INSA Toulouse, Applied Mathematics department
Supplementary material to J. Monnier's course

Exercises on differential calculus and the Riez-Frechet representation theorem

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Contents

1 Differential calculus

1.1 Operators $F(\cdot)$

Exercice 1.1 *Let X and Y denote Banach spaces or Hilbert spaces. F denotes either a map defined from X onto Y or a map from onto $\mathbb{R}$. (In this last case, F is called a functional).*

1) Let us consider the differential operator $F(u)(x) = \nabla u(x)$.
a) Provide examples of functional spaces pairs (X, Y) such that $F : X \to Y$.
b) Calculate the 1st order differential of $F(u)$ at u_0 in the direction δu.
c) Calculate the 2nd order differential of $F(u)$ at u_0 in the direction $(\delta u, \delta v)$.

2) Let us consider the differential operator $F(k; u)(x) = div(k(x)\nabla u(x))$.
a) (To go further). Provide examples of functional spaces triplets (K, X, Y) such that $F : (K, X) \to Y$.
b) Calculate the following differentials:
- $\partial_k F(k_0; u_0) \cdot \delta k$
- $\partial_u F(k_0; u_0) \cdot \delta u$
- $dF(k_0; u_0) \cdot (\delta k, \delta u)$

3) Let us consider the integral operator $F(u) = \displaystyle\int_\Omega u^p(x)\ dx$.
a) Clarify the minimal functional space to well define F.

b) Calculate the differential at u_0 in the direction δu.

4) Let us consider the operator $F(u) = \displaystyle\int_\Omega \|\nabla u\|^2 dx$.
a) Clarify a functional space such that F is well defined.
b) Calculate the 1st (resp. 2nd order) differential at u_0 in the direction δu (resp. $(\delta u, \delta v)$).

Corrections

1) $F(u)(x) = \nabla u(x)$. $F : X \to Y$.
a) Let Ω be a bounded domain in $\mathbb{R}^d$. Typical examples of functional spaces (X, Y) are: $X = H^1(\Omega)$ and $Y = (L^2(\Omega))^d$, or $X = H^2(\Omega)$ and $Y = (H^1(\Omega))^d$, or $X = W^{1,\infty}(\Omega)$ and $Y = (L^\infty(\Omega))^d$.
b) The map $u \mapsto F(u)$ is linear. Its differential $F'(u)$ ($F'(u) \equiv d_u F(u)$) equals F for any $u \in X$.
For all $\delta u \in X$, $F'(u_0) \cdot \delta u = \nabla(\delta u)$.
c) Next, the 2nd order differential simply reads: for all $\delta u \in X$ and $\delta v \in X$,

$$F''(u_0) \cdot (\delta u, \delta v) = 0$$

for any element u_0.

2) $F(k; u)(x) = div(k(x)\nabla u(x))$, $F : (Z, X) \to Y$.
a) Let Ω be a bounded domain in $\mathbb{R}^d$. Typical examples of functional spaces (X, Y, Z) are: $Z = L^\infty(\Omega)$, $X = H^2(\Omega)$ and $Y = L^2(\Omega)$.
b) The two maps $u \mapsto F(\cdot; u)$ and $k \mapsto F(k; \cdot)$ are linear. Therefore their differential are trivial:
For all $\delta k \in Z$, $\partial_k F(k_0; u_0) \cdot \delta k = div(\delta k \nabla u_0)$.
For all $\delta u \in X$, $\partial_u F(k_0; u_0) \cdot \delta u = div(k_0 \nabla(\delta u))$.
The total derivative $dF(k_0; u_0) \cdot (\delta k, \delta u)$ reads:

$$dF(k_0; u_0) \cdot (\delta k, \delta u) = \partial_k F(k_0; u_0) \cdot \delta k + \partial_u F(k_0; u_0) \cdot \delta u = div(\delta k \nabla u_0) + div(k_0 \nabla(\delta u))$$

3) $F(u) = \displaystyle\int_\Omega u^p(x)\, dx$, $F : X \to \mathbb{R}$.
a) The minimal regularity for u is $u \in X$ with $X = L^p(\Omega)$, $1 \leq p < \infty$.
b) For all $\delta u \in X$,

$$F'(u) \cdot \delta u = p \int_\Omega u^{p-1}(x) \delta u(x)\, dx$$

4) $F(u) = \displaystyle\int_\Omega \|\nabla u\|^2 dx$, $F : X \to \mathbb{R}$.
a) The minimal regularity for u is $\nabla u \in (L^2(\Omega))^d$. Therefore $X = H^1(\Omega)$ is an adequate functional space.

