Cours Multi-fidélité.

$$\bar{Y}_n^{cv}(\alpha) = \bar{Y}_n - \alpha(\bar{Z}_n - \mu_2) \qquad \mu_2 = \mathbb{E}[Z].$$

$$\mathbb{E}[\bar{Y}_n^{cv}(\alpha)] = \underbrace{\mathbb{E}[\bar{Y}_n]}_{=\mathbb{E}[Y].} - \alpha\big(\underbrace{\mathbb{E}[\bar{Z}_n] - \mu_2}_{=0}\big)$$

$\{X_1, \dots, X_n\}$ iid

$\downarrow$

$\{Y_1, \dots, Y_n\}$, $Y_i = f(X_i)$ iid

$\{Z_1, \dots, Z_n\}$, $Z_i = g(X_i)$ iid

$f(X_i) \perp\!\!\!\perp g(X_j)$ pour $i \neq j$.

$$V[\bar{Y}_n^{cv}(\alpha)] = V[\bar{Y}_n] + \alpha^2 V[\bar{Z}_n] - 2\alpha\, \mathbb{C}[\bar{Y}_n, \bar{Z}_n] = J(\alpha)$$

$$J'(\alpha) = 2\alpha V[\bar{Z}_n] - 2\mathbb{C}[\bar{Y}_n, \bar{Z}_n]$$

$$\hookrightarrow \alpha^* = \frac{\mathbb{C}[\bar{Y}_n, \bar{Z}_n]}{V[\bar{Z}_n]}.$$

$$J(\alpha^*) = V(\bar{Y}_n) - \frac{\mathbb{C}^2(\bar{Y}_n, \bar{Z}_n)}{V(\bar{Z}_n)}$$

$$= V(\bar{Y}_n) + \frac{\mathbb{C}^2(\bar{Y}_n, \bar{Z}_n)}{V(\bar{Z}_n)} - 2\frac{\mathbb{C}^2(\bar{Y}_n, \bar{Z}_n)}{V(\bar{Z}_n)} < J(\alpha).$$

on a diminué le biais

$$V[\bar{Y}_n^{cv}(\alpha^*)] = V[\bar{Y}_n] \times \left(1 - \frac{\mathbb{C}^2[\bar{Y}_n, \bar{Z}_n]}{V(\bar{Y}_n)V(\bar{Z}_n)}\right)$$

$$= V(\bar{Y}_n)(1 - \rho^2).$$

$Y = f(X)$

$Z = g(X)$

$$\mathbb{C}[\bar{Y}_n, \bar{Z}_n] = \mathbb{C}\left[\frac{1}{n}\sum_{i=1}^n Y_i, \frac{1}{n}\sum_{i=1}^n Z_i\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n \underbrace{\mathbb{C}[Y_i, Z_j]}_{=\delta_{ij}\mathbb{C}[Y_i, Z_i].}$$

$$= \frac{1}{n^2}\sum_{i=1}^n \underbrace{\mathbb{C}[Y_i, Z_i]}_{=\mathbb{C}[Y_i, Z_i] = \mathbb{C}[Y, Z].}$$

$$\bar{Y}^{cv}_n(\alpha^*) = \bar{Y}_n - \alpha^*(\bar{Z}_n - \mu_Z)$$

$$C = n(c_f + c_g) = n c_f \left(1 + \frac{c_g}{c_f}\right) = n c_f (1+\omega).$$

$$\tilde{n} = \frac{c}{c_f} = n(1+\omega).$$

$$n = \frac{\tilde{n}}{1+\omega}$$

$$(1+\omega)(1-\rho^2) < 1 \qquad \rho^2 > \frac{\omega}{1+\omega}.$$

$$\mathbb{E}[\bar{Y}_n(\alpha^*)] = \mathbb{E}[\bar{Y}_n] - \mathbb{E}$$

$$\bar{Y}^{cv}_{n,N} = \bar{Y}_n - \alpha(\bar{Z}_n - \bar{Z}_N).$$

$$\mathrm{Var}[\bar{Y}^{cv}_{n,N}] = \mathrm{Var}[\bar{Y}_n] + \alpha^2 \left(\mathrm{Var}[\bar{Z}_n - \bar{Z}_N]\right) - 2\alpha\, \mathbb{C}[\bar{Y}_n, \bar{Z}_n - \bar{Z}_N].$$

$$J'(\alpha) = \cancel{2}\alpha\left(\mathrm{Var}[\bar{Z}_n] - \mathrm{Var}[\bar{Z}_n]\right) - \cancel{2}\,\mathbb{C}[\bar{Y}_n, \bar{Z}_n - \bar{Z}_N].$$

$$J'(\alpha) = 0 \iff \alpha = \frac{\mathbb{C}[\bar{Y}_n, \bar{Z}_n - \bar{Z}_N]}{\mathrm{Var}[\bar{Z}_n - \bar{Z}_N]}$$

$$\bar{Z}_n - \bar{Z}_N = \frac{1}{n}\sum_{i=1}^{n} g(X_i) - \frac{1}{N}\sum_{i=1}^{N} g(X_i)$$

$$= \frac{N}{nN}\sum_{i=1}^{n} g(X_i) - \frac{n}{nN}\sum_{i=1} g(X_i)$$

$$= \frac{1}{nN}\left[\sum_{i=1}^{n}\left(Ng(X_i) - ng(X_i)\right) - \sum_{i=n+1}^{N} n\, g(X_i)\right]$$

$$= \underbrace{\frac{(N-n)}{nN}\sum_{i=1}^{n} g(X_i)}_{A} - \underbrace{\frac{1}{N}\sum_{i=n+1}^{N} g(X_i)}_{B}$$

Comme $A \perp\!\!\!\perp B \Rightarrow \mathrm{Var}(A-B) = V(A) + V(B) = \left(\frac{(N-n)^2 \cancel{n}}{n^2 N^2} - \frac{N-m}{N^2}\right) V(Z)$

or $V(A) = \left(\frac{N-n}{nN}\right)^2 n V(Z)$ et $V(B) = \frac{1}{N^2}(N-(n+1)+1)V(Z) = \left(\frac{(N-m)^2 + n(N-n)}{nN^2}\right) V(Z)$

$$= \frac{N^2 - 2Nn + \cancel{n^2} + nN - \cancel{n^2}}{nN^2} V(Z)$$

$$= \frac{N-n}{nN} V(Z)$$

$$\mathbb{C}[\bar{Y}_n, \bar{Z}_n - \bar{Z}_N] = \mathbb{C}[\bar{Y}_n, A - B)$$
$$= \mathbb{C}(\bar{Y}_n, A) - \mathbb{C}(\bar{Y}_n, B]$$

$$\mathbb{C}(\bar{Y}_n, A) = \mathbb{C}\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i), \frac{N-n}{nN}\sum_{i=1}^{n} g(X_i)\right)$$

$$= \frac{N-n}{n^2 N} \sum_{i=1}^{n} \mathbb{C}(f(X_i), g(X_i))$$

$$= \frac{(N-n)n}{n^2 N} \mathbb{C}(Y, Z) = \frac{N-n}{nN} \mathbb{C}[Y, Z].$$

$$\mathbb{C}[\bar{Y}_n, B] = \mathbb{C}[\frac{1}{n}\sum_{i=1}^{n} f(X_i), -\sum_{i=n+1}^{N} g(X_i)) \Rightarrow \mathbb{C}[\bar{Y}_n, B] = 0$$

Comme $[\![1,n]\!] \cap [\![n+1,N]\!] = \emptyset$ $\forall i \in [\![1,N]\!]$ $\forall j \in [\![n+1,N]\!]$
$f(X_i) \perp\!\!\!\perp g(X_j) \Rightarrow \mathrm{Cov}(f(X_i), g(X_j)) = 0$

$\underline{\text{Cl}:}$ $$\frac{\mathbb{C}(\bar{Y}_n, \bar{Z}_n - \bar{Z}_N)}{V(\bar{Z}_n - \bar{Z}_N)} = \frac{\cancel{\frac{N-n}{nN}} \mathbb{C}(Y, Z)}{\cancel{\frac{N-n}{nN}} \mathrm{Var}(Z)}.$$

$$V[\bar{Y}_{n,N}(\alpha^*)] = V\left[\bar{Y}_n - \underbrace{\frac{\mathbb{C}[\bar{Y}_n, \bar{Z}_n - \bar{Z}_N]}{V[\bar{Z}_n - \bar{Z}_N]}}_{=\alpha^*} \times (\bar{Z}_n - \bar{Z}_N)\right].$$

$$= V[\bar{Y}_n] \; \cancel{+}(\alpha^*)^2 \times V[\bar{Z}_n - \bar{Z}_N].$$

$$= V[\bar{Y}_n] + (\alpha^*)^2 \times \frac{N-n}{nN} V(Z).$$

$$= V[\bar{Y}_n] + \frac{\mathbb{C}[Y, Z]^2}{V[Z]^{\cancel{2}}} \times \frac{N-n}{nN} \cancel{V[Z]}$$

$$= V[\bar{Y}_n] \times \left(1 - \frac{V[Y]}{V[\bar{Y}_n]} \times \rho^2 \frac{N-n}{nN}\right).$$

$$V(\bar{Y}_n) = \frac{1}{n} V[Y]. = \frac{1}{\tilde{n}_f} \times V(Y) \times \frac{\tilde{n}_f}{n} = V(\bar{Y}_{\tilde{n}_f}) \times \frac{\tilde{n}_f}{n}$$

$$\frac{\tilde{n}_f}{n} = 1 + \eta w$$

$$\Rightarrow V(\bar{Y}_n) = (1+\eta\omega)V(\bar{Y}_{n\eta}).$$

$$V(\bar{Y}_n^{cv}(\alpha^*)) = V(\bar{Y}_{n\eta})(1+\omega)\times(1-\rho^2)$$

$$J(\eta) = (1+\eta\omega)\times\left[1-\left[1-\frac{1}{\eta}\right]\rho^2\right].$$

$$J(\eta) = (1+\eta\omega) - (1+\eta\omega)\times\left(1-\frac{1}{\eta}\right)\rho^2$$

$$= (1+\eta\omega) - \left(1-\frac{1}{\eta}+\eta\omega-\omega\right)\rho^2.$$

$$J'(\eta) = \omega + \left(-\frac{1}{\eta^2}-\omega\right)\rho^2.$$

$$J'(\eta) = 0 \quad \Rightarrow \quad \omega + \left(-\frac{1}{\eta^2}-\omega\right)\rho^2 = 0.$$

$$\Rightarrow \quad \omega = \left(\frac{1}{\eta^2}+\omega\right)\rho^2$$

$$\Rightarrow \quad \frac{\omega}{\rho^2} - \omega = \frac{1}{\eta^2} \qquad \frac{\omega}{\rho^2} - \frac{\omega\rho^2}{\rho^2}$$

$$\frac{\omega-\omega\rho^2}{\rho^2} = \frac{1}{\eta^2}$$

$$\Rightarrow \quad \eta^2 = \frac{\rho^2}{\omega-\omega\rho^2}$$

$$\Rightarrow \quad \eta = \sqrt{\frac{\rho^2}{\omega(1-\rho^2)}}$$

$$\eta^* > 1, \quad \sqrt{\frac{\rho^2}{\omega(1-\rho^2)}} > 1 \quad \Rightarrow \frac{\rho^2}{\omega(1-\rho^2)} > 1$$

$$\Rightarrow \rho^2 > \omega(1-\rho^2)$$

$$\Rightarrow \rho^2 > \omega - \omega\rho^2$$

$$\Rightarrow \rho^2(1+\omega) > \omega$$

$$\Rightarrow \rho^2 > \frac{\omega}{1+\omega}.$$

$$\Leftarrow \rho^2 > \frac{\omega}{1+\omega} \quad \Rightarrow \quad \rho^2(1+\omega) > \omega$$

$$\Rightarrow \quad \rho^2 > \omega - \omega\rho^2 \Rightarrow \rho^2 > \omega(1-\rho^2) \Rightarrow \frac{\rho^2}{\omega(1-\rho^2)} > 1$$

$$\sqrt{\frac{\rho^2}{\omega(1-\rho^2)}} > \sqrt{1} \Rightarrow \eta^* > 1.$$

Pour un $\tilde{n}$ donné, $n^* = \frac{\tilde{n}}{1+\eta^*\omega}$ et $N^* = \eta^* n$.