Exploration Statistique Multidimensionnelle

# Multiple Correspondence Analysis

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## Main features

- Generalization of CA to more than 2 variables
- Representations of the correspondences between levels
- Methodology : from $p = 2$ to $p > 2$

## Utilization

- Data analysis in presence of qualitative data
- MCA provides a particular PCA for qualitative data
  $\rightarrow$ can be used for clustering (on the PCA coordinates)

The variables in Velib are both quantitative & qualitative.
It is easy to convert them to qualitative variables only,
by discretizating the numerical values.

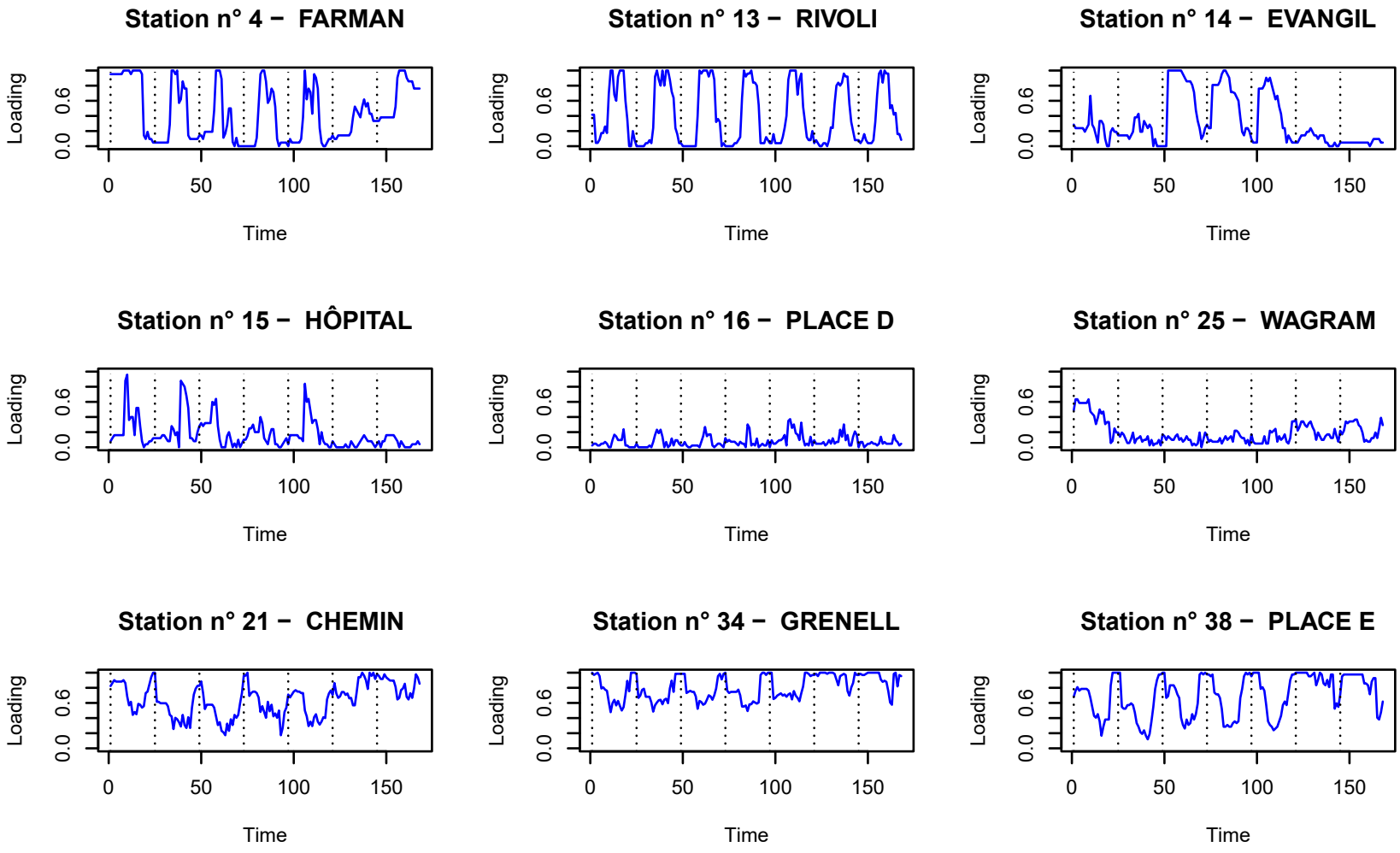



FIGURE – Selection of nine stations

Creation of two qualitative variables according to

- the daily time : day = [7h, 20h], night = [21h, 6h]
- loading values : ”-” = [0, 0.2), ”=” = [0.2, 0.5), ”+” = [0, 5, 1]

| | loadDay | loadNight |
|---|---|---|
| FARMAN | day+ | night= |
| RIVOLI | day+ | night- |
| EVANGIL | day= | night= |
| HÔPITAL | day= | night- |
| PLACE D | day- | night- |
| WAGRAM | day- | night- |
| CHEMIN | day+ | night+ |
| GRENELL | day+ | night+ |
| PLACE E | day+ | night+ |

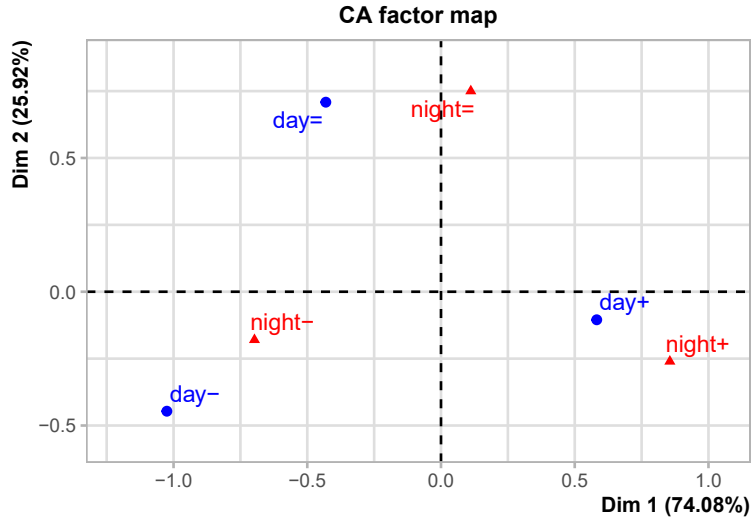
| | night- | night= | night+ |
|---|---|---|---|
| day- | 2 | 0 | 0 |
| day= | 1 | 1 | 0 |
| day+ | 1 | 1 | 3 |

Row profiles :

| | night- | night= | night+ |
|---|---|---|---|
| day- | | | |
| day= | | | |
| day+ | | | |

Column profiles :

| | night- | night= | night+ |
|---|---|---|---|
| day- | | | |
| day= | | | |
| day+ | | | |



