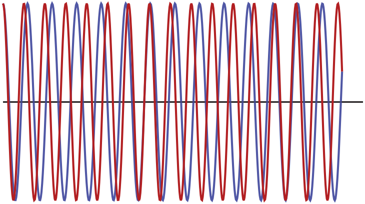
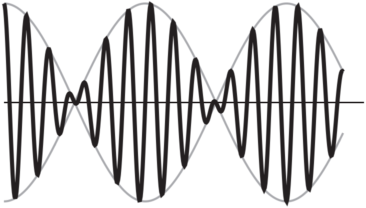
Beat frequencies
Damped HM
Driven HM
DDHO: damped, driven harmonic oscillator
phase angle $\delta(\omega)$
Amplitude $|A(\omega)|$
Quality factor Q
Phase through resonance
Complex amplitude
Low frequency limit
High frequency limit
Energy in DDHO
Power dissipated
Instantaneous power
FWHM $\Delta\omega$
LRC circuit example

Beat frequencies

We talked about the addition of two waves with slightly different frequencies, leading to an average frequency $\bar{\theta}$ with beats at the difference frequency $d\theta$, or $Ae^{i\alpha} + Ae^{i\beta} = 2Ae^{i\bar{\theta}}\cos(d\theta)$:





An example of this effect while tuning a guitar can be heard from `beats_guitar_tuning.mp4` or `beats_guitar_tuning.wav`

Damped harmonic motion I

At the end of lecture 2, we considered damped harmonic motion:

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \tag{1}$$

Using $\omega_0 \equiv \sqrt{k/m}$ and $\gamma \equiv b/m$, we rewrote this as

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{2}$$

and we found that the motion goes like

$$x = Ae^{-(\gamma/2)t}e^{i(\omega t+\alpha)} \tag{3}$$

with

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \omega_0^2 - \frac{b^2}{4m^2} \tag{4}$$

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Damped harmonic motion II

In the limit of weak damping, we can approximate the damped resonant frequency as

$$\begin{aligned} \omega &= \left[\omega_0^2(1 - \frac{\gamma^2}{4\omega_0^2})\right]^{1/2} \\ &\simeq \omega_0(1 - \frac{\gamma^2}{8\omega_0^2}) = \omega_0(1 - \frac{1}{8Q^2}) \end{aligned} \tag{5}$$

where we have used the binomial expansion of $(1+x)^n \simeq 1 + nx$ for $x \ll 1$ which is just the leading term of a Taylor series expansion, and defined what is called a quality factor $Q \equiv \omega_0/\gamma$.

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Damped harmonic motion III

- We have seen that the resonant frequency ω undergoes a slight decrease when damping is brought into play; since frequencies are often the things that can be most easily measured in a physical experiment, one can have good sensitivity to damping.
- This is used in some types of scanning probe microscope, where the resonant frequency of a tip is observed to shift when the tip's motion is damped by proximity to a surface.
- The damping given above is for the amplitude of motion; since for $F = -kx$ the energy at extreme points of motion is given by $E = (1/2)kx^2$, so the energy is damped according to x^2 or $e^{-\gamma t}$.
- If we want to consider larger values of damping than $\gamma \leq 2\omega_0$, we have a complex value of ω in Eq. 3 of $x = Ae^{-(\gamma/2)t}e^{i(\omega t+\alpha)}$ so we have additional damping. It's not hard to do, but we'll not grind through it here. In mechanics you'll look at underdamping, critical damping, and overdamping in more detail; it's also discussed briefly on pp. 68-70 in French.

