NODAL ANALYSIS

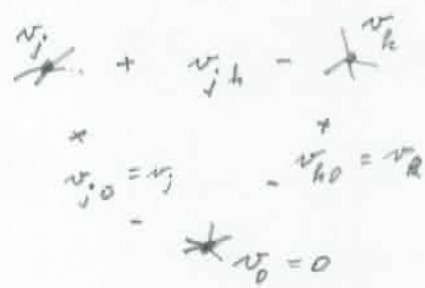
This is the most common method of analyzing a "linear" electrical network when using a computer.
"SPICE" is the name of a popular computer-based language for making such an analysis - but I won't discuss "SPICE" here.

Consider a network with $n+1$ nodes.
Number the nodes $0, 1, \ldots, n$, but let 0 be the number for the ground node. Consider three nodes, numbered $0, j, k$.
There will be node voltages defined as follows.

v_m = the node voltage at node m.

For the ground node (numbered 0), we set $v_0 = 0$.
To get v_m $(m \neq 0)$, connect a voltmeter with negative lead at node 0 and positive lead at node m, and measure v_m.
(v_m may turn out to be a negative number.)
So for 3 nodes we have the node voltages $v_0 = 0$, v_j, and v_k:

Fig 1.



Define $v_{jk} = v_j - v_k$ (Then, $v_{kj} = v_k - v_j = -v_{jk}$)

By Kirchhoff's voltage law (note the polarities here):

$$-v_{j0} + v_{jk} + v_{k0} = 0 \quad \text{or} \quad -v_j + v_{jk} + v_k = 0$$

Thus,

$$v_{jk} = v_j - v_k$$

Thus, if there is a branch connected between node j and node k,

we get that the branch voltage $v_{jk} = v_j - v_k$,
that is, the branch voltage is the difference between the node voltages.
Similarly, $v_{kj} = v_k - v_j = -v_{jk}$.

Important point:

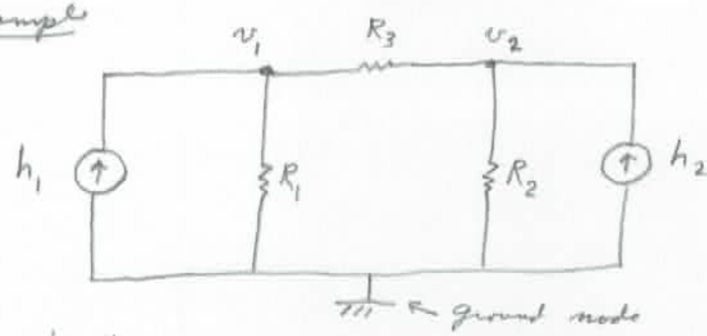
Let's add the branch voltages of Fig 1 around the loop from 0 to j to k to 0. This sum should be 0 by KVL.
It is, automatically! Indeed,

$$
\begin{aligned}
& v_{0j} + v_{jk} + v_{k0} \\
&= \cancel{v_0} - \cancel{v_j} + \cancel{v_j} - \cancel{v_k} + \cancel{v_k} - \cancel{v_0} \\
&= 0.
\end{aligned}
$$

The point is that if we set branch voltages equal to differences in their node voltages, Kirchhoff's voltage law (KVL) will be automatically satisfied, no matter how we arbitrarily assign the node voltages! So, we need only apply Kirchhoff's current law (KCL) at all the nodes — along with Ohm's law — to get a set of equations that will determine the node voltages, and thereby the branch voltages and branch currents. We will now do some examples to illustrate the method.

A comment: We need apply KCL at all the nodes other than the ground node. KCL at the ground node will then be superfluous — that is, it will be a linear combination of some or all of the other KCL equations.

An example



Here, h_1, h_2, R_1, R_2, R_3 are all given.

v_1 and v_2 are unknown node voltages — to be determined.