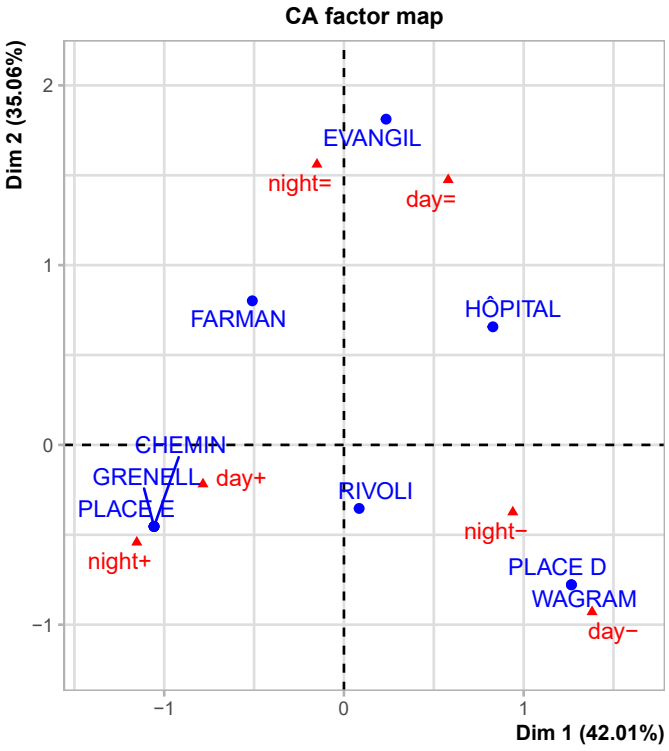

Idea 1. To apply CA on the contingency table T.

| | day- | day= | day+ | night- | night= | night+ |
|---|---|---|---|---|---|---|
| FARMAN | 0 | 0 | 1 | 0 | 1 | 0 |
| RIVOLI | 0 | 0 | 1 | 1 | 0 | 0 |
| EVANGIL | 0 | 1 | 0 | 0 | 1 | 0 |
| HÔPITAL | 0 | 1 | 0 | 1 | 0 | 0 |
| PLACE D | 1 | 0 | 0 | 1 | 0 | 0 |
| WAGRAM | 1 | 0 | 0 | 1 | 0 | 0 |
| CHEMIN | 0 | 0 | 1 | 0 | 0 | 1 |
| GRENELL | 0 | 0 | 1 | 0 | 0 | 1 |
| PLACE E | 0 | 0 | 1 | 0 | 0 | 1 |

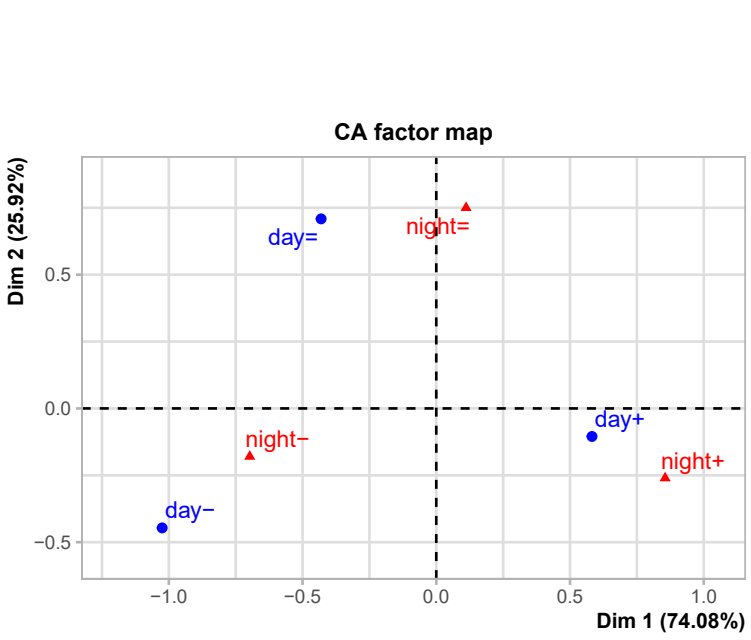
Idea 2. Create a ”disjunctive table” D with dummy variables.

*Q : What can you say of the table of row profiles ?*

| | day- | day= | day+ | night- | night= | night+ |
|---|---|---|---|---|---|---|
| FA | 0 | 0 | 1 | 0 | 1 | 0 |
| RI | 0 | 0 | 1 | 1 | 0 | 0 |
| EV | 0 | 1 | 0 | 0 | 1 | 0 |
| HÔ | 0 | 1 | 0 | 1 | 0 | 0 |
| P. D | 1 | 0 | 0 | 1 | 0 | 0 |
| WA | 1 | 0 | 0 | 1 | 0 | 0 |
| CH | 0 | 0 | 1 | 0 | 0 | 1 |
| GR | 0 | 0 | 1 | 0 | 0 | 1 |
| P. E | 0 | 0 | 1 | 0 | 0 | 1 |




Idea 2. To apply CA on the disjunctive table D.
*Q : Interpret the results. Compare with idea 1.*



