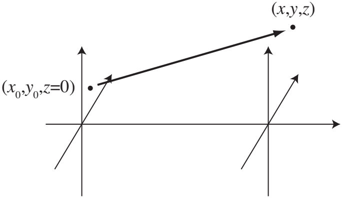
Adding up point sources

- Recall that we start with a wavefield at an input plane $(x_0, y_0, z = 0)$. We treat it as a bunch of Huygens point sources, each with magnitude and phase modulated by $\tilde{g}(x_0, y_0)$.
- To get the field at a downstream position (x, y, z), add up the contribution from the input plane sources:

$$\psi(x, y, z) = \psi_0 \frac{\lambda}{A} \int_{x_0} \int_{y_0} \tilde{g}(x_0, y_0) \frac{\exp[-ikr]}{r} \cos\theta. \tag{1}$$

- The radius r is given by

$$\begin{aligned} r &= \sqrt{z^2 + (x - x_0)^2 + (y - y_0)^2} \\ &= z\sqrt{1 + \frac{(x - x_0)^2}{z^2} + \frac{(y - y_0)^2}{z^2}} \end{aligned} \tag{2}$$



Fresnel approximation

We then did a Taylor series expansion of r from the form in Eq. 2 to obtain

$$r \simeq z\left[1+\frac{(x-x_0)^2}{2z^2}+\frac{(y-y_0)^2}{2z^2}-\frac{(x-x_0)^4}{8z^4}-\frac{(y-y_0)^4}{8z^4}+\ldots\right] \quad (3)$$

The *Fresnel approximation* involved discarding terms like $(x-x_0)^4/(8z^4)$ in phase, or saying

$$z^3 \gg \rho^4/(2\lambda) \qquad \text{(Fresnel approximation)}$$

with $\rho=\sqrt{(x-x_0)^2+(y-y_0)^2}$. We also said that $1/r \Rightarrow 1/z$ for the magnitude term.

This led to the Fresnel-Kirchoff diffraction integral:

$$\begin{aligned}\psi(x,y,z) &= \psi_0 \frac{\lambda}{z}\frac{1}{A}\exp\left[-i\frac{2\pi z}{\lambda}\right]\exp\left[-i\pi\frac{x^2+y^2}{\lambda z}\right] \\ & \int_{x_0}\int_{y_0}\tilde{g}(x_0,y_0)\exp\left[-i\pi\frac{x_0^2+y_0^2}{\lambda z}\right]\exp\left[i2\pi\frac{xx_0+yy_0}{\lambda z}\right]\end{aligned} \quad (4)$$

Fraunhofer approximation

We also considered the *Fraunhofer approximation*, which assumes the Fresnel approximation plus saying that $(x_0^2 + y_0^2)/(\lambda z) \ll 1$ or

$$z \gg 4\frac{x_0^2 + y_0^2}{\lambda} \qquad \text{(Fraunhofer approximation)}$$

This (plus disregarding the out-of-integral phase factors, since they cancel out when calculating intensities) leads to the Fraunhofer diffraction integral:

$$\psi(x,y,z) \simeq \psi_0 \frac{\lambda}{z}\frac{1}{A}\int_{x_0}\int_{y_0} \tilde{g}(x_0,y_0)\exp\left[i2\pi\left(x_0 f_x + y_0 f_y\right)\right] dx_0\, dy_0 \qquad (5)$$

where $f_x = x/(\lambda z) = \theta_x/\lambda$ and $f_y = y/(\lambda z) = \theta_y/\lambda$ are *spatial frequencies*. We recognized this last expression as a Fourier transform.

Diffraction from a slit: analytical

We start by considering diffraction from a slit of width b in the Fraunhofer approximation:

$$\begin{aligned} \psi &= \psi_0 \frac{\lambda}{z} \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{g}(x_0, y_0) e^{i2\pi\left(\dfrac{xx_0}{\lambda z} + \dfrac{yy_0}{\lambda z}\right)} \, dx_0 \, dy_0 && (6) \\ &= \psi_0 \frac{\lambda}{z} \frac{1}{A} \int_{-b/2}^{b/2} e^{i2\pi \dfrac{xx_0}{\lambda z}} \, dx_0 . && (7) \end{aligned}$$

where we have used $\tilde{g}(x_0) = 1$ for $-b/2 \geq x \leq b/2$. Because $e^{i\theta} = \cos\theta + i \sin\theta$, the integral can be written as

$$\begin{aligned} & \int_{-b/2}^{b/2} \cos\left(2\pi \frac{xx_0}{\lambda z}\right) dx_0 + i \int_{-b/2}^{b/2} \sin\left(2\pi \frac{xx_0}{\lambda z}\right) dx_0 \\ = & \int_{-b/2}^{b/2} \cos(Bx_0) \, dx_0 + i \int_{-b/2}^{b/2} \sin(Bx_0) \, dx_0 && (8) \end{aligned}$$

with $B \equiv 2\pi x/(\lambda z) = 2\pi f_x$.

