Photons? Waves? Rays?

- You've had PHY 251, so you know all about photons from Planck and Einstein. Light comes in discrete packets of energy $E = hc/\lambda$ and momentum $p = Ec$.
- But then we stepped back a few decades, and learned from Maxwell that light can be described in terms of electromagnetic waves.
- But if we step back even further, physicists described light propagation in terms of ray optics. What's with *that*?
- How do waves travel in straight lines? The answer: they only kinda sorta do.
- But before we address this question, let's consider some issues of adding up waves.

Adding waves

- Back in Lecture 2, we learned a few things about adding waves up. A movie to illustrate how phase affects wave addition is here.
- If we add N waves $|A|e^{i\omega+\theta_0}$ with identical magnitudes $|A|$ and frequencies ω but random phase offsets θ_0 (*incoherent* superposition), we found a net or resultant magnitude $|R|$ of

$$|R|_{\text{incoherent}} = \sqrt{N \cdot A_i^2} = \sqrt{N}|A|. \tag{1}$$

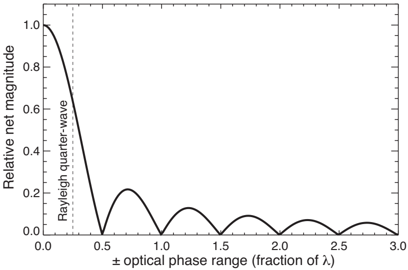
- If all the waves have identical frequencies and identical phase offsets (*coherent* superposition), we found

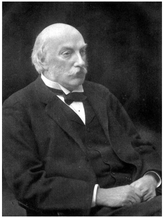
$$|R|_{\text{coherent}} = \sqrt{R^2} = \sqrt{N^2A^2} = N|A|. \tag{2}$$

- What about intermediate cases, where the phases are not completely random but are not absolutely identical either?

Adding waves II

- Let's now consider the addition of waves where the phase is neither uniform, nor completely random.
- Instead, let's consider phases that vary uniformly about zero over a uniform range.
- How is the sum of many waves affected by the $\pm$ range of starting phases? See at right.
- We see that if the phase is within about $\pm\lambda/4$, the net effect is only moderately reduced. This is known as the *Rayleigh quarter wave criterion*.

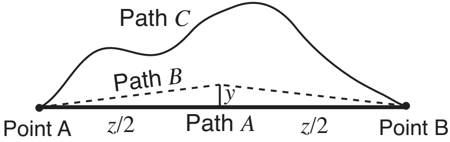




John William Strutt, 3rd Baron Rayleigh (1842–1919)

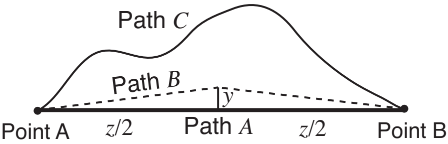
Rays and Fermat's principle

- Consider light traveling from point A to point B. If we consider point A to be emitting spherical waves, why do we draw a straight line ray path to point B?



- Let's think about two simple cases: a wave traveling a distance z along a straight line (path A), and a wave traveling through a point y off of the straight line (path B).
- In each case the wave propagates according to $e^{-in\vec{k}\cdot\vec{x}}$, where $\vec{k}$ diverges to all directions in the case of a spherical wave, while $\vec{x}$ describes the vector path.

Rays and Fermat II

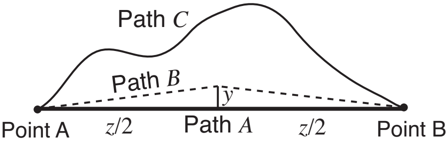


- Again, we're considering the straight line path A versus the off-axis path B in $e^{-in\vec{k}\cdot\vec{x}}$.
- Let's handle the path by saying that we have spherical waves (all directions in $\vec{k}$) and a distance ℓ along a particular path. That is, $n\vec{k}\cdot\vec{x} \Rightarrow nk\ell$ where ℓ measures the distance along a path and n is the refractive index, telling us the speed of wave front propagation.
- We refer to $n\ell$ as the *optical path length*, and $nk\ell$ as the accumulated optical phase.
- For the straight line path A, the distance traveled is $\ell_A = z$.
- For the off-axis path B, the distance traveled is

$$\ell_B = 2\sqrt{(\frac{z}{2})^2 + y^2} = z\sqrt{1 + (\frac{2y}{z})^2} \simeq z\left(1 + 2(\frac{y}{z})^2\right)$$

where we have made use of the binomial expansion.

