INSA de Toulouse
Département GMM
Processus de Poisson et Application en actuariat et fiabilité - 5 ModIA
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Feuille de TD

Exercise 1 (Reinsurance contract) The basic principle is that the reinsurer covers the losses above a fixed loss $K > 0$. The total claim amount of the reinsurer is thus of the form $X = \sum_{i=1}^{N}(C_i - K)^+$. We assume that the number of claims N is a Poisson random variable with parameter $\lambda > 0$ and that the costs C_i are independent and identically distributed random variables, with common distribution F_C, independent of N.

1. Compute the moment generating function φ_{Y_1} of $Y_1 = (C_1 - K)^+$ when F_C is exponentially distributed, with parameter $\gamma > 0$ for a certain domain to be specified. Derive the expectation $\mathbb{E}[Y_1]$.

2. Recall the expression for the moment generating function φ_N of N and compute the moment generating function φ_X of X as a function of Y.

3. Compute the premium $\pi(X)$ based on the expected value principle, for a safety loading $\rho > 0$, that is
$$\mathbb{E}[X](1 + \rho) = \pi(X).$$

4. Let N_K be the number of claims with a cost greater or equal to K. Compute the moment generating function φ_{N_K} of N_K. Derive that the law of N_K is a Poisson distribution. Identify its parameter.

5. Let
$$\tilde{X} = \sum_{i=1}^{N_K} \tilde{C}_i,$$
where $(\tilde{C}_i, i \geq 1)$ is a sequence of independent random variables with common exponential distribution with parameter $\gamma > 0$, independent of N_K. Compute the moment generating function $\varphi_{\tilde{X}}$ of $\tilde{X}$. Conclude that the law of $\tilde{X}$ is the same as the law of X when the C_i's are distributed according to an exponential distribution with parameter $\gamma > 0$.

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Exercise 2. Let $N = (N_t, t \geq 0)$ be a standard Poisson process with intensity $\lambda > 0$. Let $f : [0, \infty) \to [0, \infty)$ be a locally bounded Borel function. Set

$$N(f)_t = \sum_{i \geq 1} f(T_i) \mathbf{1}_{\{T_i \leq t\}} \quad \text{for} \quad t \geq 0,$$

where the $(T_i)_{i\geq 1}$ are the jump times of N.

1. Show that for all $t \geq 0$, we have $N(f)_t < \infty$ almost-surely.
2. If $f(s) = \mathbf{1}_{(a,b]}(s)$ where $[a, b] \subset [0, t]$, what is the distribution of $N(\mathbf{1}_{(a,b]})_t$?
3. Show that for $u \geq 0$, we have

$$\mathbb{E}\left[e^{-uN(f)_t} \middle| N_t = n\right] = \frac{1}{t^n} \left(\int_0^t e^{-uf(s)} ds \right)^n.$$

4. Derive $\mathbb{E}\left[e^{-uN(f)_t}\right]$ ad find back the result of Question 2.
5. Compute $\mathbb{E}\left[N(f)_t\right]$ and $\mathrm{Var}[N(f)_t]$.
6. Prove that $N(f)_t - \lambda \int_0^t f(s) ds$ is a martingale.

Exercise 3. The total claim amount of a portfolio for a year is modelled by

$$X = \sum_{j=1}^{N} C_j$$

where N is the number of claims in the year and C_j is the cost of the j-th claim. Assume that N follows a mixed Poisson distribution with random parameter Λ, *i.e.* the conditional distribution of N given $\Lambda = \lambda$ is Poisson(λ). Assume moreover that Λ is distributed according to a $\Gamma(b, b)$ distribution, for some $b > 0$, that is a continuous r.v. on $\mathbb{R}_+^*$ with density :

$$f_{b,b}(x) := \frac{x^{b-1} b^b e^{-bx}}{\Gamma(b)}, \quad x > 0.$$

Assume that the cost of the claims $(C_j)_{j\geq 1}$ are independent and identically distributed random variables, independent of N.

1. Compute $\mathbb{E}(\Lambda)$ and $\mathrm{Var}(\Lambda)$.
2. Compute $\mathbb{E}(N)$ and $\mathrm{Var}(N)$.
3. We assume that $C_1 \sim$ Exponential (α) for some $\alpha > 0$. Show that the conditional law of X given N is a Gamma distribution and identify its parameters. What is the pure premium?
4. Show that the conditional law of Λ given (X, N) is independent of X and that it is a Gamma distribution. Identify its parameters.