Ant Colony Optimization

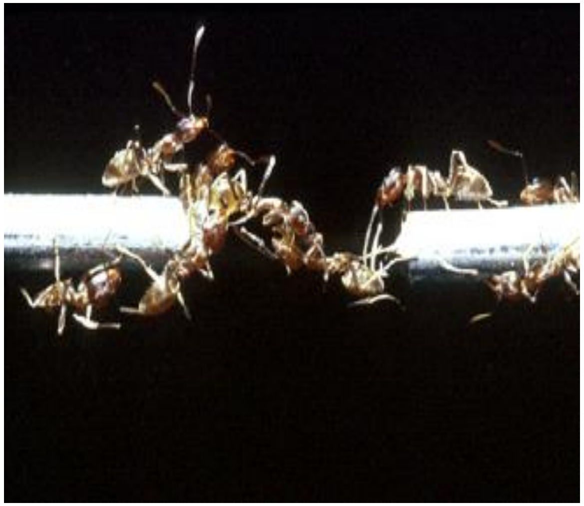
Winter 2024

Motivation

- Inspired by the behavior of **ants** in finding paths from the colony to food
- Known for their highly organized **colonies**, which may consist of millions of individuals.
- Ant colony works as a unified entity

Better Together

- Individually, ants may be not so smart, but put enough of them together and they manage **cooperatively** (decentralized and self-organized) to build bridges, grow fungus as food, **milk aphids**, and **weave their own shelters**



Pheromone-driven Search

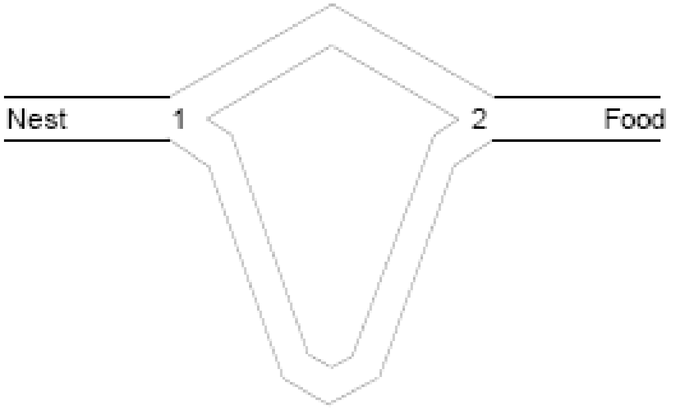
- When ants forage, they randomly wander the forest or jungle floor and lay a trail for nest-mates to lead them to a source of food.
- Many individual ants may discover different routes to the same food but the **shortest path** that leads to it will have the strongest concentration of **pheromone**, a chemical indicator laid down by the ants.

Binary Bridge Experiment

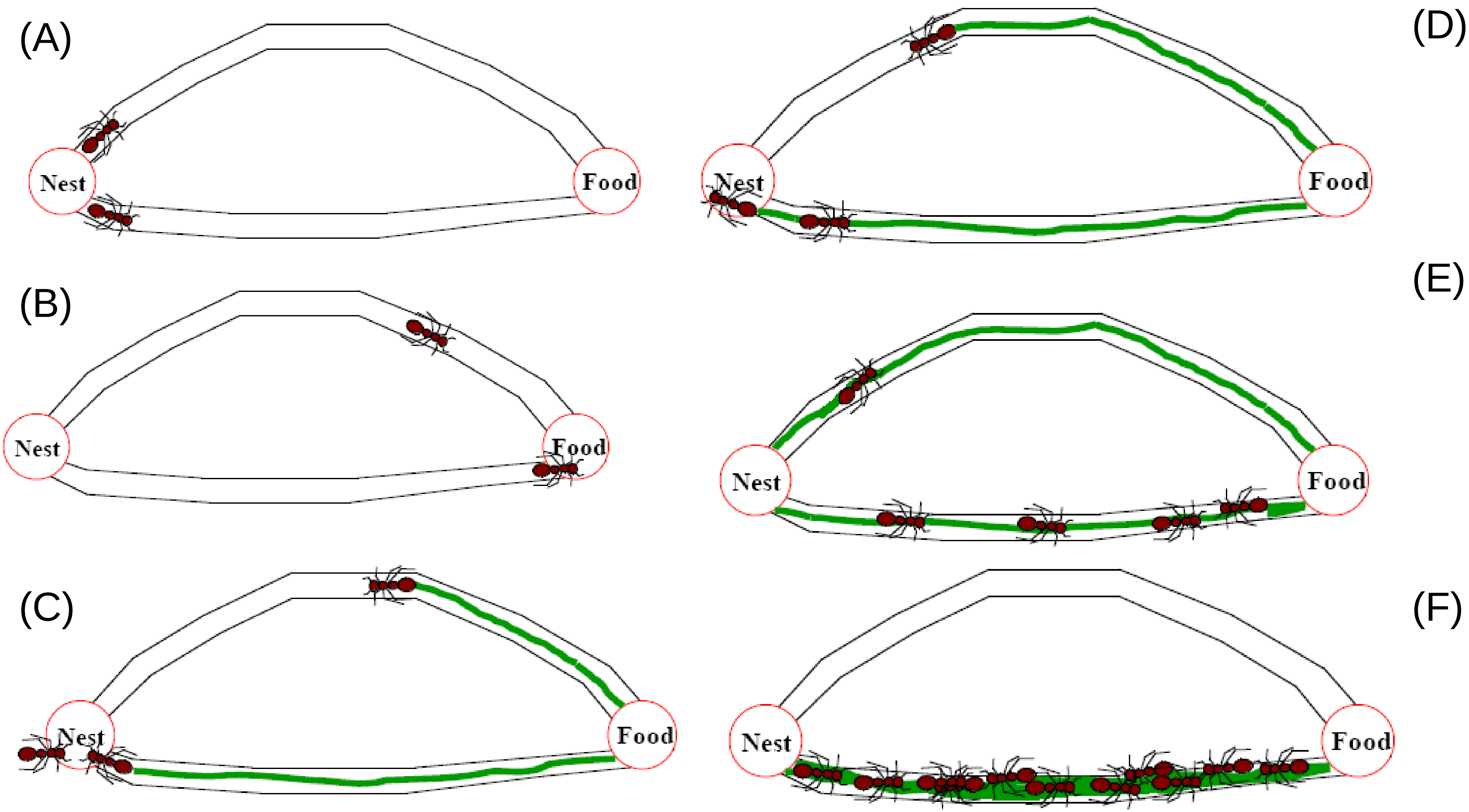
- After t time units, m_1 ants had used the first bridge and m_2 the second one, the probability p_1 for the $(m_1+m_2+1)th$ ant to choose the first bridge can be given by