KCL at the v_1 node: $\dfrac{v_1}{R_1} + \dfrac{v_1 - v_2}{R_3} = h_1$

KCL at the v_2 node: $\dfrac{v_2 - v_1}{R_3} + \dfrac{v_2}{R_2} = h_2$

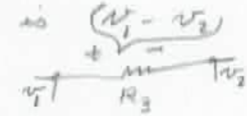
These equation can be rearranged into:

$$v_1\left(\frac{1}{R_1} + \frac{1}{R_3}\right) - \frac{v_2}{R_3} = h_1$$

$$-\frac{v_1}{R_3} + v_2\left(\frac{1}{R_2} + \frac{1}{R_3}\right) = h_2$$

These equations can be solved to get v_1 and v_2 and thereby all the branch voltages and branch currents.

For instance, the branch voltage on R_3 is $(v_1 - v_2)$



and the current flowing toward the right is $\dfrac{v_1 - v_2}{R_3}$.

Using matrix methods and Cramer's rule (CRAMER'S) (See Appendix A of book):

$$v_1 = \frac{\begin{vmatrix} h_1 & -\frac{1}{R_3} \\ h_2 & \frac{1}{R_2} + \frac{1}{R_3} \end{vmatrix}}{\begin{vmatrix} \frac{1}{R_1} + \frac{1}{R_3} & -\frac{1}{R_3} \\ -\frac{1}{R_3} & \frac{1}{R_2} + \frac{1}{R_3} \end{vmatrix}}$$

In the example on page 3, we had only the independent current sources h_1 and h_2 and the resistors R_1, R_2, and R_3.

Nodal analysis gets more complicated when we have independent voltage sources and dependent sources of all four kinds.

When we have a voltage source as a branch, we cannot use Ohm's law to get the current in the branch.

Instead, we put a closed line around the branch

(I call it a "balloon." The book calls it a "generalized node.")

and then write two equations

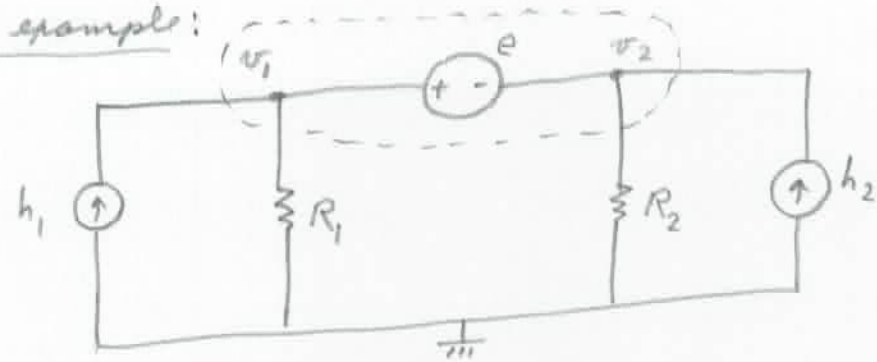
← e is given

Inside the balloon: $v_k - v_j = e$

On the balloon: Add up all the currents coming out of the balloon and set the sum equal to zero.

(This is KCL on the balloon)

An example:



h_1, h_2, R_1, R_2 and e are given.
Find v_1 and v_2

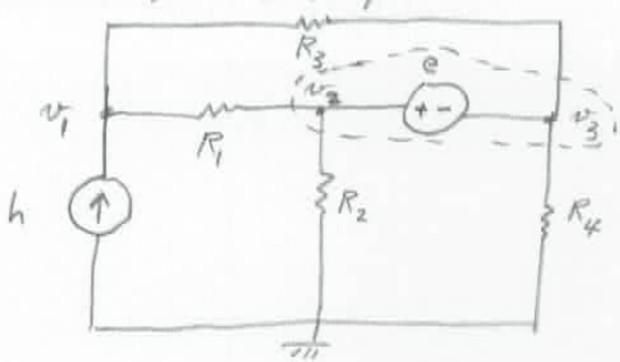
Inside the balloon: $v_1 - v_2 = e$

KCL on the balloon:

$$-h_1 + \frac{v_1}{R_1} + \frac{v_2}{R_2} - h_2 = 0$$

These two equations determine v_1 and v_2.

