INSA Toulouse 4th year, Applied Mathematics
Finte Element Modeling course

Non linear PDEs: correction

Exercise 1

The boundary value problem reads :

$$\begin{cases} -div(\lambda(x)\nabla u(x)) \ + c(x)\ u^3(x) & = \ f(x) \qquad x \in \Omega \\ u(x) & = \ u_d(x) \qquad x \in \partial\Omega \end{cases} \tag{1}$$

1. This differential equation is scalar (the unknown u is a scalar function), second order, non-linear ; it is an elliptic differential equation.

2. One can show that this equation admits a unique solution u in X_t, $X_t = X\oplus$ Dirichlet b.c., with $X = V \cap L^\infty(\Omega)$, $V = H^1(\Omega)$.
This assertion is admitted.

The weak formulation reads as follows.
Find $u \in X_t$ such that : $\forall v \in X_0$, $X_0 = H_0^1(\Omega)$,

$$\int_\Omega \lambda(x)\nabla u(x)\nabla v(x)\ dx \ + \int_\Omega c(x)\ u^3(x)v(x)\ dx \ = \int_\Omega f(x)v(x)\ dx \tag{2}$$

By setting as usual the corresponding linear forms $a(\cdot,\cdot)$ and $l(\cdot)$, this reads :

$$a(u,v) = l(v) \ \ \forall v \in X_0$$

The application $u \in X \mapsto a(u,\cdot)$ is non linear.

Let V_h be the usual P_1-Lagrange FE space. The corresponding discrete weak formulation reads as follows. We have $V_h \subset X_h$.
Find $u_h \in V_{th}$ satisfying :

$$a(u_h, v_h) = l(v_h) \ \ \forall v_h \in V_{0h}$$

3. Numerical solver.

a) Since the PDE is non linear, a potential efficient strategy to compute its solution u_h is to use the Newton-Raphson algorithm. The latter is based on the linearization of the PDE. If converging, its convergence order is quadratic.

Another option is to consider the more simple fixed point method. It si observed that the latter is more robust than the Newton method, however it converges at order 1 only (when converging).

Let us detail here the Newton-Raphson method only. To do so, we need to linearize the equation.
We have :

$$\partial_u a(u,v) \cdot w \;=\; \int_\Omega \lambda(x)\nabla w(x)\nabla v(x)\;dx \;+\; \int_\Omega c(x)\;3u^2(x)w(x)v(x)\;dx$$

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The Newton-Raphson algorithm reads as follows.

— u^0 given.
Nb. A good choice of u^0 may be the solution of the "semi-linearized" problem consisting to replace the non-linear term u^3 by $u_{ref}^2 u$ with u_{ref} an approximatively physically-consistent field.

— Given $u^{(k)}$, compute δu solution of the linear problem :

$$\partial_u a(u^{(k)},v)\cdot \delta u \;=\; -a(u^{(k)},v) \;+\; l(v) \quad \forall v \in V \tag{3}$$

Set : $u^{(k+1)} = u^{(k)} + \delta u$.

— Convergence criteria satisfied ?
e.g. $\dfrac{u^{(k+1)} - u^{(k)}}{u^{(k)}} < \varepsilon \approx 10^{-8}$.

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Let us denote the unknown vector by $U^{(k)}$, $U^{(k)} = (u_1^{(k)}, \cdots, u_{NN}^{(k)})$. In matrix form, Eqn (3) reads as follows :

$$M(U^{(k)})\cdot \delta U \;=\; b^{(k)} \tag{4}$$

with the following matrix coefficients :

$$m_{ij} \;=\; \int_\Omega \lambda(x)\nabla\varphi_j(x)\nabla\varphi_i(x)\,dx + 3\int_\Omega c(x)\left(\sum_l \varphi_l(x)u_l^{(k)}\right)^2 \varphi_j(x)\varphi_i(x)\,dx$$

The i-th equation in the RHS reads :

$$b_i \;=\; \int_\Omega f(x)\varphi_i(x)\,dx - \int_\Omega \lambda(x)\left(\sum_l \nabla\varphi_l(x)u_l^{(k)}\right)\nabla\varphi_i(x)\,dx + \int_\Omega c(x)\,\left(\sum_l \varphi_l(x)u_l^{(k)}\right)^3 \varphi_i(x)\,dx$$

b) The matrix M is sparse, symmetric. The most adequate linear solver may be either the preconditioned conjugate gradient or a multi-frontal direct method (eg. MUMPS developed at IRIT).

Exercise 2

The boundary value problem reads :

$$\begin{cases} -div(\lambda(u(x))\nabla u(x)) & = & f(x) \quad & x \in \Omega \\ u(x) & = & u_d(x) \quad & x \in \partial\Omega \end{cases} \tag{5}$$

with $\lambda(u)$ such that etc

Correction to be typed soon...