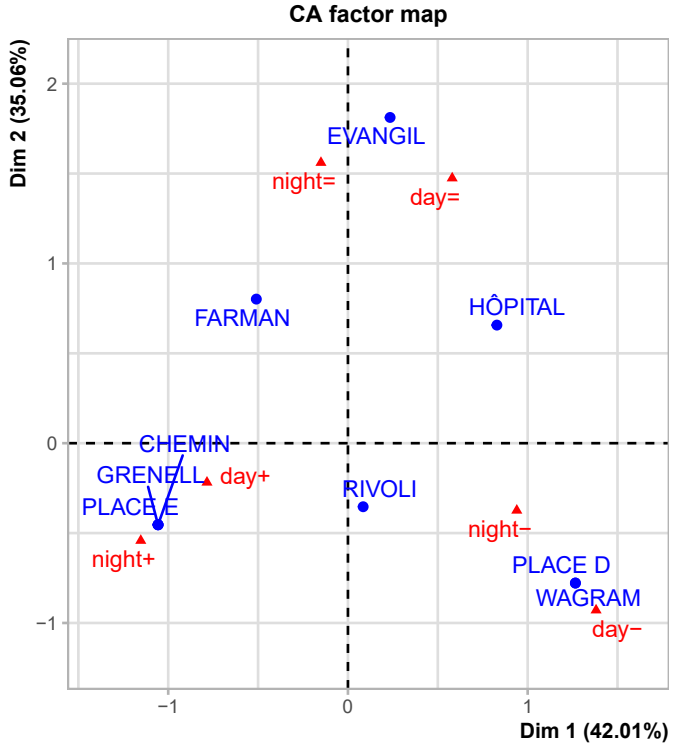



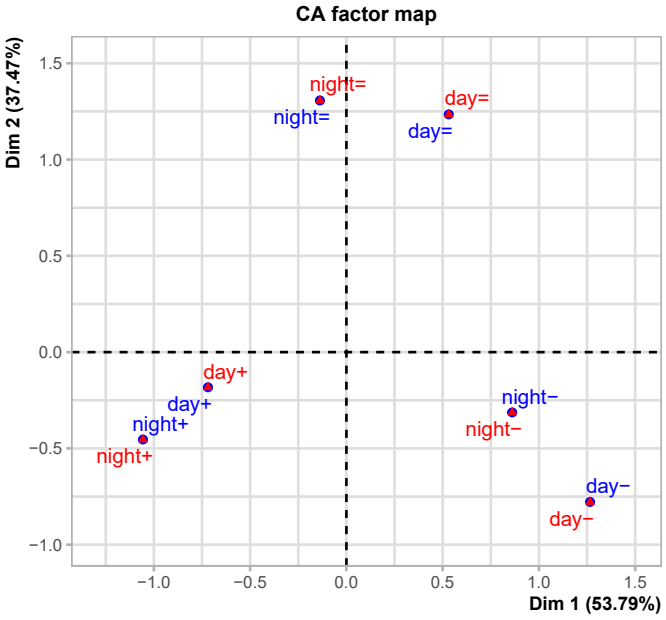

FIGURE – Comparison of the two CA, on T or on D

| | day- | day= | day+ | night- | night= | night+ |
|---|---|---|---|---|---|---|
| day- | 2 | 0 | 0 | 2 | 0 | 0 |
| day= | 0 | 2 | 0 | 1 | 1 | 0 |
| day+ | 0 | 0 | 5 | 1 | 1 | 3 |
| night- | 2 | 1 | 1 | 4 | 0 | 0 |
| night= | 0 | 1 | 1 | 0 | 2 | 0 |
| night+ | 0 | 0 | 3 | 0 | 0 | 3 |

Idea 3. Create the "Burt" table, $B = D^\top D$.

*Q : Interpret each block in terms of contingency. Where is T ?*

| | day- | day= | day+ | night- | night= | night+ |
|---|---|---|---|---|---|---|
| day- | 2 | 0 | 0 | 2 | 0 | 0 |
| day= | 0 | 2 | 0 | 1 | 1 | 0 |
| day+ | 0 | 0 | 5 | 1 | 1 | 3 |
| night- | 2 | 1 | 1 | 4 | 0 | 0 |
| night= | 0 | 1 | 1 | 0 | 2 | 0 |
| night+ | 0 | 0 | 3 | 0 | 0 | 3 |



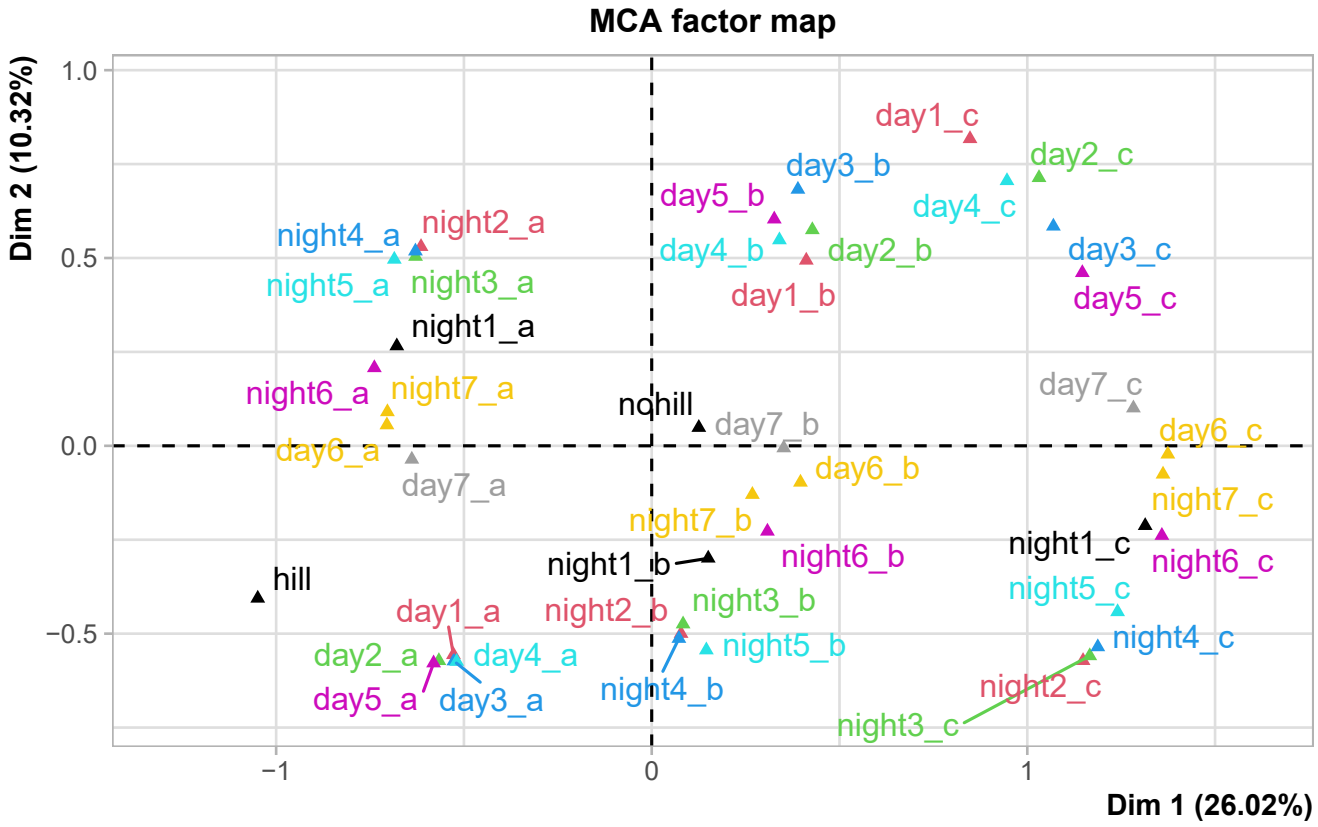

Idea 3. To apply CA on the Burt table B.