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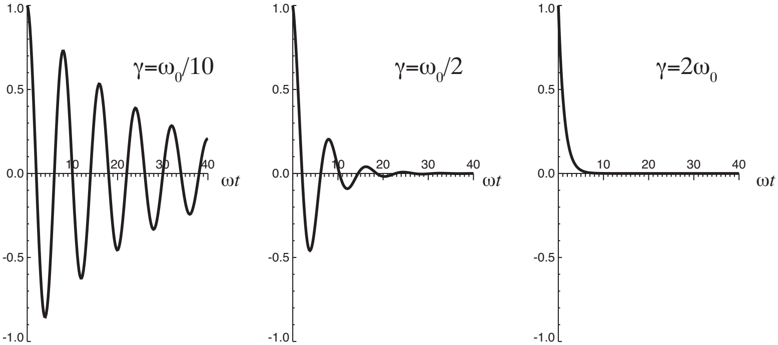
Damped harmonic motion IV

Our equation of motion from Eq. 3 is $x = Ae^{-(\gamma/2)t}e^{i(\omega t+\alpha)}$ with ω given from Eq.4 as $\omega^2 = \omega_0^2 - \gamma^2/4$ and the limit (see above Eq. 4) of $\gamma \leq 2\omega_0$.



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The driven harmonic oscillator

No, by "driven" we do not mean "determined..." Let's consider an object with a linear restoring force and an extra force which is driving it in an oscillatory fashion:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -kx + F_0 e^{i\omega t} \\ m\frac{d^2x}{dt^2} + kx - F_0 e^{i\omega t} &= 0 \end{aligned} \tag{6}$$

While the undriven object might have its own resonance frequency $\omega_0 = \sqrt{k/m}$, we're exciting its motion at a (possibly) different frequency ω so we'll seek solutions involving this frequency, or trial solutions of $x = Ce^{i\omega t+\varphi}$. Putting this into Eq. 6, we have

$$(-\omega^2 m + k - \frac{F_0}{C})Ce^{i\omega t+\varphi} = 0. \tag{7}$$

This must be true at all times t, which means the terms in parenthesis must be zero or

$$-\omega^2 m + k - \frac{F_0}{C} = 0 \tag{8}$$

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Driven harmonic oscillator II

Again, we required Eq. 8 or

$$-\omega^2 m + k - \frac{F_0}{C} = 0$$

so that motion with a driving force of $Ce^{i\omega t+\varphi}$ has an amplitude

$$C = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2}. \tag{9}$$

Are we happy with this result?

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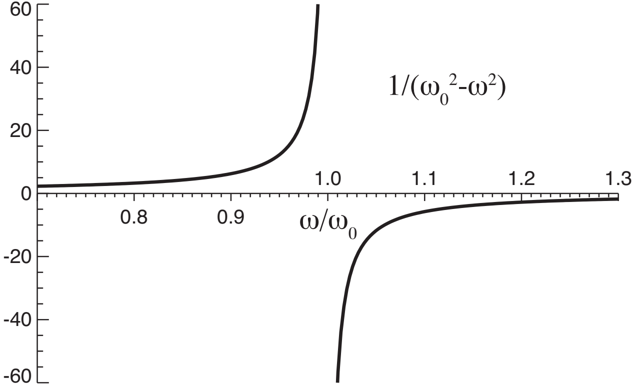
Driven harmonic oscillator III

Again, we've found that motion with a driving force $F_0 e^{i\omega t+\varphi}$ is given by Eq. 9 as

$$C = \frac{F_0/m}{\omega_0^2 - \omega^2}.$$

What does this function look like?



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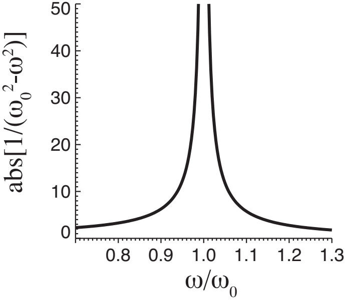
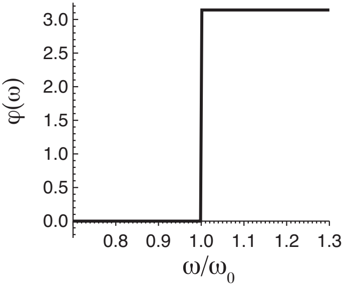
Driven harmonic oscillator IV

We can also describe Eq. 9 as

$$|C| = \frac{F_0/m}{|\omega_0^2 - \omega^2|} \qquad \text{with} \qquad \varphi(\omega) \tag{10}$$

where $\varphi(\omega < \omega_0) = 0$ and $\varphi(\omega > \omega_0) = \pi$.





