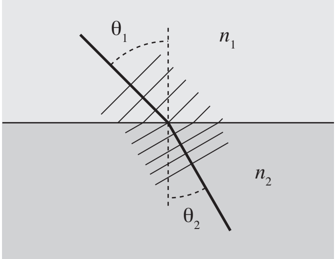
Snell's law

- Rays in optics represent the paths of least optical phase. Only paths quite near the ray path lead to constructive wave superposition.
- Fermat's principle: light follows the path of least time, or smallest optical path length $n\ell$.
- This let us derive Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

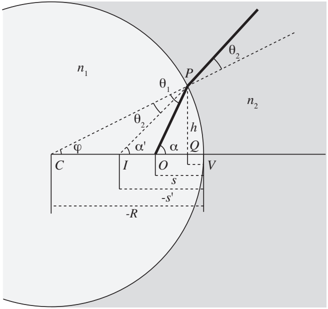


We use a darker shading for n_2 to indicate $n_2 > n_1$.

Refraction at a curved interface

For rays at a curved refractive interface, and in the paraxial approximation (not too far off of optical axis), we found

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} \tag{1}$$



which is Fowles Eq. 10.4. Notice that we defined the distances s' and R to be negative, consistent with our sign convention:

- s positive on left of interface
- s', R, f positive on right of interface

Two surfaces III

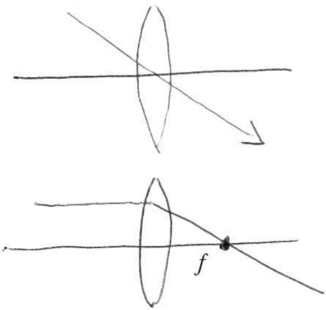
We then derived the lensmaker's equation:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \qquad \text{with} \qquad \frac{1}{f} \equiv \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2}$$

This equation assumes $+R$ for centers of curvature located "downstream" of the lens, so for a double-convex lens we have $R_1 = +|R_1|$ and $R_2 = -|R_2|$.

Ray tracing: three simple rules

1. **Rays through the center of a lens are undeviated.** With a truly thin lens, they enter and exit from parallel surfaces with no distance separation.
2. **Rays parallel to the optical axis are refracted to the focal length.** Because $s \to \infty$, $s' \to f$.
3. **Rays from the focal length are refracted parallel to the optical axis.** Just turn Rule 2 around!



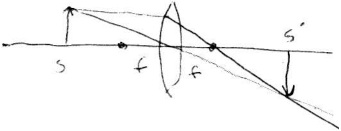
For diverging lenses, focal points are on opposite sides; that is, $f = -|f|$.

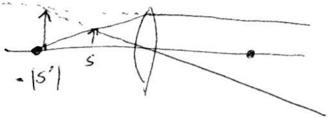
Real and virtual images

- When $s > f$ and $f = +|f|$, we get a *real image* (intensity on a screen) which is inverted. Let heights be h and h' respectively; ray through optical axis shows similar triangles so $h/s = -h'/s'$. Magnification m is then

$$m = \frac{h'}{h} = \frac{-s'}{s} \qquad (3)$$

- When $s < f$, rays never converge onto a point to produce an image on a screen. Instead, they appear to be coming from a *virtual image*. Magnification is still given by Eq. 3.

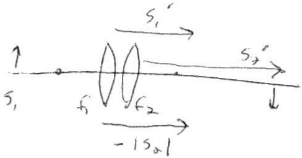




Virtual image: if we were to look back into the lens, the object appears to be at s' rather than its actual location s.

Combining lenses

Thin lens law gives



$$\frac{1}{s_1'} = \frac{1}{f_1} - \frac{1}{s_1}.$$

but $s_1' = -s_2$. Therefore

$$\frac{1}{s_1} + \frac{1}{s_2'} = \left(\frac{1}{f_1} - \frac{1}{s_1'}\right) + \left(\frac{1}{f_2} - \frac{1}{s_2}\right) = \frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{-s_2} - \frac{1}{s_2} = \frac{1}{f_1} + \frac{1}{f_2}$$

so we have $1/s_1 + 1/s_2' = 1/f_{\text{net}}$ with $1/f_{\text{net}} = 1/f_1 + 1/f_2$. This rule can be generalized: we can add up reciprocal focal lengths if lenses are placed close to each other.

