THE LAPLACE TRANSFORMATION $\mathcal{L}$

Let $f(t)$ be a Riemann-integrable function defined for $0 < t < \infty$.

Its Laplace transform is

(1) $\mathcal{L} f(t) = F(s) = \int_0^\infty f(t) e^{-st} dt$ ← where $s = \sigma + j\omega$ is a complex variable (σ and ω are real)

so long as this improper integral converges absolutely for some s.

If the integral converges for some particular s_0, it will converge for all s such that $\operatorname{Re} s > \operatorname{Re} s_0$ because

$$e^{-st} = e^{-\sigma t} e^{-j\omega t} = e^{-\sigma t}(\cos \omega t - j \sin \omega t)$$

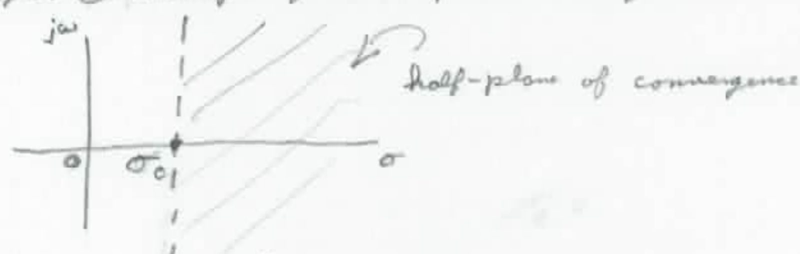
and thus $|e^{-st}| = e^{-\sigma t}$.

Consequently, $\int_0^\infty |f(t)| |e^{-st}| dt = \int_0^\infty |f(t)| e^{-\sigma t} dt < \int_0^\infty |f(t)| e^{-\sigma_0 t} dt < \infty$

($\sigma_0 = \operatorname{Re} s_0$) ← assumed convergence at $s = s_0$

In general, there will be a half-plane $\{s : \operatorname{Re} s > \sigma_c\}$ wherein the integral (1) converges for the Laplace-transformable $f(t)$.

s-plane:



Once $F(s)$ is so defined for $\{s : \operatorname{Re} s > \sigma_c\}$,
it can usually (always for the functions we will consider)
be extended by its formula over the entire s-plane except at certain singular points.

called "analytic continuation"

Here is an example

Let $f(t) = e^{bt}$ (b is any real number)

Then, $F(s) = \int_0^\infty e^{bt} e^{-st}\,dt = \int_0^\infty e^{-(s-b)t}\,dt$

But, $|e^{-(s-b)t}| = e^{-(\mathrm{Re}\,s - b)t}$ is integrable on $0 < t < \infty$ when $\mathrm{Re}\,s > b$.

So we get absolute convergence for $\mathrm{Re}\,s > b$

drawn for $b < 0$

In this case, $F(s) = \left. \dfrac{e^{-(s-b)t}}{-(s-b)} \right|_0^\infty = 0 - \dfrac{1}{-(s-b)} = \dfrac{1}{s-b}$

when $\mathrm{Re}\,s > b$.

But, $\frac{1}{s-b}$ exists for all s except when $s = b$.

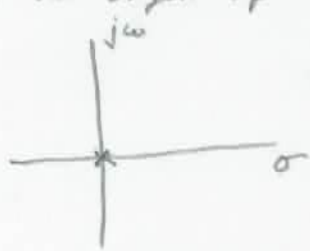
So, by analytic continuation we say that $\mathcal{L}\, f(t)$ is $\dfrac{1}{s-b}$

and $\mathcal{L}\, f(t)$ has a singularity, called a "pole", at $s = b$.

For the special case where $b = 0$, we have $e^{bt} = 1$ for $0 < t < \infty$,

and $\mathcal{L}\, f(t) = \frac{1}{s}$

and the pole is at the origin of the s-plane:



The Tranform of a Derivative

Assume $\mathcal{L}\, f(t) = F(s) = \int_0^\infty f(t) e^{-st} dt \quad (\operatorname{Re} s > \sigma_c)$

Using the integration-by-parts formula, we can get

$\mathcal{L}\, f^{(1)}(t)$, where $f^{(1)}(t)$ denotes the first derivative of $f(t)$. $\left\{ \frac{df}{dt} \right.$

$$\mathcal{L} f^{(1)}(t) = \int_0^\infty f^{(1)}(t) e^{-st} = f(t) e^{-st} \Big|_{0+}^{\infty} - \int_0^\infty f(t) \frac{d}{dt} e^{-st} dt$$

$$= \underbrace{0}_{\text{For } \operatorname{Re} s > \sigma_c} - f(0+) + s \int_0^\infty f(t) e^{-st} dt = s F(s) - f(0+)$$

We then have the important formula

(2) $\quad \mathcal{L}\, f^{(1)}(t) = s F(s) - f(0+)$ where again analytic continuation of $F(s)$ beyond $\{s : \operatorname{Re} s > \sigma_c\}$ is assumed.

Also, $f(0+) = \lim_{t \to 0+} f(t)$.

Thus, the difficult operation of <u>differentiation</u> is transformed into the easy operation of <u>multiplication by s and subtraction of $f(0+)$.</u>

Integrating by parts twice, we get

$$\mathcal{L}\, f^{(2)}(t) = f^{(1)}(t) e^{-st} - \int_0^\infty f^{(1)}(t) \frac{d}{dt} e^{-st} dt$$

$$= \underbrace{0}_{(\text{For } \operatorname{Re} s > \sigma_c)} - f^{(1)}(0) + s \int_0^\infty f^{(1)}(t) e^{-st} dt \qquad \text{(as above)}$$

$$= -f^{(1)}(0) + s\left(-f(0+) + s \int_0^\infty f(t) e^{-st} dt \right)$$

$$= s F(s) - s f(0+) - f^{(1)}(0+)$$

Continuing in this way with repeated integrations by parts we get the general formula: For $n = 1, 2, 3, \cdots$,

$$\mathcal{L}\, f^{(n)}(t) = s^n F(s) - s^{n-1} f(0+) - s^{n-2} f^{(1)}(0+) - \cdots - s f^{(n-2)}(0+) - f^{(n-1)}(0+)$$

$$\boxed{f^{(k)}(t) = \frac{d^k}{dt^k} f(t)}$$

The Transform of an Integral

Again assume $\mathcal{L}\, f(t) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \quad (\mathrm{Re}\, s > \sigma_c)$

We can use the integration-by-parts formula in a different way to get

$$\mathcal{L} \int_0^t f(x)\,dx = \int_0^\infty \int_0^t f(x)\,dx\; e^{-st}\,dt$$

$$= \int_0^t f(x)\,dx\, \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty f(t)\, \frac{e^{-st}}{-s}\,dt$$

Let us now assume that the upper limit here is 0.

That is, $\lim_{t\to\infty} \left|\int_0^t f(x)\,dx\right| e^{-\sigma_0 t} = 0$ for some σ_0.

Then, in some half-plane $\{s : \mathrm{Re}\, s > \sigma_d\}$, we have

$$\mathcal{L} \int_0^t f(x)\,dx = 0 - 0 + \frac{1}{s}\int_0^\infty f(t)\, e^{-st}\,dt.$$

Again by analytic continuation, we see that

③ $$\mathcal{L} \int_0^t f(x)\,dx = \frac{F(s)}{s}$$

Thus the difficult operation of <u>integration</u> is transformed into the easy operation of <u>division by s</u>.

Similarly, for repeated integrations,

$$\mathcal{L} \int_0^t dx_1 \int_0^{t_1} dx_2 \cdots \int_0^{t_{n-1}} f(x_n)\,dx_n = \frac{F(s)}{s^n}, \quad n = 1, 2, 3, \ldots$$

Here are some examples of Laplace transforms:

Example:

We have that $\mathcal{L}\, e^{-at} = \dfrac{1}{s+a}$ (See page LaP2)

$$\mathcal{L}\, \frac{d}{dt} e^{-at} = \mathcal{L}(-a e^{-at}) = \frac{-a}{s+a}$$

The same result can be obtained from ② on page LaP3:

$$\mathcal{L}\, \frac{d}{dt} e^{-at} = s\,\frac{1}{s+a} - 1 = \frac{s-s-a}{s+a} = \frac{-a}{s+a}$$

Also,

$$\mathcal{L} \int_0^t e^{-ax}\,dx = \mathcal{L}\, \frac{e^{-ax}}{-a}\Big|_0^t = \mathcal{L}\left[\frac{1}{-a}(e^{-at}-1)\right] = \frac{1}{-a}\left(\frac{1}{s+a} - \frac{1}{s}\right)$$

$$= \frac{s-s-a}{-a(s+a)s} = \frac{1}{s(s+a)}$$

The same result is obtained from ③ on page LaP4:

$$\mathcal{L} \int_0^t e^{-ax}\,dx = \frac{1}{s(s+a)}$$

Example:

$$\mathcal{L}\, \cos kt = \int_0^\infty \frac{e^{jkt} + e^{-jkt}}{2}\, e^{-st}\,dt = \frac{1}{2}\cdot \frac{e^{-(s-jk)t}}{-(s-jk)}\Big|_0^\infty + \frac{1}{2}\cdot\frac{e^{-(s+jk)t}}{-(s+jk)}\Big|_0^\infty$$

k is any real number

$$= \frac{1}{2}\left(\frac{1}{s-jk} + \frac{1}{s+jk}\right)$$

$\boxed{\text{convergence occurs when } \operatorname{Re} s > 0}$

$$= \frac{s}{s^2+k^2}$$

Similarly, $\mathcal{L}\, \sin kt = \dfrac{k}{s^2+k}$

(This time use $\sin kt = \dfrac{e^{jkt} - e^{-jkt}}{2j}$)

In the following, we use the notation $u(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0. \end{cases}$

However, since the Laplace transformation is an integration over $0 < t < \infty$, we could simply write 1 instead of $u(t)$ when applying the Laplace transform.