$$p_1 = \frac{(m_1+k)^h}{(m_1+k)^h + (m_2+k)^h}$$

$$k \cong 20, h \cong 2$$



Search for the Shortest



Ant Colony Optimization (ACO)

- References
  - Dorigo, M. and Stützle, T. (2004). Ant Colony Optimization, MIT Press
  - Dorigo. M., Birattari, M., Stützle, T. (2006). Ant Colony Optimization-- Artificial Ants as a Computational Intelligence Technique, *IEEE Computational Intelligence Magazine*
  - http://iridia.ulb.ac.be/~mdorigo/HomePageDorigo/

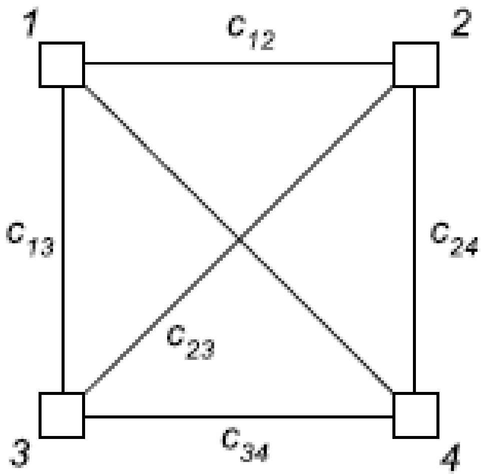
Dr. Marco Dorigo

Applications of Ant Colony

- Primarily combinatorial optimization problems (with discrete and finite solution spaces)
  - TSP
  - Assignment and facility layout
  - Scheduling of limited resources
  - Transportation and vehicle routing
  - Design optimization: design of steel frames and water distribution systems

Graphical Representation

- A number of artificial ants are simulated to move on a graph that encodes as left with vertices (nodes) and edges (links)
- The goal is to find the shortest path
- A variable called **pheromone** is associated with each edge and can be read and modified by ants.



Idea (1)

- At each iteration, each of the ants builds a solution by walking from vertex to vertex on the graph with the constraint of not visiting any vertex previously visited
- At each step of solution construction, an ant selects the next vertex according to a stochastic mechanism that is biased by the pheromone:
  an unvisited vertex j can be selected with a probability that is proportional to the pheromone associated with edge (i, j).

Idea (2)

- At the end of an iteration, on the basis of the quality of the solutions constructed by the ants, the pheromone values are modified (intensifying while evaporating) in order to bias ants in future iterations to construct solutions similar to the best ones previously constructed
- Pheromone evaporation is to avoid the convergence to a locally optimal solution

Sketch (1)

1. Initial population
   – Create a population of ants; distribute them across the vertices when the start vertex is unknown
2. Ant movement
   – Use state transition probability (will discuss later) to determine next **unvisited vertex** to reach

Sketch (2)

$$p_j = \frac{\tau(i,j)^{\alpha}\eta(i,j)^{\beta}}{\sum_{h \in \mathrm{J}} \tau(i,h)^{\alpha}\eta(i,h)^{\beta}}$$

p_j : probability of visisitng node j

$\tau(i,j)$: *tau*; level of pheromone on the edge (i,j)

