Waves in linear media

At the end of our previous lecture, we arrived at the statement that plane waves ψ travel in linear, non-ferromagnetic, non-conducting media with a phase (wavefront) velocity of

$$v_{\text{phase}} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\epsilon_0}}\frac{\sqrt{\mu_0\epsilon_0}}{\sqrt{\mu\epsilon}} \tag{1}$$

If we define a dimensionless velocity scaling parameter n of

$$n \equiv \frac{\sqrt{\mu\epsilon}}{\sqrt{\mu_0\epsilon_0}} \tag{2}$$

we can write the wavefront velocity in linear media as

$$v = \frac{c}{n} \tag{3}$$

This in fact is the index of refraction n, which we'll want to explore further!

Waves in linear media II

Let's consider electromagnetic waves in linear media to be described by $\psi = Ae^{-i(kx-\omega t)}$. How is k modified by the medium? The second derivative in time is

$$\frac{\partial^2}{\partial t^2}\psi = (i\omega)^2\psi = -\omega^2\psi$$

while the second derivative in space is

$$\frac{\partial^2}{\partial x^2}\psi = (-ik)^2\psi = -k^2\psi.$$

Insert these into our wave equation of

$$\nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

The result is

$$k^2 = \mu\epsilon\,\omega^2 \tag{4}$$

which is changed from our free-space equation of $k^2 = \mu_0\epsilon_0\,\omega^2$.

Waves in linear media III

- Again, we had from Eq. 4 the result of $k^2 = \mu\epsilon\,\omega^2$ which looked like a modification of velocity.
- Now either we have to change the distance λ that a wavefront travels in one cycle, or the time or period T for that cycle, in order to have a change in velocity $v = \lambda/T$.
- Since the energy of photons is given by $E = hf = h/T$ and we know that green photons remain green after travel through some glass (*i.e.*, their photon energy is unchanged), it's the wavefront distance λ that must change! Let's refer to this changed wavelength λ'
- This means from $k = \sqrt{\mu\epsilon}\omega$ and $k = 2\pi/\lambda$ we can write

$$v = \frac{\lambda'}{T} = \frac{2\pi/k'}{2\pi/\omega} = \frac{\omega}{k'} = \frac{1}{\sqrt{\mu\epsilon}}. \tag{5}$$

as the speed of wavefront propagation in a medium.

Waves in linear media IV

- Again, we had from Eq. 5 a wavefront speed in linear media of

$$v_{\text{phase}} = \frac{\omega}{k'} = \frac{1}{\sqrt{\mu\epsilon}}$$

  whereas we found before in Eq. 1 a phase velocity of $v_{\text{phase}} = \omega/k$ when ignoring any effects due to linear media.
- Because in a medium $v = 1/\sqrt{\mu\epsilon} = c/n$ with $n \equiv \sqrt{\mu\epsilon}/\sqrt{\mu_0\epsilon_0}$ (Eq. 2), we see that the in-media wavenumber k' is given by

$$k' = nk_0 \tag{6}$$

  where k_0 is the wavenumber $2\pi/\lambda_0$ of a wave in vacuum.
- Similarly, we can say that the wavelength λ' in a medium is related to the wavelength λ_0 in vacuum according to

$$\lambda' = \frac{\lambda_0}{n} \tag{7}$$

- In fact it is conventional to always refer to λ_0 simply as λ, and k_0 simply as k, and write nk rather than nk_0 for the wave vector in media.

Wave equation in linear medium

In conclusion, we can make the general statement that plane waves ψ travel in linear media according to

$$\psi = Ae^{-i(nkx-\omega t)} \tag{8}$$

where n is an index of refraction given by

$$\begin{aligned} n &= \frac{\sqrt{\mu\epsilon}}{\sqrt{\mu_0\epsilon_0}} \qquad \text{(Eq. 2)} \\ &= \frac{\sqrt{\mu_0(1+\chi_m)\,\epsilon_0(1+\chi_e)}}{\sqrt{\mu_0\epsilon_0}} = (1+\chi_m)^{1/2}(1+\chi_e)^{1/2} \end{aligned}$$

where we have used the linear media results of

- electric permittivity $\epsilon = \epsilon_0(1+\chi_e)$ expressed in terms of electric susceptibility χ_e, and of
- magnetic permeability $\mu = \mu_0(1+\chi_m)$ expressed in terms of magnetic susceptibility χ_m.