Reciprocal focal length: 1 diopter=1/(focal length in meters). Add up diopeters! Eye exam instrument: phoroptor.

Optics of the human eye

Let's consider a particular imaging system: the human eye. A wet and squishy picture of it is shown below.

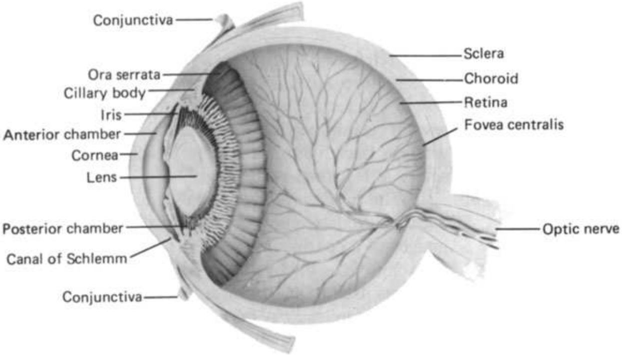


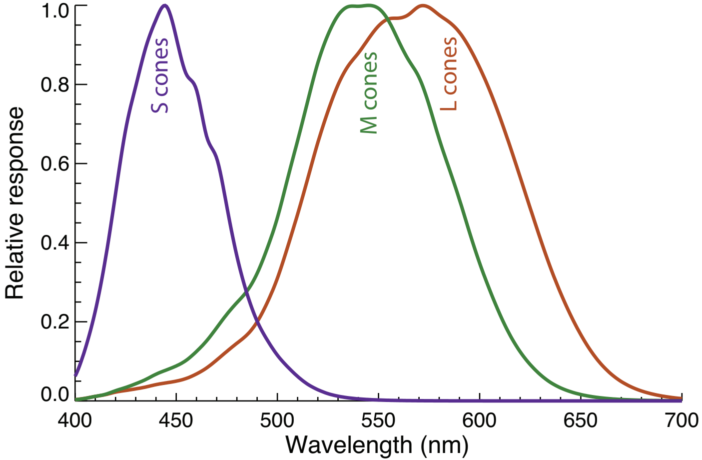
Figure 7-1 Vertical cross section of the eye. (Courtesy of Burroughs Wellcome Co.)

From *Introduction to Optics*, F. Pedrotti and L. Pedrotti (Prentice-Hall, 2nd edition, 1993).

Color vision

The "daylight vision" photoreceptor molecules (cones) on the retina come in three flavors. We perceive colors according to the ratio at which these three flavors are excited.

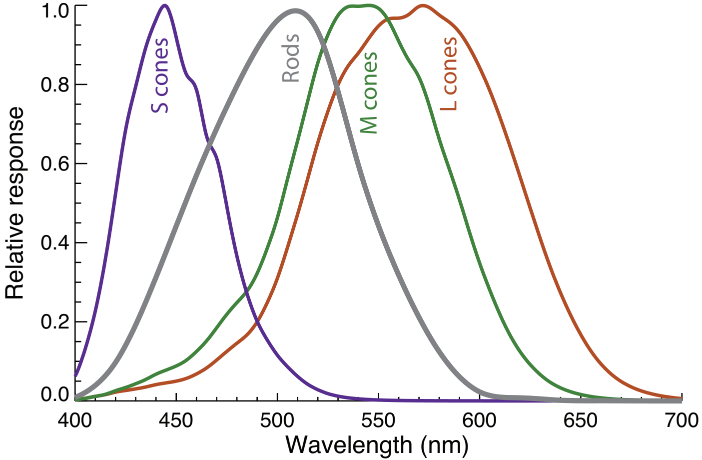
Daylight vision (cones):



Color vision

The "daylight vision" photoreceptor molecules (cones) on the retina come in three flavors. We perceive colors according to the ratio at which these three flavors are excited.

Daylight vision (cones) plus night vision (rods):



The eye II

Here's what the eye looks like according to an optical physicist.