$\eta(i,j)$: *eta*; heuristic function (e.g., inverse of the distance) on the edge (i,j); also called "visibility"

h is an unvisisted node, belonging to the succedding set J

α and β are weighting parameters

Sketch (3)

3. Ant tour
   - An ant completes its tour when it visits all the vertices (no cycle is permitted; thus need to maintain a list of unvisited vertices)
   - Evaluate the length of the entire tour based on a list of vertices in the current tour

Sketch (4)

4. Pheromone intensification and evaporation

$$\tau(i, j) = (1-\rho)\cdot\tau(i, j) + \sum_{k=1}^{m} \Delta\tau_k(i, j)$$

$\rho : rho;$ evapoeration rate

$\Delta\tau_k(i, j)$: quantity of pheromone on edge (i, j) laid by ant k

$$\Delta\tau_k(i, j) = \begin{cases} Q / L_k & \text{if ant } k \text{ used edge } (i, j) \text{ in its tour} \\ 0 & \text{otherwise} \end{cases}$$

Q : constant; L_k : tour length

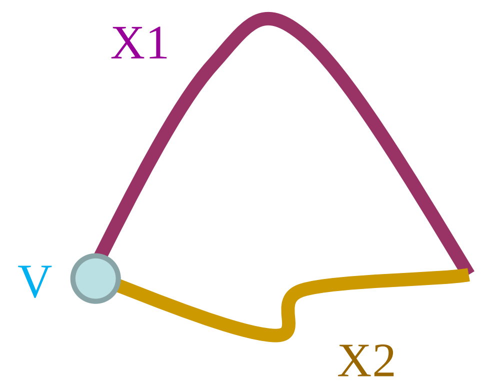
Sketch (5)

5. Start next iteration, pass on pheromone information
6. Clear memory of ants (create a new list of unvisited vertices)
7. Place ants on vertices
8. Ants work on new tours according to the updated pheromone and the visibility

Sample Iteration (1)

- At iteration t, two ants Z1 and Z2 started at vertex V; each takes a different path: X1 and X2
- The pheromone level on X1 and X2 is currently 0.4 and 0.6
- After reaching the end, Z1 traveled 20 steps whereas Z2 traveled 10 steps
- Other parameters are

  Q = 10; α =1; β =2; ρ =0.3



Sample Iteration (2)

- After this iteration, pheromone becomes

For X1, τ_1 = (1-0.3)×0.4 + 10/20 = 0.78

For X2, τ_2 = (1-0.3)×0.6+ 10/10 = 1.42

- Restart ants at vertex V

$$p_1 = \frac{\tau(i,j)^{\alpha}\eta(i,j)^{\beta}}{\sum_{h \in \mathrm{J}} \tau(i,h)^{\alpha}\eta(i,h)^{\beta}} = \frac{0.78^1 \times (1/20)^2}{0.78^1 \times (1/20)^2 + 1.42^1 \times (1/10)^2} = 0.121$$

$$p_2 = \frac{1.42^1 \times (1/10)^2}{0.78^1 \times (1/20)^2 + 1.42^1 \times (1/10)^2} = \boxed{0.879}$$

- Ants are more likely to pick X2

P.18
P.19 P.20

Sample Iteration (3)

- Suppose both ants pick X2

For X1, $\tau_1 = (1\text{-}0.3)\times 0.78 = 0.546$

For X2, $\tau_2 = (1\text{-}0.3)\times 1.42 + 10/10 + 10/10 = 2.994$

- Restart ants at vertex V

$$p_1 = \frac{\tau(i,j)^{\alpha}\eta(i,j)^{\beta}}{\sum_{h\in J}\tau(i,h)^{\alpha}\eta(i,h)^{\beta}} = \frac{0.546^1\times(1/20)^2}{0.546^1\times(1/20)^2+2.994^1\times(1/10)^2} = 0.044$$

$$p_2 = \frac{2.994^1\times(1/10)^2}{0.546^1\times(1/20)^2+2.994^1\times(1/10)^2} = \boxed{0.956}$$

- The probability of choosing X2 is much higher than 0.879

Sample Iteration (3’)

P.18

P.19 P.20

- Suppose one ant picks X1 while the other X2

For X1, $\tau_1 = (1\text{-}0.3)\times 0.78 + 10/20 = 1.046$

For X2, $\tau_2 = (1\text{-}0.3)\times 1.42 + 10/10 = 1.994$

- Restart ants at vertex V

$$p_1 = \frac{\tau(i,j)^{\alpha}\eta(i,j)^{\beta}}{\sum_{h \in J}\tau(i,h)^{\alpha}\eta(i,h)^{\beta}} = \frac{1.046^{1}\times(1/20)^{2}}{1.046^{1}\times(1/20)^{2}+1.994^{1}\times(1/10)^{2}} = 0.116$$

$$p_2 = \frac{1.994^{1}\times(1/10)^{2}}{1.046^{1}\times(1/20)^{2}+1.994^{1}\times(1/10)^{2}} = \boxed{0.884}$$

