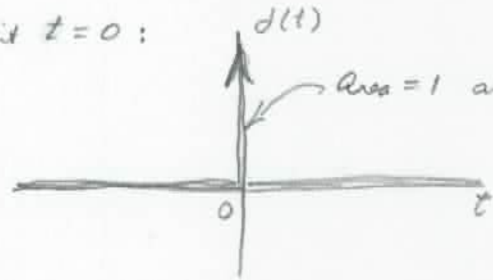
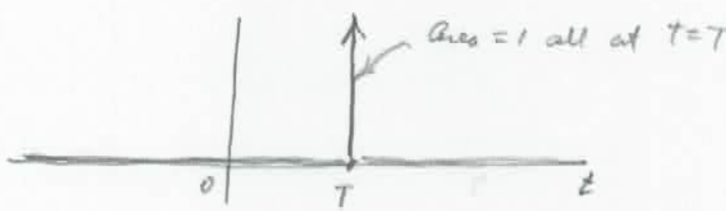
THE DELTA FUNCTION AND ITS LAPLACE TRANSFORM

Its symbol is $\delta(t)$.

We think of it as being 0 for all $t \neq 0$, but with an area of 1 all concentrated over the point $t=0$:



Also, $\delta(t-T)$ is the delta function shifted to the point $t=T$



So, it "seems" acceptable to integrate $\delta(t-T)f(t)$ to get

$$\int_a^b \delta(t-T)\, f(t)\, dt = f(T)$$

whenever $f(t)$ is a function that is continuous at $t=T$, and $a < T < b$.

Actually, Riemann integration cannot yield these results.

In fact, $\delta(t-T)$ is not a conventional function mapping real numbers into real numbers.

Instead, we think of $\delta(t-T)$ as being an "operator" that assigns the value $f(T)$ to $f(t)$.

But we continue to write

$$\int_a^b \delta(t-T)\, f(t) = f(T)$$

Another notation that is sometimes used is

$$\langle \delta(t-T), f(t) \rangle = f(T)$$

where it is understood that $f(t)$ is continuous at $t=T$.

Here's a special case:

The Laplace transform of $\delta(t-T)$:

$$\mathcal{L}\,\delta(t-T) = \int_0^\infty \delta(t-T)\,e^{-st}\,dt = e^{-sT}$$

where $s = \sigma + j\omega$ is a complex variable.

(σ and ω are real variables)

e^{-sT} is a function of the complex variable s

$$e^{-st} = e^{-\sigma T} e^{-j\omega T} = e^{-\sigma T}(\cos\omega t - j\sin\omega t)$$

We also allow the case $T = 0$, where now we think of the lower limit 0 as being replaced by a "vanishingly small" negative number, denoted as $0-$.

$$\int_{0-}^\infty \delta(t)\,e^{-st}\,dt = 1$$

"The Laplace transform of $\delta(t)$ is the constant function 1."

The derivatives of the delta function

We get these by using the formula for integration by parts, even though we do not have Riemann integrals:

We use the short-hand notation $f^{(k)}(t) = \frac{d^k}{dt^k} f(t)$ to denote the kth derivative of $f(t)$.

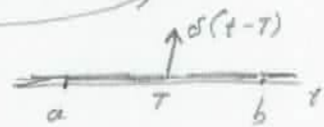
– And similarly for $\delta^{(k)}(t-T) = \frac{d^k}{dt^k}\delta(t-T)$.

Remember the integration-by-parts formula. We now use it as follows, even though this cannot be justified by Riemann integration:

For $a < T < b$:

$$\int_a^b \delta^{(1)}(t-T)\, f(t)\, dt = \delta(b-T)\, f(b) - \delta(a-T)\, f(a) - \underbrace{\int_a^b \delta(t-T)\, f^{(1)}(t)\, dt}_{f^{(1)}(T)}$$

These are 0:



So, we get

$$\langle \delta^{(1)}(t-T), f(t)\rangle = \int_a^b \delta^{(1)}(t-T)\, f(t)\, dt = -f^{(1)}(T) = -\frac{d}{dt} f(t)\Big|_{t=T}.$$

This assumes that $f(t)$ has a continuous first derivative at $t=T$.

For the Laplace transform, we get for $T>0$

$$\mathcal{L}\,\delta^{(1)}(t-T) = \int_0^\infty \delta^{(1)}(t-T)\, e^{-st}\, dt = -\frac{d}{dt} e^{-st}\Big|_{t=T} = -(-s\, e^{-st})\Big|_{t=T} = s\, e^{-sT}$$

If $T=0$, we also write

$$\mathcal{L}\,\delta^{(1)}(t) = \int_{0-}^\infty \delta^{(1)}(t)\, e^{-st}\, dt = s.$$

We can continue these manipulations to higher derivatives of $\delta(t-T)$:

Again, for $a < T < b$ and for $f(t)$ having a continuous second derivative at $t = T$,

$$\int_a^b \delta^{(2)}(t-T)\, f(t)\, dt = \delta^{(1)}(b-T) f(b) - \delta^{(1)}(a-T) f(a) - \int_a^b \delta^{(1)}(t-T)\, f^{(1)}(t)\, dt$$

These are 0.

$$= -\Big(\underbrace{\delta(b-T)}_{=0} f(b) - \underbrace{\delta(a-T)}_{=0} f(a) - \int_a^b \delta(t-T)\, f^{(2)}(t)\, dt\Big)$$

$$= (-1)^2 f^{(2)}(T) = f^{(2)}(T)$$

Continuing in this way, we get for any $k = 1, 2, 3, \dots$:

For $a < T < b$ and for $f(t)$ having a continuous k th derivative at $t = T$,

$$\langle \delta^{(k)}(t-T), f(t) \rangle = \int_a^b \delta^{(k)}(t-T)\, f(t)\, dt = (-1)^k f^{(k)}(T)$$

With regard to the Laplace transform, we have

for $T > 0$,

$$\mathcal{L}\, \delta^{(k)}(t-T) = s^k e^{-sT}, \quad k = 0, 1, 2, \dots$$

and for $T = 0$ we replace $\int_0^\infty \dots dt$ by $\int_{0-}^\infty \dots dt$ to write

$$\mathcal{L}\, \delta^{(k)}(t) = s^k, \quad k = 0, 1, 2, \dots$$