*Q : Interpret the results. Compare with idea 1 & 2.*

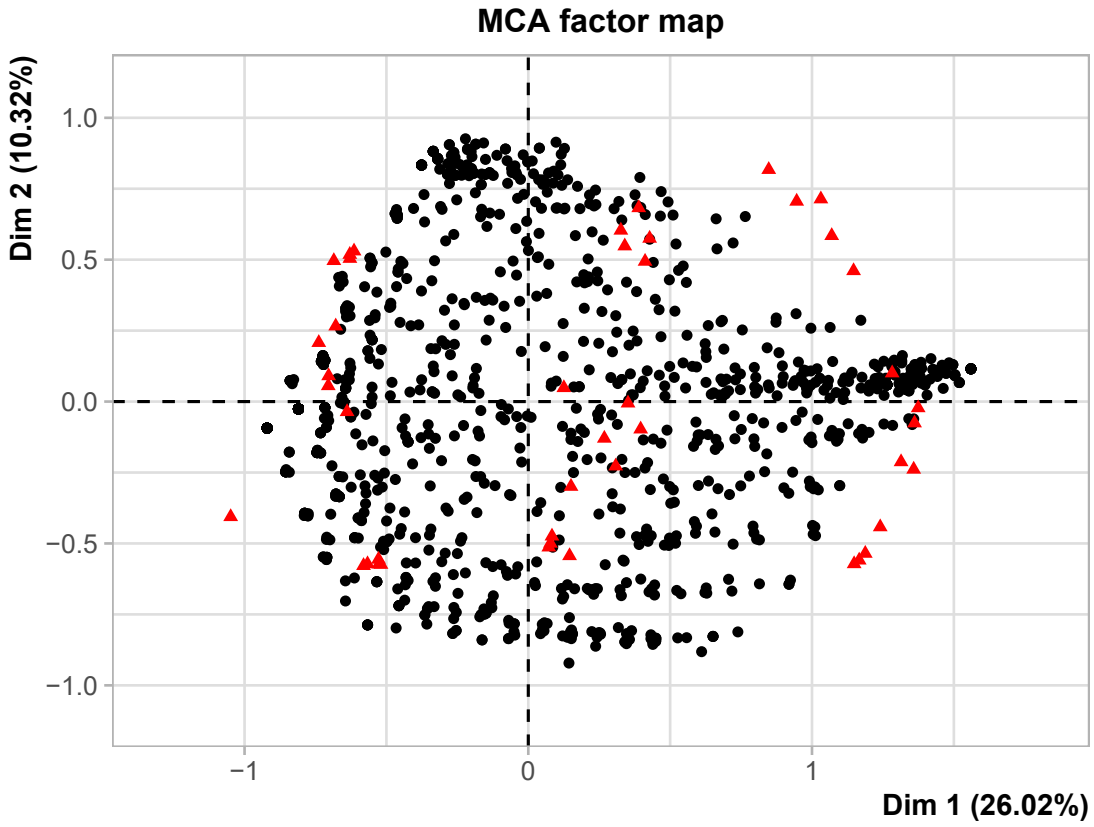
We now do a more realistic analysis, by creating 15 variables :

- hill : binary variable with levels ”hill / nohill”
- dayi $(i = 1, \ldots, 7)$ : 3 levels according to the mean loading in [7h, 20h] for day n°i, defined by [0, 1/3, 2/3, 1].
- nighti $(i = 1, \ldots 7)$ : Same thing for the nights

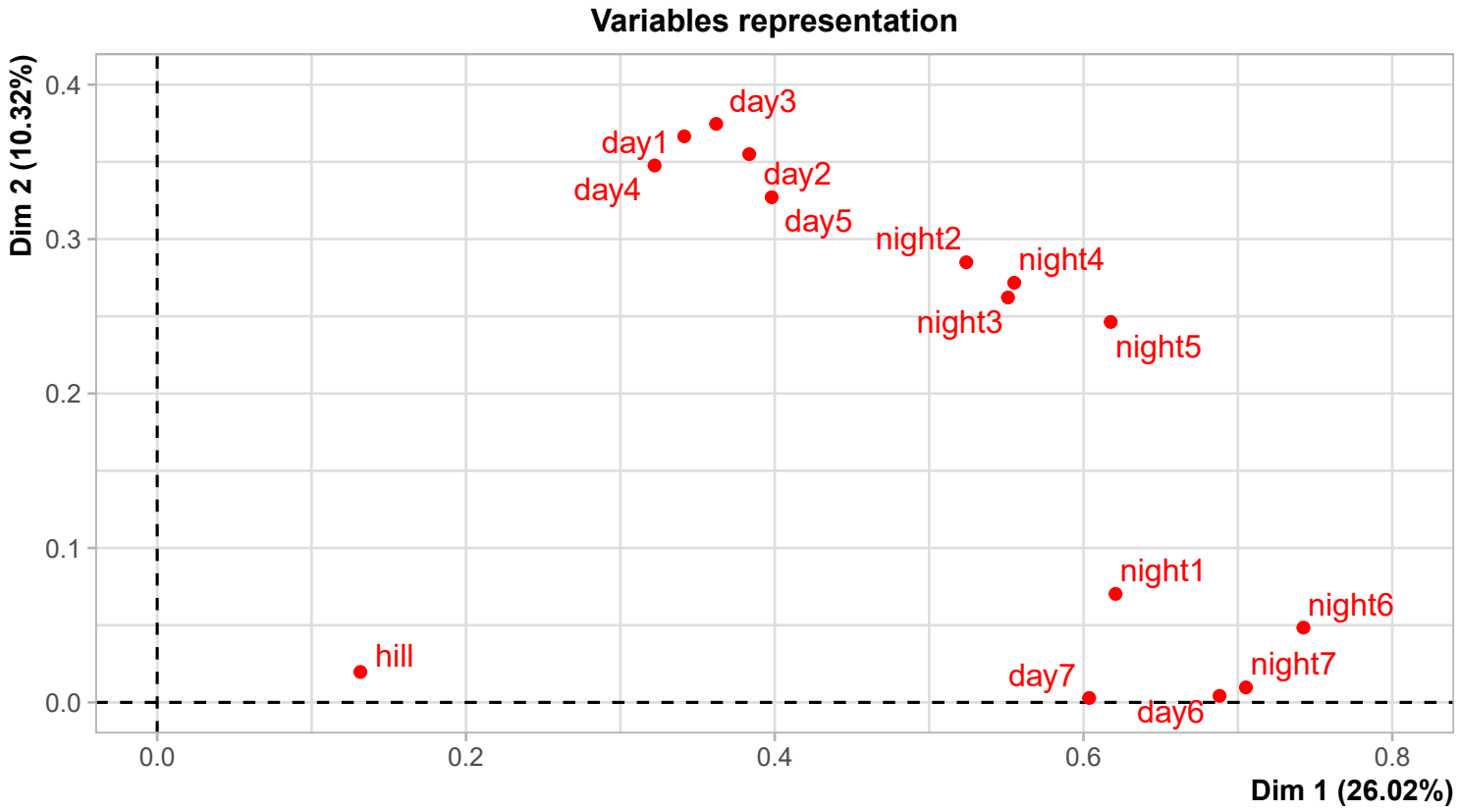
| | hill | day1 | day2 | . . . | day7 | night1 | night2 | . . . | night7 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | nohill | a | a | . . . | a | a | b | . . . | a |
| 2 | nohill | a | a | . . . | b | b | c | . . . | b |
| 3 | nohill | a | c | . . . | b | a | b | . . . | b |
| 4 | nohill | c | b | . . . | c | b | a | . . . | b |
| 5 | nohill | c | c | . . . | b | b | b | . . . | b |
| 6 | nohill | a | a | . . . | a | b | b | . . . | a |
| . . . | | | | | | | | | |




CA on the disjunctive table.