- The probability of picking X2 is slightly higher than 0.879

Suggested Parameters

- Weighting parameters

$$p_j = \frac{\tau(i,j)^\alpha \eta(i,j)^\beta}{\sum_{h \in J} \tau(i,h)^\alpha \eta(i,h)^\beta}$$

  – Usually $\beta \geq \alpha$; a bigger β may actually reduce the influence of distance if the inverse of distance is smaller than 1

- Evaporation rate

$$\tau(i,j) = (1-\rho) \cdot \tau(i,j) + \sum_{k=1}^{m} \Delta\tau_k(i,j)$$

  – Avoid locally optimal solutions
  – Usually $\rho \geq 0.4$

- Number of ants
  – Problem specific; may increase as progress

Control Growth of Probability

- If we want the probability of choosing the shorter link NOT to grow too fast
  - Increase β (when the heuristic function is the inverse of distance)
  - Decrease Q

$$p_j = \frac{\tau(i,j)^{\alpha}\eta(i,j)^{\beta}}{\sum_{h \in J}\tau(i,h)^{\alpha}\eta(i,h)^{\beta}}$$

$$\sum_{k=1}^{m} \Delta\tau_k(i,j)$$

$$\Delta\tau_k(i,j) = \begin{cases} Q/L_k & \text{if ant } k \text{ used edge } (i,j) \text{ in its tour} \\ 0 & \text{otherwise} \end{cases}$$

TSP Review

- Find, for a given set of n cities with distance d_{ij} between each pairs of cities, a shortest tour that contains every city exactly once
- Distance d_{ij} is recorded in a square matrix (symmetrical or unsymmetrical); the diagonal elements may be set to large values to prohibit cycling