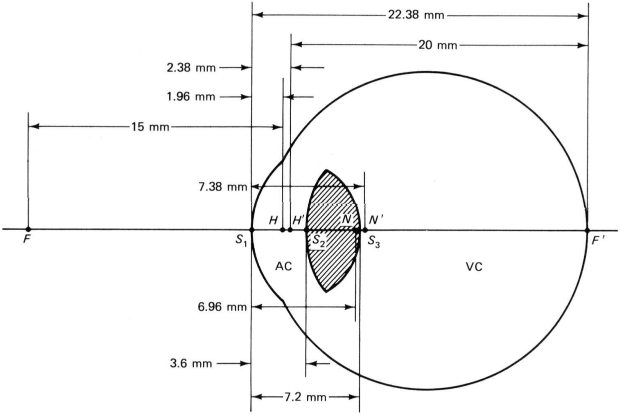


Figure 7-2 Representation of H. V. Helmholtz's schematic eye 1, as modified by L. Laurance. For definition of symbols, refer to Table 7-1. (Adapted with permission from Mathew Alpern, "The Eyes and Vision," Section 12 in *Handbook of Optics*, New York: McGraw-Hill, 1978.)

From *Introduction to Optics*, F. Pedrotti and L. Pedrotti (Prentice-Hall, 2nd edition, 1993).

The eye by the numbers

From *Introduction to Optics*, F. Pedrotti and L. Pedrotti (Prentice-Hall, 2nd edition, 1993).

TABLE 7-1 CONSTANTS OF A SCHEMATIC EYE

Optical surface or element	Defining symbol	Distance from corneal vertex (mm)	Radius of curvature of surface (mm)	Refractive index	Refractive power (diopters)
Cornea	S_1	—	+8[a]	—	+41.6
Lens (unit)	L	—	—	1.45	+30.5
Front surface	S_2	+3.6	+10[b]	—	+12.3
Back surface	S_3	+7.2	−6	—	+20.5
Eye (unit)	—	—	—	—	+66.6
Front focal plane	F	−13.04	—	—	—
Back focal plane	F'	+22.38	—	—	—
Front principal plane	H	+1.96	—	—	—
Back principal plane	H'	+2.38	—	—	—
Front nodal plane	N	+6.96	—	—	—
Back nodal plane	N'	+7.38	—	—	—
Anterior chamber	AC	—	—	1.333	—
Vitreous chamber	VC	—	—	1.333	—
Entrance pupil	E_nP	+3.04	—	—	—
Exit pupil	E_xP	+3.72	—	—	—

SOURCE: Adapted with permission from Mathew Alpern, "The Eyes and Vision," Table 1, Section 12, in *Handbook of Optics*, New York: McGraw-Hill Book Company, 1978.

[a] The cornea is assumed to be infinitely thin.

[b] Value given is for the relaxed eye. For the tensed or fully accommodated eye, the radius of curvature of the front surface is changed to +6 mm.

The eye: a simpler picture

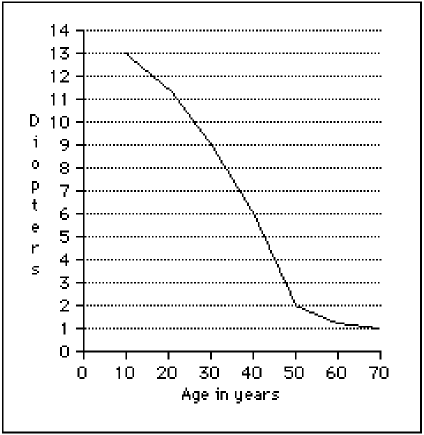
- A "standard" eye: $1/f_0 = 50$ diopters, eye length of 2.0 cm.
- Normal eye, object at infinity: $s \to \infty$ so $s' = f = 1/50 = 0.02$ m or 2 cm. Distant objects produce a real image on the retina.
- Adjustable focal length! Muscles pull on lens to change its curvature. This adds in extra focusing strength of $f_{a,\text{max}}$ for via *accommodation*. The range of accommodation for a young person is around $f_{a,\text{max}} = 12$ diopters.
- Accommodation leads to a *near point* or point of closest focus for the eye (s_{np}). For a standard eye with $f_{a,\text{max}} = 12$ diopters, we have

$$\frac{1}{s_{\text{np}}} = \frac{1}{f_0} + \frac{1}{f_{a,\text{max}}} - \frac{1}{0.02\text{ m}} = 50 + 12 - 50 = 12 = \frac{1}{0.0833}$$

or $s_{\text{np}} = 8.3$ cm.