Example: $\mathcal{L}\, t = \mathcal{L}\, t\, u(t) = \int_0^\infty t\, e^{-st}\, dt = F(s)$ ← Find $F(s)$

We have: $\mathcal{L}\left(\frac{d}{dt} t\right) = s\, F(s) - 0.$ ← By ② on page LaP 3

But $\mathcal{L}\left(\frac{d}{dt} t\right) = \mathcal{L}\, u(t) = \frac{1}{s}$

Thus $F(s) = \frac{1}{s^2}$

Similarly, let $\mathcal{L}\, t^2 = G(s)$ Find $G(s)$

$$\mathcal{L}\, \frac{d}{dt} t^2 = \mathcal{L}\, 2t = 2\, \mathcal{L}\, t = \frac{2}{s^2}$$

$$= s\, G(s) - 0$$ ← By ② on page LaP 3

So, $G(s) = \frac{2}{s^3}$

Continuing in this way, we get

$$\mathcal{L}\, t^k = \frac{k!}{s^{k+1}}$$ when $k = 1, 2, 3, \cdots$

and $k! = k(k-1)(k-2)\cdots 2 \times 1$

(This formula still holds when $k = 0$ because $0! = 1.$)

Example: $\mathcal{L}\left(2|K|e^{-\alpha t}\cos(\beta t+\theta)\right) = \frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}$

where $K = |K|\angle\theta$ is a complex number, $K^* = |K|\angle -\theta$

and α and β are real numbers.

the "complex conjugate" of K

To derive this transform, first note that

⑤→ $K e^{-(\alpha-j\beta)t} + K^* e^{-(\alpha+j\beta)t}$

$= e^{-\alpha t}\, 2\,\mathrm{Re}\left(K e^{j\beta t}\right) = e^{-\alpha t}\, 2\,\mathrm{Re}\left(|K| e^{j(\beta t+\theta)}\right)$

$= 2|K|e^{-\alpha t}\cos(\beta t+\theta)$

Remember
$K = |K|\angle\theta$
$= |K|e^{j\theta}$

Now, the formula $\mathcal{L}\, e^{bt} = \frac{1}{s-b}$ still holds when b is a complex number. The Laplace transform now converges when $\mathrm{Re}\, s > \mathrm{Re}\, b$.

See page LaP2

Now apply this formula to ⑤. We get

$$\mathcal{L}\ ⑤ = K\frac{1}{s+\alpha-j\beta} + K^*\frac{1}{s+\alpha+j\beta}$$

In conclusion:

$$\mathcal{L}\left(2|K|e^{-\alpha t}\cos(\beta t+\theta)\right) = \frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}$$

where $K = |K|\angle\theta$ ($|K|$ is the magnitude of K and θ is the angle of K)

TRANSFORMS OF $\delta^{(n)}(t)$, $t = 0, 1, 2, \ldots$

$$\mathcal{L}\,\delta(t) = \langle \delta(t), e^{-st} \rangle = e^{-st}\Big|_{t=0} = 1$$

(We write
$$\int_{0-}^{\infty} \delta(t) e^{-st}\,dt = 1)$$

$$\mathcal{L}\,\delta^{(1)}(t) = \langle \delta^{(1)}(t), e^{-st} \rangle = -\frac{d}{dt} e^{-st} = -(-s e^{-st})\Big|_{t=0} = s$$

(We write
$$\int_{0-}^{\infty} \delta^{(1)}(t)\, e^{-st}\,dt = s)$$

Continuing in this way, we get

$$\mathcal{L}\,\delta^{(2)}(t) = \langle \delta^{(2)}(t), e^{-st} \rangle = (-1)^2 \frac{d^2}{dt^2} e^{-st} = (-1)^2 (-s)^2 e^{-st}\Big|_{t=0}$$
$$= s^2$$

In general, for $n = 0, 1, 2, 3, \ldots$,

$$\mathcal{L}\,\delta^{(n)}(t) = \langle \delta^{(n)}(t), e^{-st} \rangle = (-1)^n \frac{d^n}{dt^n} e^{-st} = (-1)^n (-s)^2 e^{-st}\Big|_{t=0}$$
$$= s^n$$

Let us now note the linearity of $\mathcal{L}$: (Let $\mathcal{L}\, f(t) = F(s)$)

For any constant (i.e., real number) c, we have "homogeniety":

$$\mathcal{L}\,(c f(t)) = c F(s)$$

We also have "additivity":

$$\mathcal{L}\left(f_1(t) + f_2(t)\right) = F_1(s) + F_2(s)$$

These two properties are obvious consequences of integration.

Taken together, we have "linearity."

TRANSFORMS OF OPERATIONS

We have already derived two transforms of operations on pages LaP 3 and LaP 4.

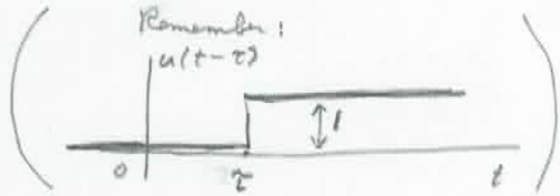
Here are some others:

$$\mathcal{L}\, e^{-\alpha t} f(t) = F(s+\alpha), \text{ where } \alpha \text{ is any real number.}$$

because $\int_0^\infty e^{-\alpha t} f(t) e^{-st}\,dt = \int_0^\infty f(t) e^{-(s+\alpha)t}\,dt$

For τ any real positive number;

$$\mathcal{L}\, f(t-\tau)\,u(t-\tau) = e^{-s\tau} F(s)$$



because

$$\int_0^\infty f(t-\tau)\,u(t-\tau)\,e^{-st}\,dt$$

$$= \int_\tau^\infty f(t-\tau) e^{-st}\,dt$$

Let $x = t-\tau$, $dx = dt$, $t = x+\tau$

$$= \int_0^\infty f(x)\, e^{-s(x+\tau)}\,d\tau = e^{-s\tau}\int_0^\infty f(x) e^{-sx}\,dx = e^{-s\tau} F(s)$$

For any real positive c,

$$\mathcal{L}\, f(ct) = \frac{1}{c} F\left(\frac{s}{c}\right)$$

because

$$\underbrace{\int_0^\infty f(ct)\, e^{-st}\,dt = \int_0^\infty f(x)\, e^{-sx/c}\frac{1}{c}\,dx}_{\text{Let } x = ct,\ t = \frac{x}{c},\ dt = \frac{1}{c}dx} = \frac{1}{c} F\left(\frac{s}{c}\right)$$

$$\mathcal{L}\, t f(t) = -\frac{d}{ds} F(s)$$

To derive this formula, start with the right-hand side.

$$-\frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\, dt$$

In general, we cannot interchange $\frac{d}{ds}$ and $\int_0^\infty \cdots dt$.

But, for the kinds of functions we will be dealing with, this will be OK.

So,

$$= -\int_0^\infty f(t)\,\frac{d}{ds} e^{-st}\, dt$$

$$= -\int_0^\infty f(t)\left(-e^{-st}\right) dt$$

$$= \int_0^\infty f(t)\, t\, e^{-st}\, dt = \mathcal{L}\, t f(t)$$

For higher order derivatives with respect to s, just keep repeating this interchange of differention and integration.

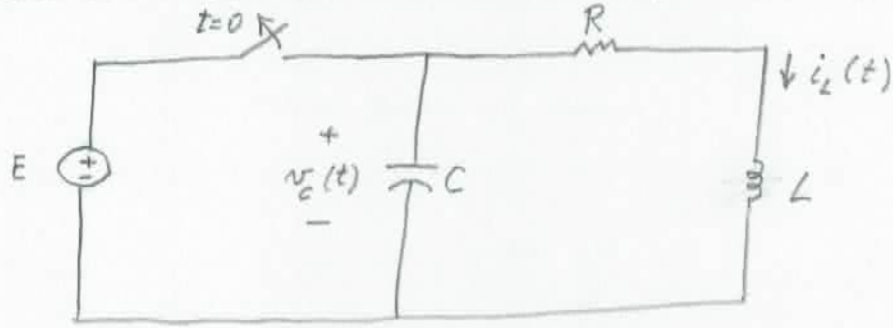
We get:

$$(-1)^n F^{(n)}(s) = (-1)^n \frac{d^n}{ds^n} F(s) = \mathcal{L}\, t^n f(t)$$

AN APPLICATION OF THE FOREGOING THEORY

Examples: Let us solve for the transient current $i(t)$ in the following circuit after the switch is opened at $t = 0$.

Assume the circuit is in the DC steady state for $t < 0$.



E is a real number (constant)

R, L, C are positive real numbers.

$v_C(t)$ and $i_L(t)$ are continuous at $t = 0$.

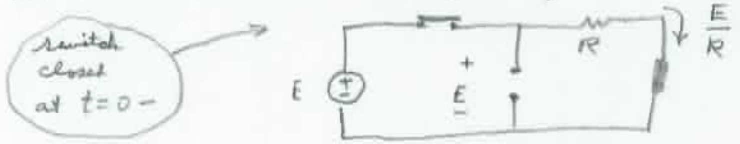
Indeed, if there was a jump in $v_C(t)$, there would be an infinite current at $t = 0$.

But, there is no source of infinite current at $t = 0+$

Also, if there was a jump in $i_L(t)$, there would be an infinite voltage at $t = 0$.

But, there is no source of infinite voltage at $t = 0+$.

So, consider the DC steady state at $t = 0-$. (C is an open circuit, and L is a short circuit.) So,



at $t = 0-$

$v_C(0-) = E$ and $i_L(0-) = \frac{E}{R}$

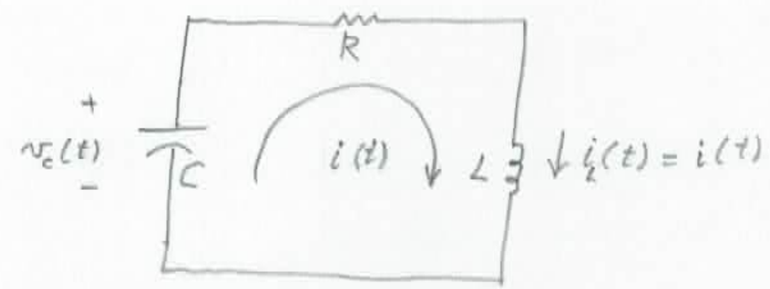
Because $v_C(t)$ and $i_L(t)$ are continuous at $t = 0$,

we have the initial conditions:

$$v_C(0+) = v_C(0-) = E$$
$$i_L(0+) = i_L(0-) = \frac{E}{R}$$

for the transients for $t > 0$.