CA on the disjunctive table, with individuals.




CA on the disjunctive table (representation of variables only).

Application to clustering : let's use the 5 first coordinates of the individuals (PCA scores of the row profiles obtained from D).

In the next slides, we compare the results obtained :

- by kMeans on the 5th first PCA principal components, on the original quantitative data (without hill)
- by kMeans on the 5th first MCA principal components, on the qualitative data (including hill)

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 137 | 15 | 7 | 9 |
| 2 | 0 | 1 | 81 | 0 | 0 | 9 | 0 |
| 3 | 0 | 1 | 55 | 0 | 0 | 12 | 49 |
| 4 | 7 | 0 | 0 | 43 | 261 | 0 | 0 |
| 5 | 0 | 1 | 0 | 17 | 8 | 7 | 96 |
| 6 | 2 | 58 | 4 | 7 | 0 | 61 | 20 |
| 7 | 168 | 14 | 0 | 10 | 10 | 18 | 0 |

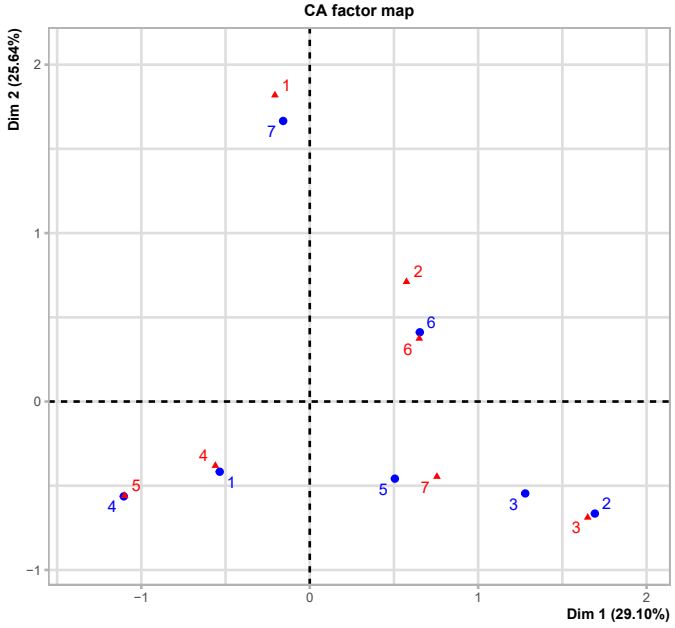
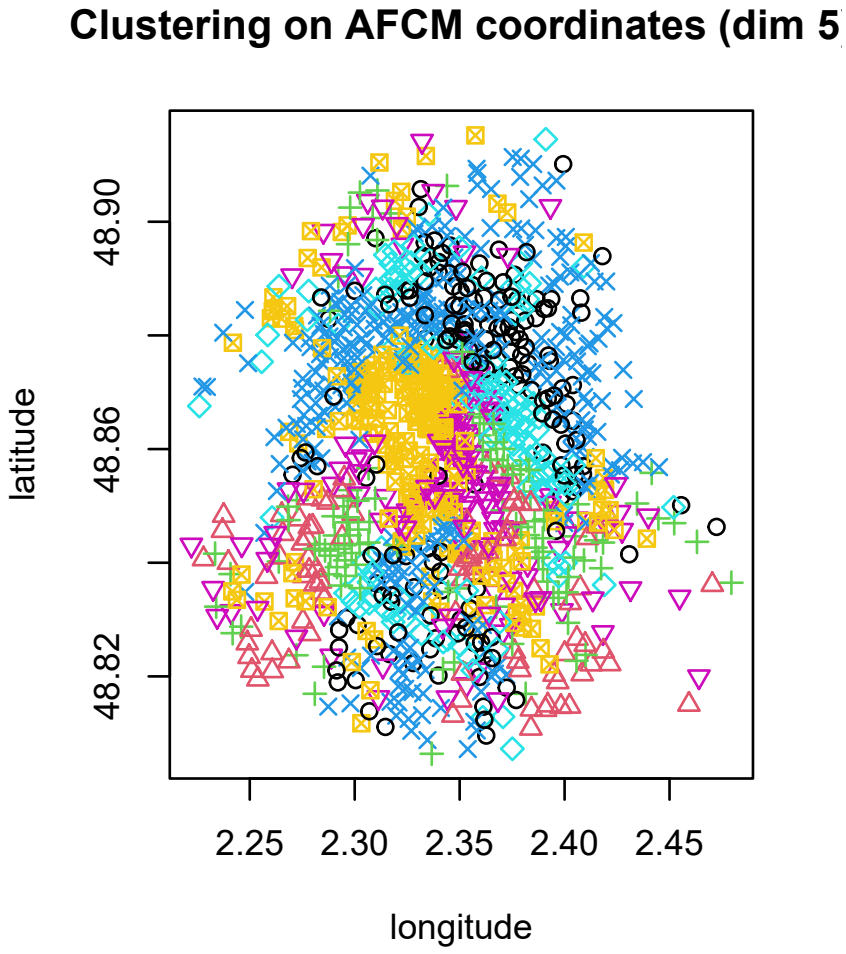
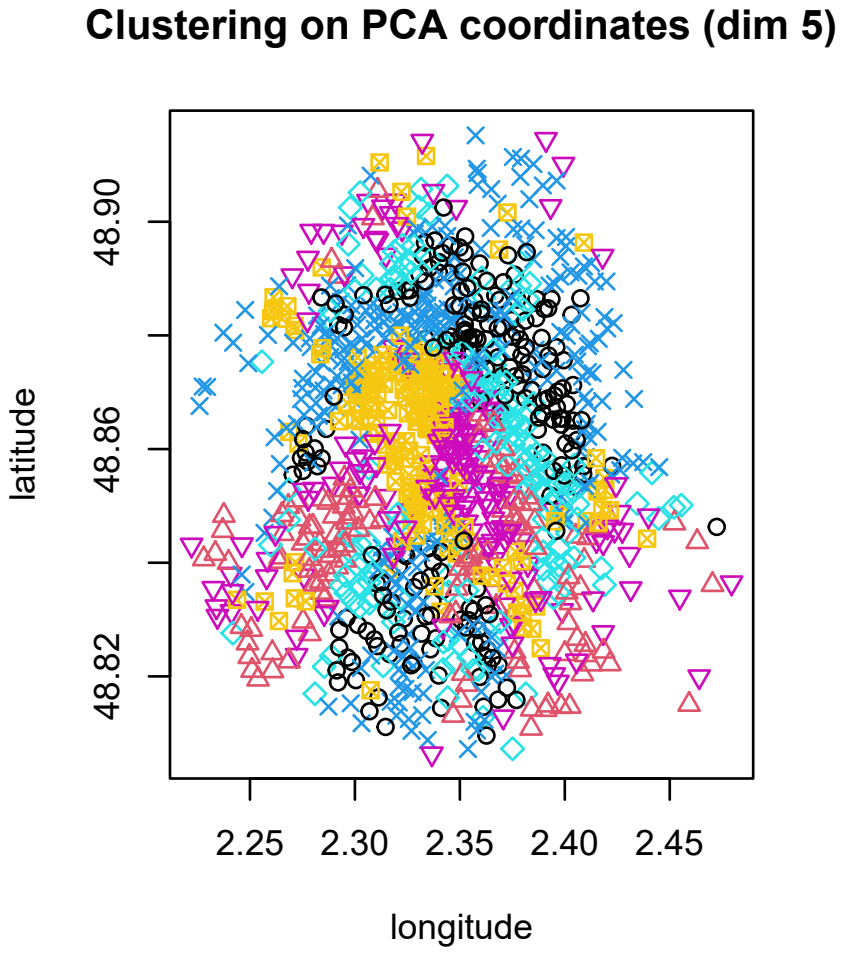
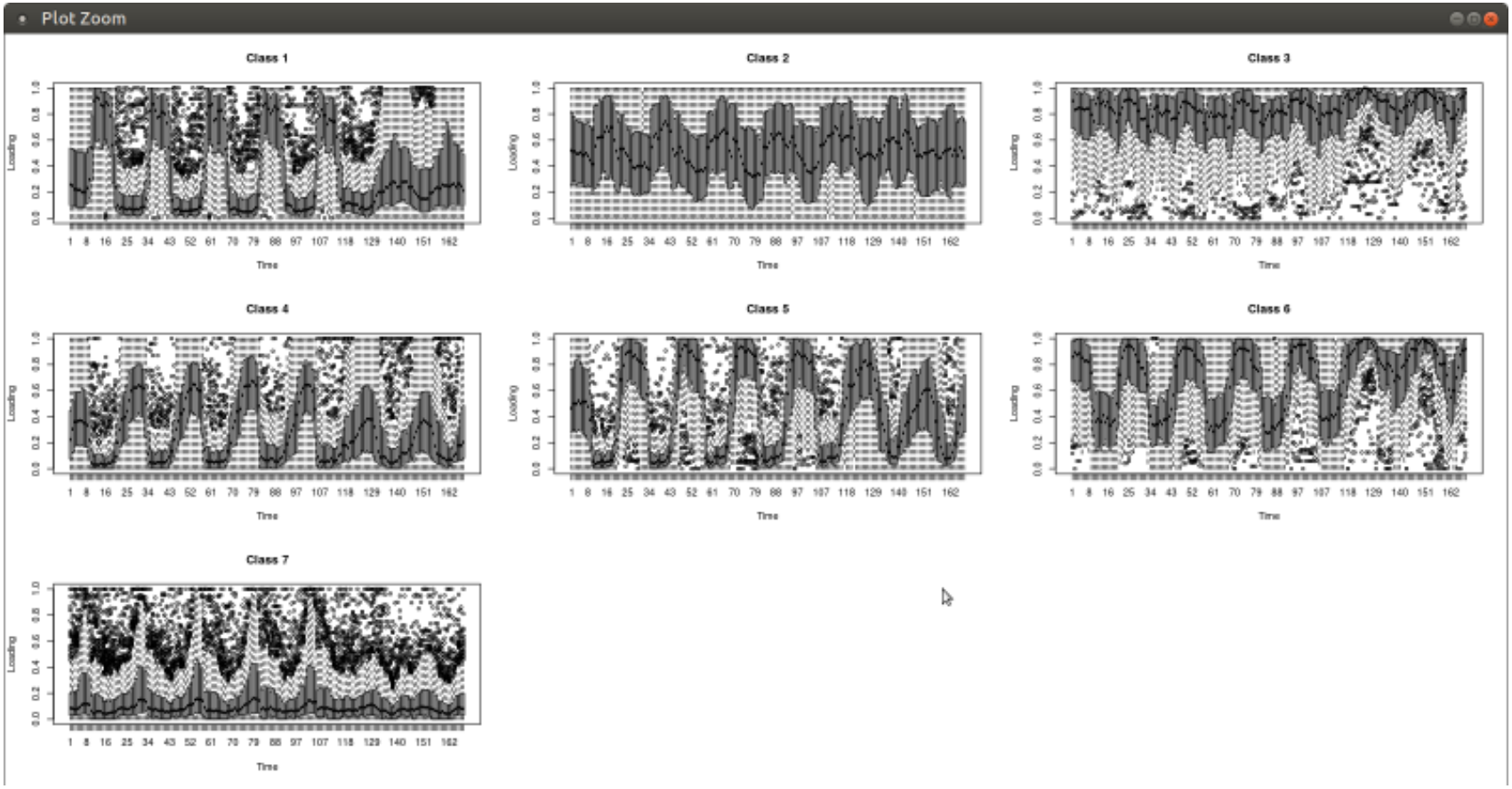