Don't trust anyone over 30 to be able to read this

Due to loss of muscle tone, and loss of plasticity in the eye's lens, our range of accommodation decreases with age: presbyopia. It's all downhill from here... This curve is from Wikipedia.



Reading glasses

Let's say you're 50 years old and your range of accomodation is 2 diopters. What's your near point?

$$s_{np} = \frac{1}{1/f_0 + 1/f_{a,\max} - 1/s'} = \frac{1}{50 + 2 - 50} = 0.5 \text{ meters}$$

so you have to hold things at arm's length to read them. Alternatively, you can buy reading glasses to read things at a more reasonable near point of 25 cm:

$$\begin{aligned}
\frac{1}{s_{np}} + \frac{1}{s'} &= \frac{1}{f_0} + \frac{1}{f_{a,\max}} + \frac{1}{f_{corr}} \\
\frac{1}{f_{corr}} &= \frac{1}{s_{np}} + \frac{1}{s'} - \frac{1}{f_0} - \frac{1}{f_{a,\max}} \\
&= \frac{1}{0.25} + \frac{1}{0.02} - 50 - 2 = 4 + 50 - 50 - 2 = +2 \text{ diopters}
\end{aligned}$$

Myopia (nearsightedness)

From Merriam-Webster: `myopia`, *n.*:

1. a condition in which the visual images come to a focus in front of the retina of the eye resulting especially in defective vision of distant objects
2. a lack of foresight or discernment : a narrow view of something

In myopia, there's a mismatch between the curvature of the cornea and the length of the eye. You have too much focusing, or too long an eyeball. You need a diverging lens to correct your vision. Example: eyeball is 2.08 cm long. Far point is with eye relaxed:

$$\frac{1}{s_{\mathrm{fp}}} = \frac{1}{f_0} - \frac{1}{s'} = 50 - \frac{1}{0.0208} = 50 - 48.08 = 1.92$$

or $s_{\mathrm{fp}} = 1/1.92 = 0.52$ meters so you can't see very clearly beyond arm's length.

Correcting for myopia

You want to see at a distance, or $s_{\mathrm{fp}} \to \infty$ in $1/s_{\mathrm{fp}} + 1/s' = 1/f_0 + 1/f_{\mathrm{corr}}$:

$$\frac{1}{f_{\mathrm{corr}}} = \frac{1}{s_{\mathrm{fp}}} + \frac{1}{s'} - \frac{1}{f_0} = 0 + \frac{1}{0.0208} - 50 = 48.08 - 50 = -1.92$$

For cosmetic reasons, your eyeglasses will be slightly convex on the outside and strongly concave on the inside; will demagnify your eyes and temples; and mess with your near point!

Correcting for myopia II

Again, $1/f_{\mathrm{corr}} = -1.92$ diopters to set far point to infinity. What about near point? Without glasses it was

$$\begin{aligned} \frac{1}{s_{\mathrm{np}}} &= \frac{1}{f_0} + \frac{1}{f_{a,\mathrm{max}}} - \frac{1}{s'} \\ &= 50 + 2 + (-1.92) - \frac{1}{0.0208} = 50 + 2 - 48.08 = 3.92 \\ &\quad \text{giving } s_{\mathrm{np}} = 0.255 \text{ meters} \end{aligned}$$

With glasses, it now is

$$\begin{aligned} \frac{1}{s_{\mathrm{np}}} &= \frac{1}{f_0} + \frac{1}{f_{a,\mathrm{max}}} + \frac{1}{f_{\mathrm{corr}}} - \frac{1}{s'} \\ &= 650 + 2 + (-1.92) - \frac{1}{0.0208} = 50 + 2 - 1.92 - 48.08 \\ &\quad \text{giving } s_{\mathrm{np}} = 0.5 \text{ meters} \end{aligned}$$

Hyperopia (farsightedness)