Rays and Fermat III

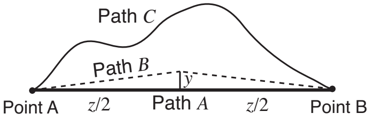


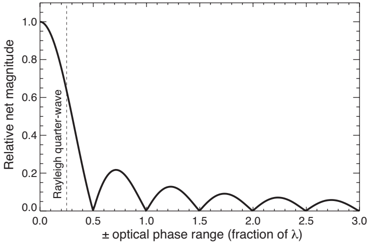
- Again, we're considering the straight line path A versus the off-axis path B in $e^{-in\vec{k}\cdot\vec{x}}$.
- Path A has an optical path length of $n\ell_A = nz$.
- Path B has an optical path length of $n\ell_B \simeq nz(1+2y^2/z^2)$
- These paths, plus those "bounded inside," reinforce each other if their net optical path lengths are within $\pm\lambda/4$:

$$\begin{aligned} \ell_B - \ell_A \simeq z\left(1 + 2\left(\frac{y}{z}\right)^2\right) - z &\leq \frac{\lambda}{4} \\ \frac{2y^2}{z} &\leq \frac{\lambda}{4} \\ |y| &\leq \sqrt{\frac{\lambda z}{8}} \end{aligned} \tag{3}$$

Rays and Fermat IV

- From Eq. 3 we found that paths with $|y| \leq \sqrt{\lambda z/8}$ satisfy the *Rayleigh quarter wave criterion*: they add up nearly coherently.
- Consider $\lambda = 532$ nm (green laser pointer), and $z = 5$ meters: this says $|y| \leq \sqrt{(532 \times 10^{-9})(5)/8}$ or $|y| \leq 0.58$ mm.
- So waves that travel within a millimeter of the straight line path satisfy the quarter-wave criterion, and add up nearly coherently.
- With $z = 1$ mm, we have $|y| \leq 8\ \mu$m.

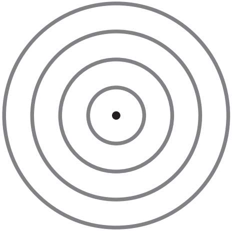




Rays and Fermat V

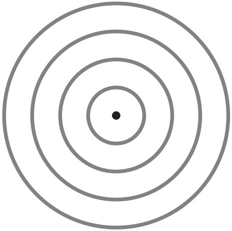
- By the way, the Rayleigh quarter wave criterion is related to the method of stationary phase in complex integrals.
- Also, we'll see later on that we can do the "adding up waves" calculation using Fourier transforms, and find that the pattern we calculated is equivalent to the diffraction pattern of a rectangular aperture: $[\sin(x)/x]^2$.
- And finally, what we have shown is Fermat's principle, which is that light travels the path of least time (because that's the minimum optical path length $n\ell$ or minimum phase $nk\ell$ path, which does not get canceled out by other paths).

The Huygens construction

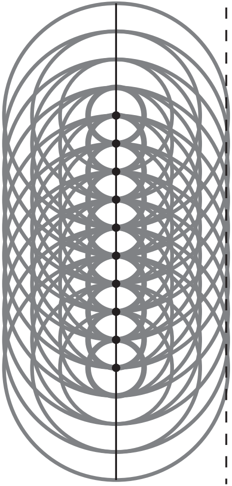


Christiaan Huygens (1629-1695)

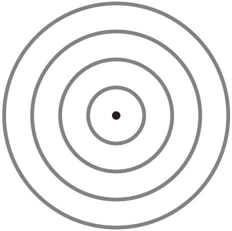
The Huygens construction



Christiaan Huygens (1629-1695)