Relating $\vec{E}$ and $\vec{B}$

- For electrostatically neutral material, Gauss' law gives $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0$ which, for waves $\vec{E} = \vec{E}_0 e^{-i(kx-\omega t)}$ and $\vec{B} = \vec{B}_0 e^{-i(kx-\omega t)}$ traveling in the $\hat{x}$ direction, means that $(\vec{E}_0)_{\hat{x}} = (\vec{B}_0)_{\hat{x}} = 0$. Therefore for plane waves the $\vec{E}$ and $\vec{B}$ fields oscillate *transverse* to the direction of wave propagation.
- From Faraday's law of $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$, one obtains $-k(\vec{E}_0)_{\hat{z}} = \omega(\vec{B}_0)_{\hat{y}}$ and $k(\vec{E}_0)_{\hat{y}} = \omega(\vec{B}_0)_{\hat{z}}$. Since multipication by i represents a rotation by 90° transverse to the $\hat{x}$ direction, this gives $\vec{B}_0 = (k/\omega)(\vec{i} \times \vec{E}_0)$ so $|B_0| = (k/\omega)|E_0| = (n/c)|E_0|$, and $\vec{E}$ and $\vec{B}$ are mutually perpendicular.
- See this figure from Griffiths' E&M book for linear polarization:

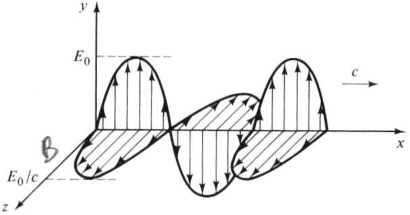


Figure 8.13

Energy and Poynting's theorem

- To gauge the relative role of electric and magnetic responses, we wish to consider the energy in EM waves.
- The work needed to assemble a static charge distribution against the Coulomb force is

$$W_E = \frac{\epsilon_0}{2} \int_{\text{volume}} E^2 \, d\tau \tag{9}$$

- The work needed to get currents going and create a magnetic field (against back EMF) is

$$W_B = \frac{1}{2\mu_0} \int_{\text{volume}} B^2 \, d\tau \tag{10}$$

- Suggests that energy stored in EM fields is

$$W_{EB} = \frac{1}{2} \int_{\text{volume}} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau \tag{11}$$

- In your E&M course you'll learn that the Poynting vector describes the energy flow, related to irradiance I (power/area):

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) \text{ with } \langle \vec{S} \rangle = I = \frac{1}{2} \epsilon_0 E_0^2 = \frac{1}{2\mu_0} B_0^2 \tag{12}$$

Energy and Poynting II

- From Eq. 12, we can find mean values for the electric and magnetic fields in linear, nonconducting media:

$$\langle E \rangle = \left(\frac{\mu}{\epsilon}\right)^{1/4}\sqrt{I} \text{ and } \langle B \rangle = (\mu^3\epsilon)^{1/4}\sqrt{I} \tag{13}$$

- Consider a 10 μm diameter focus of a 5 mW HeNe laser beam. The irradiance is $I = 6.4 \times 10^7$ W/m^2. The resulting mean field values in vacuum are

$$\langle E \rangle = 1.5 \times 10^5 \text{ Volts/m} \qquad \text{and} \qquad \langle B \rangle = 5.2 \times 10^{-4} \text{ Tesla}$$

- The value of $\langle E \rangle$ is large; air sparks under a static field of about 8×10^5 Volts/meter!
- The value of $\langle B \rangle$ is small; by comparison, the earth's magnetic field varies from 3×10^{-5} to 6×10^{-5} Tesla on the planet's surface.
- From this, and from $B_0 = (n/c)E_0$, we conclude that dielectric effects are much stronger than magnetic effects when EM waves propagate through linear media.

