INSA de Toulouse
Département Génie Mathématique et Modélisation
4 GMM
2019-2020

An introduction to Risk Measures

Anthony Réveillac[1]

[1] anthony.reveillac@insa-toulouse.fr, bureau 111

Contents

1 Motivation and general notations . . . 3
2 Risk measures . . . 4
2.1 Definition . . . 4
2.2 Acceptance set . . . 7
2.3 Robust representation of risk measures . . . 9
3 Quantile functions and risk measures . . . 10
3.1 Introduction . . . 10
3.2 Quantile functions and Value at Risk . . . 11
3.3 Generalisations of $V@R_\lambda$ . . . 12

Bibliography 14

The material presented in these notes can be found in [2, 3].

1 Motivation and general notations

Throughout these notes $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and T a finite positive real number.

Definition 1.1 (Set of financial positions)**.** *A financial position is any bounded random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$. We set $\mathcal{X}$ the set of financial positions :*

$$\begin{aligned}\mathcal{X} &:= \{X \in L_\infty(\Omega, \mathcal{F}, \mathbb{P})\} \\ &= \left\{X : \Omega \to \mathbb{R},\ \textit{such that } X \textit{ is } \mathcal{F}-\textit{measurable and } \|X\|_\infty := \sup_{\omega\in\Omega} |X(\omega)| < +\infty\right\}.\end{aligned}$$

Hence, X represents here the payoff[2] associated to a financial position or its return[3]. Before going further we give two examples of financial positions.

Examples 1.2. *1. Consider a financial market composed of a risky asset $S := (S_0, S_T)$ and a riskless asset $S^0 := (S^0_0, S^0_T)$ with S_t (resp. S^0_t) the value of the asset S (resp. S^0) at time t ($t = 0$ or $t = T$). Riskless asset means here that S^0 is associated to an interest rate $r \geq 0$ as follows :*

$$S^0_0 := 1, \quad S^0_T := 1 + r.$$

For the risky asset, $S_0 > 0$ is a positive number and $S_T > 0$ is a $\mathcal{F}$-measurable bounded random variable. A portfolio (which represents an investment on this market) is a pair of real numbers (x, π) with $x \geq 0$ and $\pi \in \mathbb{R}$ which represents the number of risky assets S in the portfolio, so that the wealth associated $X^{(x,\pi)} := (X^{(x,\pi)}_0, X^{(x,\pi)}_T)$ to a portfolio (x, π) is defined as :

$$X^{(x,\pi)}_0 := x, \quad X^{(x,\pi)}_T := \pi S_T + (x - \pi S_0)(1 + r).$$

Note that x is the capital initial and that π could be negative which corresponds to a short sale[4]. Note as well that the value which is not invested on the risky asset is invested on the reiskless asset.

$X^{(x,\pi)}_T$ for some portfolio (x, π) is an example of a financial position. In that case note that X is non-negative. Another way to look at such an investment is to consider the financial position X defined below and which is the return[5] of the discounted value[6]

[2] Fr. Gain, c'est à dire la valeur à la maturité T

[3] Fr. Rendement

[4] Fr. : Vente à découvert

[5] Fr. : Rendement

[6] The discounted value of $X^{(x,\pi)}_T$ is defined as : $\frac{X^{(x,\pi)}_T}{1+r}$

of $X_T^{(x,\pi)}$ associated to this portfolio:

$$X := \frac{(1+r)^{-1}X_T^{(x,\pi)} - X_0^{(x,\pi)}}{X_0^{(x,\pi)}} = \frac{\pi((1+r)S_T - S_0)}{x},$$

which is not necessarily a positive random variable.

2. *The second example is the one of a cumulated loss R_T in Insurance in the Cramér-Lundberg model (with $c = 0$) as follows :*

$$R_T = u - \sum_{i=1}^{N_T} Y_i,$$

where N_T is a Poisson random variable with parameter $\lambda T > 0$ and $(Y_i)_{i\geq 1}$ is a sequence of iid bounded random variables. In that case X can be takes as : $X = R_T \wedge p$ for p in $\mathbb{N}$ (so that X is bounded).

What we aim in these notes is to study a tool (that we will call a risk measure) which allows one to "measure" the risk associated to a given financial position. We mention that the modern theory of risk measures has been introduced in [1].

2 Risk measures

2.1 Definition

We aim at defining a tool, that is a map $\rho : \mathcal{X} \to \mathbb{R}$ which associates to any financial position X a real number $\rho(X)$ that models the risk carried out by the position X. More precisely, given a financial position X, $\rho(X)$ represents the *monetary measure of risk of the financial position X*.

Remark 2.1. *What does the word "monetary" mean ? To answer that question, we need to explain what the real number $\rho(X)$ stand for. In this model, $\rho(X)$ will be seen as the quantity of cash (positive or negative) such that $X + \rho(X)$ is "acceptable". In particular, assume for simplicity that "Y is acceptable if and only if $Y \geq 0$, $\mathbb{P}$-a.s.", then :*

- *$\rho(X) > 0$ means that X is risky, as one needs to add a positive amount of cash ($\rho(X)$) to X so that $X + \rho(X)$ is acceptable. Hence in that case, $\rho(X)$ represents a reserve that covers (partly) the losses of the financial position X.*

- *$\rho(X) < 0$ means that X is riskless, as one can withdraw a positive amount of cash ($-\rho(X)$) so that $X + \rho(X)$ is acceptable. Hence in that case, $\rho(X)$ can be seen as an extra amount of cash that can be consumed.*

What is a "good" measure of risk ? We can consider some examples :

Examples 2.2.

Set $\rho_1 : \mathcal{X} \to \mathbb{R}$ *as follows :*

$$\rho_1(X) := \mathbb{E}[-X].$$

Set $\rho_2 : \mathcal{X} \to \mathbb{R}$ *as follows :*

$$\rho_2(X) := \sup_{\omega\in\Omega} -X(\omega) = -\inf_{\omega\in\Omega} X(\omega).$$

Set $\rho_3 : \mathcal{X} \to \mathbb{R}$ *as follows :*

$$\rho_3(X) := \mathbb{E}^{\mathbb{Q}}[-X] = \mathbb{E}\left[-X\frac{d\mathbb{Q}}{d\mathbb{P}}\right],$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ *is a positive* $L^1(\Omega, \mathcal{F}, \mathbb{P})$*-random variable with* $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$*.*

One sees that there are many ways of defining a risk measure. In order to be useful, we would like that a risk measure to satisfy some axioms (which make sense from the economics point of view). As an example, if a position is less risky than another one then the associated risk should be ordered in a logical way. Another example, is the one of diversification : it is an economics principle that diversification should reduce the risk. One may like to have a risk measure that satisfies such principles (that we will call axioms).

Definition 2.3 (Axioms)**.** *Let* $\rho : \mathcal{X} \to \mathbb{R}$ *be a map. We say that :*

1. ρ *is monotonic if*

$$\forall X, Y \in \mathcal{X},\ [X \le Y,\ \mathbb{P} - a.s. \Rightarrow \rho(X) \ge \rho(Y)],$$

2. ρ *is translation (or cash) invariant if*

$$\forall X \in \mathcal{X},\ \forall m \in \mathbb{R}, \quad \rho(X+m) = \rho(X) - m,$$

3. ρ *is normalized if*

$$\rho(0) = 0,$$

4. ρ *is convex if*

$$\forall X, Y \in \mathcal{X}, \forall\lambda \in [0,1], \quad \rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y),$$

5. ρ *is positive homogeneous if*

$$\forall X \in \mathcal{X}, \forall\beta \ge 0, \quad \rho(\beta X) = \beta\rho(X),$$

6. ρ *is subadditive if*

$$\forall X, Y \in \mathcal{X}, \quad \rho(X+Y) \le \rho(X) + \rho(Y).$$

We comment on the meaning of these axioms.

Remarks 2.4. *1. $X \leq Y$, $\mathbb{P}$-a.s. means that X is riskier than Y (take for example $Y = 0$ so X only leads to losses) so the amount of cash that needs to be added to X so that it becomes admissible (for instance non-negative) is higher than the one needed for Y.*

2. *Translation invariance just means that if one adds a (deterministic) amount of cash to X, then the amount of cash needed to make it acceptable is substracted by m (if $m > 0$, the risk is reduced, the risk is increased otherwise).*

3. $\rho(0) = 0$ *is just a convention.*

4. *The convex feature is the mathematical translation of the diversification principle in Economics.*

5. *The positive homogeneous principle, means that the risk is somehow proportional but only for positive factors. Indeed, if one would ask this property to be true for any β then it would imply that ρ is linear. So we would end up with examples 1 and 3 in Examples 2.2 only.*

Definition 2.5 (Risk measure)**.** *A risk measure is a map $\rho : \mathcal{X} \to \mathbb{R}$ which is cash invariant and monotonic.*

We will always assume that a risk measure is normalised.

Definition 2.6. *A risk measure ρ is said to be a*

1. *convex risk measure if it is convex,*

2. *coherent risk measure if it is convex and positive homogeneous.*

Exercise 2.7. *For each risk measure of Examples 2.2, list the properties (of Definition 2.3) that are satisfied.*

Exercise 2.8. *Let ρ be a risk measure. Prove that if ρ satisfy two of the three properties below then it satisfies the third one :*

- *Convexity*

- *Positive homogeneous*

- *Subadditive.*

Proposition 2.9. *Let ρ be a risk measure. Then ρ is a Lipschitz function with respect to the sup norm that is :*

$$|\rho(X) - \rho(Y)| \leq \|X - Y\|_\infty, \quad \forall X, Y \in \mathcal{X},$$

where we recall that $\|X\|_\infty = \sup_{\omega \in \Omega} |X(\omega)|$.

Proof. Let $X, Y \in \mathcal{X}$. We have that either $\rho(X) - \rho(Y) > 0$ or $\rho(X) - \rho(Y) \leq 0$ (these two cases being symmetric). Assume $\rho(X) - \rho(Y) \leq 0$. We have that :

$$X - Y \leq \|X - Y\|_\infty, \ \mathbb{P} - \text{a.s.},$$

so

$$X \leq Y + \|X - Y\|_\infty, \ \mathbb{P} - \text{a.s.}.$$

So by monotonicity and cash invariance, we have that

$$\rho(X) \geq \rho(Y + \|X - Y\|_\infty) = \rho(Y) - \|X - Y\|_\infty.$$

So

$$0 \geq \rho(X) - \rho(Y) \geq -\|X - Y\|_\infty,$$

and thus $|\rho(X) - \rho(Y)| \leq \|X - Y\|_\infty$. The case where $\rho(X) - \rho(Y) > 0$ is treated by exchanging the role of X and Y.

□

2.2 Acceptance set

We have seen that for a given risk measure ρ, and a financial position X, $\rho(X)$ denotes an amount of cash such that $X + \rho(X)$ is acceptable. But what does that mean ?

Definition 2.10 (Acceptance set)**.** *Let ρ be a risk measure. The acceptance set $\mathcal{A}_\rho$ associated to ρ is defined as :*

$$\mathcal{A}_\rho := \{X \in \mathcal{X}, \ \rho(X) \leq 0\}.$$

In other words, a position X is acceptable (from the perspective of ρ) if its monetary measure of risk $\rho(X)$ is non-positive. In other words, it is not a reserve but a consumption. Why does it make sense ? Consider a risk measure ρ (recall it is normalised). By cash invariance, we have that :

$$\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0.$$

So $X + \rho(X)$ belongs to $\mathcal{A}_\rho$. In addition, by monotonicity, $\rho(X)$ seems to be the minimal value which when added to X makes it acceptable. We will prove this claim in the result below.

Theorem 2.11. *Let ρ be a risk measure. Let $\mathcal{A}_\rho$ the acceptance set associated to ρ. We have :*

1. *$\mathcal{A}_\rho$ is non-empty, sequentially closed with respect to the sup norm.*
2. *Let X in $\mathcal{A}_\rho$ and Y in $\mathcal{X}$ such that $Y \geq X$, $\mathbb{P}$-a.s. then Y belongs to $\mathcal{A}_\rho$.*
3. *For any X in $\mathcal{X}$,*

$$\rho(X) = \inf\{m \in \mathbb{R}, \ X + m \in \mathcal{A}_\rho\}.$$

4. *For any X in $\mathcal{A}_\rho$, $X + \rho(X)$ belongs to $\mathcal{A}_\rho$.*

5. *ρ is convex if and only if $\mathcal{A}_\rho$ is convex.*

6. *ρ is positive positive homegeneous if and only if $\mathcal{A}_\rho$ is cone.*

7. *ρ is coherent if and only if $\mathcal{A}_\rho$ is convex cone.*

Proof. 1. By normalisation 0 belongs to $\mathcal{A}_\rho$. In addition, the fact that $\mathcal{A}_\rho$ is closed is a consequence of Proposition 2.9.

2. As $Y \geq X$, $\mathbb{P}$-a.s. and $\rho(X) \leq 0$, we have that $\rho(Y) \leq \rho(X) \leq 0$, so Y belongs to $\mathcal{A}_\rho$.

3. We have :

$$\begin{aligned}
\inf\{m \in \mathbb{R},\ X + m \in \mathcal{A}_\rho\} &= \inf\{m \in \mathbb{R},\ \rho(X + m) \leq 0\} \\
&= \inf\{m \in \mathbb{R},\ \rho(X) - m \leq 0\} \\
&= \inf\{m \in \mathbb{R},\ \rho(X) \leq m\} \\
&= \rho(X).
\end{aligned}$$

4. Follows from the fact that for any X in $\mathcal{A}_\rho$, $\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0$.

5. Assume $\mathcal{A}_\rho$ is convex. Let X, Y in $\mathcal{A}_\rho$ and λ in $[0,1]$. We have that $X + \rho(X)$ and $Y + \rho(Y)$ belong to $\mathcal{A}_\rho$. As $\mathcal{A}_\rho$ is assumed to be convex, we have that $\lambda(X + \rho(X)) + (1 - \lambda)(Y + \rho(Y))$ belongs to $\mathcal{A}_\rho$. Hence :

   $$0 \geq \rho(\lambda(X+\rho(X))+(1-\lambda)(Y+\rho(Y))) = \rho(\lambda X+(1-\lambda)Y)-(\lambda\rho(X) + (1 - \lambda)\rho(Y))\,,$$

   which proves that ρ is convex.

   Assume ρ is convex. Let X, Y in $\mathcal{A}_\rho$ and λ in $[0,1]$. We have :

   $$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y) \leq 0.$$

   So $\lambda X + (1 - \lambda)Y$ belongs to $\mathcal{A}_\rho$.

6. Left as an exercise.

7. Follows from 5. and 6..

□

Note that the acceptance set $\mathcal{A}_\rho$ completely define ρ and *vice-versa*.

2.3 Robust representation of risk measures

We have seen some examples of risk measures. The question we ask ourselves is : "Can we give the form of a risk measure ? " In other words, are there other risk measures than those we have cited ? For instance, the Daniell-Stone Theorem (which is related to the Riesz' representation theorem) states that for any real-valued continuous (in some sense) linear map ℓ on $\mathcal{X}$, there exists a probability measure $\mathbb{Q}$ such that : $\ell(X) = \mathbb{E}^{\mathbb{Q}}[X]$ for any X in $\mathcal{X}$. In other words, any linear map on $\mathcal{X}$ is an integral (or an expectation). So the question is : what happens if one replaces "linear" by "coherent" or "convex" for a risk measure ?

We need a notation. Let[7] :

$$\mathcal{M} := \{\mathbb{Q} \text{ probability measure on } (\Omega, \mathcal{F})\},$$

and

$$\mathcal{M}_f := \{\mathbb{Q} \text{ finitely additive probability measure on } (\Omega, \mathcal{F})\}.$$

Naturally, $\mathcal{M}_f \subset \mathcal{M}$ (but the converse is not true).

We start with the following proposition.

Proposition 2.12. *Let $\mathcal{Q} \subset \mathcal{M}$.*

1. *Set :*

$$\rho(X) := \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[-X], \quad \forall X \in \mathcal{X}.$$

Then, ρ is a coherent risk measure.

2. *Let $\gamma : \mathcal{Q} \to \mathbb{R}$ such that* $\sup_{\mathbb{Q} \in \mathcal{Q}} \gamma(Q) < +\infty$. *Set*

$$\rho(X) := \sup_{\mathbb{Q} \in \mathcal{Q}} \left(\mathbb{E}^{\mathbb{Q}}[-X] - \gamma(\mathbb{Q})\right), \quad \forall X \in \mathcal{X}.$$

Then, ρ is a convex risk measure.

Proof. 1. Left as an exercise.

2. For the second part, there is a trap to be avoid. Indeed, given X and Y in $\mathcal{X}$, the sup used for computing $\rho(X)$ may not be attained. But if it is, there is no reason that it is attained for the same element $\mathbb{Q}$ for $\rho(X)$ and $\rho(Y)$. So one needs to work a little bit. One way is to use the definition of supremum.

   Let X, Y in $\mathcal{X}$ and λ in $[0, 1]$. Let $\mathbb{Q}$ in $\mathcal{Q}$. We have that :

$$\mathbb{E}^{\mathbb{Q}}[-(\lambda X + (1 - \lambda)Y)] - \gamma(\mathbb{Q})$$

[7] A finitely additive probability measure $\mathbb{Q}$ is a map $\mathbb{Q} : \mathcal{F} \to [0, 1]$, such that $\mathbb{Q}[\Omega] = 1$ and for any $n \in \mathbb{N}^*$, and any $(A_i)_{i=1,\cdots,n} \subset \mathcal{F}$ pairwise disjoint, $\mathbb{Q}[\cup_{i=1}^n A_i] = \sum_{i=1}^n \mathbb{Q}[A_i]$.

$$= \lambda\left(\mathbb{E}^{\mathbb{Q}}[-X] - \gamma(\mathbb{Q})\right) + (1-\lambda)\left(\mathbb{E}^{\mathbb{Q}}[-Y] - \gamma(\mathbb{Q})\right)$$
$$\leq \lambda\rho(X) + (1-\lambda)\rho(Y).$$

As the supremum is the least bigger element, we get that :

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y).$$

□

What can be said on the converse statement. Unfortunately, there is a slight technical issue to be considered. More precisely, we have that :

Theorem 2.13 (Robust representation)**.** *Let ρ be a risk measure.*

1. *ρ is a coherent risk measure if and only if there exists a set $\mathcal{Q} \subset \mathcal{M}_f$ such that*
$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[-X], \quad \forall X \in \mathcal{X};$$
2. *ρ is a convex risk measure if and only if there exists a set $\mathcal{Q} \subset \mathcal{M}_f$ such that*
$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}^{\mathbb{Q}}[-X] - \alpha_{min}(\mathbb{Q})\right), \quad \forall X \in \mathcal{X},$$
with
$$\alpha_{min}(\mathbb{Q}) := \sup_{X\in\mathcal{A}_\rho} \mathbb{E}^{\mathbb{Q}}[-X],\ \mathbb{Q} \in \mathcal{M}.$$

3 Quantile functions and risk measures

3.1 Introduction

We have seen some examples in the previous section and the "general" form of a risk measure (coherent or convex). We aim in studying here a class of risk measures which take (*a priori*) a different form. Indeed, historically, measuring risk has been seen as estimating the probability that a financial claim leads to losses (that is the set $\{X < 0\}$ or $\{X < -a\}$, for some $a > 0$). In other words, given a level $\lambda \in (0,1)$ a position could be considered to be acceptable if :

$$\mathbb{P}[X < 0] \leq \lambda.$$

As we have seen, defining a set of acceptable claims is equivalent to define a risk measure and we define the risk measure $V@R_\lambda$ (called the value at Risk at level λ) as follows :

$$V@R_\lambda(X) := \inf\{m \in \mathbb{R},\ \mathbb{P}[m + X < 0] \leq \lambda\}, \quad X \in \mathcal{X}. \tag{3.1}$$

Our goal in this section is to study this (historical) risk measure and to provide related risk measures which will cope against the drawbacks of $V@R_\lambda$. To perform our analysis we will express more clearly $V@R_\lambda$ as a quantile. To this end we recall in the next section some elements on quantile functions.

Exercise 3.1. *Prove that $V@R_\lambda$ is a positive homogeneous risk measure.*

Exercise 3.2. *Let $X \sim \mathcal{N}(\mu, \sigma^2)$ (note that X is not a bounded random variable so strictly speaking it is not a financial position in the way we have defined it). We have*

$$V@R_\lambda(X) = -\mu + \sigma\Phi^{-1}(1-\lambda),$$

where Φ is the CDF of the $\mathcal{N}(0,1)$, that is, $\Phi(x) := \int_{-\infty}^{x} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$.

$V@R_\lambda$ is one of the most used risk measure in practise (maybe due to this explicit form for Gaussian random variables, which can be extended to log-Normal random variables). However, it has several huge drawbacks.

Remark 3.3. *$V@R_\lambda$ is not a convex (so it does not encourage diversification). Another main drawback lies in the fact that $V@R_\lambda$ indicates the level from which one will face losses but it does not give information on the amplitude of these losses.*

3.2 Quantile functions and Value at Risk

Throughout this section, X denotes an element of $\mathcal{X}$ and F denotes its CDF ($F(x) := \mathbb{P}[X \leq x]$, $x \in \mathbb{R}$).

Definition 3.4 (Quantile)**.** *Let λ in $(0,1)$. A λ-quantile of X is an real number $q_X(\lambda)$ such that :*

$$\mathbb{P}[X \leq q_X(\lambda)] \geq \lambda \quad \text{and} \quad \mathbb{P}[X < q_X(\lambda)] \leq \lambda.$$

Proposition 3.5. *Let λ in $(0,1)$. The set of λ-quantile of X is the interval $[q_X^-(\lambda), q_X^+(\lambda)]$ with :*

$$q_X^-(\lambda) := \sup\{z \in \mathbb{R},\ \mathbb{P}[X < z] < \lambda\} \quad q_X^+(\lambda) := \inf\{z \in \mathbb{R},\ \mathbb{P}[X \leq z] > \lambda\}.$$

Remarks 3.6. *Note that :*

1. $$q_X^-(\lambda) = \inf\{z \in \mathbb{R},\ \mathbb{P}[X \leq z] \geq \lambda\}, \quad q_X^+(\lambda) := \sup\{z \in \mathbb{R},\ \mathbb{P}[X < z] \leq \lambda\}.$$

2. *If F is increasing*[8] *("strictement croissante"), then $q_X^-(\lambda) = q_X^+(\lambda)$ and we recover the uniqueness of the quantile.*

3. *If F admits intervals of constancy (that is : there exists $[a,b]$ such that $F(x) = F(a)$ for any x in $[a,b]$) then $a = q_X^-(F(a))$ and $b = q_X^+(F(a))(= q_X^+(F(b)))$. This fact is annoying as for instance this happens for random variables : $X = N_T \sim \mathcal{P}(\lambda T)$ or for $X = R_T$ (as in 2. of Example 1.2).*

[8] even if it has jumps

As mentioned in the previous section, $V@R_\lambda$ amounts to a quantile as proved below.

Proposition 3.7. *Let $\lambda \in (0,1)$. We have that :*

$$V@R_\lambda(X) = -q_X^+(\lambda) = q_{-X}^-(1-\lambda), \quad \forall X \in \mathcal{X}.$$

Proof. Let $\lambda \in (0,1)$ and X in $\mathcal{X}$. We have by definition that $V@R_\lambda(X) = \inf\{m \in \mathbb{R},\ \mathbb{P}[m+X<0] \leq \lambda\}$. Hence

$$\begin{aligned} V@R_\lambda(X) &= \inf\{m \in \mathbb{R},\ \mathbb{P}[m+X<0] \leq \lambda\} \\ &= \inf\{m \in \mathbb{R},\ \mathbb{P}[X<-m] \leq \lambda\} \\ &= -\sup\{m \in \mathbb{R},\ \mathbb{P}[X<m] \leq \lambda\} \\ &= -q_X^+(\lambda) \\ &= \inf\{m \in \mathbb{R},\ 1-\mathbb{P}[X \geq -m] \leq \lambda\} \\ &= \inf\{m \in \mathbb{R},\ \mathbb{P}[X \geq -m] \geq 1-\lambda\} \\ &= \inf\{m \in \mathbb{R},\ \mathbb{P}[-X \leq m] \geq 1-\lambda\} \\ &= q_{-X}^-(1-\lambda). \end{aligned}$$

□

3.3 Generalisations of $V@R_\lambda$

In this section, we do not provide proofs of the main results as they are rather technical.

According to Remark 3.3, $V@R_\lambda$ is not a convex risk measure (so it is not a coherent risk measure). In addition, it does not give information on the amount of losses. We introduce below two risk measures who address these drawbacks.

Definition 3.8 (Average value at risk)**.** *The Average Value at Risk at level $\lambda \in (0,1)$ ($AV@R_\lambda$) is defined as :*

$$AV@R_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda V@R_\gamma(X)d\gamma, \quad X \in \mathcal{X}. \tag{3.2}$$

Proposition 3.9. *$AV@R_\lambda$ is a coherent risk measure (in particular it is convex).*

Remark 3.10. *Sometimes $AV@R_\lambda$ is called Conditional Value at Risk or Expected shortfall but as we will see later this is misleading.*

Note that :

$$AV@R_\lambda(X) = -\frac{1}{\lambda}\int_0^\lambda q_X^+(\gamma)d\gamma, \quad X \in \mathcal{X},$$

and formally

$$AV@R_1(X) = -\int_0^1 q_X^+(\gamma)d\gamma = \mathbb{E}[-X], \quad X \in \mathcal{X},$$

$$AV@R_0(X) = -\inf_{\omega\in\Omega} X(\omega), \quad X \in \mathcal{X}.$$

We have seen that any coherent risk measure can be represented as in Theorem 2.13. In fact we have that :

Proposition 3.11. *Let λ in $(0,1)$. We have that :*

$$AV@R_\lambda(X) = \sup_{\mathbb{Q}\in\mathcal{Q}_\lambda} \mathbb{E}^{\mathbb{Q}}[-X], \quad \forall X \in \mathcal{X},$$

where $\mathcal{Q}_\lambda := \left\{\mathbb{Q} << \mathbb{P},\ \sup_{\omega\in\Omega}\left|\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega)\right| \leq \frac{1}{\lambda}\right\}$.

Proposition 3.12 (Lemma 4.46 in [2])**.** *Let λ in $(0,1)$. We have that :*

$$AV@R_\lambda(X) = \lambda^{-1}\mathbb{E}[(q_X^+ - X)_+] - q_X^+, \quad \forall X \in \mathcal{X},$$

where $x_+ := \max(x,0)$.

We now define two other risk measures which are related to $AV@R_\lambda$.

Definition 3.13 (WCE_λ)**.** *We set for λ in $(0,1)$, WCE_λ (Worst Conditional Expectation et level λ) the map*

$$WCE_\lambda(X) := \sup_{A\in\mathcal{F},\ \mathbb{P}[A]>\lambda} \mathbb{E}[-X|A], \quad X \in \mathcal{X}.$$

Proposition 3.14. *For any λ in $(0,1)$, WCE_λ is a coherent risk measure.*

We now come back to Remark 3.10. We have that :

Proposition 3.15. *Let λ in $(0,1)$. For any X in $\mathcal{X}$, we have that :*

1.
$$\begin{aligned} AV@R_\lambda(X) &\geq WCE_\lambda(X) \\ &\geq \mathbb{E}[-X| -X \geq V@R_\lambda(X)] \\ &\geq V@R_\lambda(X). \end{aligned}$$

2. *If in addition $\mathbb{P}[X \leq q_X^+(X)] = \lambda$, then*
$$AV@R_\lambda(X) = WCE_\lambda(X) = \mathbb{E}[-X| -X \geq V@R_\lambda(X)].$$

The second item of the previous proposition, is quite important. Indeed, the property $\mathbb{P}[X \leq q_X^+(X)] = \lambda$ is true (for any λ in $(0,1)$) when the CDF of X does not admit levels of constancy. This is the case for instance for Gaussian random variables. This has lead to a misuse of the terminology (see Remark 3.10) but in Actuarial sciences making such a difference is quite important, as for instance for $X = R_T$ (see 2. of Example 1.2) it is not true for any λ that $\mathbb{P}[X \leq q_X^+(X)] = \lambda$ (as an exercice consider for instance a Poisson random variable). These quantities are interesting as they account for the expected losses beyond $V@R_\lambda$.

References

[1] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Math. Finance*, 9(3):203–228, 1999.

[2] H. Föllmer and A. Schied. *Stochastic finance*. Walter de Gruyter & Co., Berlin, extended edition, 2011. An introduction in discrete time.

[3] N. Gantert. Lecture notes : Actuarial risk theory summer term 2013, 2013.