Diffraction from a slit II

Now

$$\begin{array}{ll} d\sin(Bx_0) = B\cos(Bx_0)dx_0 & d[-\cos(Bx_0)] = B\sin(Bx_0)dx_0 \\ \frac{1}{B}d[\sin(Bx_0)] = \cos(Bx_0)dx_0 & -\frac{1}{B}d[\cos(Bx_0)] = \sin(Bx_0)dx_0 \end{array}$$

so the integral of Eq. 8 becomes

$$\begin{aligned} & \int_{-b/2}^{b/2} \cos\left(Bx_0\right) dx_0 + i\int_{-b/2}^{b/2} \sin\left(Bx_0\right) dx_0 \qquad \text{with} \qquad B \equiv \frac{2\pi x}{\lambda z} \\ = \quad & \frac{1}{B}\left[\sin(Bx_0)\Big|_{-b/2}^{b/2} - i\cos(Bx_0)\Big|_{-b/2}^{b/2}\right] \\ = \quad & \frac{\lambda z}{2\pi x}\left[\sin(\pi\frac{xb}{\lambda z}) - \sin(-\pi\frac{xb}{\lambda z}) - i\cos(\pi\frac{xb}{\lambda z}) + i\cos(-\pi\frac{xb}{\lambda z})\right] \qquad (9) \\ = \quad & \frac{\lambda z}{\pi x}\sin(\pi\frac{xb}{\lambda z}) \qquad (10) \end{aligned}$$

because $\sin(-\theta) = -\sin\theta$, and $\cos(-\theta) = \cos\theta$.

Diffraction from a slit III

From Eq. 10, we have for the farfield wavefield the result

$$\psi = \psi_0 \frac{\lambda}{z} \frac{1}{A} \frac{b\lambda z}{\pi x b} \sin(\pi \frac{xb}{\lambda z}). \tag{11}$$

Now let

$$\beta \equiv \pi \frac{xb}{\lambda z}. \tag{12}$$

With this we have

$$\psi = \psi_0 \frac{\lambda}{z} \frac{b}{A} \frac{\sin \beta}{\beta} \tag{13}$$

with a limit as $\beta \rightarrow 0$ given by L'Hopital's rule as

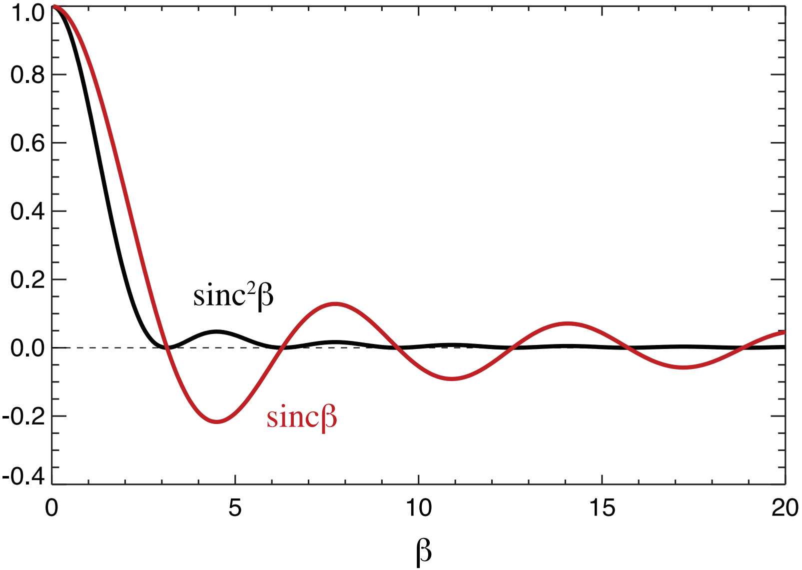
$$\lim_{\beta \rightarrow 0} \frac{\sin \beta}{\beta} = \frac{\cos \beta}{1} |_{\beta \rightarrow 0} = \frac{\cos(0)}{1} = 1. \tag{14}$$