FIGURE – CA on the contingency table of the two clusterings : kMeans on the 5th first coordinates of PCA or MCA.


Clustering on AFCM coordinates (dim 5
latitude
48.90
48.86
48.82
2.25
2.30
2.35
2.40
2.45
longitude


Clustering on PCA coordinates (dim 5)
latitude
48.90
48.86
48.82
2.25
2.30
2.35
2.40
2.45
longitude


Plot Zoom
Class 1
Class 2
Class 3
Class 4
Class 5
Class 6
Class 7
Loading
Time

Some conclusions from this example.

- With 2 qualitative variables, three CA give the same conclusions but the CA on D also includes individuals :)
- Thus the disjunctive table D is used in general.
- Realistic interpretation for the velib data.
- Realistic results for clustering with mixed quantitative / qualitative data

## Indicator functions of levels

- $X$ : qualitative variable with $c$ levels.
- Indicator variable of the $k^{\text{th}}$ level of $X$ (size $n_k$) :

$$X_{(k)}(i) = \begin{cases} 1 & \text{if } X(i) = \mathcal{X}_k, \\ 0 & \text{else,} \end{cases}$$

- Matrix of indicators (or 'dummy' variables) $\mathbf{X}(n \times c)$ :

$$x_i^k = X_{(k)}(i). \quad \text{with} \quad \sum_{k=1}^{c} x_i^k = 1, \forall i \ \text{ and } \sum_{i=1}^{n} x_i^k = n_k.$$

## Disjunctive table

- $X^j; j = 1, \ldots, p$, $p$ qualitative variables
  - $X^j$ with $c_j$ levels : $c = \sum_{j=1}^{p} c_j$
  - $\mathbf{X}_j$ the matrix of indicators of $X^j$
- Disjunctive table :

$$\mathbf{X} = [\mathbf{X}_1 | \cdots | \mathbf{X}_p] \quad \text{with} \quad \sum_{k=1}^{c} x_i^k = p, \forall i \text{ and } \sum_{i=1}^{n} \sum_{k=1}^{c} x_i^k = np$$