b) For all $\delta u \in X$,

$$F'(u) \cdot \delta u = 2 \int_\Omega (\nabla u(x), \nabla(\delta u)(x)) \; dx$$

and

$$F''(u) \cdot (\delta u, \delta v) = 2 \int_\Omega (\nabla(\delta u)(x), \nabla(\delta v)(x)) \; dx$$

1.2 Forms $a(\cdot,\cdot)$

Exercice 1.2 *Let V be a subspace of $H^1(\Omega)$. Let $a(.,.)$ be a form defined from $V \times V$ into $\mathbb{R}$.*
Write the differential expression $\partial_u a(u_0, v) \cdot \delta u$, in the following cases:
1) $a(u,v) = \displaystyle\int_\Omega u^p(x) v(x) \; dx$, $1 \le p < \infty$.
2) $a(u,v) = \displaystyle\int_\Omega \nabla u(x) \nabla v(x) \; dx$.
3) $a(u,v) = \displaystyle\int_\Omega \lambda(u(x)) \nabla u(x) \nabla v(x) \; dx$.
You will clarify the necessary minimal regularity for $\lambda(u)$.

Correction.
For all $\delta u \in V$,

1) $\partial_u a(u_0, v) \cdot \delta u = p \displaystyle\int_\Omega u_0^{p-1} \delta u v \; dx$

2) $\partial_u a(u_0, v) \cdot \delta u = \displaystyle\int_\Omega \nabla(\delta u) \nabla v \; dx$

3) The map $u \mapsto a(u,\cdot)$ is non-linear. To be differentiable, $\lambda(u)$ has to be differentiable.
Moreover, to have the integral of $a(u,v)$ well defined, $\lambda(u)$ has to be in $L^\infty(\Omega)$.
For all $\delta u \in V$, we have:

$$\partial_u a(u_0, v) \cdot \delta u = \int_\Omega \big(\lambda(u_0) \nabla(\delta u) \nabla v + \lambda'(u_0) \cdot \delta u \nabla(u_0 \nabla v)\big) \; dx$$

To have the integral above well defined, $\lambda'(u_0)$ has to be in $L^\infty(\Omega)$. Therefore, $\lambda(u)$ has to be in $W^{1,\infty}(\Omega)$.

Exercice 1.3 *Let V be a Hilbert space, let $a(.,.)$ be a bilinear form defined from $V \times V$ into $\mathbb{R}$; let l be a linear form defined from V into $\mathbb{R}$.*
Let j be the functional defined by: $j(u) = \frac{1}{2}a(u,u) - l(u)$.
Show that:

$$j'(u)(v) \equiv j'(u) \cdot v = a(u,v) - l(v) \quad \forall u \in V, \ \forall v \in V$$

$$j''(u)(v,w) = a(v,w) \quad \forall u, v, w \in V$$

Recall. The functional j corresponds to the *energy* of the system: $a(u,v) = l(v)\ \forall v$.

Correction. *The functional $j(u)$ is quadratic (since it defined from the bilinear symmetric form $a(.,.)$). Therefore it is C^∞ in V. Its derivatives are straightforward to calculate: $\forall u \in V, \ \forall v \in V$,*

$$j'(u) \cdot v = \frac{1}{2}a(u,v) + \frac{1}{2}a(v,u) - l(v)$$

Hence the result.
Next, $\forall w \in V$,

$$j''(u) \cdot (v,w) = a(v,w)$$

2 Linear forms & application of the Riez-Frechet theorem

The Riez-Frechet theorem provides a relationship between a Hilbert space H and its dual space H'. This result is very useful to handle PDE models in particular to linearize them (e.g. in view to implement the Newton-Raphson algorithm) and to derive the corresponding adjoint operator employed e.g. in variational data assimilation.

Notations.
Let H be a Hilbert space. Let L be an element of H', the dual of H.
In the following, we denote: $< L, v >_{H' \times H} \equiv L(v) \ , \ \forall v \in H$.