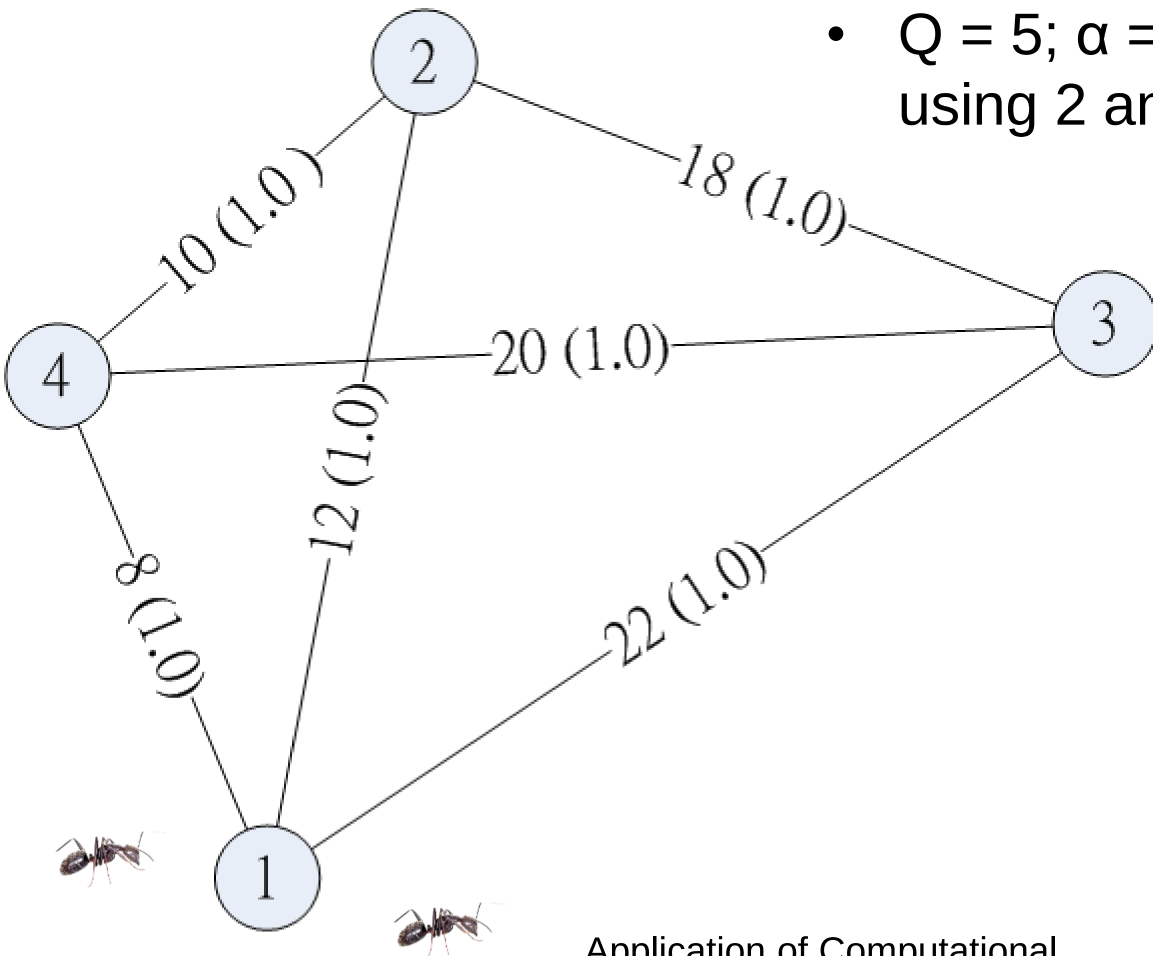
ACO applied to TSP

- Pheromone information is stored in a square matrix; each element denotes the quantity of pheromone on an edge (i,j)
- A city list is kept for each ant at the start of each iteration. When a city is chosen, it will be removed from the list

Pseudocode for ACO in TSP

```
Initialize pheromone values
Repeat
    For each ant k
        S = {1, 2, 3,…n}
        Choose city i randomly
        S = S \ {i}
        Repeat
            Choose city j with p_ij
            S = S \ {j}
            i = j
        Until S = ∅
    End For
    For all (i,j)
        Pheromone Intensification
        and Evaporation
    End For
Until stopping criterion is met
```

Numerical Example



- Start from Location 1
- Q = 5; α =1; β =2; ρ =0.2; using 2 ants

Example (1)

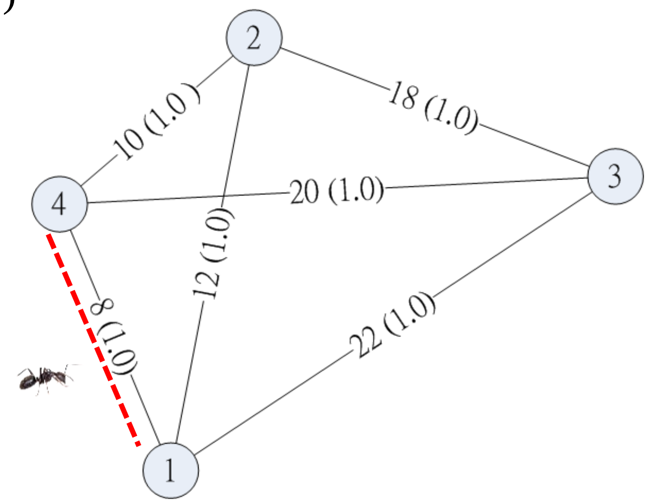
1
1-3
0.916
1-2
0.634
1-4
0

Iteration 1, Ant 1 starts at Location 1

$$p_{1-4} = \frac{\tau(i,j)^{\alpha}\eta(i,j)^{\beta}}{\sum_{h\in J}\tau(i,h)^{\alpha}\eta(i,h)^{\beta}} = \frac{1.0^1\times(1/8)^2}{1.0^1\times(1/8)^2+1.0^1\times(1/12)^2+1.0^1\times(1/22)^2} = 0.634$$

$$p_{1-2} = \frac{1.0^1\times(1/12)^2}{1.0^1\times(1/8)^2+1.0^1\times(1/12)^2+1.0^1\times(1/22)^2} = 0.282$$

$$p_{1-3} = \frac{1.0^1\times(1/22)^2}{1.0^1\times(1/8)^2+1.0^1\times(1/12)^2+1.0^1\times(1/22)^2} = 0.084$$



Generate random number=0.622

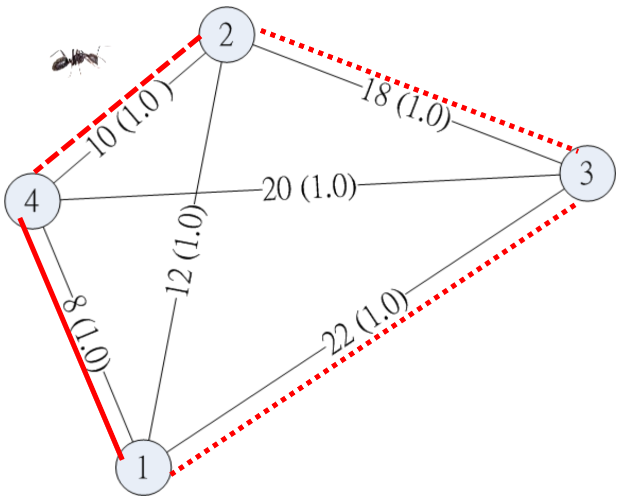
0.622 < 0.634, so choose edge 1-4

Example (2)

Iteration 1, Ant 1 is at Location 4

$$p_{4-2}=\frac{\tau(i,j)^{\alpha}\eta(i,j)^{\beta}}{\sum_{h\in J}\tau(i,h)^{\alpha}\eta(i,h)^{\beta}}=\frac{1.0^{1}\times(1/10)^{2}}{1.0^{1}\times(1/10)^{2}+1.0^{1}\times(1/20)^{2}}=0.8$$

$$p_{4-3}=\frac{1.0^{1}\times(1/20)^{2}}{1.0^{1}\times(1/10)^{2}+1.0^{1}\times(1/20)^{2}}=0.2$$