The Huygens construction



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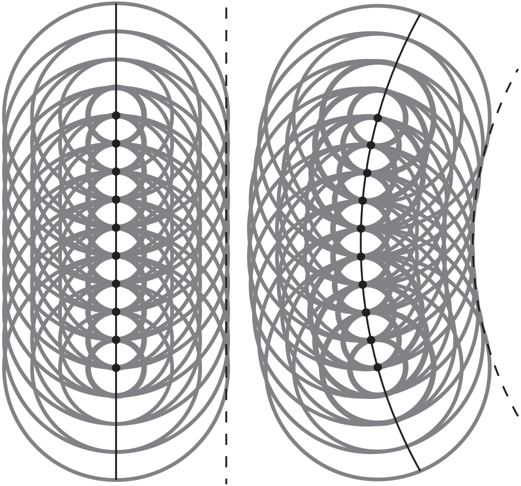
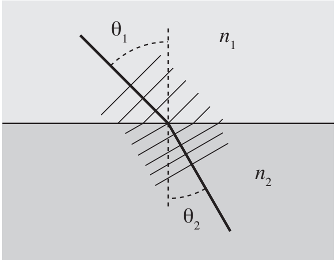


Huygens II

- The Huygens construction shows us that we can make a plane wave from lots of little spherical wave point sources.
- A plane wave travels in a particular direction, like a ray does. . .
- So we have yet another way of justifying light traveling as rays!

Snell's law

- Apply Fermat's principle to a ray reaching an interface between two different refractive indices n_1 and n_2.
- This was done by several people, including Ibn Sahl in 984, Thomas Harriot in 1602, the Dutch mathematician Willebrord Snellius in 1621, René Descartes in 1637... Most of them before Fermat's principle!
- We know it today as Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.



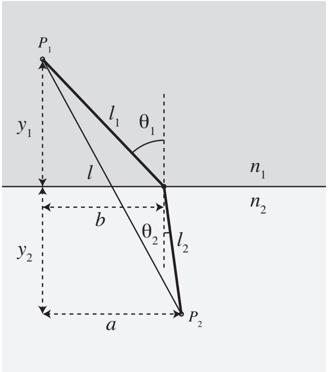
We use a darker shading for n_2 to indicate $n_2 > n_1$.

Deriving Snell's law

We have shown that light travels according to the principal of stationary phase, or least time. Let's consider how light goes from point P_1 to P_2 as it travels across a refractive boundary. The time is distance over velocity, or

$$t = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{l_1 n_1}{c} + \frac{l_2 n_2}{c} \qquad (4)$$

where we have used the phase velocity $v_p = c/n$. Light will travel along a straight line within one medium, but what about at an interface? What value of b minimizes the travel time/produces a path of stationary phase?



Snell's law II

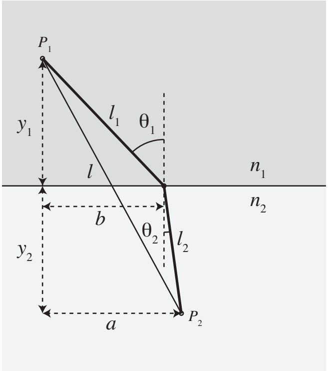
To find the path of least time, let's write the lengths l_1 and l_2 in terms of b:

$$l_1 = \sqrt{y_1^2 + b^2} \qquad l_2 = \sqrt{y_2^2 + (a-b)^2}$$

The travel time of Eq. 4 then becomes

$$t = \frac{n_1[y_1^2 + b^2]^{1/2}}{c} + \frac{n_2[y_2^2 + (a-b)^2]^{1/2}}{c} \tag{5}$$

To minimize the time, let's set the derivative as we vary b equal to zero: $dt/db = 0$.