## Burt table $(c \times c)$

$$\mathcal{B} = \mathbf{X}'\mathbf{X}$$

$$\mathcal{B} = [\mathcal{B}_{j\ell}] \quad (j = 1, \ldots, p; \ell = 1, \ldots, p);$$

where $\mathcal{B}_{j\ell}$, $(c_j \times c_\ell)$ is the contingency table :

$$\mathcal{B}_{j\ell} = \mathbf{X}_j'\mathbf{X}_\ell$$

$$\mathcal{B}_{jj} = \text{ diag } (n_1^j, \ldots, n_{c_j}^j)$$

$\mathcal{B}$ is symmetric, with marginal totals $n_\ell^j p$ and grand total $np^2$

## Notations for the MCA of $\mathbf{X}$

$X^1$ and $X^2$ with $r$ and $c$ levels.

$$
\begin{aligned}
\overline{\mathbf{T}} &= \mathbf{X} = [\mathbf{X}_1 | \mathbf{X}_2] \; ; \\
\overline{\mathbf{D}}_r &= \frac{1}{n} \mathbf{I}_n \; ; \\
\overline{\mathbf{D}}_c &= \frac{1}{2} \begin{bmatrix} \mathbf{D}_r & 0 \\ 0 & \mathbf{D}_c \end{bmatrix} = \frac{1}{2} \boldsymbol{\Delta} \; ; \\
\overline{\mathbf{A}} &= \frac{1}{2n} \overline{\mathbf{T}}' \overline{\mathbf{D}}_r^{-1} = \frac{1}{2} \mathbf{X}' \; ; \\
\overline{\mathbf{B}} &= \frac{1}{2n} \overline{\mathbf{T}} \overline{\mathbf{D}}_c^{-1} = \frac{1}{n} \mathbf{X} \boldsymbol{\Delta}^{-1}.
\end{aligned}
$$

MCA = PCA of row and column profiles.

## PCA of the row profiles of $\mathbf{X}$

That PCA leads to the spectral decomposition of the $\overline{\mathbf{D}}_c^{-1}$–symmetric and p.s.d. matrix : $\overline{\mathbf{A}}\,\overline{\mathbf{B}} = \frac{1}{2}\begin{bmatrix} \mathbf{I}_r & \mathbf{B} \\ \mathbf{A} & \mathbf{I}_c \end{bmatrix}$

- $r+c$ eigenvalues of $\overline{\mathbf{A}}\,\overline{\mathbf{B}}$ : $\mu_k = \frac{1 \pm \sqrt{\lambda_k}}{2}$ ($\lambda_k$, eigenvalue of $\mathbf{AB}$)
- $\overline{\mathbf{D}}_c^{-1}$–orthonormal eigenvectors : $\overline{\mathbf{V}} = \frac{1}{2}\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}$ with $\mathbf{U}$ (resp. $\mathbf{V}$) = $\mathbf{D}_r^{-1}$–ortho. (resp. $\mathbf{D}_c^{-1}$) eigenvectors of $\mathbf{BA}$ (resp. $\mathbf{AB}$)
- Principal components : $\overline{\mathbf{C}}_r = \frac{1}{2}\left[\mathbf{X}_1\mathbf{C}_r + \mathbf{X}_2\mathbf{C}_c\right]\mathbf{\Lambda}^{-1/2}$, with $\mathbf{C}_r$ and $\mathbf{C}_c$ : principal components of CA

## Caution

- CA provide additional non-zero eigenvalues, without statistical signification

## PCA of colums profils of $\mathbf{X}$

That PCA leads to the spectral decomposition of the $\overline{\mathbf{D}}_r^{-1}$–symmetric and p.s.d. matrix :

$$\overline{\mathbf{B}}\,\overline{\mathbf{A}} = \frac{1}{2n}\left[\mathbf{X}_1\mathbf{D}_r^{-1}\mathbf{X}_1' + \mathbf{X}_2\mathbf{D}_c^{-1}\mathbf{X}_2'\right].$$

- $\mu_k$ : $r+c$ non-zero eigenvalues of $\overline{\mathbf{B}}\,\overline{\mathbf{A}}$
- $\overline{\mathbf{D}}_r^{-1}$–orthonormal eigenvectors : $\overline{\mathbf{U}} = \frac{1}{n}\overline{\mathbf{C}}_r\mathbf{M}^{-1/2}$
- Principal components : $\overline{\mathbf{C}}_c = \begin{bmatrix} \mathbf{C}_r \\ \mathbf{C}_c \end{bmatrix} \mathbf{\Lambda}^{-1/2}\mathbf{M}^{1/2}$

## Notations of MCA for $\mathcal{B}$

$\mathcal{B}$ is symmetric $\Rightarrow$ row profiles = column profiles

$$\widetilde{\mathbf{T}} = \mathcal{B} = \begin{bmatrix} n\mathbf{D}_r & \mathbf{T} \\ \mathbf{T}' & n\mathbf{D}_c \end{bmatrix};$$

$$\widetilde{\mathbf{D}}_r = \widetilde{\mathbf{D}}_c = \frac{1}{2}\begin{bmatrix} \mathbf{D}_r & 0 \\ 0 & \mathbf{D}_c \end{bmatrix} = \frac{1}{2}\boldsymbol{\Delta} = \overline{\mathbf{D}}_c \ ;$$

$$\widetilde{\mathbf{A}} = \widetilde{\mathbf{B}} = \frac{1}{2}\begin{bmatrix} \mathbf{I}_r & \mathbf{B} \\ \mathbf{A} & \mathbf{I}_c \end{bmatrix} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}.$$

## PCA of the row (or column) profiles of $\mathcal{B}$

That PCA leads to the spectral decomposition of the $\widetilde{\mathbf{D}_c}^{-1}$–symmetric and p.s.d. matrix :

$$\widetilde{\mathbf{A}}\widetilde{\mathbf{B}} = \left[\overline{\mathbf{A}}\,\overline{\mathbf{B}}\right]^2$$

- $\widetilde{\mathbf{D}_c}^{-1}$–orthonormal eigenvectors

$$\widetilde{\mathbf{U}} = \widetilde{\mathbf{V}} = \overline{\mathbf{V}} = \frac{1}{2}\begin{bmatrix}\mathbf{U}\\ \mathbf{V}\end{bmatrix}$$

- Eigenvalues : $\nu_k = \mu_k^2$
- Principal components : $\widetilde{\mathbf{C}_r} = \widetilde{\mathbf{C}_c} = \begin{bmatrix}\mathbf{C}_r\\ \mathbf{C}_c\end{bmatrix}\mathbf{\Lambda}^{-1/2}\mathbf{M}$

## Comparison

- The three MCA of $\mathbf{T}, \mathbf{X}, \mathcal{B}$, give homothetic representations of levels $\Rightarrow$ same interpretation
- The MCA of $\mathbf{X}$ and $\mathcal{B}$ have non-zero eigenvalues without signification
- The MCA of $\mathbf{X}$ provides a representation of individuals

## Notations of MCA

- $\{X^j ; j = 1, \ldots, p\}$, $p$ qualitative variables
- $n_k^j$ counts for the $k^{\text{th}}$ level of $X^j$
- $\mathbf{D}_j = \frac{1}{n}$ diag $(n_1^j, \ldots, n_{c_j}^j)$
- $\mathbf{\Delta} =$ diag $(\mathbf{D}_1 \ldots \mathbf{D}_p)$ (squared, $c \times c$)
- $\mathbf{X} = [\mathbf{X}_1 | \cdots | \mathbf{X}_p]$ : disjunctive table
- $\mathcal{B} = \mathbf{X}'\mathbf{X}$ : Burt table

## Definition

We call Multiple Correspondence Analysis (MCA) of the variables $(X^1, \ldots, X^p)$ the CA done either on $\mathbf{X}$ or on $\mathcal{B}$

## Limitation

Mind that only 2nd order interactions are considered, but not the link between triplets (or more) of variables.

$$\begin{aligned}
\overline{\mathbf{T}} &= \mathbf{X}\,; \\
\overline{\mathbf{D}}_r &= \frac{1}{n}\mathbf{I}_n\,; \quad \overline{\mathbf{D}}_c = \frac{1}{p}\boldsymbol{\Delta}\,; \\
\overline{\mathbf{A}} &= \frac{1}{p}\mathbf{X}'\,; \quad \overline{\mathbf{B}} = \frac{1}{n}\mathbf{X}\boldsymbol{\Delta}^{-1}.
\end{aligned}$$

## Row profiles of $\mathbf{X}$

- Diagonalize : $\overline{\mathbf{A}}\,\overline{\mathbf{B}} = \frac{1}{np}\boldsymbol{\mathcal{B}}\boldsymbol{\Delta}^{-1}$
- $m \leq c - p$ eigenvalues $\mu_k$, in $\mathbf{M}$
- Eigenvectors : $\overline{\mathbf{V}} = \begin{bmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_p \end{bmatrix}$
- Principal components : $\overline{\mathbf{C}}_r = \sum_{j=1}^{p} \mathbf{X}_j \mathbf{D}_j^{-1} \mathbf{V}_j$
- Caution : The eigenvectors of the blocks $\mathbf{V}_j$ are not the $\mathbf{D}_j^{-1}$–orthonormal eigenvectors of a known matrix.

## PCA of the column profiles of $\mathbf{X}$

- Diagonalize : $\overline{\mathbf{B}}\,\overline{\mathbf{A}} = \frac{1}{np}\mathbf{X}\boldsymbol{\Delta}^{-1}\mathbf{X}' = \frac{1}{np}\sum_{j=1}^{p}\mathbf{X}_j\mathbf{D}_j^{-1}\mathbf{X}_j'$
- Eigenvectors : $\overline{\mathbf{U}} = \overline{\mathbf{B}\mathbf{V}}\mathbf{M}^{-1/2}$
- Principal components :

$$\overline{\mathbf{C}}_c = p\boldsymbol{\Delta}^{-1}\overline{\mathbf{V}}\mathbf{M}^{1/2} = \begin{bmatrix} \mathbf{C}_1 \\ \vdots \\ \mathbf{C}_p \end{bmatrix}$$

## Properties of the MCA of $\mathcal{B}$

$\mathcal{B}$ is symmetric $\Rightarrow$ row profiles = column profiles.

$$\begin{aligned}
\widetilde{\mathbf{T}} &= \mathcal{B} \, ; \\
\widetilde{\mathbf{D}_r} &= \widetilde{\mathbf{D}_c} = \frac{1}{p}\boldsymbol{\Delta} = \overline{\mathbf{D}}_c \, ; \\
\widetilde{\mathbf{A}} &= \widetilde{\mathbf{B}} = \frac{1}{np}\mathcal{B}\boldsymbol{\Delta}^{-1} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}.
\end{aligned}$$

## PCA of the row (or column) profiles of $\mathcal{B}$

- Diagonalize :

$$\widetilde{\mathbf{A}}\widetilde{\mathbf{B}} = \left[\overline{\mathbf{A}}\,\overline{\mathbf{B}}\right]^2$$

- Eigenvectors : $\widetilde{\mathbf{U}} = \widetilde{\mathbf{V}} = \overline{\mathbf{V}}$
- Eigenvalues : $\nu_k = \mu_k^2$
- Principal components :

$$\widetilde{\mathbf{C}_r} = \widetilde{\mathbf{C}_c} = \overline{\mathbf{C}}_c\mathbf{M}^{1/2} = \begin{bmatrix} \mathbf{C}_1 \\ \vdots \\ \mathbf{C}_p \end{bmatrix} \mathbf{M}^{1/2}$$

## Interpretation

- proximities and oppositions of the levels of different variables, preferring levels far from the origin
- mind to levels with small counts
- the $\%$ of explained inertia and other indicators are not easy to interpret anymore with the original data

# Table de contingence complète

| | | | Histologie | | | |
|---|---|---|---|---|---|---|
| | | | Inflam minime | | Grande inflam | |
| Centre | Âge | Survie | Maligne | Bénigne | Maligne | Bénigne |
| Tokyo | $< 50$ | non | 9 | 7 | 4 | 3 |
| | | oui | 26 | 68 | 25 | 9 |
| | $50 - 69$ | non | 9 | 9 | 11 | 2 |
| | | oui | 20 | 46 | 18 | 5 |
| | $> 70$ | non | 2 | 3 | 1 | 0 |
| | | oui | 1 | 6 | 5 | 1 |
| Boston | $< 50$ | non | 6 | 7 | 6 | 0 |
| | | oui | 11 | 24 | 4 | 0 |
| | $50 - 69$ | non | 8 | 20 | 3 | 2 |
| | | oui | 18 | 58 | 10 | 3 |
| | $> 70$ | non | 9 | 18 | 3 | 0 |
| | | oui | 15 | 26 | 1 | 1 |
| Glamor. | $< 50$ | non | 16 | 7 | 3 | 0 |
| | | oui | 16 | 20 | 8 | 1 |
| | $50 - 69$ | non | 14 | 12 | 3 | 0 |
| | | oui | 27 | 39 | 10 | 4 |
| | $> 70$ | non | 3 | 7 | 3 | 0 |
| | | oui | 12 | 11 | 4 | 1 |

We will study this example in a computer lab session :

- The direct analysis only considers second-order interactions on Survival $\Rightarrow$ not enough to see the interaction between $\{$Center, Age, Inflam.$\}$ on Survival.
- A solution is to create new variables and to redo the analysis.