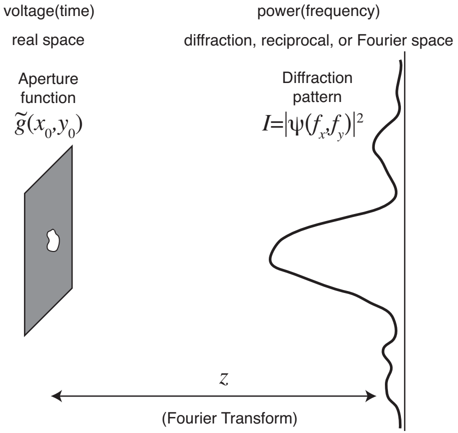
Propagating wavefields

Our goal was to calculate a downstream wavefield $\psi(x, y, z)$ based on an "aperture" function $\tilde{g}(x_0, y_0, z = 0)$ which might modify both magnitude and phase:



Propagating wavefields

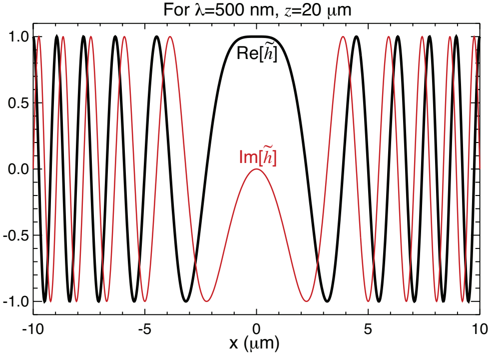
We found that we could write the Fresnel-Kirchoff diffraction equation as

$$\begin{aligned}
\psi(x,y,z) &= \psi_0 \frac{\lambda}{A} \int_{x_0} \int_{y_0} \tilde{g}(x_0,y_0) \frac{\exp[-ikr]}{r} \cos\theta \\
&= \psi_0 \frac{\lambda}{z} \frac{1}{A} e^{-i2\pi\frac{z}{\lambda}} \\
&\quad \int_{x_0} \int_{y_0} \tilde{g}(x_0,y_0) e^{-i\pi\frac{(x-x_0)^2}{\lambda z}} e^{-i\pi\frac{(y-y_0)^2}{\lambda z}} \qquad (1) \\
&= \psi_0 \frac{\lambda}{z} \frac{1}{A} e^{-i2\pi\frac{z}{\lambda}} \left\{ \tilde{g}(x_0,y_0) * \tilde{h}(x_0,y_0) \right\} \qquad (2)
\end{aligned}$$

and that this involved a convolution of $\tilde{g}(x_0,y_0)$ with a propagator function $h(x_0,y_0) = \exp[-i\pi(x_0^2+y_0^2)/(\lambda z)]$

The propagator function

The propagator function $h(x_0, y_0)$ has a magnitude of 1 everywhere. Here's how its real and imaginary part varies:



Also, the Fourier transform of the propagator is a propagator:

$$H(f) = \sqrt{\lambda z}\exp\left[i\pi\lambda z f^2\right] = \sqrt{\lambda z}\exp\left[i\pi x^2/(\lambda z)\right] \quad (3)$$

Wavefield propagation: final result

Whew! Now that we know that $H(f) = \sqrt{\lambda z} e^{i\pi\lambda z f^2}$, we can re-write the Fresnel-Kirchoff diffraction integral in the Fresnel approximation (Eq. 2) as

$$\begin{aligned} \psi &= \psi_0 \frac{\lambda}{z} \frac{1}{A} e^{-i2\pi\frac{z}{\lambda}} \left\{ \tilde{g}(x_0, y_0) * \tilde{h}(x_0, y_0) \right\} \\ &= \psi_0 \frac{\lambda^3}{A} \exp\left[-i2\pi \frac{z}{\lambda} \right] \mathcal{F}^{-1} \left\{ \mathcal{F}\left\{ \tilde{g}(x_0, y_0) \right\} \cdot e^{i\pi\lambda z (f_x^2 + f_y^2)} \right\} \qquad (4) \\ &= \psi_0 (\lambda^3/A) \exp[-i2\pi z/\lambda] \mathcal{F}^{-1} \left\{ \mathcal{F}\left\{ \tilde{g}(x_0, y_0) \right\} \cdot e^{i\pi\lambda z (f_x^2 + f_y^2)} \right\} \end{aligned}$$

It now becomes something we can calculate numerically:

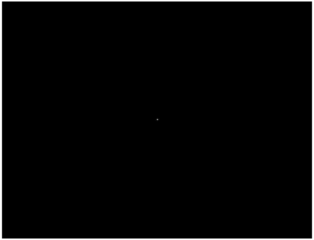
1. Take the Fourier transform of the input wavefield $\mathcal{F}\{\tilde{g}(x_0, y_0)\}$
2. Multiply by the Fourier transform of a propagator $\exp[i\pi\lambda z(f_x^2 + f_y^2)]$
3. Inverse transform the result $\mathcal{F}^{-1}\{\}$

Example: numerical wavefield propagation

Let's illustrate this by doing some numerical Fourier transforms on a digitized image of Jon Stewart, host of *The Daily Show*:

1. Take the square root of the image to convert it from intensity to amplitude
2. Take the Fourier transform (remember that we shift by $N/2$ on the input and output, as we illustrated in 1D!)
3. Square the result to look at the resulting far-field intensity

$\Rightarrow$

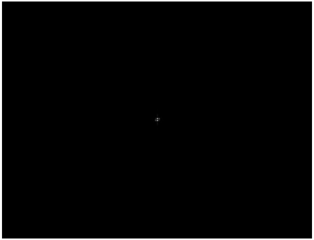


What the $#%?? Just a bright dot in the center? The Fourier transform of what gives a spike?

Removing the DC value

Aha! The Fourier transform of a flat function gives a delta function. So let's do this:

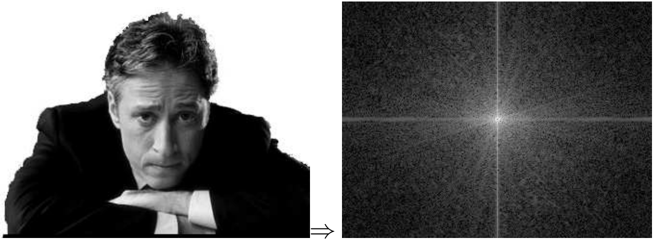
1. Take the square root of the image to convert it from intensity to amplitude
2. Subtract the average value of the image from the image, to give it a "DC" (direct current in circuits) value of zero.
3. Now take the Fourier transform and square it for intensity

 $\Rightarrow$ 

Well, this is still not terribly informative. Maybe the intensity scaling needs tweaking?

We need a log scale

Examining the values of our image, we find that the intensity varies from $10^{-1.1}$ to $10^{10.7}$! Therefore maybe it's better to display the image on a logarithmic scale, and show eight decades worth of intensity:



This is looking better—but what about those horizontal and vertical streaks?

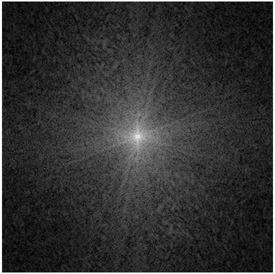
Repeated objects and edge ringing

It turns out that the discrete Fourier transform, with its finite "integration" bounds, has a property of assuming that the object repeats in an array:

Notice the discontinuities at the edges? They will diffract strongly!

Computational tricks

Solution: roll the edges off with a gaussian (here with $\sigma = 4$ pixels), and embed the image in a larger array (here we go from 322×246 to 512×512 since FFTs are fast with $N = 2^n$ pixels):

$\log\left(|\mathcal{F}\{$  $\}|^2 \right) =$ 

OK, now we have a version of our image that is well-behaved for numerical Fourier transform calculations!

Making Jon fuzzy

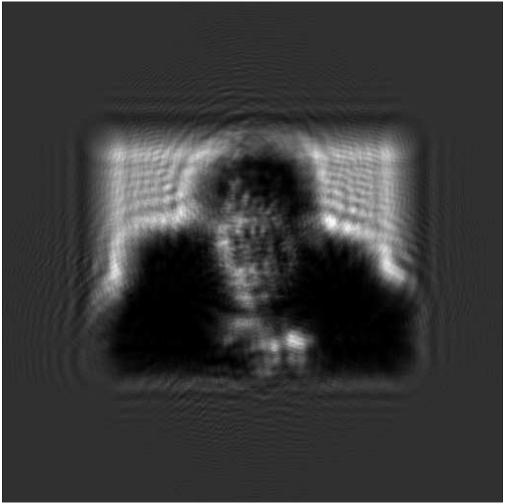
Now we can follow our perscription of wavefield propagation. Let's assume $\lambda = 500$ nm, a micrograph pixel size of 1 μm, and propagation distances of 10, 100, and 1000 μm:

We'll blow these individual pictures up in subsequent slides.
These pictures of the intensity of the propagated wavefields involve simply the information of the original object, and a well-defined mathematical transform. This suggests that one can reconstruct images from these propagated waveforms, but that's the subject for the next lecture!

Jon at 10 μm

Jon at 100 μm

Jon at 1000 μm

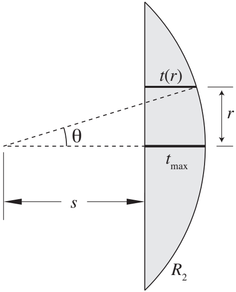


Imaging via propagation

- Let's put propagation to use. Let's see how it provides another way to describe how a lens works.
- To do so, we will consider another way to look at a lens: it has a radially-dependent phase function.
- We will propagate a wavefield by an object distance s, alter its phase by the lens, and propagate it by a distance s' to the image plane.

Phase function of a lens

- Consider half of a lens as shown at right. The second optical surface has a radius of curvature R_2 which we will write as a positive number $|R_2|$
- Its thickness at the center is $t_{\max} = |R_2| - s$.
- The thickness $t(r)$ at a radius r off the optical axis is given by



$$\begin{aligned} t(r) &= |R_2|\cos\theta - s \\ &= |R_2|\cos\theta - (|R_2| - t_{\max}) \\ &= t_{\max} - |R_2|(1-\cos\theta). \qquad (5) \end{aligned}$$

In the limit $r \ll |R_2|$, we have $1 - \cos\theta \simeq \theta^2/2$ and $\theta \simeq r/|R_2|$, so that

$$t(r) \simeq t_{\max} - \frac{r^2}{2|R_2|}. \qquad (6)$$

Phase function of a lens II

Again, we had from Eq. 6 an approximate thickness of the glass of $t(r) \simeq t_{\max} - r^2/(2|R_2|)$, and the thickness of "not glass" is $t_{\max} - t(r)$. We can then calculate the phase function $\varphi(r)$ corresponding to this thickness:

$$\begin{aligned} \varphi(r) &= \Big[-nkt(r)\Big] + \Big[-k\big(t_{\max} - t(r)\big)\Big] \\ &= \Big[-nkt_{\max}\Big] + \Big[nk\frac{r^2}{2|R_2|}\Big] + \Big[-kt_{\max}\Big] + \Big[kt_{\max}\Big] + \Big[-k\frac{r^2}{2|R_2|}\Big] \\ &= \Big[-nkt_{\max}\Big] + \Big[(n-1)k\frac{r^2}{2|R_2|}\Big]. \end{aligned} \tag{7}$$

Ignoring the constant $[-nkt_{\max}]$ term, then, the phase shift $\exp[i\varphi(r)]$ relative to there being no lens at all is given by

$$\varphi(r) = \Big[(n-1)k\frac{r^2}{2|R_2|}\Big]. \tag{8}$$

Phase function of a lens III

Again, we had from Eq. 8 the result

$$\varphi(r) = \left[(n-1)k\frac{r^2}{2|R_2|}\right] = \left[(n-1)\frac{2\pi}{\lambda}\frac{r^2}{2|R_2|}\right] = \left[(n-1)\frac{\pi r^2}{\lambda|R_2|}\right]$$

Now the plano-convex lens we sketched had $|R_2| = -R_2$. If we repeat the same calculation for a convex-plano lens with $|R_1| = +R_1$, and add the results together, we have

$$\varphi(r) = \left[(n-1)\frac{\pi r^2}{\lambda}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right] = \frac{\pi r^2}{\lambda f} \tag{9}$$

if we use the usual definition for the focal length of a thin lens of $1/f \equiv (n-1)(1/R_1 - 1/R_2)$.

An optical system via propagators

- Let's now describe an optical system using propagators and lens phase functions. From Eq. 4 we have

$$\psi = \psi_0 \frac{\lambda^3}{A} e^{-i2\pi\frac{z}{\lambda}} \mathcal{F}^{-1}\left\{\mathcal{F}\left\{\tilde{g}(x_0, y_0)\right\} \cdot \tilde{H}(f_x, f_y)\right\} \qquad (10)$$

  and we'll drop $(\lambda^3/A)e^{-i2\pi z/\lambda}$ in what follows.
- If $\tilde{g}(x_0, y_0)$ is a δ-function, then its Fourier transform is 1 at all frequencies: $\mathcal{F}\{\delta\} = 1$.
- We then have

$$\begin{aligned} \psi &= \psi_0 \mathcal{F}^{-1}\left\{1 \cdot \tilde{H}(f_x, f_y)\right\} = \psi_0\, h(x, y, z = s) \\ &= \psi_0 \exp\left[-i\pi\frac{x^2 + y^2}{\lambda s}\right] \end{aligned} \qquad (11)$$

- We then multiply by the phase function of the lens $e^{i\varphi(r)}$ (using Eq. 9), giving

$$\psi = \psi_0 \exp\left[-i\pi\frac{x^2 + y^2}{\lambda s}\right] \exp\left[i\pi\frac{x^2 + y^2}{\lambda f}\right] \qquad (12)$$

The solution thus far

- Continuing from Eq. 12, we have

$$\begin{aligned} \psi &= \psi_0 \exp\left[-i\pi\frac{x^2+y^2}{\lambda s}\right]\exp\left[i\pi\frac{x^2+y^2}{\lambda f}\right] \\ &= \psi_0 \exp\left[i\pi\frac{x^2+y^2}{\lambda}\left(\frac{1}{f}-\frac{1}{s}\right)\right] \end{aligned} \tag{13}$$

- If $f = s$, then the entire quadratic phase term $\exp[i\pi(x^2+y^2)/(\lambda z)]$ becomes 1. In that case, the lens has taken a point source and turned it into a plane wave, just as we would expect from geometric optics with a point source a focal length away from a lens.
- If we were to place the Dirac δ function not on but off the optical axis, the Shift Theorem of Fourier transforms would give

$$\mathcal{F}\{\delta(x,y)\} = 1 \cdot \exp\left[-i2\pi(xf_x + yf_y)\right] \tag{14}$$

so that positions (x, y) would be changed to spatial frequencies or diffraction angles (f_x, f_y).
- A lens gives the Fourier transform of the plane located a focal length away.

Propagating further

- Again, we had from Eq. 13 the result

$$\psi = \psi_0 \exp\left[i\pi \frac{x^2+y^2}{\lambda}\left(\frac{1}{f}-\frac{1}{s}\right)\right] = \psi_0 \exp\left[i\pi \frac{x^2+y^2}{\lambda A}\right]$$

[with $\frac{1}{A} \equiv \left(\frac{1}{f}-\frac{1}{s}\right)$] for light leaving the plane of the lens.

- Now let's propagate by some further distance $s' + \Delta$, again using a propagator $h(x,y,z)$:

$$\begin{aligned}\psi &= \psi_0 \exp\left[i\pi \frac{x^2+y^2}{\lambda A}\right] * \exp\left[-i\pi \frac{x^2+y^2}{\lambda(s'+\Delta)}\right] \\ &= \psi_0\, \mathcal{F}^{-1}\Big\{\exp\left[+i\pi\lambda A(f_x^2+f_y^2)\right] \\ &\qquad \cdot \exp\left[-i\pi\lambda(s'+\Delta)(f_x^2+f_y^2)\right]\Big\} \qquad (15)\end{aligned}$$

Propagating further II

- Again, we had

$$\psi = \psi_0 \, \mathcal{F}^{-1} \Big\{ \exp\left[+i\pi\lambda A(f_x^2 + f_y^2)\right] \cdot \exp\left[-i\pi\lambda(s' + \Delta)(f_x^2 + f_y^2)\right] \Big\}$$

- Consider the case when $\Delta = 0$ and $s' = A$. In that case, In that case, the complex exponentials inside $\{\}$ cancel each other out, so we are left with

$$\psi = \psi_0 \mathcal{F}^{-1}\,\{1\} = \psi_0\,\delta(0,0). \tag{16}$$

Wavefield when $A = s'$

- We just found that for $\Delta = 0$ and $A = s'$ we wind up with a wavefield of $\psi_0\, \delta(0,0)$. That is, we have imaged from a point to point!
- This happens when $A = s'$, or $1/A = 1/s'$, or $1/A - 1/s' = 0$, or

$$\left(\frac{1}{f} - \frac{1}{s}\right) - \frac{1}{s'} = 0 \qquad \text{or} \qquad \frac{1}{f} = \frac{1}{s} + \frac{1}{s'} \tag{17}$$

  which is simply the thin lens imaging equation!
- What happens when we go out of focus? That corresponds to $\Delta \neq 0$. We then propagate a point source (the Dirac δ function we got in the image plane) with a propagator, giving

$$\psi(\Delta) = \psi_0 \exp\left[-i\pi \frac{x^2 + y^2}{\lambda\Delta}\right] \tag{18}$$

- This may seem a bit nonsensical, since it suggests that light at the out-of-focus plane D extends over the entire plane (there's no limit to x or y in the above equation). In fact if we put in a finite lens we get the diffraction pattern of a pinhole superimposed on the focus: the Airy pattern!

Defocus is an aberration!

- On slide 23 of l11.pdf, we found that the term in lowest order of h in the aberration function $a(Q)$ was

$$\frac{h^2}{2}\left[\left(\frac{n_1}{s}+\frac{n_2}{s'}\right)-\left(\frac{n_2-n_1}{R}\right)\right]$$

and this vanished at the single refractive interface imaging condition.

- If we add to the terms in $[\,]$ an extra term of $1/\Delta$, we will have remaining h^2-dependent aberration term of

$$\varphi=\frac{2\pi a(Q)}{\lambda}=\frac{2\pi}{\lambda}\frac{h^2}{2}\frac{1}{\Delta}=\pi\frac{h^2}{\lambda\Delta} \tag{19}$$

which is of the same form as Eq. 18.

- We therefore see that defocus is an aberration.

Diffraction gratings

- You have learned about diffraction from slits, and what the intensity pattern looks like.
- A grating can be used with a slit to pass only a certain wavelength of light in a *monochromator*.
- If instead you collect information on a range of wavelengths at once (such as with a piece of film, or a diode array), you have a *spectrometer*.
- There are several subtleties to diffraction gratings. If light comes in over a range of λ_{low} to λ_{high}, you do not want to have the $(n+1)^{\mathrm{th}}$ diffraction order from λ_{high} overlap with the n^{th} order from λ_{low}. This allows you to work out something called the *free spectral range* of a monochromator or spectrometer.
- There is one other important limit: the *Fourier transform* limit:

$$\frac{\lambda_1 - \lambda_2}{\frac{1}{2}(\lambda_1 + \lambda_2)} = \frac{1}{x}$$

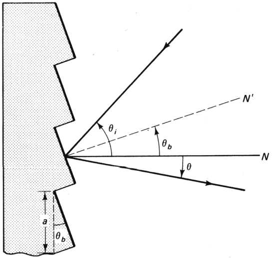
requires $x = nN$ where n is the diffraction order and N is the number of slits in your grating.

Blazed gratings

- A grating made of slits is inefficient; light goes into many diffraction orders simultaneously, and half the amplitude/a quarter of the intensity goes straight through without being diffracted.
- A *blazed grating* seeks to cure this by combining diffraction with reflection to put most of the light into one diffracted order.
- The reflection condition for a blazed grating is

$$\theta_i - \theta_b = \theta + \theta_b \qquad (20)$$

where θ_b is the blaze angle of the grating.



(Figure from Pedrotti & Pedrotti)

Blazed gratings II

- Again we have the reflection condition from Eq. 20 as

$$\theta_i - \theta_b = \theta + \theta_b \qquad \Rightarrow \qquad \theta_b = \frac{\theta_i - \theta}{2} \tag{21}$$

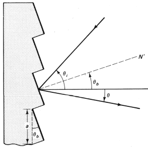
- We also have the condition of constructive interference of

$$m\lambda = a\left(\sin\theta_i + \sin\theta\right) = a\left(\sin\theta_i + \sin(2\theta_b - \theta_i)\right) \tag{22}$$

where in the second expression we have used a sign convention to reflect the fact that positive angles are used for the "normal" case of the diffracted beam coming out on the other side of the surface normal from the incident beam.

- One way to used blazed gratings is in the *Littrow mount*, where $\theta_i = \theta_b$. In this case, Eq. 22 becomes

$$m\lambda = 2a\sin\theta_b \tag{23}$$



(Figure from Pedrotti & Pedrotti)

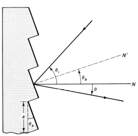
Blazed gratings III

- Another way to use a blazed grating is with light incident along the normal N to the grating. In this case $\theta_i = 0$.
- From Eq. 20 of $\theta_i - \theta_b = \theta + \theta_b$ and our sign convention we have

$$\theta = -2\theta_b. \tag{24}$$

- From Eq. 22 of $m\lambda = a\left(\sin\theta_i + \sin(2\theta_b - \theta_i)\right)$ we have

$$m\lambda = a\sin(2\theta_b) \tag{25}$$



(Figure from Pedrotti & Pedrotti)