For $t>0$, we have the following circuit to the right of the open switch. Let us determine $i(t)$



By KVL, we have the integrodifferential equation:

$$\frac{1}{C}\int_0^t i(x)\,dx - E + Ri(t) + L\frac{d}{dt}i(t) = 0$$

$v_c(0+) = E$

Upon applying $\mathcal{L}$, we get:

$i(0+) = i_L(0+) = \frac{E}{R}$

$$\frac{I(s)}{Cs} - \frac{E}{s} + RI(s) + L\left(sI(s) - \frac{E}{R}\right) = 0$$

Upon solving for $I(s)$, we get

$$I(s) = \frac{EC + \frac{LCE}{R}s}{LCs^2 + RCs + 1} = \frac{EC + \frac{LCE}{R}s}{LC(s-p_1)(s-p_2)}$$

where p_1 and p_2 are roots of the denominator quadratic (called the "poles" of $I(s)$)

We now examine some numerical results:

Case 1: Let $E=1$, $R=2$, $C=2$, $L=\frac{1}{4}$.

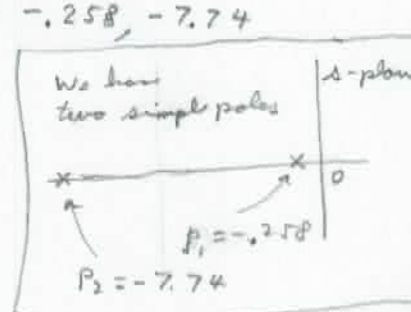
Then, $$I(s) = \frac{2+\frac{s}{4}}{\frac{1}{2}(s-p_1)(s-p_2)} = \frac{2+\frac{s}{4}}{\frac{1}{2}s^2 + 4s + 1}$$

$$p_1, p_2 = \frac{-4 \pm \sqrt{16-2}}{2 \times \frac{1}{2}} = -4 \pm 3.74 = -.258,\ -7.74$$

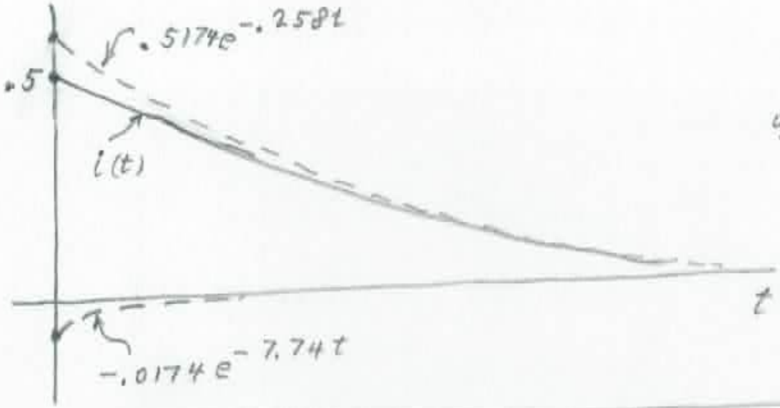
By a partial fraction expansion, we get

$$I(s) = \frac{.5174}{s+.258} - \frac{.0174}{s+7.74}$$

Therefore, $i(t) = .5174e^{-.258t} - .0174e^{-7.74t}$



Case 1 continued: Plotting the exponential functions, we get



This is an overdamped transient.

Case 2: Let $E = 1$, $R = \frac{1}{2}$, $C = 2$, $L = \frac{1}{4}$ (Only R is changed - made smaller.)

$$I(s) = \frac{2+s}{\frac{1}{2}s^2 + s + 1} = \frac{s+2}{\frac{1}{2}(s-p_1)(s-p_2)}$$

$$\left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} = \frac{-1 \pm \sqrt{1-2}}{2 \times \frac{1}{2}} = -1 \pm j \quad , \quad p_1 = -1 + j, \quad p_2 = -1 - j$$

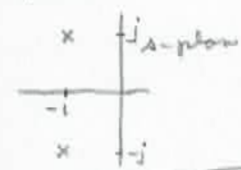
So, $$I(s) = \frac{A}{s+1-j} + \frac{A^*}{s+1+j}$$

$\alpha = 1$

$\beta = 1$

$$A = \frac{-1+j+2}{\frac{1}{2}(-1+j+1+j)} = \frac{1+j}{j} = \frac{\sqrt{2}\,\angle 45^\circ}{1\angle 90^\circ} = \sqrt{2}\,\angle -45^\circ$$

We have a pair of complex-conjugate poles.



So, by the formula in the Table of Laplace Transforms,

$$i(t) = 2|A|\,e^{-\alpha t}\cos(\beta t - \arg A) = 2\sqrt{2}\,e^{-t}\cos(t - 45^\circ)$$

Plotting this, we get:

This is an underdamped transient.

Case 3: We now consider a critically damped transient.

It lies at the border between the overdamped and underdamped transient.

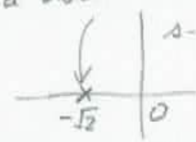
One way to get it is to set $E = 1$, $R = \frac{1}{\sqrt{2}}$, $C = 2$, $L = \frac{1}{4}$

Now,
$$I(s) = \frac{2 + \frac{s}{\sqrt{2}}}{\frac{1}{2}(s - p_1)(s - p_2)} = \frac{2 + \frac{s}{\sqrt{2}}}{\frac{1}{2}s^2 + \sqrt{2}\,s + 1}$$

$$\left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} = \frac{-\sqrt{2} \pm \sqrt{2 - 2}}{\frac{1}{2} \times 2} = -\sqrt{2}.$$

That is, $p_1 = p_2 = -\sqrt{2}$

We have a "double pole" at $s = -\sqrt{2}$.



Thus,
$$I(s) = \frac{\sqrt{2}\,s + 4}{(s + \sqrt{2})^2}$$

The partial fraction expansion of this is:

$$I(s) = \frac{A_1}{(s + \sqrt{2})^2} + \frac{A_2}{s + \sqrt{2}}$$

$$A_1 = (\sqrt{2}\,s + 4)\Big|_{s = -\sqrt{2}} = 2$$

$$A_2 = \frac{d}{ds}(\sqrt{2}\,s + 4)\Big|_{s = -\sqrt{2}} = \sqrt{2}$$

So $i(t) = 2t\,e^{-\sqrt{2}\,t} + \sqrt{2}\,e^{-\sqrt{2}\,t}$ ← (See the formula in the Table of Laplace transforms)

Plotting this, we get

The transient is just on the verge of dipping below the t axis

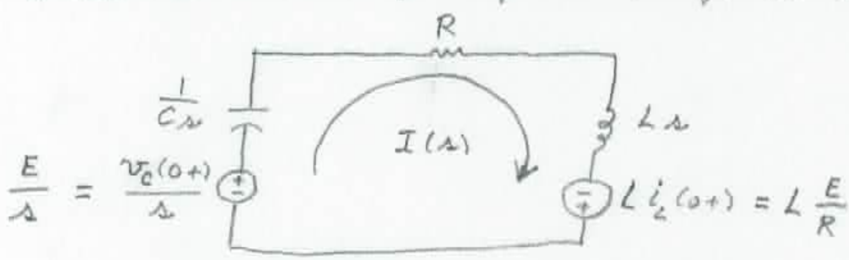
"The critically damped" case.

Laplace – Transformed Circuits

In AC steady-state analysis, we replaced the time-domain circuit by its phasor circuit. Similarly, for general transient analysis, we can replace the time-domain circuit by its Laplace-transformed circuit. Upon analyzing the latter, we will obtain the same transformed equations as those obtained by applying the Laplace transformation to the time-domain integrodifferential equations.

Thus, we have two ways of doing a transient analysis by means of the Laplace transformation. This is illustrated on the next page: La P-16. To use this other method (shown on the right-hand side of page La P-16), we have to transform the elements of the circuit, given the initial currents in the inductors and the initial voltages on the capacitors. The formulas and circuits for the transformed elements in the s-domain are shown on the right-hand side of page La P-17.

Example. Let us transform the circuit on page La P-12. Since we will be writing KVL around a single mesh, it is convenient to use the Thevenin form of the transformed elements. This yields:

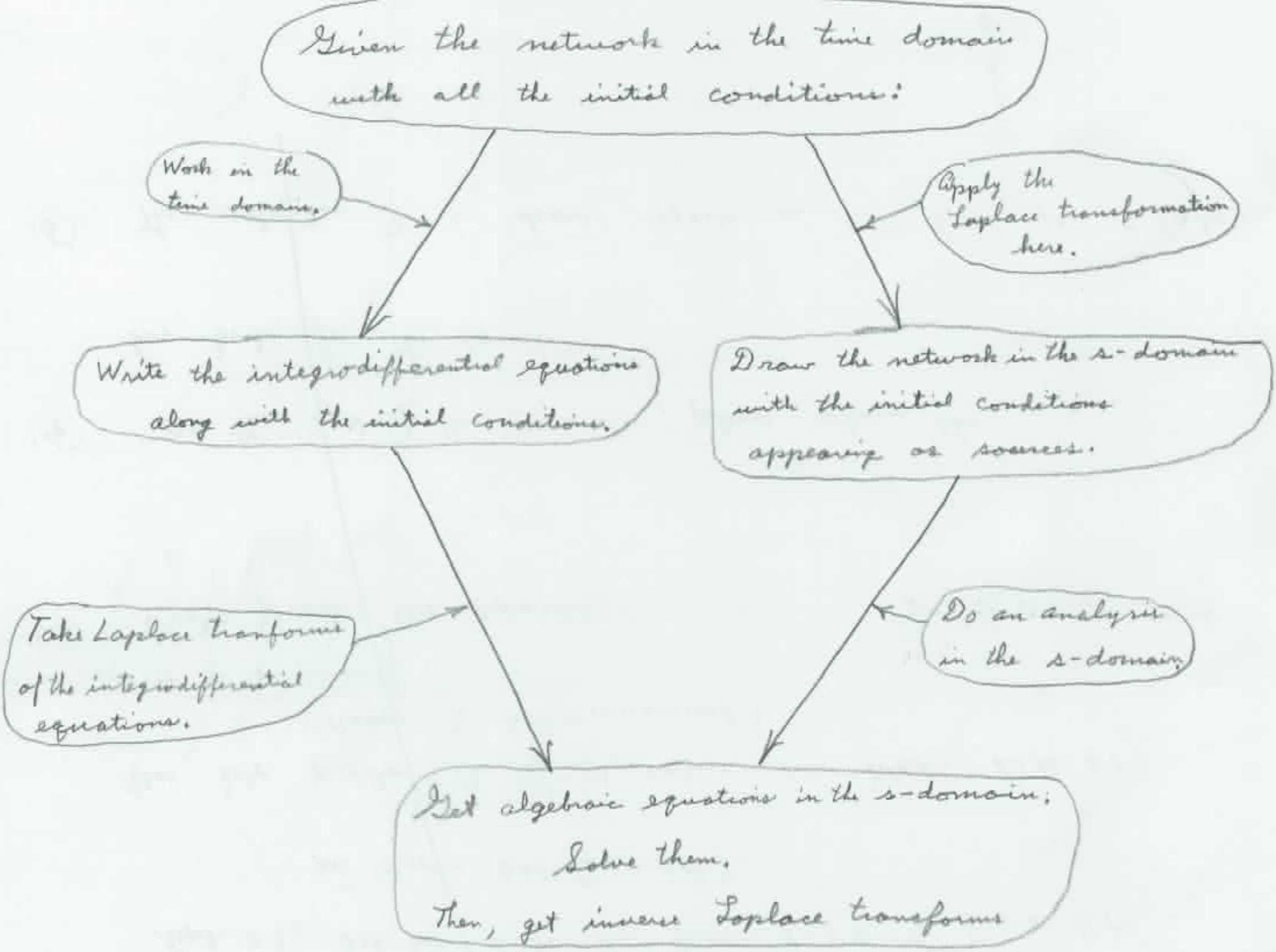


By KVL,

$$-\frac{E}{s} + \frac{1}{Cs}I(s) + RI(s) + Ls\,I(s) - L\frac{E}{R} = 0$$

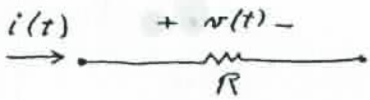
This is the same equation as that on page La P-12, which was obtained by applying the Laplace transformation to the integrodifferential equation of the time-domain circuit.

Two ways to solve for transients in electrical networks.

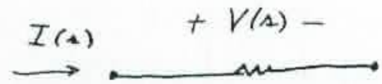


TRANSFORMS OF ELEMENTS (BRING THESE NOTES TO THE EXAMS)

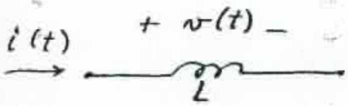
TIME DOMAIN | **s - DOMAIN**



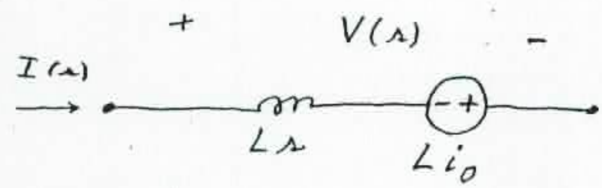
$v(t) = R\, i(t)$



$V(s) = R\, I(s)$

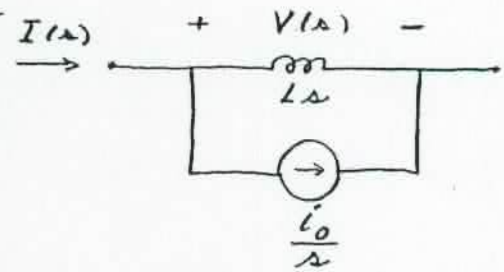


$$\left.\begin{aligned} v(t) &= L\,\frac{di}{dt} \\ i(0+) &= i_0 \end{aligned}\right\} \longleftrightarrow$$

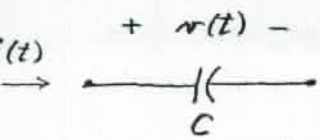


$V(s) = Ls\, I(s) - L\, i_0$

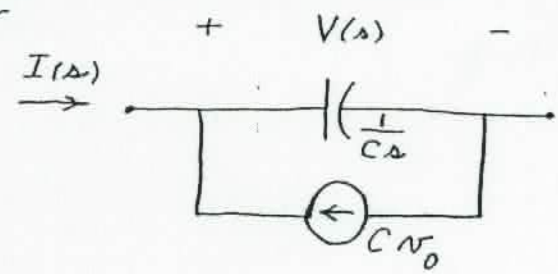
$i(t) = \frac{1}{L}\int_0^t v(x)\,dx + i_0$



$I(s) = \frac{V(s)}{Ls} + \frac{i_0}{s}$

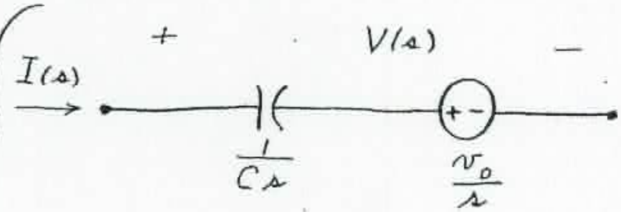


$$\left.\begin{aligned} i(t) &= C\,\frac{dv}{dt} \\ v(0+) &= v_0 \end{aligned}\right\} \longleftrightarrow$$



$I(s) = Cs\, V(s) - C v_0$

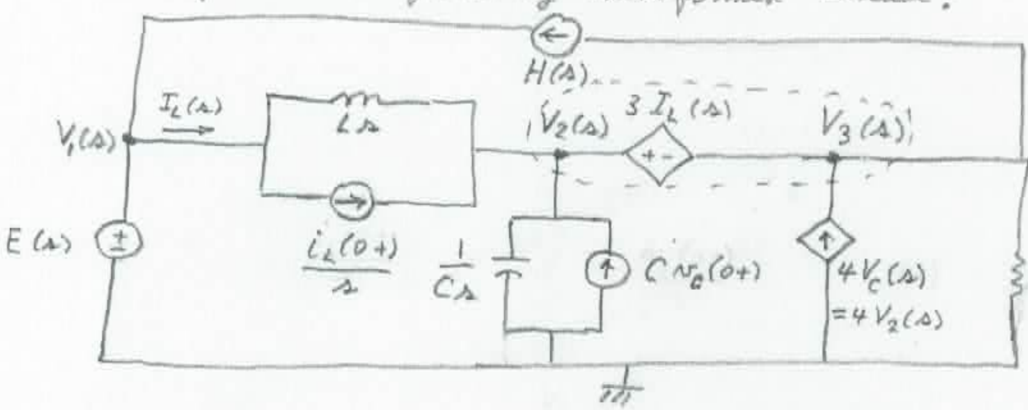
$v(t) = \frac{1}{C}\int_0^t i(x)\,dx + v_0 \longleftrightarrow$



$V(s) = \frac{1}{Cs}\, I(s) + \frac{v_0}{s}$

In J J H + L
see Section 13.2
pages 525-529
for these formulas

Example: As another example, let us transform the circuit on page 1D-2. Let us also choose to do a nodal analysis. Since we will be using KCL, it is more convenient to use the Norton circuits for the elements, as shown on page LaP-17. This yields the following transformed circuit:



The transformed nodal equations are obtained from this circuit:

At the $V_1(s)$ node:

$$V_1(s) = E(s)$$

Inside a balloon around the $3I_L(s)$ dependent source:

$$V_2(s) - V_3(s) = 3I_L(s) = 3\left(\frac{V_1(s) - V_2(s)}{Ls} + \frac{i_L(0+)}{s}\right)$$

KCL on that balloon:

$$\frac{V_2(s) - V_1(s)}{Ls} - \frac{i_L(0+)}{s} + \frac{V_2(s)}{\frac{1}{Cs}} - Cv_C(0+) - 4V_2(s) + \frac{V_3(s)}{R} + H(s) = 0$$

These are the same equations as those on page 1D-4.

Partial-Fraction Expansions

The final step in a Laplace transform analysis of a circuit is to convert the solution in Laplace-transform form into its corresponding time-domain function. It often occurs that the Laplace-transform form is a rational function. Such a function can always be written as the ratio of two polynomials in s:

$$\text{(1)} \qquad F(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0} = \frac{N(s)}{D(s)}$$

One way of getting the time function corresponding to this is to expand $F(s)$ into sums of "partial fractions" of the forms:

$$\frac{A}{s-p}, \quad \frac{B}{(s-p)^k} \text{ where } k = 2, 3, 4, \ldots, \text{ and } C s^k \text{ where } k = 0, 1, 2, \ldots$$

where A, B, and C are constants and p is either a real or complex number.

Then, the following correspondences occur when $\mathcal{L}$ denotes the Laplace transformation

$$\frac{A}{s-p} \xleftarrow{\mathcal{L}} A e^{pt} \quad (A \text{ and } p \text{ can be real or complex})$$

$$\frac{B}{(s-p)^k} \xleftarrow{\mathcal{L}} B \frac{t^{k-1}}{(k-1)!} e^{pt} \quad (B, p \text{ real or complex},\ k = 1, 2, 3, \ldots)$$

$$\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta} \xleftarrow{\mathcal{L}} 2|K| e^{-\alpha t} \cos(\beta t + \theta)$$

$$\frac{K}{(s+\alpha-j\beta)^k} + \frac{K^*}{(s+\alpha+j\beta)^k} \xleftarrow{\mathcal{L}} 2|K| \frac{t^{k-1}}{(k-1)!} e^{-\alpha t} \cos(\beta t + \theta)$$

$$\left\{ \begin{array}{l} K \text{ real or complex}, \\ K = |K| \angle \theta \\ \alpha, \beta \text{ real} \\ k = 1, 2, 3, \ldots \end{array} \right.$$

$$C \xleftarrow{\mathcal{L}} C\,\delta(t)$$

$$C s^k \xleftarrow{\mathcal{L}} C\,\delta^{(k)}(t) \quad (k = 1, 2, 3, \ldots)$$

So, once we get the partial fractions, we can write down the time functions

We consider several cases:

Case 1:

② $$F(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + \cdots + a_0}{b_n (s-p_1)(s-p_2)\cdots(s-p_n)}$$

{ Here we have factored $D(s)$ by means of its roots.

where the roots p_k of $D(s)$ are all distinct and $m < n$.