Exercice 2.1
1) Write the standard equation $a(u, v) = l(v)$ for all v in $H = H^1(\Omega)$, in an equation which holds in H'.

*2) Let L be a linear form defined from $H = L^2(\Omega)$ into $\mathbb{R}$.
We set:*

$$L(v) = \int_\Omega f v \; dx$$

with $f \in L^2(\Omega)$.

*Apply the Riesz-Fréchet representation theorem to L.
(We recall that $\left(L^2(\Omega)\right)' = L^2(\Omega)$).*

3) Let H be a subspace of $H^1(\Omega)$. Let $a(.,.)$ be the form defined from $H \times H$ into $\mathbb{R}$ by:

$$a(u, v) = \int_\Omega \lambda \nabla u \nabla v \; dx + \int_\Omega c u^3 v \; dx$$

with λ and c given in $L^\infty(\Omega)$.

Apply the Riesz-Fréchet representation theorem to this case.

4) What can you note in the case $c = 0$ a.e. $\in \Omega$?

Correction.

2) The mapping $v \mapsto L(v)$ *is indeed linear from* $H = L^2(\Omega)$ *into* $\mathbb{R}$.
In other words $L \in (L^2)' = L^2$ *and* $L(v) \equiv < L, v >_{(L^2)' \times L^2}$, $\forall v \in L^2(\Omega)$.
The Riesz-Fréchet representation theorem states that it exists an unique $f \in L^2(\Omega)$ *such that:*

$$< L, v >_{L^2 \times L^2} = (f, v)_{L^2}$$

Moreover: $\|L\|_{L^2} = \|f\|_{L^2}$.

Moreover, let us recall that: $(f, v)_{L^2} = \int_\Omega f v dx$.

3) The mapping $v \mapsto a(u, v)$ *is a linear form from* H *into* $\mathbb{R}$.

This linear mapping defines an element $A(u) \in H'$ *such that:*

$$< A(u), v >_{H' \times H} = a(u, v) \quad \forall v \in H$$

The Riesz-Fréchet representation theorem states that it exists an unique $a_u \in H$ *such that:*

$$(a_u, v)_H = < A(u), v >_{H' \times H}$$

Furthermore: $\|A(u)\|_{H'} = \|a_u\|_H$.

4) Here, in addition the mapping $u \mapsto a(u, v)$ *is linear. Therefore* $a(u, v)$ *is a bilinear of* $H \times H$ *onto* $\mathbb{R}$.
Compared to the previous case, the map $u \in H \mapsto A(u) \in H'$ *is in addition linear.*

Bilinear symmetric form case $a(.,.)$

Exercice 2.2 *Let* $H = L^2(\Omega)$. *Let* $a(.,.)$ *be the bilinear symmetric form defined on* $H \times H$ *by:* $a(u,v) = \int_\Omega uv dx$.
Let $l(.)$ *be the linear form defined by:* $l(v) = \int_\Omega fv dx$, *with* $f \in L^2(\Omega)$.
We set:

$$j(u) = \frac{1}{2}a(u,u) - l(u)$$

Prove that:

$$j'(u) = (u - f) \text{ in } L^2(\Omega)$$

Correction. *First, let us recall that* $j(u)$ *represents the system energy. Indeed, it has been previously shown that the unique solution of* $a(u,v) = l(v)$ *for all* v, *minimizes the functional* $j(u)$ *(see the course material on the INSA Moodle page).*

The functional j *is defined from* H *into* $\mathbb{R}$ *therefore* $j' \in L(H,\mathbb{R}) \equiv H' = L^2(\Omega)$. *And* $\forall \delta u \in H$,

$$j'(u) \cdot \delta u \equiv < j'(u), \delta u >_{H' \times H} = a(u, \delta u) - l(\delta u) = \int_\Omega (u - f)\delta u \, dx = (u - f, \delta u)_{L^2}$$

In vertu of the Riesz-Frechet representation theorem, it follows that:

$$j'(u) \cdot \delta u = < (u - f), \delta u >_{H' \times H}$$

Hence the result: $j'(u) = (u - f)$ *in* $H' = (L^2)' = L^2$.

Comment: The gradient of the energy functional is nothing else than $(u - f)$. *This expression may be employed to numerically minimize the energy therefore obtaining the solution* u^* *of the "state" equation* $a(u,v) = l(v) \;\; \forall v$.