Beat frequencies
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LRC circuit example

DDHO: damped, driven harmonic oscillator

When the harmonic oscillator is driven at its resonance frequency, its amplitude tends towards infinity. Do we honestly believe that this will accurately describe real life? Probably not. A more realistic situation will be to have both a driving force and a damping force, or

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -kx + -b\frac{dx}{dt} + F_0 e^{i\omega t+\varphi} \\ \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x &= \frac{F_0}{m} e^{i\omega t} \end{aligned} \tag{11}$$

where in the second expression we have again assumed $\omega_0^2 \equiv k/m$ and $\gamma \equiv b/m$. Again, we'll assume that the solution is of the form

$$x = \mathrm{Re}[Ae^{i\omega t-\delta}] \tag{12}$$

where we will allow for a time-independant phase retardation δ in our solution.

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DDHO II

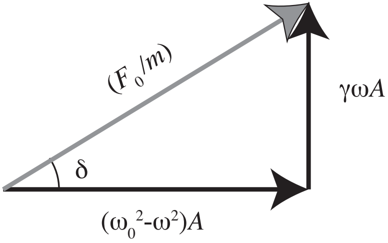
Placing our assumed solution form of Eq. 12 into our differential equation of Eq. 11, we have

$$(-\omega^2 A + i\gamma\omega A + \omega_0^2 A)e^{i\omega t-\delta} = \frac{F_0}{m}e^{i\omega t} \tag{13}$$

Again, this must be true for any time t, so we require

$$(\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m}e^{i\delta} \tag{14}$$

This has the same form as $x + iy = Re^{i\theta}$:



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DDHO III

The geometrical interpretation of Eq. 14

$$(\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m} e^{i\delta}$$

implies the following:

$$(\omega_0^2 - \omega^2)A = \frac{F_0}{m}\cos\delta \tag{15}$$

$$\gamma\omega A = \frac{F_0}{m}\sin\delta \tag{16}$$

The ratio of Eq. 16 over Eq. 15 tells us the frequency-dependent phase angle $\delta(\omega)$:

$$\tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2} \tag{17}$$

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DDHO IV

Let's again consider Eq. 14:

$$(\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m}e^{i\delta}$$

If we square both sides by multiplying by their respective complex conjugates, we have

$$(\omega_0^2 - \omega^2)^2 A^2 + (\gamma\omega)^2 A^2 = \left(\frac{F_0}{m}\right)^2$$

from which we obtain

$$|A(\omega)| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \tag{18}$$

as the frequency-dependent pure-real representation of the amplitude (with $e^{i\delta(\omega)}$ providing the phase).

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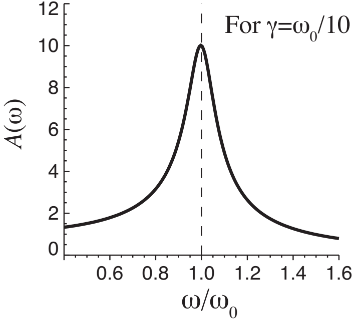
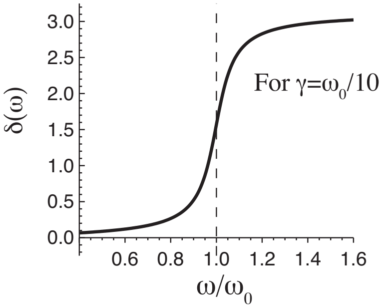
LRC circuit example

DDHO V

Let's take a look at our solutions for the amplitude and phase of motion for the damped, driven harmonic oscillator:





Note that the motion is in phase with the driving force at low frequencies and 180° out of phase at high frequencies.