In this case,

$$F(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots + \frac{A_n}{s-p_n}$$

where $A_k = (s-p_k)F(s)\Big|_{s=p_k} = \dfrac{N(s)}{b_n(s-p_1)\cdots\cancel{(s-p_k)}\cdots(s-p_n)}\Bigg|_{s=p_k}$

$k = 1, 2, \cdots, n$

Here, we remove $(s-p_k)$ before substituting $s = p_k$.

Example: $$F(s) = \frac{2}{(s+1)(s+2)(s+3)}$$

$$A_1 = \frac{2}{(s+2)(s+3)}\Bigg|_{s=-1} = \frac{2}{1\times 2} = 1$$

$$A_2 = \frac{2}{(s+1)(s+3)}\Bigg|_{s=-2} = \frac{2}{(-1)\times 1} = -2$$

$$A_3 = \frac{2}{(s+1)(s+2)}\Bigg|_{s=-3} = \frac{2}{(-2)(-1)} = 1$$

So $$F(s) = \frac{1}{s+1} - \frac{2}{s+2} + \frac{1}{s+3}$$

and

$$f(t) = e^{-t} - 2e^{-2t} + e^{-3t} \qquad (t > 0)$$

Case 2: Some of the poles are "multiple."

That is, some of the P_k in ② on page LaP-20 have the same value.

We can represent this case by

$$F(s) = \frac{N(s)}{(s-p)^q\, B(s)}$$

where q is a natural number ≥ 2 and p is not a root of the polynomial $B(s)$.

The partial-fraction expansion of $F(s)$ is now:

$$F(s) = \frac{A_1}{(s-p)^q} + \frac{A_2}{(s-p)^{q-1}} + \cdots + \frac{A_q}{s-p} + \text{partial fractions at the roots of } B(s).$$

Then, $A_1 = (s-p)^q F(s)\Big|_{s=p} = \dfrac{N(s)}{B(s)}\Big|_{s=p}$

$1! = 1$ →

$$A_2 = \frac{1}{1!}\frac{d}{ds}(s-p)^q F(s)\Big|_{s=p} = \frac{d}{ds}\frac{N(s)}{B(s)}\Big|_{s=p}$$

$2! = 2\times 1$ →

$$A_3 = \frac{1}{2!}\frac{d^2}{ds^2}(s-p)^q F(s)\Big|_{s=p} = \frac{1}{2}\frac{d^2}{ds^2}\frac{N(s)}{B(s)}\Big|_{s=p}$$

$\vdots$

$(q-1)! = (q-1)(q-2)\cdots 2\times 1$

$$A_q = \frac{1}{(q-1)!}\frac{d^{q-1}}{ds^{q-1}}(s-p)^q F(s)\Big|_{s=p} = \frac{1}{(q-1)(q-2)\cdots 2\times 1}\frac{d^{q-1}}{ds^{q-1}}(s-p)^q F(s)\Big|_{s=p}$$

Then,

$$\mathcal{L}^{-1}F(s) = A_1\frac{t^{q-1}}{(q-1)!}e^{-pt} + A_2\frac{t^{q-2}}{(q-2)!}e^{-pt} + \cdots + A_q e^{-pt}$$

Example. Here is an example for case 2:

Find $\mathcal{L}^{-1} \dfrac{2s+1}{(s+2)^3 s}$

We write:

$$\frac{2s+1}{(s+2)^3 s} = \frac{A_1}{(s+2)^3} + \frac{A_2}{(s+2)^2} + \frac{A_3}{s+2} + \frac{C}{s}$$

$$A_1 = \left.\frac{2s+1}{s}\right|_{s=-2} = \frac{-3}{-2} = \frac{3}{2}$$

$$A_2 = \left.\frac{d}{ds}\frac{2s+1}{s}\right|_{s=-2} = \left.\frac{s\,2-(2s+1)}{s^2}\right|_{s=-2} = \left.\frac{-1}{s^2}\right|_{s=-2} = -\frac{1}{4}$$

$$A_3 = \left.\frac{1}{2}\frac{d^2}{ds^2}\frac{2s+1}{s}\right|_{s=-2} = \left.\frac{1}{2}\frac{d}{ds}\left(\frac{-1}{s^2}\right)\right|_{s=-2} = \left.\frac{1}{2}\cdot\frac{(-1)(-2)}{s^3}\right|_{s=-2} = -\frac{1}{8}$$

$$B = \left.\frac{2s+1}{(s+2)^3}\right|_{s=0} = \frac{1}{8}$$

Thus,

$$\mathcal{L}^{-1}\frac{2s+1}{(s+2)^3 s} = \frac{3}{2}\cdot\frac{t^2}{2!}e^{-2t} - \frac{1}{4}\cdot\frac{t}{1!}e^{-2t} - \frac{1}{8}e^{-2t} + \frac{1}{8} \qquad \text{for } t>0$$

$$= \frac{3}{4}t^2e^{-2t} - \frac{1}{4}te^{-2t} - \frac{1}{8}e^{-2t} + \frac{1}{8}$$

Case 3: The poles of $F(s)$ can be simple and/or multiple, but now $m \geq n$ (see ① on page La P-19).

In this case we use long division starting with the highest degree terms in $N(s)$ and $D(s)$ to convert $F(s)$ into the following form.

$$F(s) = P(s) + \frac{Q(s)}{D(s)}$$

where $P(s) = C_{m-n} s^{m-n} + C_{m-n-1} s^{m-n-1} + \cdots + C_1 s + C_0 + \frac{Q(s)}{D(s)}$

and the C_k are constants, $Q(s)$ is a polynomial of degree less than n.

n = degree of $D(s)$

Actually, $C_{m-n} = \frac{a_m}{b_n}$, but the $C_{m-n-1}, \cdots, C_1, C_0$ must be obtained by long division.

We can now apply $\mathcal{L}^{-1}$ to this expanded form of $F(s)$ to get

$$f(t) = C_{m-n} \delta^{(m-n)}(t) + C_{m-n-1} \delta^{(m-n-1)}(t) + \cdots + C_1 \delta^{(1)}(t) + C_0 \delta(t) + g(t)$$

where $g(t)$ is obtained from the inverse Laplace transform of a partial-fraction expansion of $\frac{Q(s)}{D(s)}$ just as in Cases 1 and 2.

Example: Here is an example for Case 3:

$$F(s) = \frac{4s^4 + 10s^3 + 14s^2 + 6s + 3}{2s^2 + 6s + 4}$$

By long division:

$$\begin{array}{r} 2s^2 - s + 6 \\ 2s^2+6s+4 \,\overline{)\, 4s^4 + 10s^3 + 14s^2 + 6s + 3} \\ \underline{4s^4 + 12s^3 + 8s^2 } \\ -2s^3 + 6s^2 + 6s + 3 \\ \underline{-2s^3 - 6s^2 - 4s } \\ 12s^2 + 10s + 3 \\ \underline{12s^2 + 36s + 24} \\ -26s - 21 \end{array}$$

So,

$$F(s) = 2s^2 - s + 6 - \frac{26s + 31}{2s^2 + 6s + 4}$$

$$= 2s^2 - s + 6 - \frac{26s + 31}{2(s+1)(s+2)}$$

$$= 2s^2 - s + 6 + \frac{5/2}{s+1} - \frac{31/2}{s+2}$$

This is a "Case 1" partial-fraction expansion of the remainder term: $-\dfrac{26s+31}{2s^2+6s+4}$

Finally,

$$f(t) = 2\,\delta^{(2)}(t) - \delta^{(1)}(t) + 6\,\delta(t) + \frac{5}{2}e^{-t}u(t) - \frac{31}{2}e^{-2t}u(t)$$

THE INITIAL AND FINAL VALUE THEOREMS

These state how a transient $f(t) = \mathcal{L}^{-1}F(s)$ starts and finishes, given a rational function $F(s)$ such as ① on page LaP-19.

The initial-value theorem:

① $$f(0+) = \lim_{s\to\infty} s\,F(s)$$ where s goes to ∞ along the real axis.

This holds so long as $m < n$ in ① page LaP-19.

Proof: We know that

$$sF(s) - f(0+) = \int_0^\infty f^{(1)}(t)\, e^{-st}\,dt \qquad \left(f^{(1)}(t) = \frac{d}{dt} f(t)\right)$$

Because $m<n$, $f^{(1)}(t)$ has no delta function or derivative of such at $t=0$. So, $f^{(1)}(t)$ is an ordinary function.

Since it has a Laplace transform for all Re s sufficiently large, we have that

$$\lim_{s\to\infty} sF(s) - f(0+) = \lim_{s\to\infty}\int_0^\infty f^{(1)}(t)\, e^{-st}\,dt$$

$$= \int_0^\infty f^{(1)}(t)\underbrace{\lim_{s\to\infty} e^{-st}}_{=0}\,dt = 0$$

This yields ①.

This interchange of $\lim_{s\to\infty}$ and $\int_0^\infty \cdots dt$ is OK for the kinds of Laplace-transformable functions we will be dealing with.