You have too little focusing, or too short an eyeball. You need a converging lens to correct your vision, or you need to crank on your power of accommodation. Example: eyeball is 1.88 cm long. Can you focus at a distance? Required accommodation:

$$\frac{1}{f_a} = \frac{1}{s_{\text{fp}}} + \frac{1}{s'} - \frac{1}{f_0} = 0 + \frac{1}{0.0188} - 50 = 0 + 53.2 - 50 = 3.2$$

So a young person can do it, but with constant eye strain. An old person can't do it at all! Correction for relaxed viewing at a distance:

$$\frac{1}{f_{\text{corr}}} = \frac{1}{s_{\text{fp}}} + \frac{1}{s'} - \frac{1}{f_0} = 0 + \frac{1}{0.0188} - 50 = 53.2 - 50 = 3.2 \text{ diopters}$$

This is a positive lens, which really works like a magnifying lens so the person's eyes look really big.

Near point with hyperopia

For a reasonably young person with $1/f_{a,\mathrm{max}} = 8$:

$$\begin{aligned}
\text{With glasses: } s_{\mathrm{np}} &= \frac{1}{f_0} + \frac{1}{f_{a,\mathrm{max}}} + \frac{1}{f_{\mathrm{corr}}} - \frac{1}{s'} \\
&= 50 + 8 + 3.2 - \frac{1}{0.0188} = 50 + 8 + 3.2 - 53.2 \\
&\quad \text{giving } s_{\mathrm{np}} = 0.125 \text{ meters} \\
\text{Without glasses: } s_{\mathrm{np}} &= \frac{1}{f_0} + \frac{1}{f_{a,\mathrm{max}}} - \frac{1}{s'} \\
&= 50 + 8 - \frac{1}{0.0188} = 50 + 8 - 53.2 = 4.8 \\
&\quad \text{giving } s_{\mathrm{np}} = 0.208 \text{ meters}
\end{aligned}$$

so you have to hold things farther away to see them.

The vision thing

- If you need glasses, your eyes aren't bad; you just have a mismatch between cornea curvature and eye length. You could well have high visual acuity (ability to see fine detail) with corrected vision.
- You can use corrective lenses to shift $1/f_0 + 1/f_{\mathrm{corr}}$ to see at $s_{\mathrm{fp}} \to \infty$ with the relaxed eye. The correction can be done by eyeglasses, by contact lenses, or by reshaping of the cornea to adjust its curvature (laser surgery).
- Shifting your far point to $s_{\mathrm{fp}} \to \infty$ also affects your near point.
- Laser surgery does *nothing* for your range of accommodation. Old geezers still need reading glasses for close-up vision, even if they've had laser surgery.
- In fact the cornea might be slightly out-of-round. This can give two different focal distances along two different axes; this is called *astigmatism* and can be corrected for in eyeglasses, some contact lenses, and laser surgery.

Matrix optics

We want a way to deal with more complicated optical systems. What are the three things most common things that can happen to light rays?

1. Travel in a straight line within a uniform refractive medium.
2. Refract at an interface between two refractive media, where we have (in the small angle approximation) $n'\theta' = n_2\theta_2$.
3. Reflect at a smooth surface with $\theta' = -\theta_2$.

The easy way to handle this in optics is through the use of matrices.

Matrix multiplication: a review

Multiply rows by columns:

$$\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_2A_1 + B_2C_1 & A_2B_1 + B_2D_1 \\ C_2A_1 + D_2C_1 & C_2B_1 + D_2D_1 \end{bmatrix}$$

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \alpha \\ y \end{bmatrix} = \begin{bmatrix} A\alpha + By \\ C\alpha + Dy \end{bmatrix}$$

Matrices for rays

Start at (angle, position) of (α, y) and go to (α', y'). In general we might have

$$\begin{aligned} \alpha' &= A \cdot \alpha + B \cdot y \\ y' &= C \cdot \alpha + D \cdot y. \end{aligned}$$

The matrix equation for this is

$$\begin{bmatrix} \alpha' \\ y' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \alpha \\ y \end{bmatrix}.$$

In this notation, our starting condition is at the right, and the matrix acts upon it to create the new result.