Snell's law III

Again, we want to minimize the time of Eq. 5 of

$$t = \frac{n_1[y_1^2 + b^2]^{1/2}}{c} + \frac{n_2[y_2^2 + (a-b)^2]^{1/2}}{c}$$

by setting $dt/db = 0$:

$$\begin{aligned} \frac{dt}{db} = 0 &= \frac{1}{2}\frac{n_1[y_1^2 + b^2]^{-1/2}}{c}2b + \frac{1}{2}\frac{n_1[y_2^2 + (a-b)^2]^{-1/2}}{c}2(a-b)(-1) \\ 0 &= n_1\frac{b}{\sqrt{y_1^2 + b^2}} - n_2\frac{a-b}{\sqrt{y_2^2 + (a-b)^2}} \end{aligned} \quad (6)$$

Snell's law IV

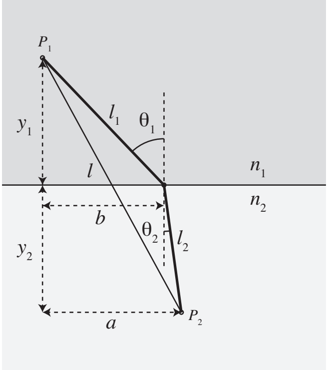
Again our condition for minimizing the time was given in Eq. 6:

$$0 = n_1 \frac{b}{\sqrt{y_1^2 + b^2}} - n_2 \frac{a-b}{\sqrt{y_2^2 + (a-b)^2}}$$

If we consult again our original diagram, we see that we can also express this as

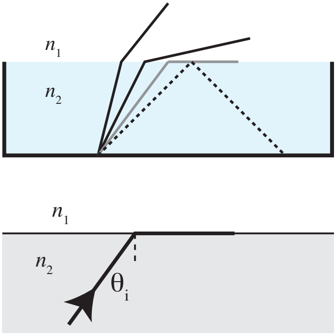
$$\begin{aligned} 0 &= n_1 \sin\theta_1 - n_2 \sin\theta_2 \\ \text{giving } n_1 \sin\theta_1 &= n_2 \sin\theta_2 \qquad (7) \end{aligned}$$

so we have proved Snell's law.



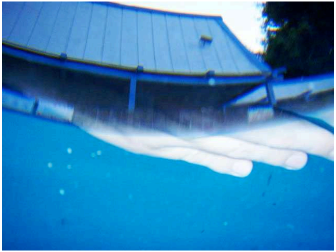
TIR: total internal reflection

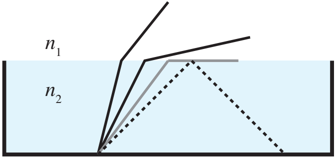
- Again, with $n_2 > n_1$, we can get total internal reflection.
- As θ_2 increases, θ_1 goes to 90° and then you get total internal reflection!
- Known as the critical angle θ_c: $n_1 \sin 90^\circ = n_2 \sin \theta_c$, or $\sin \theta_c = n_1 / n_2$.



TIR for swimming pools

- Remember Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.
- Sit at the bottom of a swimming pool (water: $n_2 = 1.33$) and look up at the sky, where $n_1 = 1$. That is, $n_2 > n_1$.
- As θ_2 increases, θ_1 goes to 90° and then you get total internal reflection!

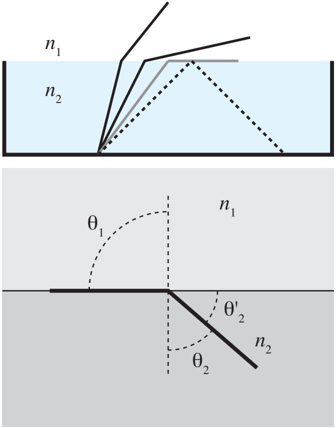




Pool photo: http://lifshitz.ucdavis.edu/~dmartin/phy7/7C/Refraction/Refraction.html

TIR for X rays

- Reminder: we get Total Internal Reflection (TIR) within n_2 at a critical angle $\sin\theta_c = n_1/n_2$ when $n_2 > n_1$.
- But for X rays $n = 1 - \delta - i\beta$, or just $n = 1 - \delta$ if we ignore absorption.
- Therefore we have $n_2 > n_1$ if $n_2 = 1$ for vacuum/air, and $n_1 = 1 - \delta$ for a medium.
- That is, external reflection for X rays is really *total internal reflection*!
- The critical angle is $\sin\theta_c = 1 - \delta$ with θ_c from the surface normal, or $\cos\theta_c' = 1 - \delta$ from the surface.