Proof:
Use $$\int_0^\infty |f^{(1)}(t)|\, e^{-\operatorname{Re} s\, t}\,dt$$

$$\le \underbrace{\int_0^T |f^{(1)}(t)|\,dt}_{\le\, \varepsilon} + \underbrace{\int_T^\infty |f^{(1)}(t)|\, e^{-\operatorname{Re} s\, t}\,dt}_{\le \int_T^\infty |f^{(1)}(t)|\,dt\; \underbrace{e^{-\operatorname{Re} s\, T}}_{\to 0}}$$

for T small enough

for all Re s > 0

as Re s → ∞

The initial-slope theorem:

② $$f^{(1)}(0+) = \lim_{s\to\infty} s\left(F(s) - f(0+)\right)$$ ← (This holds if $m<n$ in ① page LaP-19.)

Proof: For the same reasons, we can let $s \to \infty$ along the real positive axis using the formula for $\mathcal{L}\, f^{(2)}(t)$ to write

$$\lim_{s\to\infty}\left(s^2F(s) - s\,f(0+) - f^{(1)}(0+)\right) = \lim_{s\to\infty}\int_0^\infty f^{(2)}(t)\, e^{-st}\,dt = 0$$

again as indicated on page LaP-25

This yields ②.

The initial curvature theorem:

$$f^{(2)}(0+) = \lim_{s\to\infty} s\left(s\left(s\,F(s) - f^{(1)}(0+)\right) - f^{(1)}(0+)\right)$$

(Similar proof.)

We can also get the initial values of the higher derivatives in a similar way, but these values are not of much interest.

The final-value theorem:

③ $$f(\infty) = \lim_{s \to 0+} s F(s)$$

Here, $f(\infty) = \lim_{t \to \infty} f(t)$. Also, s tends to the origin along the real positive axis

This theorem holds for a rational function such as ① on page LaP-19 so long as all the poles of $F(s)$ lie in the left-half s-plane except for a possible simple pole at the origin ($s = 0$)

($m > n$ is now allowed.)

If all poles are in the left-half s-plane (no pole at $s=0$), $f(\infty) = 0$

If there is a simple pole at $s = 0$, $f(\infty)$ is a number.

Proof of ③:

$$\lim_{s \to 0+} \left(s F(s) - f(0+) \right) = \lim_{s \to 0+} \int_0^\infty f^{(1)}(t) e^{-st} \, dt$$

This interchange can also be justified in a similar fashion

$$= \int_0^\infty f^{(1)}(t) \underbrace{\lim_{s \to 0+} e^{-st}}_{=1} \, dt$$

$$= \int_0^\infty f^{(1)}(t) \, dt = \lim_{t \to \infty} f(t) - f(0+)$$

Cancelling the $f(0+)$, we get ③.

Example of the use of the initial and final value theorems:

Let $F(s) = \dfrac{3s+6}{s^2+4s+3} = \dfrac{3s+6}{(s+1)(s+3)}$ poles are at $s=-1$ and $s=-3$.

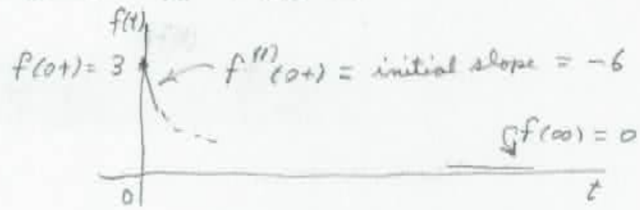
So, $f(0+) = \lim\limits_{s\to\infty} \dfrac{3s^2+6s}{s^2+4s+3} = 3$

$$f^{(1)}(0+) = \lim_{s\to\infty} s\left(\frac{3s^2+6s}{s^2+4s+3} - 3\right)$$

$$= \lim_{s\to\infty} s\left(\frac{\cancel{3s^2}+6s-\cancel{3s^2}-12s-9}{s^2+4s+3}\right)$$

$$= \lim_{s\to\infty} \frac{-6s^2-9s}{s^2+4s+3} = -6$$

$$f(\infty) = \lim_{s\to 0+} \frac{3s^2+6s}{s^2+4s+3} = 0$$



As an example of a case where the final value is not 0, consider:

$$G(s) = \frac{3s+6}{s(s^2+4s+3)} = \frac{3s+6}{s(s+1)(s+3)}$$

$$\lim_{t\to\infty} g(t) = \lim_{s\to 0+} s\,G(s) = \lim_{s\to 0+} \frac{3s+6}{s^2+4s+3} = 2$$

CONVOLUTION

An intuitive approach:

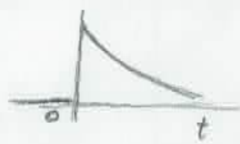
{Consider an input-output system

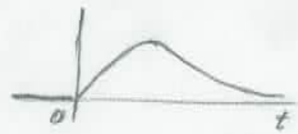
When $f(t) = \delta(t)$, $w(t) = h(t)$ ← The delta-function response.

Then, for any $f(t)$, $w(t) = f(t) * h(t)$ ← "Convolution"

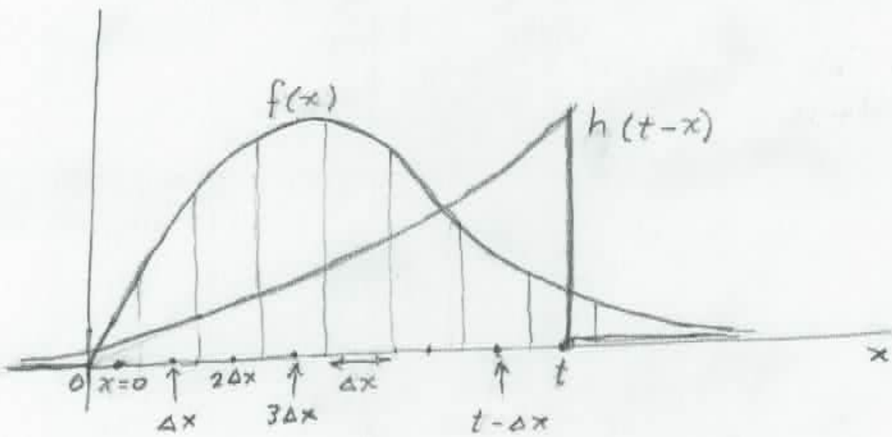
$$w(t) = \int_0^t f(x)\,h(t-x)\,dx$$

To see this, assume $h(t) = e^{-t}$ and $f(t) = te^{-t}$ where $t > 0$.





Consider the response $w(t)$ to $f(x)$ at a fixed time $t > 0$, where $x < t$.



So, at time t, $w(t) \approx \Delta x\, f(0)h(t-0) + \Delta x\, f(\Delta x)\,h(t-\Delta x)$

$$+ \Delta x\, f(2\Delta x)\,h(t-2\Delta x) + \cdots +$$

$$+ \Delta x\, f(t-\Delta x)\underbrace{h(t-(t-\Delta x))}_{h(\Delta x)} + \Delta x\, f(t)\underbrace{h(t-t)}_{h(0)}$$

This is a Riemann sum as an approximation of an integral.

As $\Delta x \to 0$, we get

$$w(t) = \int_0^t f(x)\,h(t-x)\,dx = f(t) * h(t)$$

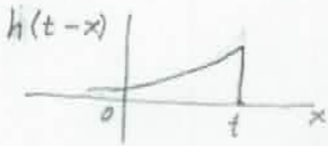
This is the convolution integral.

This is another symbol for convolution. It is also written as $f * g$.

Convolution can also be thought of as "flip," "shift," "multiply," "integrate":

$h(x)$  $f(x)$ 

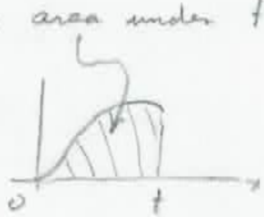
flip: $h(-x)$ 

shift: 

multiply: $f(x)\,h(t-x) \rightarrow$ 

integrate: $\int_0^t f(x)\,h(t-x)\,dx$ = area under $f(x)\,h(t-x)$

$= f(t) * h(t)$



<u>Note</u>: It does not matter which function you flip and shift:

Indeed:

$$f(t) * h(t) = \int_0^t f(x)\, h(t-x)\, dx$$

a change of variable (t is fixed.) Let $y = t - x$, $x = t - y$, $dx = -dy$

$$f(t) * h(t) = -\int_t^0 f(t-y)\, h(y)\, dy = \int_0^t h(y)\, f(t-y)\, dy = h(t) * f(t).$$

The "Exchange Formula" for Convolution

Consider the definition of convolution when both $f(t)$ and $g(t)$ are zero for all $t < 0$:

$$w(t) = (f * g)(t) = \int_0^t f(x)\, g(t-x)\, dx \quad , \text{ when } t > 0.$$

$$w(t) = 0 \, , \text{ when } t < 0$$

Apply the Laplace Transformation:

$$W(s) = \int_0^\infty \int_0^t f(x)\, g(t-x)\, dx\, e^{-st} dt = \int_{t=0}^\infty \int_{x=0}^\infty f(x)\, g(t-x)\, dx\, dt$$

Here, the upper limit t can be replaced by ∞ because $g(t-x) = 0$ when $x > t$.

Switch OK

$$= \int_0^\infty f(x) \int_0^\infty g(t-x)\, e^{-st} dt\, dx$$

$$= \int_0^\infty f(x)\, G(s)\, e^{-sx} dx$$

$$= G(s) \int_0^\infty f(x)\, e^{-sx} dx$$

$$= G(s)\, F(s)$$

Thus, we have the "exchange formula"

$$\mathcal{L}(f * g) = G(s)\, F(s)$$

Obviously, $G(s) F(s) = F(s) G(s)$

Therefore, $\mathcal{L}(f * g) = \mathcal{L}(g * f)$

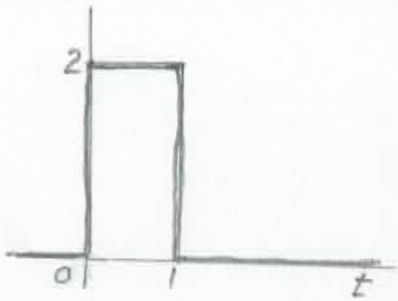
By the uniqueness theorem of the Laplace transform $\mathcal{L}$, convolution "commutes":

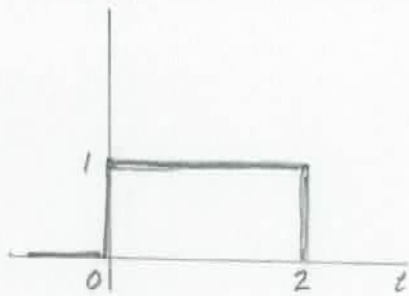
$$f * g = g * f$$

(We have already derived this fact at the bottom of page LaP-30.)