Here's another example:



h, e, and the R's are given.

Find the node voltages v_1, v_2, v_3

Solution:

Inside balloon: $v_2 - v_3 = e$

KCL on balloon:

$$\frac{v_2 - v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_4} + \frac{v_3 - v_1}{R_3} = 0$$

KCL at v_1 node:

$$-h + \frac{v_1 - v_3}{R_3} + \frac{v_1 - v_2}{R_1} = 0$$

These three equations determine v_1, v_2, and v_3.

COMMENT: When a network consists only of positive resistors and independent sources (where the independent voltage sources do not form a loop, and the independent current sources do not form a "cutset"), a nodal analysis will yield a unique voltage-current regime – that is, a unique set of branch voltages and branch currents.

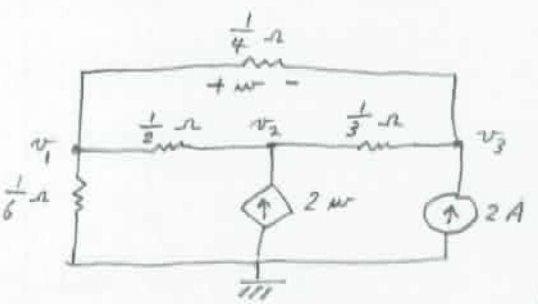
I'll explain this someday

This fact requires some amount of derivation – which I will skip.

However, if there are dependent sources, this uniqueness need not hold. But, in most cases it will. So, here's how to proceed when there are dependent sources.

An example:

FIG. 7



The dependent source $2w$ produces a current $2w$ that is determined by the branch voltage w.
So "2" has the units of SIEMENS because it is a conductance

The nodal analysis:

KCL at the v_1 node: $\dfrac{v_1 - 0}{\frac{1}{6}} + \dfrac{v_1 - v_2}{\frac{1}{2}} + \dfrac{v_1 - v_3}{\frac{1}{4}} = 0$

(1) Rearranging: $\boxed{12v_1 - 2v_2 - 4v_3 = 0}$

KCL at the v_2 node: $\dfrac{v_2 - v_1}{\frac{1}{2}} - 2w + \dfrac{v_2 - v_3}{\frac{1}{3}} = 0$ But $w = v_1 - v_3$

(2) So we get $\boxed{-4v_1 + 5v_2 - v_3 = 0}$

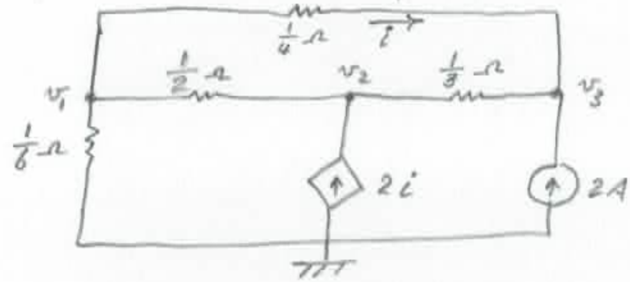
KCL at the v_3 node: $\dfrac{v_3 - v_1}{\frac{1}{4}} + \dfrac{v_3 - v_2}{\frac{1}{3}} - 2 = 0$

(3) This gives $\boxed{-4v_1 - 3v_2 + 7v_3 = 2}$

These three equations can be solved to get v_1, v_2, and v_3 as numbers, uniquely.
(because they happen to be linearly independent)

For instance, we can solve for v_2 to get $v_2 = .2545$ volts.

Another example: We can replace the determining voltage v in the current source of Fig 7 by a current, say i, as follows



In this case, 2 is a "current gain" and is dimensionless.

