Energy Storage Elements

A little later on, we will be talking about AC (alternating current) circuits.

In that case, two new kinds of elements arise:

Capacitors and Inductors.

These elements store energy, but do not dissipate them as do resistors.

So, let's talk about them first.

Capacitors, C:

> The straight and curved lines are not significant.
> We can also write
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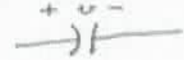
> 
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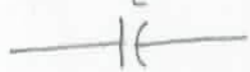
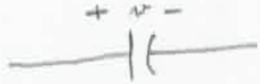
Symbol:



Basic formula:



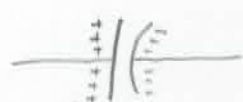
"electrical charge" → "charge" → $q = C\,v$ ← "voltage"

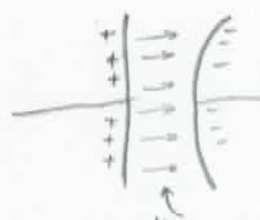
C = a constant, called "capacitance" or "capacitor"

Units: coulombs, farads, volts

$q = q(t)$ and $v = v(t)$ can vary with time.

$i(t) = \dfrac{dq(t)}{dt}$ is the current. It equals the rate of change of q.

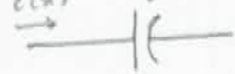




The "electric field" inside the capacitor stores electrical energy.

Energy storage in a capacitor.

Assuming $v(x)$ and $i(x)$ are both 0 for all sufficiently small x, the energy $w_C(t)$ stored in C at time t is



$$w_C(t) = \int_{-\infty}^{t} \underbrace{v(x)\, i(x)}_{\text{"power" going into } C} \, dx$$

$$i(x) = \frac{dq(x)}{dx}$$

$$w_C(t) = \int_{-\infty}^{t} v(x)\, C \frac{dv(x)}{dx}\, dx$$

$$= C \int_{0}^{v(t)} v\, dv \qquad \leftarrow \text{Change of variable for the integration.}$$

$$= C \frac{v^2}{2} \text{ joules} \qquad \text{(Joules is the units for energy)}$$

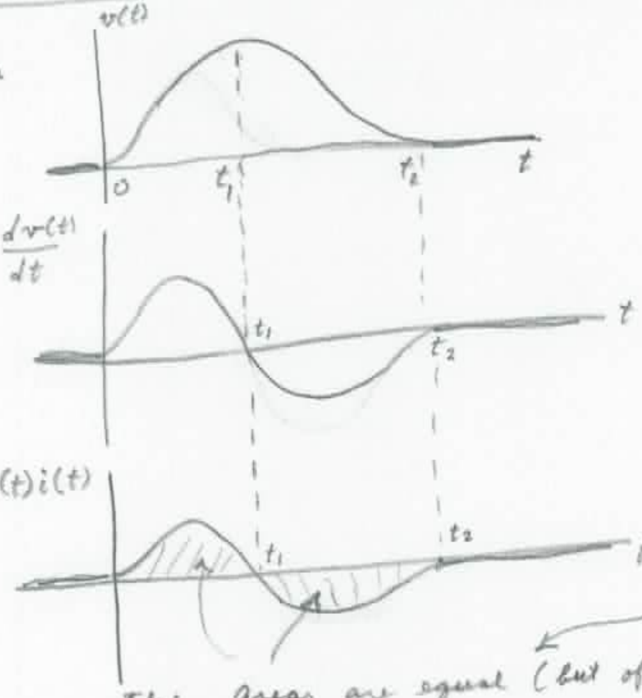
$$= \frac{q^2}{2C} \text{ joules} \qquad \leftarrow \text{But } q = Cv,\ v = \frac{q}{C}$$

Example: Assume

$v(t)$

$i(t) = C \frac{dv(t)}{dt}$

$p(t) = v(t)\, i(t)$



because $w_c(t_2) = C \frac{v(t_2)^2}{2} = 0$

Total areas are equal (but of opposite sign)

On the "upswing" between 0 and t_1, the capacitor is "charging up" — its energy storage is increasing.