Example:

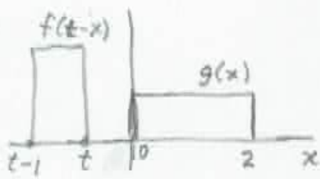
Let $f(t) = 2u(t) - 2u(t-1)$ $\qquad$ $g(t) = u(t) - u(t-2)$





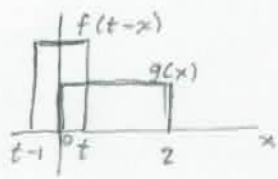
Consider $f(t) * g(t) = \int_0^t f(t-x)\, g(x)\, dx$

For $t < 0$,



$f(t) * g(t) = 0$

For $0 < t < 1$,

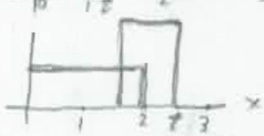


$f(t) * g(t) = 2t$

For $1 < t < 2$,

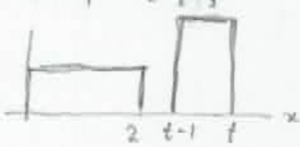
$f(t) * g(t) = 2$

For $2 < t < 3$



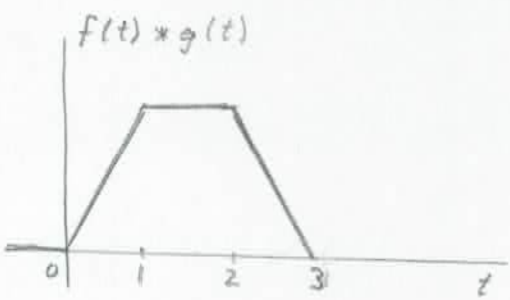
$f(t) * g(t) = 2(2-(t-1)) = 2(3-t)$

For $3 < t$



$f(t) * g(t) = 0$

So, we have



Example (continued)

Another way to get the result on the bottom of page LaP- 32 is to use the exchange formula:

$$F(s) = \frac{2}{s} - \frac{2}{s} e^{-s}$$

$$G(s) = \frac{1}{s} - \frac{1}{s} e^{-2s}$$

So,

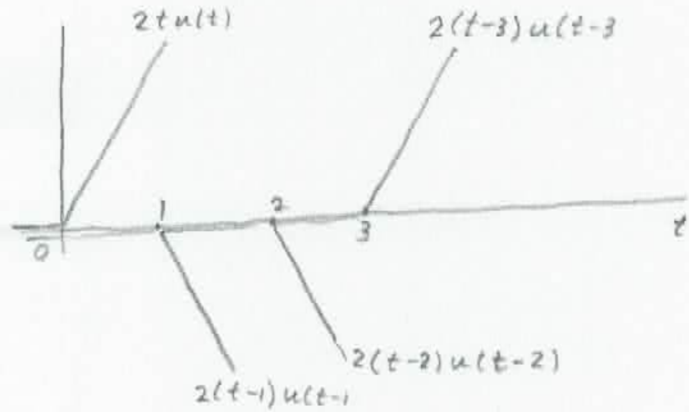
$$F(s)\,G(s) = \left(\frac{2}{s} - \frac{2}{s} e^{-s}\right)\left(\frac{1}{s} - \frac{1}{s} e^{-2s}\right)$$

$$= \frac{2}{s^2}\left(1 - e^{-s} - e^{-2s} + e^{-3s}\right)$$

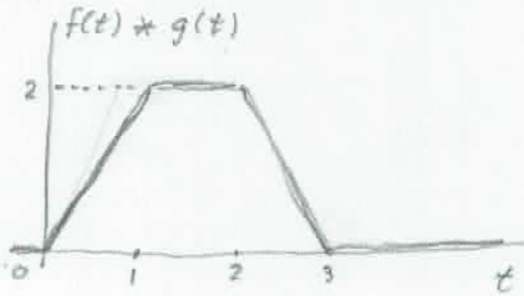
Apply $\mathcal{L}^{-1}$ term by term:

$$f(t) * g(t) = 2t\,u(t) - 2(t-1)u(t-1) - 2(t-2)u(t-2) + 2(t-3)u(t-3)$$

Plotting:



Adding terms, we get

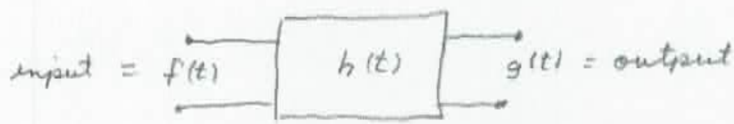


A convolution is one form of an integral equation:

A convolution equation:

(1) $$\int_0^t f(x)h(t-x)\,dx = g(t) \quad \text{or} \quad f * h = g$$

One interpretation of this (but not the only one) is to think in terms of an input-output system:



where $h(t)$ is the response to a delta-function input. That is, when $f(t) = \delta(t)$, we have $g(t) = h(t)$ because $\int_{0-}^{t} \delta(x)h(t-x)\,dx = h(t)$.

Whether or not we use this interpretation, we have the following problem concerning (1):

Given any two of the functions: $f(t)$, $h(t)$, $g(t)$, find the third one.

In general, a facile way of doing this is to use the Laplace transformation $\mathcal{L}$ on (1). We get

$$F(s)\,H(s) = G(s).$$

So, given $f(t)$ and $h(t)$, we use $G(s) = F(s)H(s)$.

Given $f(t)$ and $g(t)$, we use $H(s) = G(s)/F(s)$.

Given $h(t)$ and $g(t)$, we use $F(s) = G(s)/H(s)$.

We then apply the inverse Laplace transformation $\mathcal{L}^{-1}$.

Example: Here's an example with regard to the input - output system shown on page LaP-34.

Case 1. When $f(t) = e^{-t}u(t)$, we get $g(t) = \left(2e^{-t} + 3e^{-2t}\right)u(t)$ ←(assumed)

Case 2. Question: What is $g(t)$, when $f(t) = \sin t\ u(t)$

Solution: In Case 1, we have

$$G(s) = F(s)\,H(s)$$

$$G(s) = \frac{2}{s+1} + \frac{3}{s+2} = \frac{1}{s+1}H(s)$$

$$\text{So, } H(s) = \frac{\frac{2}{s+1} + \frac{3}{s+2}}{\frac{1}{s+1}} = \frac{5s+7}{s+2}$$

In Case 2,

$$G(s) = \frac{1}{s^2+1}\cdot\frac{5s+7}{s+2} = \frac{5s+7}{(s+2)(s-j)(s+j)}$$

$$= \frac{A}{s+2} + \frac{B}{s-j} + \frac{B^*}{s+j}$$

$$A = \left.\frac{5s+7}{s^2+1}\right|_{s=-2} = -\frac{3}{5}$$

$$B = \left.\frac{5s+7}{(s+2)(s+j)}\right|_{s=j} = .3 - j1.9 = 1.924\angle{-81.03^\circ}$$

Hence, $g(t) = -\frac{3}{5}e^{-2t} + 3.848\cos(t - 81.03^\circ)$, where $t > 0$.

$$\boxed{\begin{array}{l} 2|B|e^{-\alpha t}\cos(\beta t + \theta) \\ \text{where } 2|B| = 3.848,\ \theta = -81.03^\circ \\ \alpha = 0,\ \beta = 1 \end{array}}$$

Here's another question:
What is the delta-function response $h(t)$ in Case 1?

Answer: $h(t) = \mathcal{L}^{-1}\frac{5s+7}{s+2} = \mathcal{L}^{-1}\left(5 - \frac{3}{s+2}\right) = 5\delta(t) - 3e^{-2t}u(t)$

Transfer Functions

For an input-output system:

the ratio of the Laplace transform $V_2(s)$ of the output $v_2(t)$ over the Laplace transform $V_1(s)$ of the input $v_1(t)$ is the transfer function:

$$H(s) = \frac{V_2(s)}{V_1(s)}$$

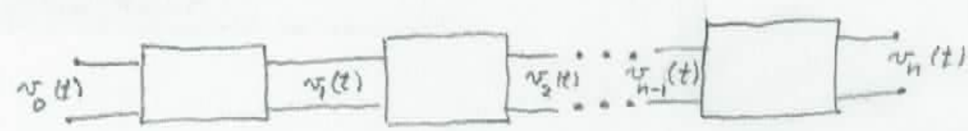
H(s) is also the Laplace transform of the delta-function response

That is, $H(s) = \mathcal{L}\, v_2(t)$ when $v_1(t) = \delta(t)$

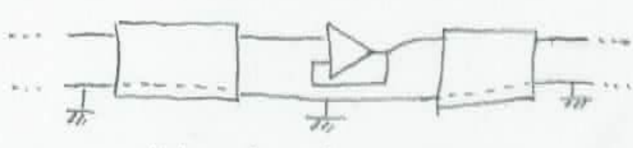
This idea is useful for a variety of electrical systems consisting of several input-output subsystems.

For example, consider a "cascade" of n input-output systems:

We call these "blocks"



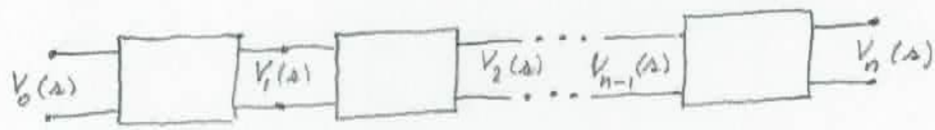
(✩) Assume that the output of each block does not depend upon what the next block is. This can be assured by connecting voltage followers between blocks, as for instance:



where the lower line is "ground"

But, this assumed condition often holds in other ways as well.