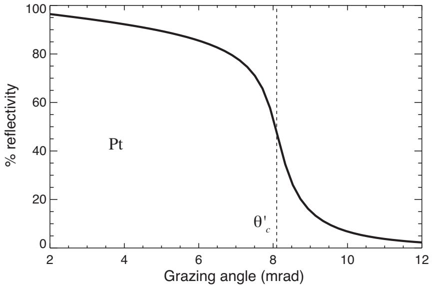


X-ray critical angle θ'_c

- Again, for X rays we have $\cos\theta'_c = 1 - \delta$ with θ'_c measured from the surface.
- Since δ is small, let's use an approximation:

$$1 - \frac{(\theta'_c)^2}{2} \simeq \cos\theta'_c = 1 - \delta \qquad \Rightarrow \qquad \theta'_c \simeq \sqrt{2\delta} = \lambda\sqrt{2\alpha f_1}$$

- Since δ is small, X rays only reflect at small grazing angles!

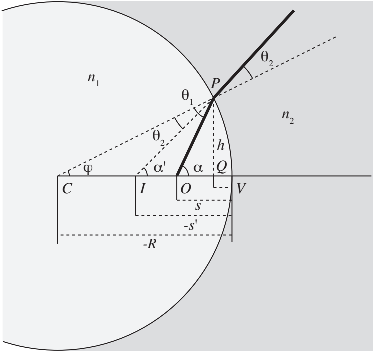


Refraction at a curved interface

Consider refraction at an interface with a radius of curvature $-R$. We'll follow a ray from the point O as it travels at some angle α and hits the interface at a point P. Now Snell's law tells us

$$\begin{aligned} n_1 \sin\theta_1 &= n_2 \sin\theta_2 \\ n_1\theta_1 &\simeq n_2\theta_2 \\ n_1(\alpha-\varphi) &\simeq n_2(\alpha'-\varphi) \qquad (8) \end{aligned}$$

where we have used the small angle approximation. We get the relationship $\theta_1 = \alpha - \varphi$ from considering the triangles PCQ, CPO, and POQ. We get the relationship $\theta_2 = \alpha' - \varphi$ from considering the triangles PCQ, CPI, and PIQ.



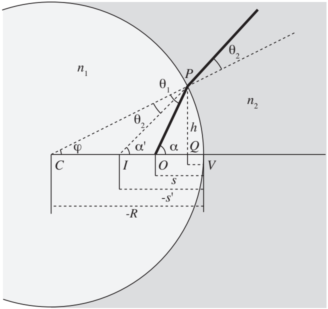
Refraction II

Again, we had from Eq. 8 the relationship

$$n_1(\alpha - \varphi) = n_2(\alpha' - \varphi)$$

in the small angle approximation. Now let's express these angles in terms of distances on the diagram, where we ignore the distance QV:

$$\begin{aligned} n_1\left(\frac{h}{s} - \frac{h}{-R}\right) &= n_2\left(\frac{h}{-s'} - \frac{h}{-R}\right) \\ \frac{n_1}{s} + \frac{n_2}{s'} &= \frac{n_2 - n_1}{R} \end{aligned} \qquad (9)$$



which is Fowles Eq. 10.4. Notice that we defined the distances s' and R to be negative in anticipation of what will follow. This is a general relationship for refraction at a curved interface.

Apparent depth

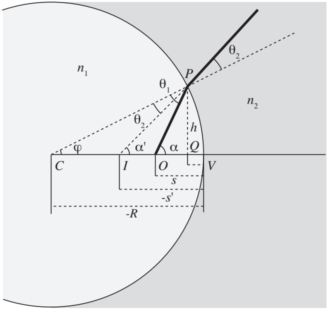
Let's consider some consequences of the expression of Eq. 9 of

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}.$$

First of all, consider standing on the side of n_2 and looking at an object at point O within n_1, and letting $R \to \infty$ for a flat interface. As you look at the object and as your brain assumes that light travels in straight lines, you will think that the object is really located at point I. That is, the apparent depth $-s'$ of the object will be at

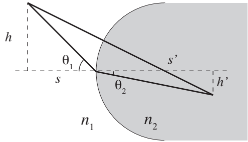
$$\begin{aligned} \frac{n_1}{s} &= -\frac{n_2}{s'} \\ s' &= -\frac{n_2}{n_1} s \end{aligned} \qquad (10)$$

so if you're trying to spear a fish you will miss!