The irradiance is given by

$$I \propto \frac{\sin^2 \beta}{\beta^2} \equiv \text{sinc}^2 \beta. \tag{15}$$

The sinc function

Here's a plot of sincβ=sin β/β, as well as sinc$^2\beta$:



Diffraction from a slit IV

The first minimum of the slit diffraction pattern is when $\beta = \pi$ which is when

$$\pi = \pi \frac{xb}{\lambda z} \text{ or } \frac{x}{z} = \frac{\lambda}{b} = \sin\theta. \tag{16}$$

Maxima are at

$$\frac{d}{d\beta}\frac{\sin\beta}{\beta} = \frac{-\sin\beta}{\beta^2} + \frac{\cos\beta}{\beta} = 0 \tag{17}$$

$$0 = -\sin\beta + \beta\cos\beta \tag{18}$$

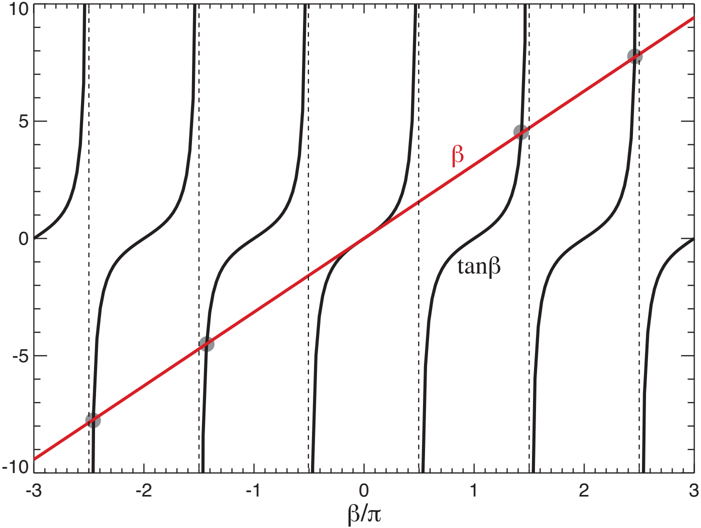
$$\beta = \frac{\sin\beta}{\cos\beta} = \tan\beta \tag{19}$$

which is why we don't quite have $(b/2)\sin\theta = \lambda$ for maxima. Finally, note that our slit function could be defined as $\mathrm{rect}(b) = \Pi(f_b)$ with a Fourier transform of

$$\mathcal{F}\{\mathrm{rect}(b)\} = \mathcal{F}\{\Pi(b)\} = \mathrm{sinc}(f_b). \tag{20}$$

Slit maxima

Here's a plot to show when $\beta = \tan\beta$, showing that the intensity maxima are *not* at $\beta = \pm(n + \frac{1}{2})\pi$ with $n = 1, 2, \ldots$

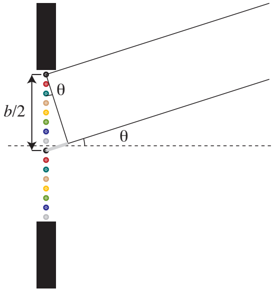


Diffraction from a slit: what you knew before

You've probably seen this construction in your first year physics course. Imagine dividing a slit up into many point sources. You can then pick, two-by-two, point sources that cancel each other when you meet the condition

$$\frac{b}{2}\sin\theta = \frac{\lambda}{2}$$
$$\sin\theta = \frac{\lambda}{b}$$

which is exactly the same condition as we found in Eq. 16.

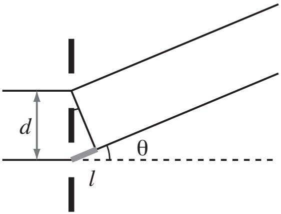


Double slit diffraction

You already know about the result for double-slit diffraction. You get constructive interference when the path length difference is equal to an integer number of wavelengths, or

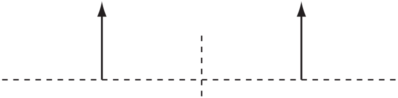
$$d \sin\theta = n\lambda$$

However, we now want to know about the overall intensity pattern using Fraunhofer diffraction which is just a Fourier transform of the aperture function.