Refractive index

- Recall that we've shown for linear media that $B_0 = (n/c)E_0$ with

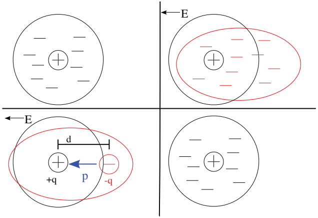
$$n \equiv \frac{\sqrt{\mu\epsilon}}{\sqrt{\mu_0\epsilon_0}} = (1+\chi_m)^{1/2}(1+\chi_e)^{1/2} \qquad \text{(Eq. 2)}$$

- This was with $\epsilon = (1+\chi_e)\epsilon_0 = K\epsilon_0$, where χ_e is the electric susceptibility χ_e and K is the dielectric constant. For solids and liquids, $K \simeq 2$–100.
- This was also with $\mu = (1+\chi_m)\mu_0$, where χ_m is the magnetic susceptibility. For most diamagnetic and paramagnetic materials, $|\chi_m| \simeq 10^{-5}$.
- Since B is "wimpy" compared to E, and χ_m is wimpy compared to χ_e, to a good approximation we should only worry about the dielectric part of the index of refraction!

$$n \simeq (1+\chi_e)^{1/2} = \sqrt{K}. \qquad (14)$$

Refractive index II

- Now let's return to the picture that led to $\epsilon = \epsilon_0(1+\chi_e)$. It came from a dielectric model with bulk polarization $\vec{P} = \epsilon 0 \chi_e \vec{E}$ according to the distortion of the electron cloud around an atom's nucleus:



- Obviously there will be a restoring force, and thus an oscillatory motion ω_0 on the electron cloud. In fact, since there are a multitude of quantum mechanical orbitals for electrons in atoms, we may have a multitude of oscillator modes ω_j where j is an index!
- For the moment, let's consider a single oscillator mode ω_j, and an incident electric field $E_0 e^{i\omega t}$.

Refractive index III

- With an incident electric field $E_0 e^{i\omega t}$, an electron cloud's position will oscillate as $x \propto \cos(\omega t)$. Therefore, we'll have $\dot{x} \propto -\omega \sin(\omega t)$, $\ddot{x} = \dot{v} \propto -\omega^2 \cos(\omega t)$, and $\ddot{v} \propto \omega^3 \sin(\omega t)$.
- Now if we make electrons oscillate, the Abraham-Lorentz formula of classical E&M tells us that we radiate power according to $\ddot{v} = \partial^3 x / \partial t^3$. That is,

$$F_{\text{radiation}} \propto \ddot{v} \propto \omega^2 \dot{x} \tag{15}$$

- In keeping with the damped, driven harmonic oscillator model, we'll bury the ω^2 dependence into an oscillator-specific damping coefficient γ_j and write

$$F_{\text{damping}} = -m_e \gamma_j \dot{x} \tag{16}$$

Refractive index as DDHO

- We will now describe the refractive index in terms of a damped, driven harmonic oscillator (DDHO):

$$\begin{aligned} F_{\text{total}} &= F_{\text{binding}} + F_{\text{damping}} + F_{\text{driving}} \\ m_e\ddot{x} &= -m_e\omega_j^2 x - m_e\gamma_j\dot{x} + q\,E_0 e^{i\omega t} \end{aligned} \tag{17}$$

- We've solved this problem! The oscillator's motion goes like

$$x_j(\omega) = \frac{q/m_e}{(\omega_j^2 - \omega^2) + i\gamma_j\omega}\mathrm{Re}[E_0 e^{i\omega t}]. \tag{18}$$

- The atom's oscillation mode thus has a time-dependent dipole moment of

$$p_j(t) = q\,x_j(t) = \frac{q^2/m_e}{(\omega_j^2 - \omega^2) + i\gamma_j\omega}\mathrm{Re}[E_0 e^{i\omega t}]. \tag{19}$$

Refractive index as DDHO II

- Now let's go from the dipole moment p_j induced on an oscillator, to the volume polarization P.
- Let's weight each oscillation mode by f_j such that the sum adds up to the number of electrons or $\sum_j |f_j| e = Z$.
- Atoms per volume n_a of

$$n_a = \frac{\rho N_A}{A} = \frac{(\text{mass/volume}) \cdot (\text{atoms/mole})}{(\text{mass/mole})} = \frac{\text{atoms}}{\text{volume}} \tag{20}$$

leads to $Zn_a = (\sum_j |f_j|) e\, n_a$ electrons per volume.