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LRC circuit example

Quality factor Q

As with the damped harmonic oscillator, let's characterize our solution in terms of a quality factor $Q \equiv \omega_0/\gamma$ and look again at the solution of Eq. 18 for $|A(\omega)|$:

$$\begin{aligned} |A(\omega)| &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \\ &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\omega_0/Q)^2}} \end{aligned} \tag{19}$$

At the point when $\omega = \omega_0$, this reduces to

$$|A(\omega_0)| = \frac{F_0/m}{\sqrt{0 + (\omega_0^2/Q)^2}} = Q\frac{F_0}{m\omega_0^2} = Q\frac{F_0}{k} \tag{20}$$

Thus we see that $Q \equiv \omega_0/\gamma$ factors directly and linearly into the maximum amplitude, so that low damping γ corresponds to large amplitude Q.

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Phase of resonant motion

- Let's also examine the phase of the resonant motion from Eq. 17 of

$$\tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

  As $\omega \to \omega_0$, the denominator goes towards zero so that the argument $\delta(\omega)$ of the tangent function tends towards $\delta(\omega_0) \to \pi/2$.
- When driven at exactly the undamped resonance frequency $\omega_0 = \sqrt{k/m}$, the motion of the object is phase-shifted by 90° relative to the driving force.
- As we noted before, at driving frequencies ω well below the resonant frequency ω_0 the motion is in-phase with the driving force: $\delta(\omega \ll \omega_0) \to 0$.
- At driving frequencies ω well above the resonance frequency ω_0 the motion becomes exactly out-of-phase with the driving force: $\delta(\omega \gg \omega_0) \to \pi$.

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Complex amplitude

Let's return to the expression of Eq. 14 of

$$(\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m}e^{i\delta}$$

except we'll now make A be complex ($A \to \tilde{A}$) so that we can throw away the term $e^{i\delta}$ as it is no longer needed for the "bookkeeping" of handling a phase angle. Let's then solve for the complex amplitude $\tilde{A}(\omega)$:

$$\tilde{A}(\omega) = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\gamma\omega} \tag{21}$$

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Low frequency limit

Again, we have Eq. 21 of

$$\tilde{A}(\omega) = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\gamma\omega}$$

Let's consider the low frequency limit, where $\omega \ll \omega_0$ and $\gamma \ll \omega_0$. In this case we have

$$\begin{aligned} \tilde{A}(\omega \ll \omega_0) &= \frac{F_0/m}{\omega_0^2\left(1 - \frac{\omega^2}{\omega_0^2} + i\frac{\gamma\omega}{\omega_0^2}\right)} \\ &\simeq \frac{F_0}{m\omega_0^2}\left(1 + \frac{\omega^2}{\omega_0^2} - i\frac{\gamma\omega}{\omega_0^2}\right) \end{aligned} \tag{22}$$

where we have used the binomial expansion (the lowest order Taylor series expansion of $(1+x)^n \simeq 1 + nx$ for $x \ll 1$) to obtain the approximate result. We see again that in the low frequency limit the amplitude is nearly pure real (in-phase motion) and it has a constant term plus a weaker term that increases with ω^2. Later on in the course we will see that this describes the refractive index for visible light, among other things.

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High frequency limit

Now let's consider the result of Eq. 21 of

$$\tilde{A}(\omega) = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\gamma\omega}$$

in the high frequency limit where $\omega \gg \omega_0$, and the limit of modest damping $\gamma \lesssim \omega_0$. In this case the leading term in the denominator is ω^2 so that the amplitude becomes

$$\tilde{A}(\omega \gg \omega_0) \simeq -\frac{F_0}{m\omega^2} \tag{23}$$

We see two things in the high frequency limit:

1. The amplitude is negative, or since $e^{i\pi} = -1$, it is 180° out of phase with the driving force; and
2. The amplitude decreases as $1/\omega^2$.