Equations ① and ③ remain the same,

but equation ② is now different:

$$\frac{v_2 - v_1}{\frac{1}{2}} - 2i + \frac{v_2 - v_3}{\frac{1}{3}} = 0 \qquad \text{But } i = \frac{v_1 - v_3}{\frac{1}{4}}$$

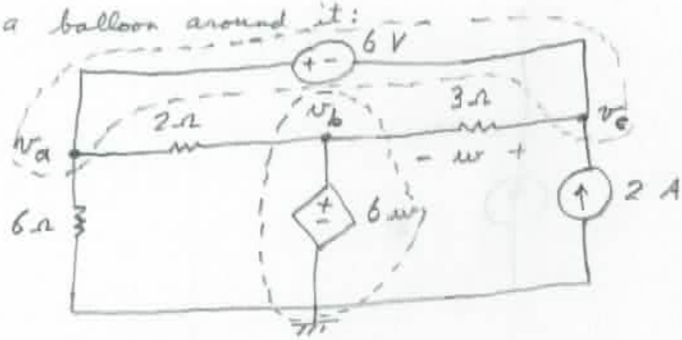
This gives $\boxed{-10v_1 + 5v_2 + 5v_3 = 0}$ ②'

①, ②', and ③ can be solved to get v_1, v_2, and v_3.

This time, $v_2 = \dfrac{\begin{vmatrix} 12 & 0 & -4 \\ -10 & 0 & 5 \\ -4 & 2 & 7 \end{vmatrix}}{\begin{vmatrix} 12 & -2 & -4 \\ -10 & 5 & 5 \\ -4 & -3 & 7 \end{vmatrix}}$

Another example:

If the dependent source is a voltage source, we treat it like an independent voltage source (in a nodal analysis). That is, we put a balloon around it:



Here the "6" in $6w$ is a voltage gain (or voltage ratio). So, it is dimensionless.

FIND w:

Inside the lower balloon: $v_b - 0 = 6w$, where $w = v_c - v_b$

Therefore $v_b = 6v_c - 6v_b$. Thus, $\boxed{7v_b - 6v_c = 0}$

Inside the upper balloon: $\boxed{v_a - v_c = 6}$

KCL for currents coming out of the upper balloon:

$$\frac{v_a}{6} + \frac{v_a - v_b}{2} + \frac{v_c - v_b}{3} - 2 = 0$$

This gives $\boxed{4v_a - 5v_b + 2v_c = 12}$

(KCL for currents coming out of the lower balloon would give the same equation.)

In matrix form:

$$\begin{bmatrix} 0 & 7 & -6 \\ 1 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 12 \end{bmatrix}$$

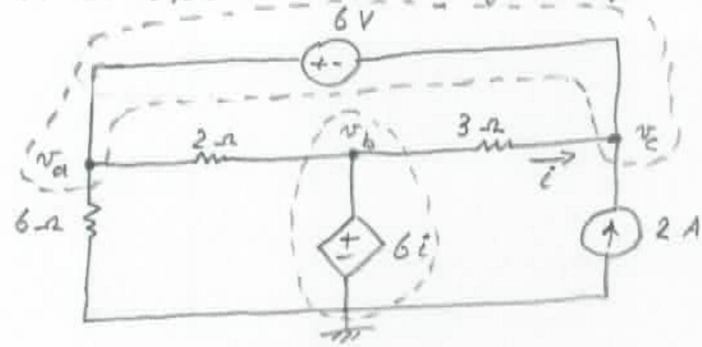
Solving: $v_b = -6$

$v_c = -7$

So, $w = v_c - v_b = -1$.

Another example:

We can replace the determining voltage w by a current i as follows.



Now, inside the lower balloon, we get:

$$v_b - 0 = 6i \quad \text{where } i = \frac{v_b - v_c}{3}$$

So $$\boxed{-v_b + 2v_c = 0}$$

This replaces the first equation on page NA 9:

So in matrix form the three equations are:

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 12 \end{bmatrix}$$

This can be solved for v_a, v_b, and v_c

and for any branch voltage and any branch current.

For instance, $i = \dfrac{v_b - v_c}{3}$, as above.