- This leads to a volume polarization of

$$\vec{P} = \frac{n_a e^2}{m_e} \left(\sum_j \frac{f_j}{(\omega_j^2 - \omega^2) + i\gamma\omega} \right) \text{Re}[E_0 e^{i\omega t}], \tag{21}$$

- From this, we obtain a complex value for the electric susceptability $\chi_e = \vec{P}/(\epsilon_0 \vec{E})$ of

$$\chi_e = \frac{n_a e^2}{m_e \epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2) + i\gamma_j\omega}. \tag{22}$$

Refractive index as DDHO III

- Back in Eq. 14 we said $n \simeq (1+\chi_e)^{1/2}$ and from Eq. 22 we have

$$\chi_e = \frac{n_a e^2}{m_e \epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2) + i\gamma_j \omega}.$$

- If we assume χ_e is small for any particular oscillator mode, we can use the binomial expansion $n \simeq 1 + \chi_e/2$ and find

$$n = 1 + \frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2) + i\gamma_j \omega} \tag{23}$$

- Multiplying the top and bottom by $(\omega_j^2 - \omega^2) - i\gamma_j\omega$ gives

$$n = 1 - \frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \left[(\omega^2 - \omega_j^2) + i\gamma_j \omega \right] \tag{24}$$

Refractive index as DDHO IV

- We can also write Eq. 24 of

$$n = 1 - \frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \left[(\omega^2 - \omega_j^2) + i \gamma_j \omega \right]$$

in terms of separate real and imaginary parts:

$$\mathrm{Re}[n] = 1 - \frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{(\omega^2 - \omega_j^2) f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \tag{25}$$

$$\mathrm{Im}[n] = -\frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{\gamma_j \omega f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}. \tag{26}$$

- Since a wave propagates in real space as e^{-inkx}, this gives separately an absorption and a phase shift:

$$e^{-inkx} = e^{-i\,\mathrm{Re}[n]\,kx} e^{-i\,(-i|\mathrm{Im}[n]|)\,kx} = e^{-ikx} e^{-i(\mathrm{Re}[n]-1)kx} e^{-|\mathrm{Im}[n]|\,kx} \tag{27}$$

Wave phase shift and attenuation

- Again, we have shown that the wave is phase shifted according to

$$e^{-inkx} = e^{-i\,\mathrm{Re}[n]\,kx}\, e^{-|\mathrm{Im}[n]|\,kx}$$

- If we write the amplitude attenuation as $e^{-\mu/(2x)}$, the intensity is attenuated as $e^{-\mu x}$ with a characteristic length of

$$\frac{1}{\mu} = \frac{1}{\frac{n_a e^2}{2m_e \epsilon_0} \sum_j \frac{\gamma_j \omega f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}} \tag{28}$$

- The wave attenuation depends on the damping coefficient γ_j, as it should.
- Note, however, that in Eqs. 25 and 26, we've assumed that f_j is pure real which might not be the case!

The Kramers-Kronig relations

- If you look carefully at Eq. 25 for $\mathrm{Re}[n]$ and Eq. 26 for $\mathrm{Im}[n]$, you will see that the same parameters (f_j, ω_j, γ_j) enter into *both* equations:

$$\begin{aligned}
\mathrm{Re}[n] &= 1 - \frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{(\omega^2 - \omega_j^2)\, f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \\
\mathrm{Im}[n] &= -\frac{n_a e^2}{2 m_e \epsilon_0} \sum_j \frac{\gamma_j \omega\, f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}.
\end{aligned}$$

- Therefore, one should be able to calculate $\mathrm{Re}[n(\omega)]$ from complete knowledge of $\mathrm{Im}[n(\omega)]$, and vice versa.
- This is a powerful statement, for $\mathrm{Im}[n(\omega)]$ is just a measure of attenuation at different wavelengths, and is relatively easy to come by!

Kramers-Kronig II

It turns out that with a few basic assumptions (such as $\chi_e \rightarrow 0$ as $\omega \rightarrow \infty$, and a requirement for causality in that charge displacements can lag but not lead the application of an electric field), one can relate the real and imaginary parts of the permittivity with the Kramers-Kronig relationships:

$$\mathrm{Re}[\epsilon(\omega)] = 1 + \frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{\omega' \,\mathrm{Im}[\epsilon(\omega')]}{\omega'^2 - \omega^2} d\omega' \tag{29}$$

$$\mathrm{Im}[\epsilon(\omega)] = -\frac{2\omega}{\pi}\mathcal{P}\int_0^\infty \frac{\mathrm{Re}[\epsilon(\omega') - 1]}{\omega'^2 - \omega^2} d\omega' \tag{30}$$

where $\mathcal{P}$ refers to the principal part of a complex integral. This of course can be carried over to the refractive index using $n \simeq \sqrt{\epsilon(\omega)/\epsilon_0}$.