As we will see later on in the course, this describes the refractive index for x rays.

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Energy in the DDHO

Recall a few fun facts about the damped, driven harmonic oscillator:

$$\text{Driving force: } F = F_0 e^{i\omega t} \qquad \text{position: } x = \mathrm{Re}[Ae^{i(\omega t - \delta)}] \tag{24}$$

$$\text{Amplitude: } |A(\omega)| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\omega_0/Q)^2}} \tag{25}$$

gives the amplitude of the motion [and in fact the phase δ in Eq. 24 is a frequency-dependent phase $\delta(\omega)$]. This was for a velocity-dependant damping force of

$$F_{\text{damping}} = -b\frac{dx}{dt} \qquad \text{where} \qquad \gamma = \frac{b}{m}, \tag{26}$$

$$\text{and quality factor } Q = \frac{\omega_0}{\gamma} \qquad \text{with} \qquad \omega_0^2 = \frac{k}{m} \tag{27}$$

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Power dissipated

With that reminder, let's consider power dissipated in the system:

$$P = \frac{dW}{dt} \qquad \text{but} \qquad dW = (F_{\text{friction}})dx = (-b\frac{dx}{dt})dx = -(\gamma m\frac{dx}{dt})dx \tag{28}$$

where we have recalled that the damping/frictional force is as given by Eq. 26. The power dissipated is

$$P = \frac{dW}{dt} = \frac{-\gamma m\frac{dx}{dt}dx}{dt} = -\gamma m(\frac{dx}{dt})^2 \tag{29}$$

OK, so what's dx/dt? We can get that from Eq. 24 for the position:

$$x = Ae^{i(\omega t-\delta)} \text{ so } \frac{dx}{dt} = i\omega Ae^{i(\omega t-\delta)} \tag{30}$$

Therefore

$$(\frac{dx}{dt})^2 = -\omega^2A^2e^{i(2\omega t-2\delta)}$$

$$\text{and} \qquad P = -\gamma m(-\omega^2A^2e^{i(2\omega t-2\delta)}) = \gamma m\omega^2A^2e^{i(2\omega t-2\delta)} \tag{31}$$

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Power dissipated II

OK, we have from Eq. 31 the power dissipated as

$$P = \gamma m \omega^2 A^2 e^{i(2\omega t - 2\delta)}$$

so let's look at A^2 using Eq. 25:

$$A^2 = \frac{F_0^2}{m^2\left[(\omega_0^2 - \omega^2)^2 + (\omega\omega_0/Q)^2\right]} = \frac{F_0^2}{m^2\omega^2\omega_0^2\left[(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2}\right]}$$

We then can write the power dissipated as

$$P = \gamma m \omega^2 \frac{F_0^2}{m^2\omega^2\omega_0^2\left[(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2}\right]} e^{i(2\omega t - 2\delta)} \tag{32}$$

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Power dissipated III

Again, we have from Eq. 32

$$P = \gamma m\omega^2 \frac{F_0^2}{m^2\omega^2\omega_0^2\left[(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2}\right]} e^{i(2\omega t - 2\delta)}$$

but $\omega_0^2 = k/m$ and $\gamma = \omega_0/Q$, so $1/(m\omega_0^2) = 1/k$ and we can write Eq. 32 as

$$P = \frac{\omega_0 F_0^2}{kQ} \frac{1}{(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2}} e^{i(2\omega t - 2\delta)}. \tag{33}$$

This gives the instantaneous power dissipated at any particular time t. To get the overall trend, let's average this over one cycle of oscillation of the position $e^{i(\omega t - \delta)}$ which means a time that goes from $\omega t = 0$ to $\omega t = 2\pi$, or $t = 0$ to $t = 2\pi/\omega$. The only time-dependent part is $e^{i(2\omega t - 2\delta)}$ and furthermore let's recall that our measureables are the real part of the complex exponential.