Generate random number=0.268,

0.268 < 0.8, so choose edge 4-2

the entire path is 1-4-2-3-1

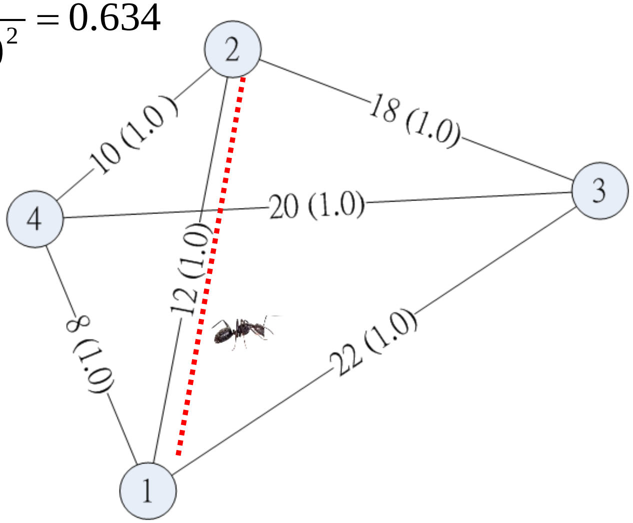
Example (3)

Iteration 1, Ant 2 starts at Location 1

$$p_{1-4}=\frac{\tau(i,j)^{\alpha}\eta(i,j)^{\beta}}{\sum_{h\in J}\tau(i,h)^{\alpha}\eta(i,h)^{\beta}}=\frac{1.0^1\times(1/8)^2}{1.0^1\times(1/8)^2+1.0^1\times(1/12)^2+1.0^1\times(1/22)^2}=0.634$$

$$p_{1-2}=\frac{1.0^1\times(1/12)^2}{1.0^1\times(1/8)^2+1.0^1\times(1/12)^2+1.0^1\times(1/22)^2}=0.282$$

$$p_{1-3}=\frac{1.0^1\times(1/22)^2}{1.0^1\times(1/8)^2+1.0^1\times(1/12)^2+1.0^1\times(1/22)^2}=0.084$$



Generate random number=0.785,

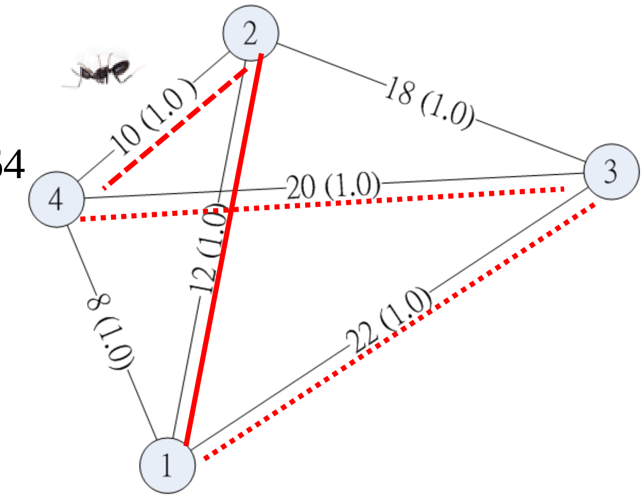
0.785 < (0.634+0.282) , so choose edge 1-2

Example (4)

Iteration 1, Ant 2 is at Location 2

$$p_{2-4} = \frac{\tau(i,j)^{\alpha}\eta(i,j)^{\beta}}{\sum_{h \in J}\tau(i,h)^{\alpha}\eta(i,h)^{\beta}} = \frac{1.0^{1} \times (1/10)^{2}}{1.0^{1} \times (1/10)^{2} + 1.0^{1} \times (1/18)^{2}} = 0.764$$

$$p_{2-3} = \frac{1.0^{1} \times (1/18)^{2}}{1.0^{1} \times (1/10)^{2} + 1.0^{1} \times (1/18)^{2}} = 0.236$$



Generate random number=0.491,

0.491 < 0.764, so choose edge 2-4

the entire path is 1-2-4-3-1

Example (5)