Translation matrix

Consider a simple translation by a distance L from (α_0, y) to (α', y'). We have

$$\begin{aligned} \alpha' &= 1 \cdot \alpha + 0 \cdot y \\ y' &= L \cdot \alpha + 1 \cdot y \end{aligned}$$

Thus $$\begin{bmatrix} \alpha' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ y \end{bmatrix} = \mathcal{T} \begin{bmatrix} \alpha \\ y \end{bmatrix} \tag{4}$$

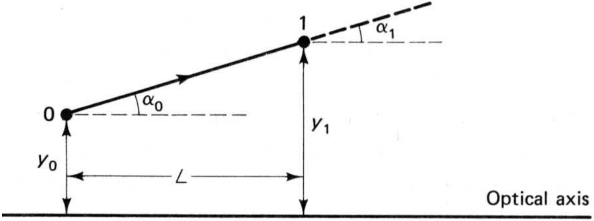


Figure 4-5 Simple translation of a ray.

Refraction matrix

Consider a refractive index boundary in the small angle approximation:

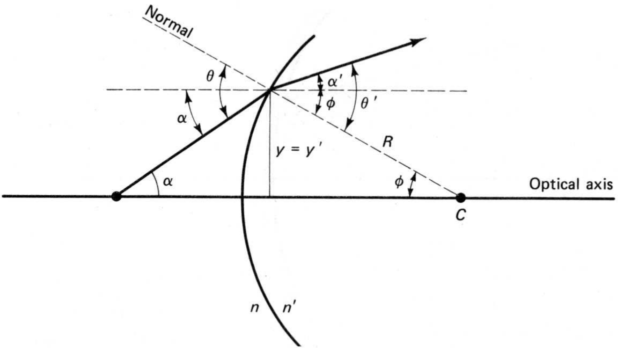


Figure 4-6 Refraction of a ray at a spherical surface.

The position of the ray remains unchanged, giving

$$y' = 1 \cdot y + 0 \cdot \alpha \tag{5}$$

Refraction matrix II

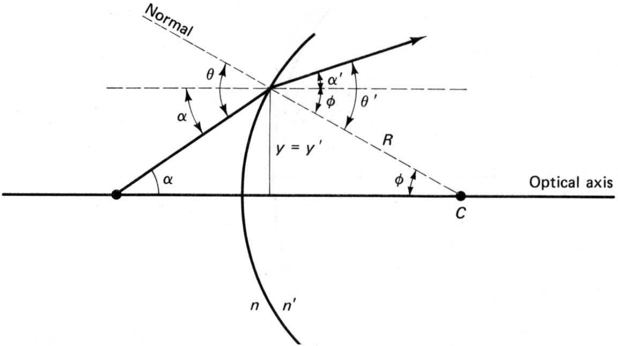


Figure 4-6 Refraction of a ray at a spherical surface.

We have $\theta = \alpha + \phi \simeq \alpha + y/R$, and $\theta' = \alpha' + \phi \simeq \alpha' + y/R$ giving $\alpha' \simeq \theta' - y/R$.

Refraction matrix III

- Again, in the small angle approximation we have $\theta = \alpha + y/R$ and $\alpha' = \theta' - y/R$. We also have $n\theta = n'\theta'$ from Snell's law.
- Our goal is to find an expression for α' in terms of y and α:

$$\begin{aligned}\alpha' &= \theta' - \frac{y}{R} = \frac{n}{n'}\theta - \frac{y}{R} = \frac{n}{n'}\alpha + \frac{n}{n'}\frac{y}{R} - \frac{y}{R} \\ &= \frac{1}{R}\left(\frac{n}{n'} - 1\right)y + \frac{n}{n'}\alpha \qquad (6)\end{aligned}$$

- Together with the result of Eq. 5 of $y' = 1 \cdot y + 0 \cdot \alpha$, this means refraction can be described by a matrix of

$$\begin{bmatrix}\alpha' \\ y'\end{bmatrix} = \begin{bmatrix}\frac{n}{n'} & \frac{1}{R}\left(\frac{n}{n'} - 1\right) \\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha \\ y\end{bmatrix} = \mathcal{R}\begin{bmatrix}\alpha \\ y\end{bmatrix} \qquad (7)$$

Reflection matrix

For reflection, we know $y' = 0 \cdot \alpha + 1 \cdot y$ and $\theta' = \theta$.