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Power dissipated IV

Again, we had from Eq. 33

$$P = \frac{\omega_0 F_0^2}{kQ} \frac{1}{(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2}} e^{i(2\omega t - 2\delta)}$$

and we want to find the time-averaged power $\langle P \rangle$ which means we must find

$$\langle \mathrm{Re}[e^{i(2\omega t - 2\delta)}] \rangle|_{t=0}^{t=2\pi/\omega} = \langle \mathrm{Re}[(e^{i\omega t})^2 (e^{-i\delta})^2] \rangle|_{t=0}^{t=2\pi/\omega} = \langle \cos^2 \omega t \rangle|_{t=0}^{t=2\pi/\omega}$$

where we have realized that the length of the square of a "static" complex number $(e^{-i\delta})^2$ is 1. We then need to find

$$\langle \cos^2 \omega t \rangle|_{t=0}^{t=2\pi/\omega} = \int_0^{2\pi/\omega} \cos^2 \omega t \, dt = \frac{1}{2}$$

We can then use this along with Eq. 33 to write (see French Eq. 4-23)

$$\langle P \rangle = \frac{\omega_0 F_0^2}{2kQ} \frac{1}{(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2}} \tag{34}$$

Beat frequencies
Damped HM
Driven HM
DDHO: damped, driven harmonic oscillator
phase angle $\delta(\omega)$
Amplitude $|A(\omega)|$
Quality factor Q
Phase through resonance
Complex amplitude
Low frequency limit
High frequency limit
Energy in DDHO
Power dissipated
Instantaneous power
FWHM $\Delta\omega$
LRC circuit example

Power dissipated V

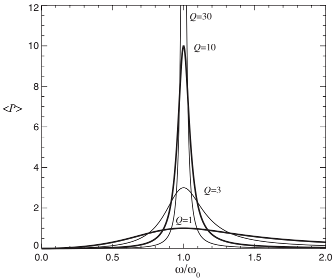
Again, from Eq. 34 we had

$$\langle P \rangle = \frac{F_0^2 \omega_0}{2kQ} \frac{1}{(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2}}$$

At resonance, where $\omega = \omega_0$, this reduces to

$$P_{\text{resonance}} = \frac{F_0^2 \omega_0}{2k} Q \qquad (35)$$

so that we see that the power dissipated at resonance scales linearly with the quality factor $Q = \omega_0/\gamma$.



$< P(\omega) >$ for $F_0/(2k) = 1$ and $\omega_0 = 1$

Beat frequencies
Damped HM
Driven HM
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High frequency limit
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Power dissipated
Instantaneous power
FWHM $\Delta\omega$
LRC circuit example

Power dissipated VI

Again, from Eq. 34 we had

$$\langle P\rangle = \frac{F_0^2\omega_0}{2kQ}\frac{1}{(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0})^2+\frac{1}{Q^2}}$$

What's the frequency range over which power is dissipated? Let's consider a small range of frequencies $\omega = \omega_0 + \Delta\omega$. The frequency-dependent part of the above expression looks like

$$\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0} = \frac{\omega_0^2}{\omega\omega_0}-\frac{\omega^2}{\omega\omega_0} = \frac{(\omega_0^2-\omega^2)}{\omega\omega_0} = \frac{(\omega_0+\omega)(\omega_0-\omega)}{\omega\omega_0}$$

Now let's define $\Delta\omega \equiv \omega - \omega_0$ and thereby make the replacement $\omega \rightarrow \omega_0 + \Delta\omega$:

$$\begin{aligned}\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0} &= \frac{(\omega_0+\omega_0+\Delta\omega)(\omega_0-\omega_0-\Delta\omega)}{(\omega_0+\Delta\omega)\omega_0} = \frac{(2\omega_0+\Delta\omega)(-\Delta\omega)}{\omega_0^2+\omega_0\Delta\omega} \\ &= \frac{-2\omega_0\Delta\omega-(\Delta\omega)^2}{\omega_0^2+\omega_0\Delta\omega} \simeq -\frac{2\Delta\omega}{\omega_0} \qquad (36)\end{aligned}$$

where in the final step we have discarded all terms of order $O(\Delta\omega^2)$.