- Pheromone intensification and evaporation

User-specified: Q = 5; ρ =0.2

$$\tau(i,j) = (1-\rho)\cdot\tau(i,j) + \sum_{k=1}^{m}\Delta\tau_k(i,j) = (1-\rho)\cdot\tau(i,j) + \sum_{k=1}^{m} Q/L_k$$

$$\tau(1,2) = (1-0.2)\cdot 1.0 + 5/12 = 1.217$$

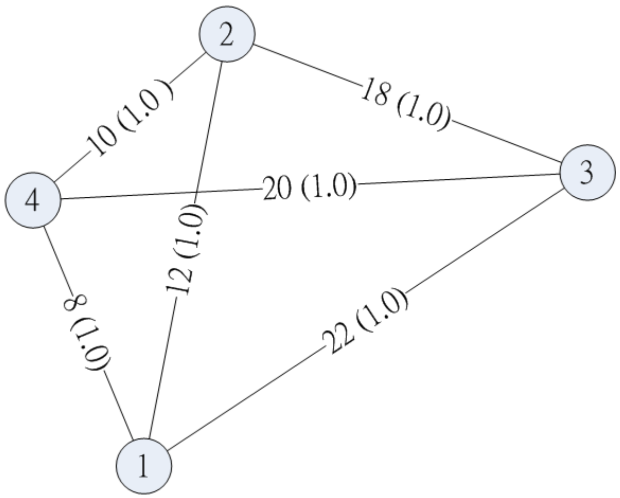
$$\underline{\tau(1,3) = (1-0.2)\cdot 1.0 + (5/22 + 5/22) = 1.255}$$

$$\tau(1,4) = (1-0.2)\cdot 1.0 + 5/8 = 1.425$$

$$\tau(2,3) = (1-0.2)\cdot 1.0 + 5/18 = 1.078$$

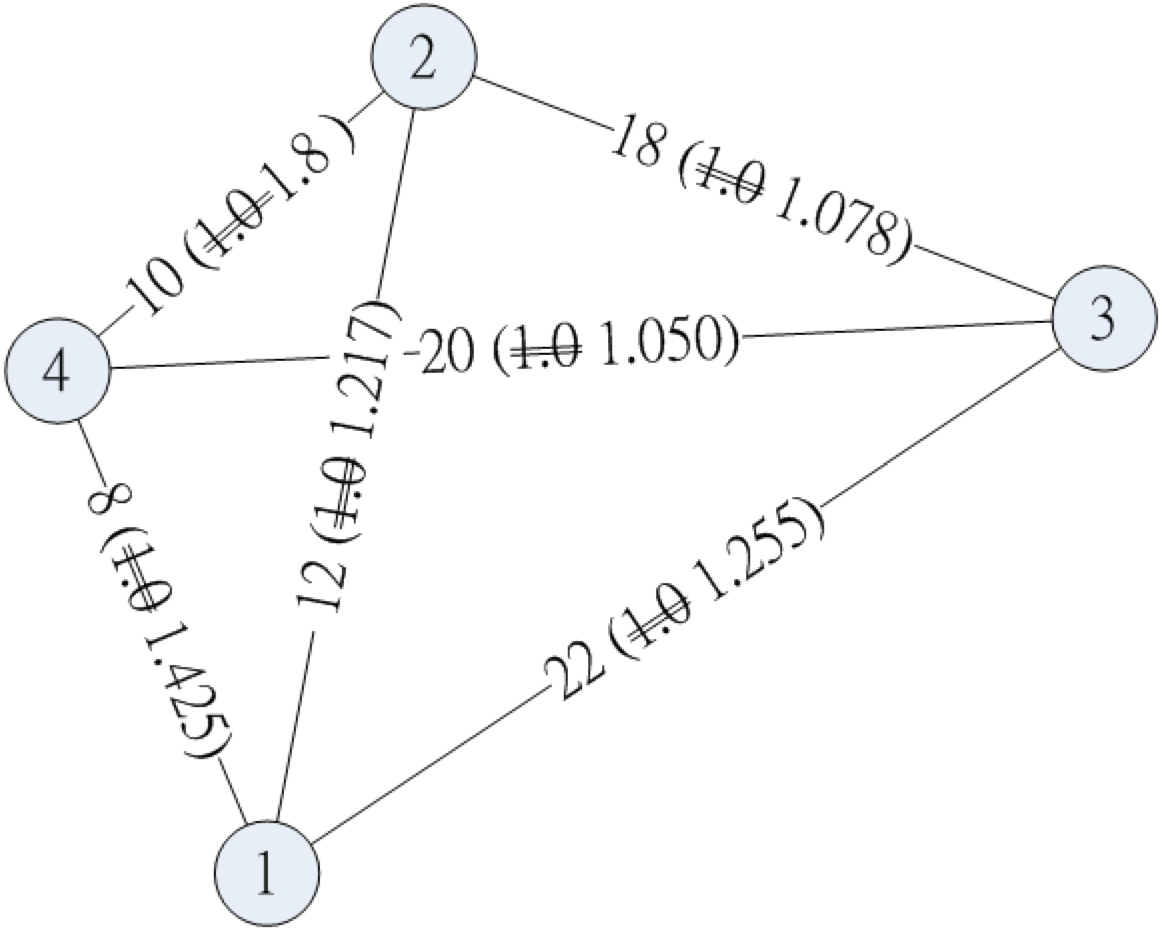
$$\underline{\tau(2,4) = (1-0.2)\cdot 1.0 + (5/10 + 5/10) = 1.800}$$

$$\tau(3,4) = (1-0.2)\cdot 1.0 + 5/20 = 1.050$$



Example (6)