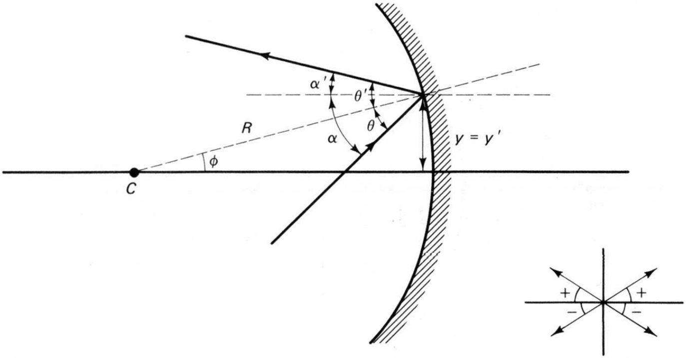


Figure 4-7 Reflection of a ray at a spherical surface. The inset illustrates the sign convention for ray angles.

From the above (for a concave lens, with negative R), we have $\alpha = \theta + \phi$ giving $\theta = \alpha - \phi \simeq \alpha - y/(-R) \simeq \alpha + y/R$, and $\theta' = \alpha' + \phi \simeq \alpha' + y/(-R)$, giving $\alpha' \simeq \theta' - y/(-R) \simeq \theta' + y/R$.

Reflection matrix II

- Again, we have $\theta \simeq \alpha + y/R$, and $\alpha' \simeq \theta' + y/R$, and $\theta' = \theta$.
- We want an expression for α' in terms of y and α:

$$\begin{aligned} \alpha' &= \theta' + \frac{y}{R} = \theta + \frac{y}{R} = \left(\alpha + \frac{y}{R}\right) + \frac{y}{R} \\ &= \alpha + \frac{2}{R}y \end{aligned} \tag{8}$$

- This means we have a matrix for reflection of

$$\begin{bmatrix} \alpha' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{R} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ y \end{bmatrix} = \mathcal{L} \begin{bmatrix} \alpha \\ y \end{bmatrix} \tag{9}$$

Multiple matrices

We have determined the basic matrices for translation $\mathcal{T}$, refraction $\mathcal{R}$, and reflection $\mathcal{L}$. How can we put them together? Well, if we have a sequence of operations, we must calculate them in order:

$$\begin{bmatrix} \alpha_1 \\ y_1 \end{bmatrix} = \mathcal{M}_1 \begin{bmatrix} \alpha_0 \\ y_0 \end{bmatrix} \qquad \text{then} \qquad \begin{bmatrix} \alpha_2 \\ y_2 \end{bmatrix} = \mathcal{M}_2 \begin{bmatrix} \alpha_1 \\ y_1 \end{bmatrix}$$

We therefore see that the rule for going through any number of matrix operations initial to final is

$$\begin{bmatrix} \alpha_N \\ y_N \end{bmatrix} = \mathcal{M}_N \mathcal{M}_{N-1} \ldots \mathcal{M}_2 \mathcal{M}_1 \begin{bmatrix} \alpha_0 \\ y_0 \end{bmatrix}$$

where we have to calculate matrix multiplications starting from the right and working our way left.

Determinants

The determinant of a matrix is in some sense a measure of its normalization. The determinant is defined as

$$\mathrm{Det}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = AD - BC.$$

We can then consider the determinants for our three primary matrices:

$$\text{Translation } \mathcal{T}\text{: Det}\begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} = 1\cdot 1 - 0\cdot L = 1$$

$$\begin{aligned}\text{Refraction } \mathcal{R}\text{: Det}\begin{bmatrix} \frac{n}{n'} & \frac{1}{R}\left(\frac{n}{n'}-1\right) \\ 0 & 1 \end{bmatrix} &= \frac{n}{n'}\cdot 1 - \frac{1}{R}\left(\frac{n}{n'}-1\right)\cdot 0 \\ &= \frac{n}{n'}\end{aligned}$$

$$\text{Reflection } \mathcal{L}\text{: Det}\begin{bmatrix} 1 & \frac{2}{R} \\ 0 & 1 \end{bmatrix} = 1\cdot 1 - \frac{2}{R}\cdot 0 = 1$$

All these matrices have the property $\mathrm{Det}\mathcal{M} = n/n'$; so should their products.