Magnification

Now let's consider multiple rays from one point to another, hitting a convex refractive surface with positive R. One ray hits the spherical surface at $\theta_1 = 0$ so that $\theta_2 = 0$. Another ray goes through the centerline of the lens so that the angle θ_1 from the source to the refractive interface is equal (in the small angle approximation) to h/s. The two rays cross at a point s' on the other side of the surface (hence our choice of sign convention earlier for s'). The second ray reaches that point at an angle $\theta_2 = h'/s'$. From Snell's law we have



$$n_1 \frac{h}{s} = n_2 \frac{h'}{s'}$$

$$\text{giving} \qquad m \equiv \frac{h'}{h} = -\frac{n_1 s'}{n_2 s} \tag{11}$$

where the negative sign accounts for the fact that the object is inverted.

Curved refractive surfaces

Consider two refractive surfaces. For the first surface we have

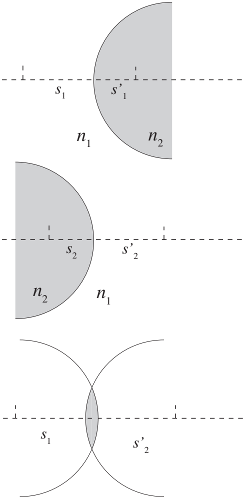
$$\frac{n_1}{s_1} + \frac{n_2}{s_1'} = \frac{n_2 - n_1}{R_1} \tag{12}$$

Consider a second surface immediately following. For it, we have the opposite ordering for n_1 and n_2, and we have the opposite radius of curvature, giving

$$\frac{n_2}{s_2} + \frac{n_1}{s_2'} = \frac{n_1 - n_2}{-R_2} \tag{13}$$

Here's the key: if the lens is truly thin, then

$$s_1' = -s_2. \tag{14}$$



Two surfaces II

We can rewrite our expression of Eq. 12 for the first refractive interface as

$$\frac{1}{s_1'} = \frac{n_2 - n_1}{n_2}\frac{1}{R_1} - \frac{n_1}{n_2}\frac{1}{s_1} \tag{15}$$

and our expression of Eq. 13 for the second refractive interface as

$$\frac{1}{s_2} = \frac{n_1 - n_2}{n_2}\frac{1}{R_2} - \frac{n_1}{n_2}\frac{1}{s_2'} \tag{16}$$

Now let's use the result of Eq. 14 of $s_1' = -s_2$ to combine these two:

$$\begin{aligned}
\frac{n_1 - n_2}{n_2}\frac{1}{R_1} + \frac{n_1}{n_2}\frac{1}{s_1} &= \frac{n_1 - n_2}{n_2}\frac{1}{R_2} - \frac{n_1}{n_2}\frac{1}{s_2'} \\
\frac{n_1}{n_2}\left(\frac{1}{s_1} + \frac{1}{s_2'}\right) &= \frac{n_1 - n_2}{n_2}\left(-\frac{1}{R_1} + \frac{1}{R_2}\right) \\
\frac{1}{s_1} + \frac{1}{s_2'} &= \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \equiv \frac{1}{f} \qquad \text{(Fowles 10.7)}
\end{aligned}$$

Two surfaces III

For the overall optical system, we'll write s for s_1 and s' for s_2'. Therefore we have derived the lensmaker's equation in Eq. 17:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \qquad \text{with} \qquad \frac{1}{f} \equiv \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

This allows us to calculate the focal length of a lens with any set of spherical surfaces. This equation assumes $+R$ for centers of curvature located "downstream" of the lens, so for a double-convex lens we have $R_1 = +|R_1|$ and $R_2 = -|R_2|$.