- Now, Ants are faced with the updated pheromone levels



Other Applications

- In addition to TSP, ACO can be applied to many other combinatorial problems
- Key is to alter the heuristic function to reflect the solution quality, i.e., objective value

$$p_j = \frac{\tau(i,j)^{\alpha}\boxed{\eta(i,j)^{\beta}}}{\sum_{h \in J} \tau(i,h)^{\alpha}\boxed{\eta(i,h)^{\beta}}}$$

- Usually, the heuristic function is to treat minimization problems: taking an inverse for maximization problems

Variants

- In the following, we discuss some variants of ACO
  1. Add Local Search
  2. Ant Colony System: 2 approaches
  3. Population-based ACO

Variant: Add Local Search

- One may perform local search after each ant constructs a solution
- For example, if the solution is {1 6 3 4 2 5}; swap two nodes randomly for a few times to seek better solutions
- Because local search may be time-consuming, allowing only a few best ants to perform local search

Variant: Ant Colony System (1-1)

- Each ant builds a tour by repeatedly applying a stochastic greedy rule (state transition rule)
- **While** constructing its tour, an ant also modifies the amount of pheromone on the visited edges by applying the local updating rule (Originally, the pheromone is updated **after** all ants complete tour)
- Once all ants have completed their tour, the amount of pheromone on edges is modified again (global update)

Variant: Ant Colony System (1-2)

- State transition rule from *i* to *j*

$$j = \begin{cases} \arg\max_{u \in S}\{[\tau(i,u)^{\alpha} \cdot \eta(i,u)^{\beta}\} \text{ if } q \le q_0 & \text{(exploition)} \\ \text{Regular ACO rule} & \text{(biased exploration)} \end{cases}$$

Greedy rule: pick the best with pre-specified probability q_0

Variant: Ant Colony System (1-3)

- Ants change pheromone level of an edge during their visit

$$\tau(i, j) = (1 - \rho) \cdot \tau(i, j) + \rho \cdot \Delta\tau(i, j)$$

$0 \le \rho \le 1$; ρ is a tuning parameter

$\Delta\tau(i, j) =$ initial pheromone level

Variant: Ant Colony System (2)

- Only globally best ant (i.e., the ant which constructed the shortest tour in all the past iterations) is allowed to deposit pheromone

$$\tau(i, j) = (1-\gamma)\cdot\tau(i, j) + \gamma\cdot\Delta\tau(i, j)$$

$0 \le \gamma \le 1; \gamma$ is a tuning parameter

$$\Delta\tau(i, j) = \begin{cases} L_{global}^{-1} & \text{if } (i, j) \in \text{global best tour} \\ 0 & \text{otherwise} \end{cases}$$

- This is done after all ants complete tours

Variant: Population-based ACO (1)

- P-ACO maintains a small population P of the *k* best solutions in past iterations: elite archive
- P is updated after every iteration; pick the best to enter P
- P's size is fixed: the oldest solution leaves when a new one enters
- The pheromone matrix is derived anew in every iteration from population P

Variant: Population-based ACO (2)

- Each pheromone value is increased by

$$\tau(i, j) = \tau(i, j) + \kappa_{ij} \cdot \Delta$$

$$\kappa_{ij} = kappa; \text{number of solutions in P with } (i, j); \text{always integer}$$

$$\Delta = \text{user} - \text{specified constant}$$

- The update of pheromone values can be done as a population update: increasing as (*i*,*j*) enters P, decreasing as (*i*,*j*) leaves P
- No evaporation

Advanced Topics (1)

- Dynamic optimization problems
  - Search space changes during time
    - Distances may change; nodes are added or removed
    - New customers join the request list as vehicles are out
    - Telecommunication networks
  - Optimization time is restricted
    - Use artificial ants to find a quick solution

Advanced Topics (2)

- Continuous optimization problems: two ways
  1. Divide the domain of each variable into a finite set of intervals; then use regular ACO to solve it
  2. Ants sample next solutions from a **continuous** probability density function (PDF), not from a **discrete** list

Socha, K. and Dorigo, M. (2008). "Ant colony optimization for continuous domains." European Journal of Operational Research, 185, pp. 1155-1173.

Advanced Topics (3)

- Parallel implementation
  - Multi-colony ACO: Subpopulations of ants are assigned to single processors and information exchange is relatively rare
  - Master-slave ACO: Central processor collects solutions, updates the pheromone matrix, and send back to other processors for future iteration

Advanced Topics (4)

- Multi-objective optimization problems
  - Multiple pheromone matrices, each for an objective function
  - Ants are divided into groups; each group focuses on only one objective function
  - The new solutions will jointly update the elite archive, which intensifies the pheromone levels

Conclusions

- ACO has been used to solve combinatorial (discrete) optimization problems with success; its extension can tackle continuous optimization problems as well.
- ACO usually performs well with graphs (networks with links)
- Similar to other meta-heuristics, ACO is still evolving to different variants