Double slits II

- We've already solved the diffraction pattern of a single slit of width b by considering the Fourier transform of $\mathrm{rect}(b) = \Pi(b)$.
- Let's now consider two infinitessimally narrow slits that are separated by a distance a, and centered on the optical axis (slits at $+a/2$, $-a/2$). We then have an impulse pair function $\mathrm{II}(a)$:



- Each impulse function (or δ function) has the property of $\int_{-\infty}^{\infty} \delta(x_0) f(x)\, dx = f(x_0)$ so the Fourier transform of an impulse pair is [using $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$]:

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{II}(a) e^{i2\pi x f_x}\, dx &= \Big[\cos(2\pi \frac{a}{2} f_x) + i\sin(2\pi\frac{a}{2} f_x) \\ &\quad + \cos(2\pi\frac{-a}{2} f_x) + i\sin(2\pi\frac{-a}{2} f_x)\Big] \\ &= 2\cos(\pi a f_x) = 2\cos\alpha \qquad (21)\end{aligned}$$

with $\alpha \equiv \pi a f_x$.

Double slits III

- We know the solution for a slit of width b, where $\mathcal{F}\left\{\Pi(b)\right\} = \sin\beta/\beta$ with $\beta = \pi b f_x$.
- We know the solution for two infintessimal slits separated by a, where $\mathcal{F}\left\{\mathrm{II}(a)\right\} = 2\cos\alpha$ with $\alpha = \pi a f_x$.
- We can put them together using convolution! The far-field diffraction pattern is

$$\begin{aligned} \psi &= \psi_0 \frac{\lambda}{z}\frac{1}{A}\int_{-\infty}^{\infty}\left[\Pi(b) * \mathrm{II}(a)\right] e^{i2\pi x_0 f_x}\, dx_0 \\ &= \psi_0 \frac{\lambda}{z}\frac{1}{A}\mathcal{F}\left\{\Pi(b)\right\}\cdot\mathcal{F}\left\{\mathrm{II}(a)\right\} \qquad \text{(Convolution theorem)} \\ &= \psi_0 \frac{\lambda}{z}\frac{1}{A}\frac{\sin\beta}{\beta}\, 2\cos\alpha \end{aligned} \tag{22}$$

- The intensity varies like

$$I = I_0 \frac{\sin^2\beta}{\beta^2}\cos^2\alpha \qquad \text{with} \qquad \beta \equiv \pi b f_x \text{ and } \alpha \equiv \pi a f_x \tag{23}$$

Double slits III

Again, we found from Eq. 23 that

$$I = I_0 \frac{\sin^2 \beta}{\beta^2} \cos^2 \alpha \qquad \text{with} \qquad \beta \equiv \pi b f_x \text{ and } \alpha \equiv \pi a f_x \tag{24}$$

and b represents the width of each individual slit while a represents a slit spacing. It's possible to wipe out certain diffraction peaks! For example, when $\alpha = n\pi$ or $af_x = n$ with $n = 1, 2, \ldots$ we will get zeroes of $\cos^2 \alpha$. Yet we always require $b < a \ldots$

Diffraction from N slits

- Let us now consider diffraction from N slits.
- We know that one slit on axis produces a diffracted wavefield (in the Fraunhofer approximation) of

$$\psi = \psi_0 \frac{\lambda}{z} \frac{b}{a} \frac{\sin\beta}{\beta} \qquad \text{with} \qquad \beta \equiv \pi b f_x. \tag{25}$$

- If we move this slit off-axis by a distance a, the shift theorem gives us

$$\mathcal{F}\{g(x-a)\} = \mathcal{F}\{g(x)\} e^{i2\pi \frac{ax}{\lambda z}}. \tag{26}$$

Let us now add up N slits:

$$\frac{\sin\beta}{\beta} \sum_{j=1}^{N} e^{i2\pi j \frac{ax}{\lambda z}} = \frac{\sin\beta}{\beta} \sum_{j=1}^{N} \left(e^{i2\alpha}\right)^j \qquad \text{with} \qquad \alpha \equiv \pi a f_x \tag{27}$$

How are we going to evaluate this sum?

