INSA Toulouse, Applied Mathematics department
Supplementary material to J. Monnier's course

Symmetric case: connection between energy minimisation and model equation

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Exercice 0.1 *Let $f \in L^2(\Omega)$, we consider the functional $j : V = H_0^1(\Omega) \to \mathbb{R}$ defined as follows:*

$$j(v) = \frac{1}{2}\int_\Omega \|\nabla v\|^2 dx - \int_\Omega f v dx$$

We seek to solve the following minimization problem:

$$\inf_{u \in V} j(u)$$

Comments. It will be shown that this functional represents the energy of a system at equilibrium.
In practice, the system may correspond to the modeling of an elastic membrane under the external force f, or the modeling of purely diffusive phenomena such as a temperature or a chemical specie and so on.

1) a) Show that the 1st order necessary optimality condition (the Euler equation) of the optimization problem above reads as:

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

b) Note that the Sobolev space $H^1(\Omega)$ is the natural energy space *for this linear second order elliptic model.*

c) Deduce that this functional $j(u)$ represents the energy at equilibrium of a system modeled by the equation above.

2) Write the classical form of the weak form equation above.

3) Prove that the energy functional is strictly convex in V.

Correction.
1) The necessary optimality condition (Euler's equation) reads: $j'(u).v = 0$

for all $v \in V$, hence the result.

2) The classical form of the model is: $-\Delta u = f$ in Ω + homogeneous Dirichlet condition on $\partial\Omega$.

3) We have: $\forall (u, v) \in V \times V, u \neq v$,

$$(j'(u) - j'(v), u - v) = \|\nabla(u - v)\|_{L^2}^2 > 0$$

hence the result.

Exercice 0.2 *Connection with the numerical solver.*

Let us consider the same BVP as in the previous exercise.

1) After discretization using a FEM, what numerical solver (what algorithm) would you use to solve this minimization problem ?

2) What algorithm would you use to solve the linear system corresponding to the variational formulation ?

Correction.

1) The functional $j(\cdot)$ is strictly convex hence the adequate algorithm to minimize it, is the conjugated gradient algorithm, preferably with preconditioning.

2) The variational formulation of the problem leads to a bilinear form V-coercitive, symmetrical. The corresponding FE matrix (the rigidity matrix) is sparse, symmetric positive define.

The right numerical solver to employ may be the same as in the previous question: it is the conjugated gradient algorithm with preconditioning.

Indeed, in large dimension, it is the most performant method in the present context: sparse, symmetric positive define linear system.
Recall that this solver consists to minimize the corresponding energy j...