Upon applying the Laplace transformation, we get the transformed cascade:



For the blocks, we have the transfer functions:

$$H_1(s) = \frac{V_1(s)}{V_0(s)}, \quad H_2(s) = \frac{V_2(s)}{V_1(s)}, \quad \cdots, \quad H_n(s) = \frac{V_n(s)}{V_{n-1}(s)}$$

Then, under the assumed condition (A), we can write the overall transfer function

$$H(s) = \frac{V_n(s)}{V_0(s)}$$

as the product of the transfer functions of the individual blocks:

$$\frac{V_n(s)}{V_0(s)} = \frac{\cancel{V_1(s)}}{V_0(s)} \cdot \frac{\cancel{V_2(s)}}{\cancel{V_1(s)}} \cdot \frac{V_3(s)}{\cancel{V_2(s)}} \cdots \frac{\cancel{V_{n-1}(s)}}{V_{n-2}(s)} \cdot \frac{V_n(s)}{\cancel{V_{n-1}(s)}}$$

Thus,

$$H(s) = H_1(s)\, H_2(s)\, H_3(s) \cdots H_n(s)$$

This result for cascades often holds for a variety of electrical systems.

<u>Note</u>: For all this to hold, the individual blocks must <u>not</u> have independent sources or nonzero initial conditions on inductors and capacitors within the blocks.

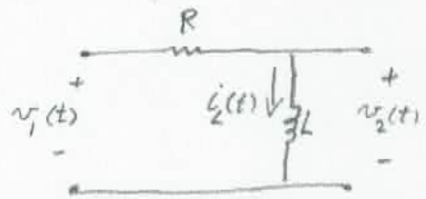
Also, a dependent source within a block may only depend upon a current or voltage within the same block, not elsewhere.

Another Way to Use a Transfer Function

The transfer function $H(s)$ of an input-output system

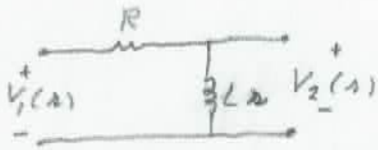
provides a convenient way to get the output $g(t)$ to any input $f(t)$. This is convenient when we wish to know the outputs for many different inputs.

Example: Consider



Assume $i_L(0+) = 0$

In terms of the transformed circuit:



we have

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{Ls}{Ls+R} = \frac{s}{s+\frac{R}{L}}$$

For $\frac{R}{L}$ "big" compared with s we have a "differentiator":

$H(s) \approx s$ (s "small")

This corresponds to approximate differentiation for t "big".

So, choosing $v_1(t) = e^{-t}u(t)$, for instance, we get

$$V_2(s) = H(s)\, V_1(s) = \frac{s}{s+\frac{R}{L}} \cdot \frac{1}{s+1}$$

For specificity, let $L=1$ and $R=2$. Then,

$$V_2(s) = \frac{s}{(s+2)(s+1)} = \frac{2}{s+2} - \frac{1}{s+1}$$

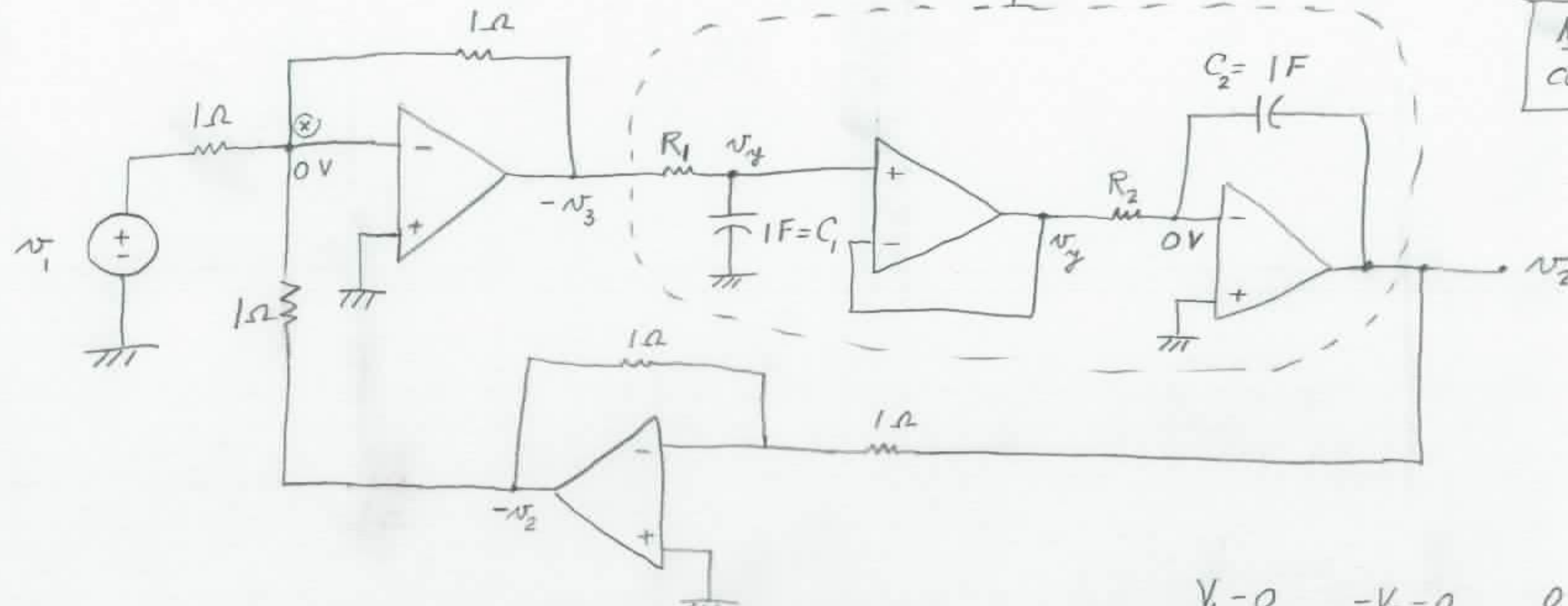
Hence, $v_2(t) = 2e^{-2t}u(t) - e^{-t}u(t)$

Note that $v_2(t)$ approaches $\frac{d}{dt}v_1(t) = -e^{-t}$ as $t \to \infty$.
That is, this system acts like a differentiator of the input when t is "large".

A "FEEDBACK" SYSTEM:

TRANSFER FUNCTION = $G(s) = \dfrac{V_2(s)}{V_3(s)}$

$\boxed{\text{NOT A CONDUCTANCE.}}$



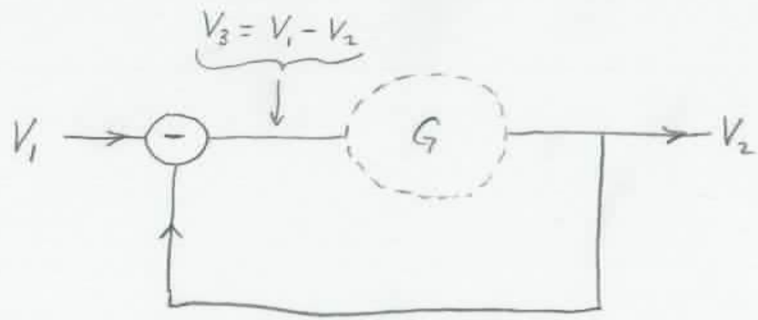
$$\frac{V_1 - 0}{1} + \frac{-V_2 - 0}{1} = \frac{0 - (-V_3)}{1}$$

THIS GIVES THIS.

$$V_2 = G\,V_3 \quad , \quad V_3 = V_1 - V_2$$

$$\therefore\ V_2 = G\,V_1 - G\,V_2$$

$$V_2 = \frac{G}{1+G}\,V_1$$



(FEEDBACK BLOCK ABSORBED INTO ERROR SENSOR)

First to establish the block equations:

Done on diagram. This gives the block diagram.

Next, to get $G(s)$:

By voltage-divider rule: $$V_y = -V_3 \frac{\frac{1}{C_1 s}}{R_1 + \frac{1}{C_1 s}} = -V_3 \frac{1}{1+sR_1C_1}$$

By KCL: $$\frac{V_y - 0}{R_2} = \frac{0 - V_2}{\frac{1}{C_2 s}} \quad \text{or} \quad \frac{V_y}{R_2} = -C_2 s V_2 \quad \text{or} \quad V_2 = \frac{V_y}{-sR_2C_2}$$

Combining $$V_2 = \frac{1}{-sR_2C_2}\left(-V_3 \frac{1}{1+sR_1C_1}\right) = V_3 \frac{1}{sR_2C_2(1+sR_1C_1)}$$

$\tau_1 = R_1 C_1$
$\tau_2 = R_2 C_2$

$$\therefore \; G = \frac{V_2}{V_3} = \frac{1}{sR_2C_2(1+sR_1C_1)} = \frac{1}{\tau_2 s(1+\tau_1 s)}$$

So, the "overall" transfer function is:

$$H = \frac{V_2}{V_1} = \frac{G}{1+G} = \frac{\frac{1}{sR_2C_2(1+sR_1C_1)}}{1+\frac{1}{sR_2C_2(1+sR_1C_1)}}$$

$$H = \frac{1}{1+sR_2C_2(1+sR_1C_1)} = \frac{1}{1+\tau_2 s(1+\tau_1 s)} = \frac{1}{\tau_1\tau_2 s^2 + \tau_2 s + 1} = \frac{\frac{1}{\tau_1\tau_2}}{s^2 + \frac{s}{\tau_1} + \frac{1}{\tau_1\tau_2}}$$

Now, let's say we want to put the poles of H at $-1 \pm j$

$$(s+1-j)(s+1+j) = (s+1)^2 + 1 = s^2 + 2s + 2$$

So we need $\frac{1}{\tau_1} = 2$ and $\frac{1}{\tau_1 \tau_2} = 2$

or $\tau_1 = \frac{1}{2}$ and $\frac{1}{\frac{1}{2}\tau_2} = 2$

or $\tau_2 = 1$

Can do by choosing $R_1 C_1 = \frac{1}{2}$ and $R_2 C_2 = 1$

Then, we have

$$H(s) = \frac{2}{s^2 + 2s + 2}$$