N slit diffraction II

Again, we have from Eq. 27:

$$\frac{\sin\beta}{\beta}\sum_{j=1}^{N}\left(e^{i2\alpha}\right)^{j} \qquad \text{with} \qquad \alpha \equiv \pi a f_x$$

Now recall something we found earlier for series $S^m = 1 + x + \ldots + x^m$:

$$S^m - 1 = \frac{x - x^{m+1}}{1-x} \tag{28}$$

so we can write our sum as

$$\frac{\sin\beta}{\beta}\,\frac{e^{i2\alpha} - e^{i2N\alpha}e^{i2\alpha}}{1-e^{i2\alpha}}. \tag{29}$$

If we multiply top and bottom by $e^{-i\alpha}$ we obtain

$$\frac{\sin\beta}{\beta}\,\frac{e^{-i\alpha}(e^{i2\alpha} - e^{i2N\alpha}e^{i2\alpha})}{e^{-i\alpha}(1-e^{i2\alpha})} = \frac{\sin\beta}{\beta}\,\frac{e^{i\alpha} - e^{i\alpha}e^{i2N\alpha}}{e^{-i\alpha}-e^{i\alpha}} = \frac{\sin\beta}{\beta}e^{i\alpha}\,\frac{1-e^{i2N\alpha}}{-2\sin\alpha} \tag{30}$$

where we've used $e^{i\theta} - e^{-i\theta} = 2\sin\theta$.

N slit diffraction III

Continuing from Eq. 30:

$$
\begin{aligned}
\frac{\sin\beta}{\beta}e^{i\alpha}\frac{1-e^{i2N\alpha}}{-2\sin\alpha} &= -\frac{\sin\beta}{\beta}e^{i\alpha}\frac{1-\cos(2N\alpha)-i\sin(2N\alpha)}{2\sin\alpha} \\
&= \frac{\sin\beta}{\beta}(-i)e^{i\alpha}\frac{(\sin(2N\alpha)-i\,(1-\cos(2N\alpha))}{2\sin\alpha}
\end{aligned}
\tag{31}
$$

The Intensity varies like

$$
I = I_0\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2 \tag{32}
$$

By varying the width of the slit a certain fringes will be enhanced and others will be cancelled out.

Diffraction by a pinhole

Now let's consider circular diffraction. We make the substitutions $x_0 \equiv r_0 \cos\theta$, $y_0 \equiv r_0 \sin\theta$, $x \equiv r\cos\varphi$, and $y \equiv r\sin\varphi$. The Fraunhofer diffraction integral then becomes

$$\psi = \psi_0 \frac{\lambda}{z}\frac{1}{A}\int_0^{2\pi} d\theta \int_0^{\infty} r_0\, dr_0\, \tilde{g}(r_0)\tilde{g}'(\theta) e^{i2\pi \frac{rr_0}{\lambda z}(\cos\theta\cos\varphi + \sin\theta\sin\varphi)}. \tag{33}$$

The trigonometric function in parenthesis is just $\cos(\theta - \varphi)$, and if we have no azimuthal angular dependence to the aperture or $\tilde{g}'(\theta) = 1$, we have

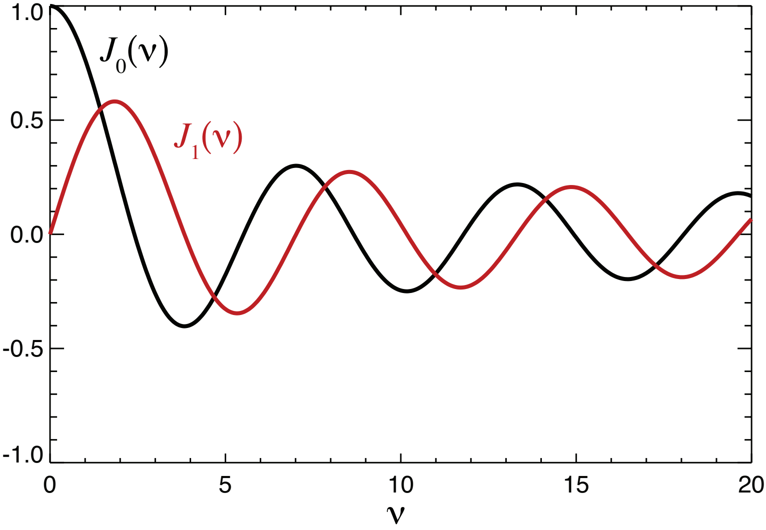
$$\psi = \psi_0 \frac{\lambda}{z}\frac{1}{A}\int_0^{\infty} r_0\, \tilde{g}\,(r_0) dr_0 \int_0^{2\pi} e^{i2\pi \frac{rr_0}{\lambda z}\cos(\theta - \varphi)}\, d\theta. \tag{34}$$