On the "downswing" between t_1 and t_2, the capacitor is "discharging" — its energy storage is decreasing.

At and after t_2, as much energy is returned to the electrical source via discharge as the capacitor received from the electrical source during the charging phase.

The basic differential equation for a capacitor C is

$$i(t) = \frac{dq(t)}{dt} = \frac{d}{dt}(C\,v(t)). \quad \text{That is,} \quad i(t) = C\,\frac{dv(t)}{dt}$$

amperes farads volts seconds

Again: $$i(t) = C\,\frac{dv(t)}{dt}$$

$$\boxed{\text{amperes} = \frac{\text{farads} \times \text{volts}}{\text{seconds}}}$$

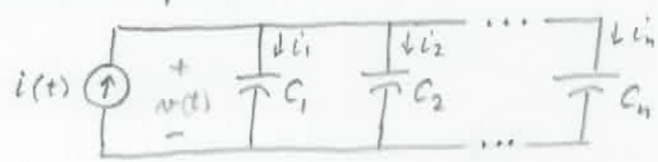
This simple differential equation can be solved to get $v(t)$ in terms of $i(t)$ if we know an initial value $v(t_0)$.

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(x)\,dx + v(t_0)$$

If $v(t_0) = 0$ at t_0, we get $v(t) = \frac{1}{C}\int_{t_0}^{t} i(x)\,dx$.

Or, if $v(t) \to 0$ as $t \to -\infty$, we get $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(x)\,dx$.

Capacitors in "parallel":

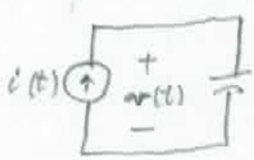


$$\text{KCL:} \quad i(t) = i_1(t) + i_2(t) + \cdots + i_n(t) = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + \cdots + C_n\frac{dv}{dt}$$

$$= (C_1 + C_2 + \cdots + C_n)\frac{dv}{dt}$$

$$= C_p\,\frac{dv}{dt}$$

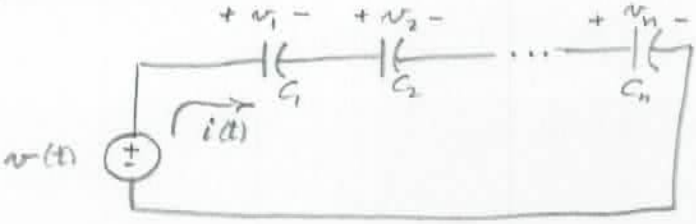
Effectively, we have an equivalent capacitor C_p for the parallel capacitors



$C_p = C_1 + C_2 + \cdots + C_n$, $\quad i(t) = C_p\,\frac{dv}{dt}$

In words, capacitors in parallel add up.

Capacitors in series:



$$v(t) = v_1(t) + v_2(t) + \cdots + v_n(t)$$

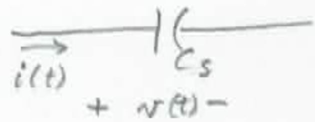
$$= \frac{1}{C_1}\int_{t_0}^{t} i(x)\,dx + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(x)\,dx + v_2(t_0) + \cdots + \frac{1}{C_n}\int_{t_0}^{t} i(x)\,dx + v_n(t_0)$$

But $v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_n(t_0)$

So effectively we have an equivalent series capacitor C_S:

$$v(t) = \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}\right)\int_{t_0}^{t} i(x)\,dx + v(t_0)$$

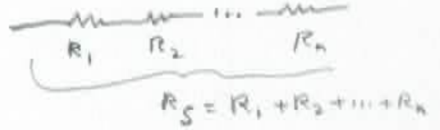
$$= \frac{1}{C_S}\int_{t_0}^{t} i(x)\,dx + v(t_0)$$



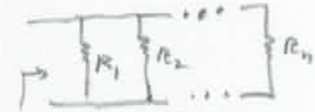
$$C_S = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}}$$

That is, capacitors in series combine according to this formula.

(Note: This is just the reverse of how resistors combine in series and in parallel.)



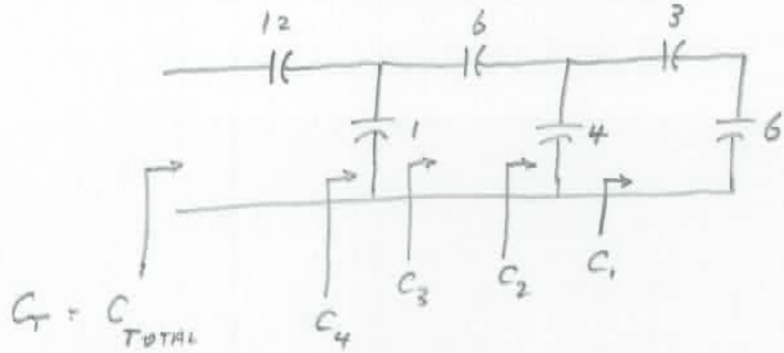
$$R_S = R_1 + R_2 + \cdots + R_n$$



$$R_p = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}}$$

Example:

A series-parallel connection of capacitors:



All capacitor units are μF → microfarads $= 10^{-6}$ farads.

(a more common capacitance value.)

$$C_1 = \frac{1}{\frac{1}{3}+\frac{1}{6}} = \frac{3 \times 6}{3+6} = 2\ \mu F$$

$$C_2 = 4 + 2 = 6$$

$$C_3 = \frac{1}{\frac{1}{6}+\frac{1}{6}} = 3$$

$$C_4 = 1 + 3 = 4$$

$$C_T = \frac{1}{\frac{1}{12}+\frac{1}{4}} = \frac{12 \times 4}{12+4} = 3\ \mu F$$

Inductors

Symbol $\quad \overset{L}{\text{—}\!\!\!\frown\!\frown\!\frown\!\!\!\text{—}}$

Basic formula: $\quad \overset{i(t)}{\longrightarrow}$ $\underset{+\ v(t)\ -}{\text{—}\!\!\!\frown\!\frown\!\frown\!\!\!\text{—}}$

Magnetic field stores magnetic energy.

Current in the coils produces a magnetic field, which is proportional to the current.

Lenz's law: Changing the current produces a change in the magnetic field, which produces a voltage that "tries" to oppose the change in the magnetic field. More concisely, we have the basic equation:

Note: This is the same equation as for a capacitor except that $v(t)$ and $i(t)$ interchange their roles, and $C \to L$.

$$v(t) = L\,\frac{di}{dt} \qquad \text{(See polarities above)}$$

Units: volts V; henries H; amperes A; seconds

$$\left(\text{volts} = \frac{\text{henries} \times \text{amperes}}{\text{seconds}}\right)$$

Energy storage in an inductor:

If $v(t)$ and $i(t)$ are 0 for all sufficiently negative time t,

the energy $w_L(t)$ stored in the magnetic field at time t is

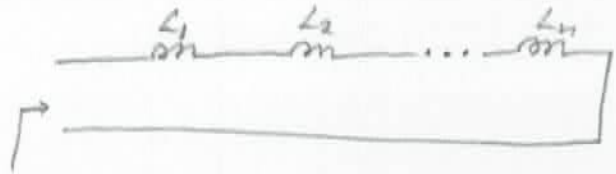
$$w_L(t) = \int_{-\infty}^{t} \underbrace{v(x)\,i(x)}_{\substack{\text{(power at } x\\ \text{in watts)}}}\,dx = \int_{-\infty}^{t} L\,\frac{di}{dx}\,i(x)\,dx = L\int_{-\infty}^{t} i\,di = L\,\frac{i(t)^2}{2}$$

That is,

$$w_L(t) = L\,\frac{i(t)^2}{2} \text{ joules}$$

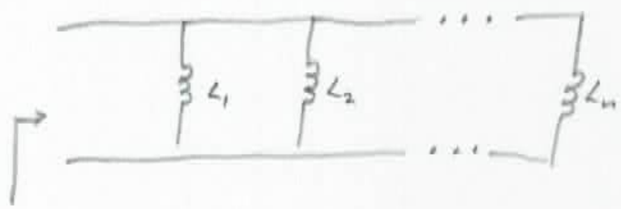
The examples and graphs on the bottom of page C-L-2 again hold if we replace $v(t)$ by $i(t)$, $i(t)$ by $v(t)$, and C by L.

Inductances combine the way resistors combine!



← Inductors in series

Total series inductance $= L_S = L_1 + L_2 + \cdots + L_n$

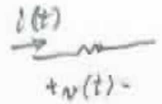


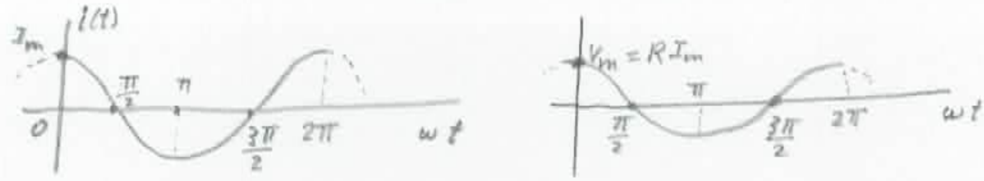
← Inductors in parallel

Total parallel inductance $= L_p = \dfrac{1}{\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}}$

Cosinusoidal signals

For a resistor R: $\quad v(t) = R\,i(t)$ or $V_m \cos\omega t = R I_m \cos\omega t$





Let f = frequency = number of cycles per second (cps)

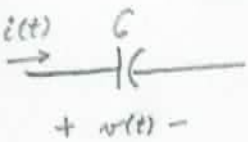
$\omega = 2\pi f$ = number of radians per second. (rad./sec)

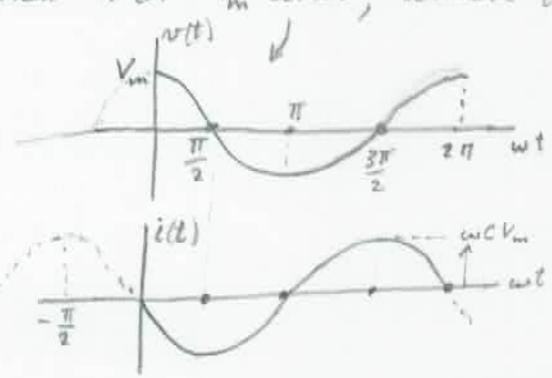
(For power work, $f = 60$ cps. So $\omega = 2\pi \times 60 = 120\pi$ rad./sec. $\cong 377$ rad./sec.)

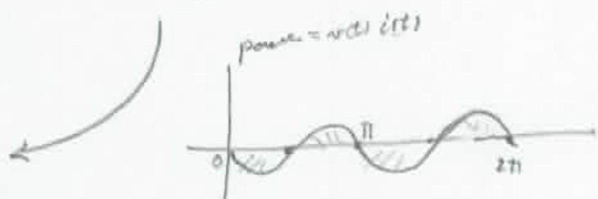
For a capacitor: $\quad i(t) = C\dfrac{dv}{dt}$

So, when $v(t) = V_m \cos\omega t$, we have $i(t) = C V_m(-\omega \sin\omega t)$

$= \omega C V_m \cos(\omega t + 90°)$







Average power = 0 (areas cancel)

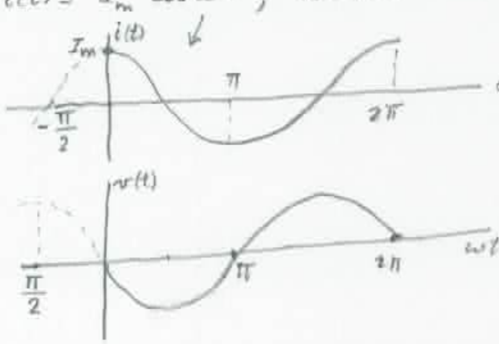
So the current "peaks" at $\frac{\pi}{2}$ radians before the voltage "peaks".

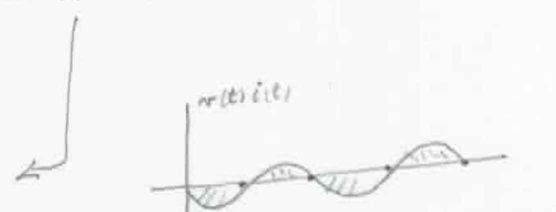
We say that, in a capacitor, "the current leads the voltage by $90°$ ($= \frac{\pi}{2}$)"

(or that "the voltage lags the current by $90°$.")

For an inductor: $\quad v(t) = L\dfrac{di}{dt}$

When $i(t) = I_m \cos\omega t$, we have $v(t) = \omega L I_m \cos(\omega t + 90°)$



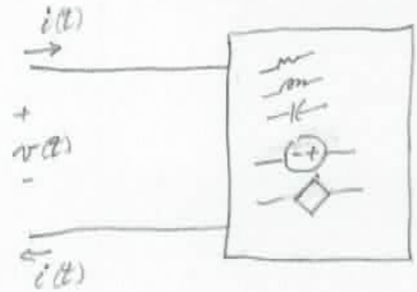


Again average power = 0 because areas cancel.

So now the voltage peaks at $\frac{\pi}{2}$ radians before the current peaks.

We say that in a inductor, "the current lags the voltage by $90°$"

(or that "the voltage leads the current by $90°$")



For a general network, we conventionally talk about the current "lagging" or "leading" the voltage by the phase angle θ. Also called the "power factor angle"

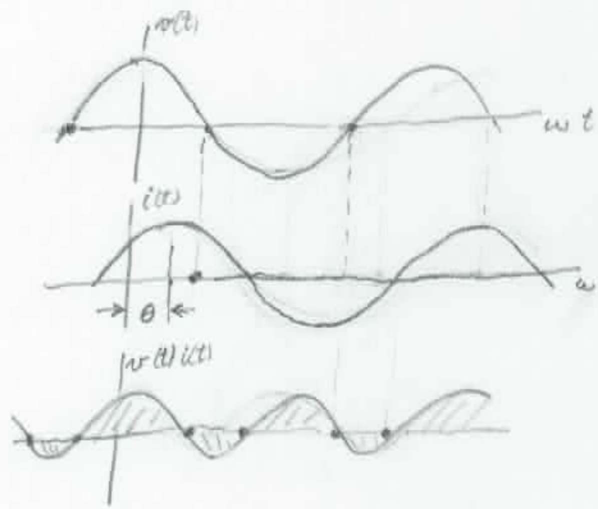
and we write

$$v(t) = V_m \cos(\omega t + \varphi_v)$$

$$i(t) = I_m \cos(\omega t + \varphi_i) \quad , \text{ where } \varphi_v - \varphi_i = \theta$$

More simply, if $v(t) = V_m \cos \omega t$, we have $i(t) = I_m \cos(\omega t - \theta)$.

So the power-factor angle is taken to be positive when $v(t)$ leads $i(t)$ by θ and is taken to be negative when $v(t)$ lags $i(t)$ by θ.



Curves drawn with $0 < \theta < 90°$

In this case, areas do not cancel and we have

P_{av} = average power

$$P_{av} = \frac{V_m I_m}{2} \cos \theta$$

We will derive this later on.

This is conventionally done.

If we use effective values for $v(t)$ and $i(t)$: $V_{eff} = \frac{V_m}{\sqrt{2}}$, $I_{eff} = \frac{I_m}{\sqrt{2}}$, the formula simplifies to $\underline{P_{av} = V_{eff} I_{eff} \cos \theta}$.