Beat frequencies
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Driven HM
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FWHM $\Delta\omega$
LRC circuit example

Dissipated power VII

With the result of Eq. 36, we can write the time-averaged power expression of Eq. 34 as

$$\langle P(\Delta\omega)\rangle = \frac{F_0^2\omega_0}{2kQ}\,\frac{1}{(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0})^2+\frac{1}{Q^2}} \simeq \frac{F_0^2\omega_0}{2kQ}\,\frac{1}{4(\frac{\Delta\omega}{\omega_0})^2+\frac{1}{Q^2}} \qquad (37)$$

At what value of $\Delta\omega$ do we have $\langle P(\Delta\omega)\rangle = \langle P(\omega_0)\rangle/2$? That is, what is the half-power point? We can find this from $(1/2) = \langle P(\Delta\omega)\rangle \,/\, \langle P(\omega_0)\rangle$, which when inverted gives

$$\begin{aligned} 2 &= \frac{4(\Delta\omega/\omega_0)^2+1/Q^2}{1/Q^2} = 4Q^2(\Delta\omega/\omega_0)^2+1 \\ 1 &= 4Q^2(\frac{\Delta\omega}{\omega_0})^2 \\ \frac{\Delta\omega}{\omega_0} &= \frac{1}{2Q} \end{aligned} \qquad (38)$$

which reproduces French Eq. 4-27 and gives the half-width at half-maximum. The full-width at half-maximum or FWHM frequency range is twice this, or ω_0/Q.

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LRC circuit example

DDHO: circuit example

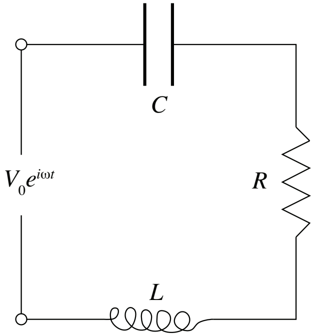
Let's consider a simple LRC circuit (like in French Fig. 4-13). Basic electrodynamics tells us

$$V_C = \frac{q}{C} \qquad V_R = iR = R\frac{dq}{dt}$$

$$V_L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

So let's add a harmonic driving voltage $V = V_0 e^{i\omega t}$ and see how it drops across the circuit elements:

$$V_0 e^{i\omega t} - \frac{1}{C}q - R\frac{dq}{dt} - L\frac{d^2q}{dt^2} = 0 \tag{39}$$



Beat frequencies
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DDHO: damped, driven harmonic oscillator
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Phase through resonance
Complex amplitude
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High frequency limit
Energy in DDHO
Power dissipated
Instantaneous power
FWHM $\Delta\omega$
LRC circuit example

LRC circuit

Again, from Eq. 39 we had

$$V_0 e^{i\omega t} - \frac{1}{C}q - R\frac{dq}{dt} - L\frac{d^2q}{dt^2} = 0$$

which can be rewritten in the form

$$\begin{aligned} \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q &= \frac{V_0}{L}e^{i\omega t} \qquad \text{(LRC circuit)} \qquad (40) \\ \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x &= \frac{F_0}{m}e^{i\omega t} \qquad \text{(DDHO)} \end{aligned}$$

(where DDHO=damped, driven harmonic oscillator). Thus we can make the associations

Damping: $\gamma = \dfrac{b}{m} \Leftrightarrow \dfrac{R}{L}$ Resonance: $\omega_0^2 = \dfrac{k}{m} \Leftrightarrow \dfrac{1}{LC}$ so $\omega_0 =$

Resonator quality factor: $Q^2 = \dfrac{\omega_0^2}{\gamma^2} \Leftrightarrow Q^2 = \dfrac{1}{LC}\dfrac{L^2}{R^2}$ or $Q = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$