The angular integral should be true for any value of φ since by saying $\tilde{g}'(\theta) = 1$ we made the problem cylindrically symmetric. This angular integral is a Bessel function, which is defined by

$$J_n(x) \equiv \frac{i}{2\pi}^{-n} \int_0^{2\pi} e^{ix\cos\alpha} e^{in\alpha}\, d\alpha \tag{35}$$

Bessel functions

Here's what $J_0(\nu)$ and $J_1(\nu)$ look like:



Diffraction by a pinhole II

Again, we had a Bessel function which is defined by

$$J_n(x) \equiv \frac{i}{2\pi}^{-n} \int_0^{2\pi} e^{ix\cos\alpha} e^{in\alpha}\, d\alpha$$

It has the property

$$\frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] = x^{n+1} J_n(x) \tag{36}$$

The cylindrically symmetrical far-field diffraction integral of Eq. 34 then becomes

$$\psi = \psi_0 \frac{\lambda}{z}\frac{1}{A}\int_0^\infty r_0\, \tilde{g}(r_0)\, J_0(2\pi\frac{rr_0}{\lambda z})\, dr_0. \tag{37}$$

For an aperture of radius a, we have $\tilde{g}(r) = 1$ for $r \leq a$, and $\tilde{g}(r) = 0$ otherwise. We then have

$$\psi = \psi_0 \frac{\lambda}{z}\frac{1}{A}\int_0^a r_0\, J_0(2\pi\frac{rr_0}{\lambda z})\, dr_0. \tag{38}$$

Diffraction by a pinhole III

Again, from Eq. 38 we have

$$\psi = \psi_0 \frac{\lambda}{z}\frac{1}{A}\int_0^a r_0\, J_0(2\pi\frac{rr_0}{\lambda z})\, dr_0.$$

Now let $r' \equiv 2\pi r r_0/(\lambda z)$ so $dr' = 2\pi r/(\lambda z)\, dr_0$. We then have

$$\psi = \psi_0 \frac{\lambda}{z}\frac{1}{A}\left(\frac{\lambda z}{2\pi r}\right)^2 \int_0^a r'\, J_0(r')\, dr'. \tag{39}$$

However, in this expression note that

$$\frac{d}{dx}\left[x^{0+1}J_{0+1}\right] = x^{0+1}J_0(x) \tag{40}$$

so the integral is $xJ_1(x)|_0^a$, giving

$$\psi = \psi_0 \frac{\lambda}{z}\frac{1}{A}\left(\frac{\lambda z}{2\pi}\right)^2 \frac{1}{r^2}\left[\frac{2\pi ar}{\lambda z}J_1(\frac{2\pi ar}{\lambda z}) - \frac{2\pi 0r}{\lambda z}J_1(\frac{2\pi 0r}{\lambda z})\right] \tag{41}$$

Diffraction by a pinhole IV

Again, we had from Eq. 41 the result which we now want to evaluate further:

$$\begin{aligned}
\psi &= \psi_0 \frac{\lambda}{z}\frac{1}{A}\left(\frac{\lambda z}{2\pi}\right)^2 \frac{1}{r^2}\left[\frac{2\pi a r}{\lambda z} J_1\left(\frac{2\pi a r}{\lambda z}\right) - \frac{2\pi 0 r}{\lambda z} J_1\left(\frac{2\pi 0 r}{\lambda z}\right)\right] \\
&= \psi_0 \frac{\lambda}{z}\frac{1}{A}\frac{\lambda^2 z^2}{4\pi^2}\frac{1}{r^2}\frac{2\pi a r}{\lambda z} J_1\left(\frac{2\pi a r}{\lambda z}\right) \\
&= \psi_0 \frac{\lambda}{z}\frac{a^2}{A}\frac{2J_1(\nu)}{\nu} \qquad \text{where} \qquad \nu \equiv \frac{2\pi a r}{\lambda z} \qquad (42)
\end{aligned}$$

where $2J_1(\nu)/\nu$ is known as an Airy function. The diffraction intensity goes like $[2J_1(\nu)/\nu]^2$, which is shown on the next slide. The first minimum is at $\nu = 1.22\pi = 3.83$.

Airy pattern

Here is the Airy pattern: