z}\right] \end{aligned} \quad (4)$$

Fraunhofer approximation

We also considered the *Fraunhofer approximation*, which assumes the Fresnel approximation plus saying that $(x_0^2 + y_0^2)/(\lambda z) \ll 1$ or

$$z \gg 4\frac{x_0^2 + y_0^2}{\lambda} \qquad \text{(Fraunhofer approximation)}$$

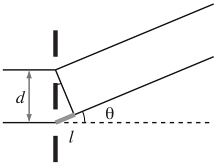
This (plus disregarding the out-of-integral phase factors, since they cancel out when calculating intensities) leads to the Fraunhofer diffraction integral:

$$\psi(x, y, z) \simeq \psi_0 \frac{\lambda}{z}\frac{1}{A}\int_{x_0}\int_{y_0} \tilde{g}(x_0, y_0) \exp\left[i2\pi\left(\frac{xx_0}{\lambda z} + \frac{yy_0}{\lambda z}\right)\right] dx_0\, dy_0 \quad (5)$$

We then asked how this might look familiar.

Diffraction from slits

- Before we reveal the secret of Eq. 5, let's remind ourselves of diffraction from some slits with spacing d. We have constructive interference when the optical path length difference l is equal to $n\lambda$, or $l = d\sin\theta_x = n\lambda$.
- In the small angle approximation $\sin\theta_x \simeq \theta_x$, and in first diffraction order with $n = 1$, we have $d\theta_x \simeq \lambda$ or

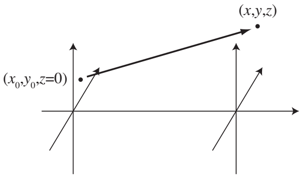


$$f_x \equiv \frac{\theta_x}{\lambda} \simeq \frac{1}{d} \qquad (6)$$

- That is, if we have a particular grating with period d, light is diffracted to (the first constructive interference intensity max is at) a wavelength-normalized angle of $f_x \equiv \theta_x/\lambda$. We can think of f_x as being a characteristic of the object. If we were to lay two sets of slits with periods d_1 and d_2 on top of each other, we would expect to have light diffracted to two wavelength-normalized angles $f_{x,1}$ and $f_{x,2}$.

Wavelength-normalized angles

- We have found that we can characterize an object in terms of its inverse periodicities, or $1/d$, by referring to a wavelength-normalized diffraction angle of $f_x \equiv \theta_x/\lambda$.
- Consider again the terms like $xx_0/(\lambda z)$ in the Fresnel (Eq. 4) and Fraunhofer (Eq. 5) diffraction integrals:



- From the above diagram, we can see that an angle θ_x in the $\hat{x}$ plane can be calculated from

$$\begin{aligned} \frac{(x-x_0)}{z} \simeq \frac{x}{z} &\simeq \theta_x \simeq f_x\lambda \\ \text{or} \quad \frac{x}{\lambda z} &\simeq f_x \end{aligned} \tag{7}$$

- Therefore we can write terms like $xx_0/(\lambda z)$ as $f_x x_0$, where we have made use of our wavelength-normalized scattering angle f_x which depends on the object (the grating).

Re-writing the Fraunhofer integral

We can now re-write the Fresnel-Kirchoff diffraction integral in the Fraunhofer approximation (Eq. 5) as

$$\begin{aligned}\psi(x,y,z) &\simeq \psi_0 \frac{\lambda}{z}\frac{1}{A}\int_{x_0}\int_{y_0} \tilde{g}(x_0,y_0)\exp\left[i2\pi\left(\frac{xx_0}{\lambda z}+\frac{yy_0}{\lambda z}\right)\right] dx_0\, dy_0 \\ &\simeq \psi_0 \frac{\lambda}{z}\frac{1}{A}\int_{x_0}\int_{y_0} \tilde{g}(x_0,y_0)\exp\left[i2\pi\left(f_x x_0+f_y y_0\right)\right] dx_0\, dy_0 \qquad (8)\end{aligned}$$

If you didn't already recognize it, this is a Fourier transform! If we were to go between a signal $g(t)$ as a function of time t, and its representation $G(f)$ in the frequency domain f, we would write it as

$$G(f) = \int_{-\infty}^{+\infty} g(t)e^{i2\pi ft}\, dt \qquad (9)$$

$$\text{Inverse:} \quad g(t) = \int_{-\infty}^{+\infty} G(f)e^{-i2\pi ft}\, df \qquad (10)$$

Based on this, our wavelength-normalized diffraction angles f_x and f_y are called *spatial frequencies*.

Fourier transforms

- The conditions for Fourier transforms to be finite and invertible are as follows:
  1. The integral of $|g(x)|$ from $-\infty$ to ∞ exists
  2. Any discontinuities in $g(x)$ are finite
  3. The function $g(x)$ must have bounded variation. This is essentially a statement that $g(x)$ must not go on forever, because that would imply infinite energy in a physical situation.

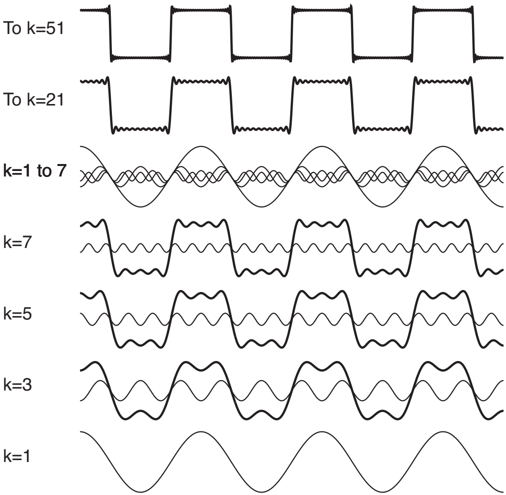
  These conditions are easily satisfied in optics; we never have infinite objects, or objects which have infinite changes in their transmission over infinitessimal distances!
- We know that $e^{i2\pi ft} = e^{i\omega t}$ represents a sine wave with frequency f. How do we represent other functions besides sine waves? By combination of many sine waves, as we shall see next.

Fourier decomposition

Let's do the Fourier decomposition of a square wave:

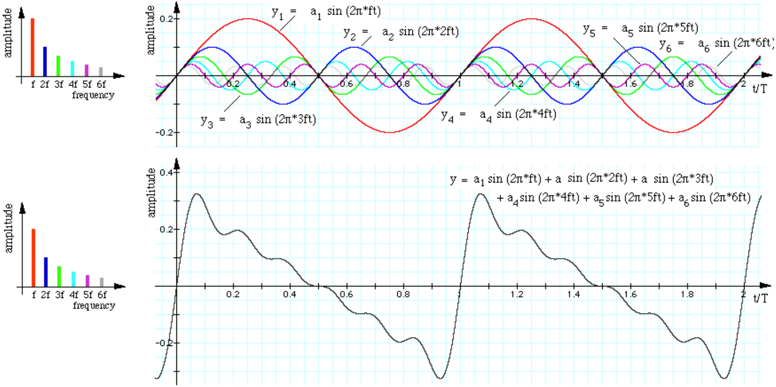
$$\sum_{k=1,3,5,\ldots} \frac{4}{k\pi} \sin kx \qquad (11)$$

Shown from bottom to top are the first seven terms in the series (thin lines) and their sum (thick lines), the $k = 1$–7 terms plotted on top of each other, and the series with terms up to $k = 21$ and $k = 51$.



Sawtooth waveform

From http://www.phys.unsw.edu.au/music/sound.spectrum.html:



Hearing Fourier transforms

- It's very useful to think of music and the *timbre* of musical instruments when thinking of Fourier transforms. The examples from the next few pages are drawn from an excellent web page on musical acoustics available from the University of New South Wales in Australia:

  http://www.phys.unsw.edu.au/music/basics.html

  Listen to the sawtooth wave being made up of its harmonics here and here.
- You'll see relative intensities expressed in decibels or dB on the pages that follow. When used as a relative scale between two intensities,

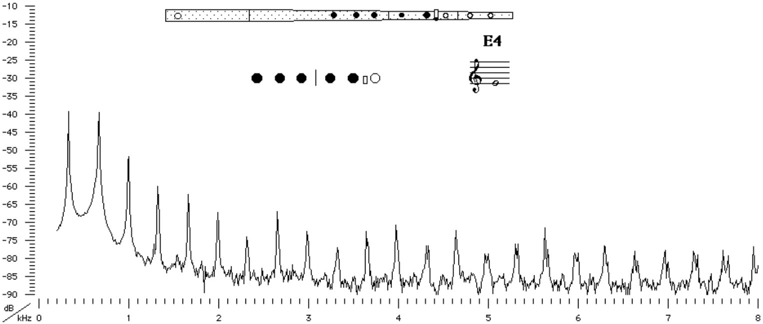
$$\mathrm{dB} = 10\log_{10}\left(\frac{I_1}{I_2}\right) \tag{12}$$

  so 1/10$^{\mathrm{th}}$ the intensity translates into -10 dB, 1/100$^{\mathrm{th}}$ the intensity translates into -20 dB, and so on. When used on an absolute scale, intensities are related to the threshold for human hearing which is 0 dB corresponding to 10^{-12} W/m^2. For info on decibels and hearing loss, see this web page:

  http://www.dangerousdecibels.org/hearingloss.cfm.

Flute

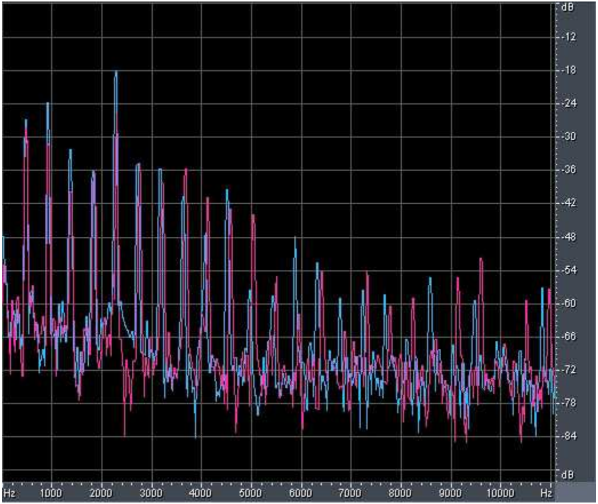
From http://www.phys.unsw.edu.au/music/flute/classicalC/E4.html:



Listen in here.

Violin

The violin body has some complicated resonances. Vibrato also plays a role, giving different frequency responses at different times.



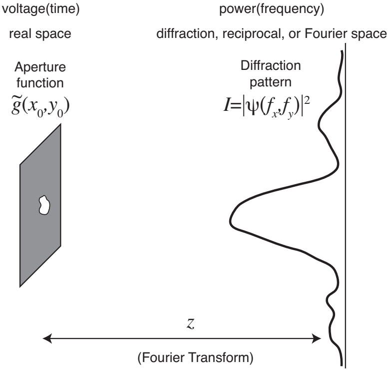
Listen in here.

Franhofer diffraction as a Fourier transform

Again, the Franhofer diffraction integral is given by Eq. 5:

$$\psi(f_x, f_y) \simeq \frac{\lambda}{z}\frac{1}{A}\int_{x_0}\int_{y_0} \tilde{g}(x_0, y_0) e^{i2\pi(f_x x_0 + f_y y_0)}\, dx_0\, dy_0$$

where $f_x = \theta_x/\lambda = x/(\lambda z)$. The far-field diffraction pattern is just a Fourier transform of the object $\tilde{g}(x_0, y_0)$.



More on Fourier transforms

- Let's change to a more compact notation:

$$G(f) = \mathcal{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x)\exp[i2\pi fx]dx \qquad (13)$$

$$g(x) = \mathcal{F}^{-1}\{G(f)\} = \int_{-\infty}^{\infty} G(f)\exp[-i2\pi fx]df. \qquad (14)$$

- The shift theorem says that if you shift an object sideways in the input plane (x_0, y_0), only the phase of the diffraction pattern changes:

$$\mathcal{F}\{g(x-a)\} = G(f)\exp[i2\pi af]. \qquad (15)$$

Since we usually look at only the intensity $I = \psi\psi^\dagger$ of the diffraction pattern, we usually can't tell if the sample has been shifted sideways.

And yet more. . .

- The similarity theorem says that if you squish an object, you expand its diffraction pattern:

$$\mathcal{F}\{g(ax)\} = \frac{G(f/a)}{|a|}. \tag{16}$$

- The addition theorem lets you add together two Fourier transforms:

$$\mathcal{F}\{g(x) + h(x)\} = G(f) + H(f). \tag{17}$$

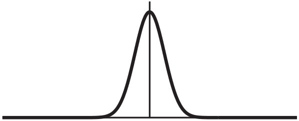
- Rayleigh's theorem can really be thought of as a statement of the conservation of energy:

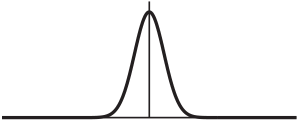
$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |G(f)|^2 df. \tag{18}$$

Fourier transform pairs

Let's look at some Fourier transform pairs. The transform of a Gaussian is a Gaussian; narrow in space goes to broad in spatial frequency, and vice versa.

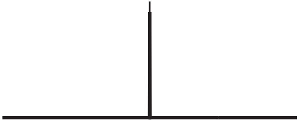
$$\int_{-\infty}^{\infty} \exp[-\pi x^2] \exp[i2\pi f x] dx = \exp[-\pi f^2] \qquad (19)$$

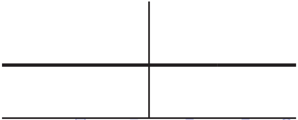




An impulse function contains all frequencies:

$$\int_{-\infty}^{\infty} \delta(x) \exp[i2\pi f x] dx = 1 \qquad (20)$$

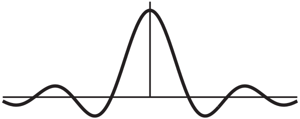


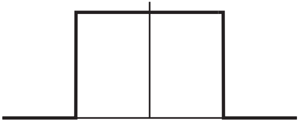


The kitchen sinc

A rect function $\Pi(f)$ gives a $\sin(x)/x$ or sinc() function and vice versa:

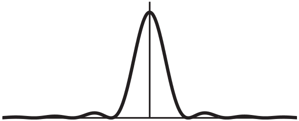
$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} \exp[i2\pi f x] dx = \Pi(f) \tag{21}$$

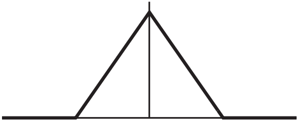




A triangle function $\Lambda(f)$ gives a $\text{sinc}^2()$ function and vice versa:

$$\int_{-\infty}^{\infty} \frac{\sin^2 \pi x}{(\pi x)^2} \exp[i2\pi f x] dx = \Lambda(f) \tag{22}$$



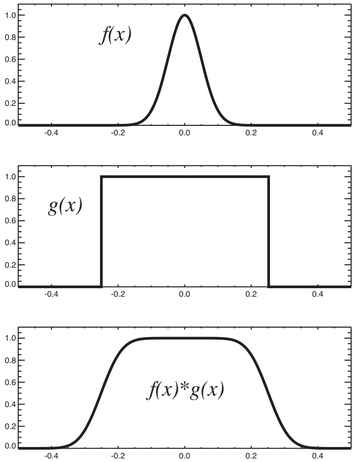


Convolution

In some ways, the most important theorem for us is the convolution theorem. Convolution can be thought of as a shift-and-multiply operation; for example, think of sliding a slit along a light table with some kind of pattern or picture underneath and simply adding up all the light getting through the slit. Mathematically, the convolution of $g(u)$ with $h(u)$ has a result $k(x)$ given by

$$\begin{aligned} k(x) &= \int_{-\infty}^{\infty} g(u)\, h(x-u)\, du \\ &= g(x) * h(x) \end{aligned} \tag{23}$$

where $*$ denotes convolution.



Convolution II

- What the convolution theorem tells us is that the Fourier transformation of the convolution of two functions is equal to the product of the Fourier transforms of the two functions, or

$$\mathcal{F}\{g(x) * h(x)\} = G(f)\,H(f). \tag{24}$$

- This is incredibly powerful because of a peculiar fact about Fourier transforms: the speed with which they can be computed. Consider the convolution integral of Eq. 23:

$$k(x) = \int_{-\infty}^{\infty} g(u)\,h(x-u)\,du$$

  If we have N points along x, and N points along u, then we have to do N calculations for each x point or N^2 calculations. The same applies to a brute-force Fourier transform with discrete sampling

$$G(f) = \mathcal{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x)\exp[i2\pi fx]dx$$

  Again, integration over N points in x for each of N points in f, giving N^2 operations.

The FFT algorithm

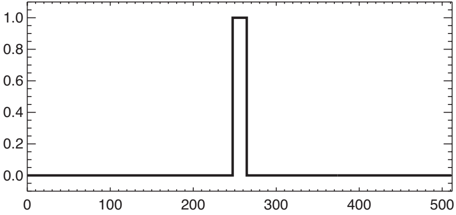
- Fortunately there is something called the Fast Fourier Transform or FFT algorithm. Consider the discrete version of the continuous integral, where $j = 1 \ldots N$:

$$\begin{aligned} G(\Delta f \cdot j) \quad = \quad & \sum_{i=1\ldots N} g(\Delta x \cdot i) \Big[\cos\big(2\pi(\Delta f \cdot j) \cdot (\Delta x \cdot i)\big) \\ & + i \sin\big(2\pi(\Delta f \cdot j) \cdot (\Delta x \cdot i)\big) \Big] \end{aligned} \tag{25}$$

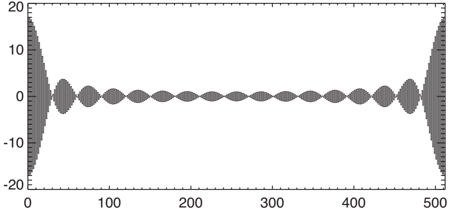
- When N is a multiple of 4, one ends up having certain easy-to-calculate results such as $\sin(0)$, $\sin(\pi/2)$, $\sin(\pi)$, and so on.
- This ends up making it much faster to calculate discrete Fourier transforms, so that it ends up taking about $N \log_2(N)$ calculations instead of N^2. (For details, see Chapter 12 of the book *Numerical Recipes* by Press *et al.*).
- Good and clever FFT algorithms can either work radix-2, or radix-3, or radix-5, or any other integer factor. One of the best implementations out there is FFTW, for Fastest Fourier Transform in the West (`www.fftw.org`).

Let's start playing!

Let's get a better feel for Fourier transforms by playing first with some FFTs. Consider a rect function $\Pi(x)$: $\Pi(x)$:



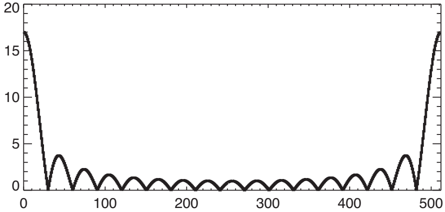
Let's pop it into an FFT algorithm and plot the real part:



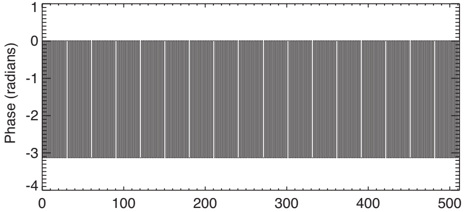
What the #&%?!

What's up with the FFT result?

Well, let's look at the magnitude:



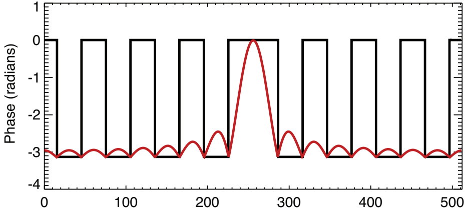
The magnitude makes sense if you swap the left and right halves! What about the phase?



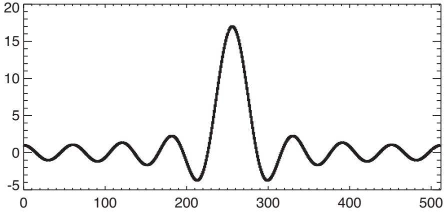
What the #&%?!

Shifty FFTs

If the output is shifted by half the width, what about shifting the input by half the width and also shifting the output?



Aha! This looks more like what we expect! Let's now look at the real part with this shift:



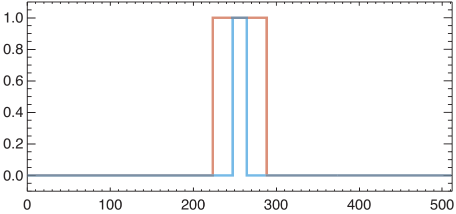
Right: $e^{-i\pi} = -1$, so this makes sense.

The shift theorem in action

- Why did we need to shift the input by half an array width? Recall the shift theorem of Eq. 15: $\mathcal{F}\{g(x-a)\} = G(f)\exp[i2\pi af]$. You can see (homework) that a shift in the center position leads to a linear phase ramp in the Fourier transform plane.
- The name of the game is that you have to be a bit careful when doing numerical Fourier transforms! There are a few other subtleties that we shall talk about in connection with 2D Fourier transforms.

Slits big and small

Recall the similarity theorem of Eq. 16: $\mathcal{F}\{g(ax)\} = \dfrac{G(f/a)}{|a|}$. Now look at slits big and small:



And their transforms: