Linearity and Superposition

For a resistor R: Ohm's law is $v = Ri$ where R is a constant.

Homogeneity: Let K be another constant. Then

$$R(Ki) = KRi$$

That is, if we double (or triple or ...), we double (or triple or ...) the resulting voltage.

Additivity: If $v_1 = Ri_1$ and $v_2 = Ri_2$, then

$$v_1 + v_2 = R(i_1 + i_2)$$

That is, the addition of two currents, leads to the addition of the corresponding voltages.

The combination of homogeneity and additivity is linearity

In particular, let K_1 and K_2 be two constants. Then,

$$R(\underbrace{K_1 i_1 + K_2 i_2}) = K_1 Ri_1 + K_2 Ri_2$$

That is, the linear combination of two currents leads to the linear combination of the corresponding voltages.

Looking ahead: Linearity also holds for inductors and capacitors:

For inductors: $L\frac{d}{dt}(K_1 i_1 + K_2 i_2) = K_1 L\frac{d}{dt} i_1 + K_2 L\frac{d}{dt} i_2.$

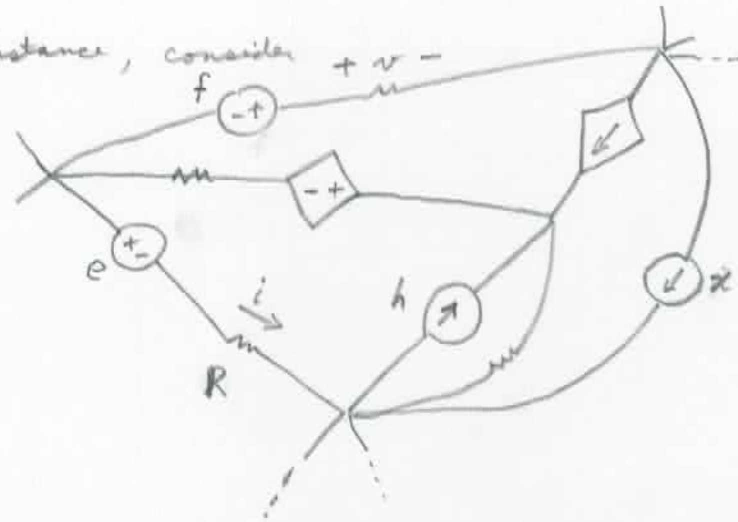
For capacitors: $\frac{1}{C}\int_{-\infty}^{t}(K_1 i_1(x) + K_2 i_2(x))dx$ where $i_1(-\infty) = i_2(-\infty) = 0$

$$= K_1 \frac{1}{C}\int_{-\infty}^{t} i_1(x)\,dx + K_2 \frac{1}{C}\int_{-\infty}^{t} i_2(x)\,dx$$

More generally:

Linearity holds for all the <u>independent</u> sources in network.

For instance, consider



If we replace all the independent sources e, f, h, x by the <u>same</u> linear combination: $K_1 e_1 + K_2 e_2$, $K_1 f_1 + K_2 f_2$, $K_1 h_1 + K_2 h_2$, $K_1 x_1 + K_2 x_2$,

then every voltage and current gets replaced by the <u>same</u> linear combination: For instance,

i gets replaced by $K_1 i_1 + K_2 i_2$

and v gets replaced by $K_1 v_1 + K_2 v_2$.

$\left\{ \begin{array}{l} \text{where } i_1 \text{ is produced by } e_1, f_1, h_1, x_1 \\ \text{and } i_2 \text{ is produced by } e_2, f_2, h_2, x_2 \end{array} \right.$

However, this does <u>not</u> work for power.

Before making the linear combination, the power in R is

$$i^2 R.$$

After making the linear combination, the power in R is

$$(K_1 i_1 + K_2 i_2)^2 R = (K_1^2 i_1^2 + 2K_1 K_2 i_1 i_2 + K_2^2 i_2^2) R$$

It is this crossproduct term that prevents linearity.

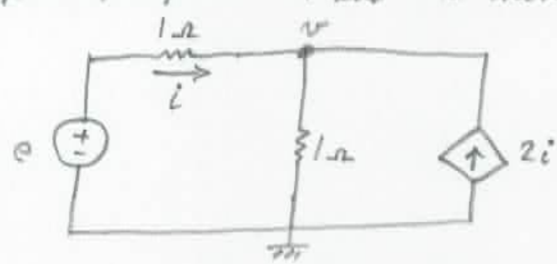
(Mistakenly invoking linearity on power we would only have
$$(K_1^2 i_1^2 + K_2^2 i_2^2) R$$
which is the wrong answer.)

Important Comment:

Linearity applies only to the independent sources.

Do not try to apply it to the dependent sources.

A simple example: Find the node voltage v in here



KCL at v:

$$\frac{v-e}{1} + \frac{v}{1} - 2i = 0 \qquad \text{But, } i = \frac{e-v}{1}$$

So $\quad v - e + v - 2e + 2v = 0$

$$v = \frac{3}{4} e.$$

Now replace e by $2e_1 + 3e_2$. Get the new v.
(Do not alter the $2i$ dependent source.)

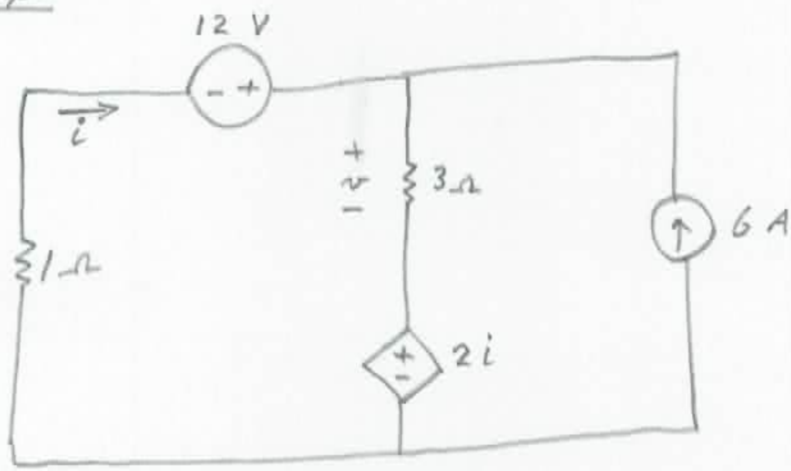
New $v = \frac{3}{4}(2e_1 + 3e_2)$.

Superposition

The current (or voltage) in any part of a general network is the sum of the currents (or voltages) due to each <u>independent</u> source acting alone.

To get the current due to one of those independent sources acting alone, open up all the other independent current sources, and short all the other independent voltage sources.

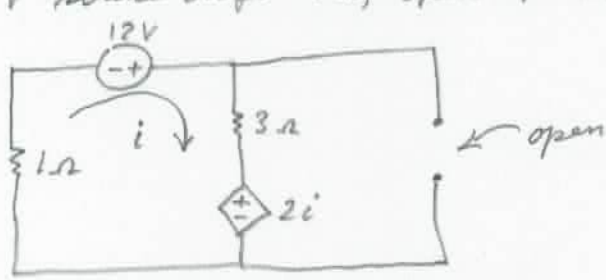
<u>Example</u>



Using superposition, find v. Then find the power in the 3-Ω resistor.

<u>Step 1</u>: Use 12 V source only. So, open up the 6 A source.

We have:



KVL around i mesh: $i - 12 + 3i + 2i = 0$

$6i = 12$

$i = 2$, So $v = 2 \times 3 = 6$ V

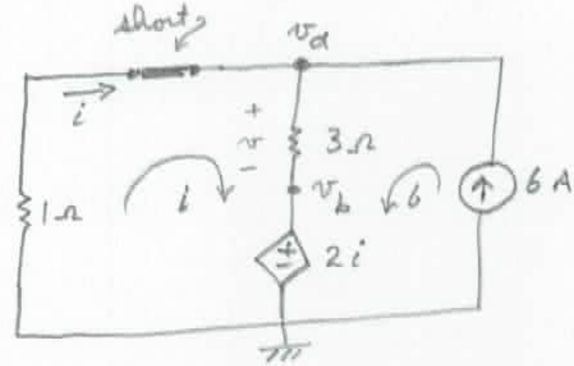
Power in 3 Ω resistor $= 2^2 \times 3 = 12$ W

But this result on power will <u>not</u> be used.

Step 2: Use 6 A source alone.

So, short the 12 V source.

We get



By a mesh analysis:

KVL around i mesh: $\quad i + (i+6)3 + 2i = 0$

$$6i = -18$$

$$i = -3$$

So, $v = (i+6)3 = (-3+6)3 = 9\ V$

Power in 3-Ω resistor is now $9^2/3 = 27$ W.
Again, this will not be useful.

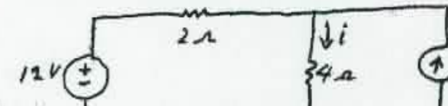
By superposition, total voltage on 3Ω with both independent sources operative:

$$v = 6 + 9 = 15\ V.$$

power in 3-Ω resistor is truly $\frac{15^2}{3} = 5 \times 15 = 75$ W.

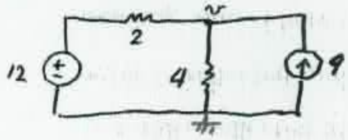
Note: The correct power is not $12\ W + 27\ W = 39\ W$
Superposition does not work for power because we lose the crossproduct term $2v_1 v_2/R = \frac{2 \times 6 \times 9}{3} = 36\ W$

Another Example:



FIND i. THEN GET THE POWER P IN THE $4\,\Omega$ RESISTO

NODAL ANALYSIS:



$$\frac{v-12}{2} + \frac{v}{4} = 9 \qquad \text{So: } 2v - 24 + v = 36$$

$$3v = 60$$

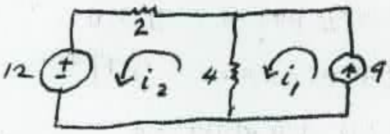
$$v = 20 \quad \Rightarrow \quad i = \frac{20}{4} = 5 \text{ A}$$

$P = i^2 R$

$P = 5^2 \times 4$

$P = 100$ W

MESH ANALYSIS:



$i_1 = 9$

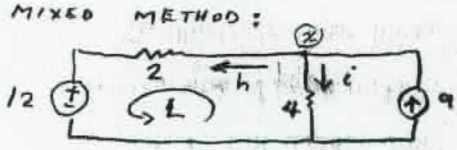
$4(i_2 - i_1) + 2i_2 + 12 = 0 \qquad$ So: $4i_2 - 36 + 2i_2 = -12$

$6i_2 = -24, \quad i_2 = 4$

$i = i_1 - i_2 = 9 - 4 = 5$

$P = 100$ W

A MIXED METHOD:



KCL AT NODE x: $h + i = 9 \qquad$ So: $h = 9 - i$

KVL AROUND LOOP L: $-4i + 2h + 12 = 0$

So: $-4i + 2(9-i) + 12 = 0$

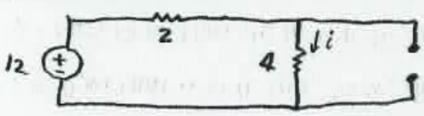
$-6i = -12 - 18 = -30$

$i = 5$

$P = 100$ W

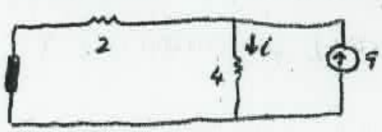
BY SUPERPOSITION:

FIRST, OPEN THE INDEPENDENT CURRENT SOURCE:



So: $i = \frac{12}{2+4} = 2 \qquad P_1 = 2^2 \times 4 = 16$ W

SECOND, SHORT THE INDEPENDENT VOLTAGE SOURCE:



By CURRENT DIVISION,

$$i = 9 \times \frac{2}{2+4} = 3$$

$P_2 = 3^2 \times 4 = 36$ W

$P = P_1 + P_2 = 16 + 36 = 52$ W

FINALLY, ADD THE TWO RESULTS:

$$i = 2 + 3 = 5$$

WRONG ANSWER!

WHAT HAPPENED?

WHEN COMPUTING POWER AT "DC" (OR AT ANY FIXED FREQUENCY), ONE CANNOT USE SUPERPOSITION.

INSTEAD, ONE MUST FIRST GET THE TOTAL CURRENT, AND THEN USE THAT TO GET THE POWER. REASON:

$$P = i^2 R = (i_1 + i_2)^2 R = (i_1^2 + 2i_1 i_2 + i_2^2) R. \neq (i_1^2 + i_2^2) R$$

DO NOT LOSE THIS TERM.