T-Axi (declarative)

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TODO

- Polymorphism.
- Type operators.
- Higher-order quantification.
- Totality checker.
- Computation.
- Inductives and records.
- Distinguish partial terms.

Quantities

Quantities:
$r ::= 0 \mid 1 \mid ? \mid + \mid *$

$*$ is the default quantity, so when there's nothing to indicate quantity, it means it's $*$.

Kinds

$$U ::= \mathtt{Type}_1 \mid \mathtt{Type}_? \mid \mathtt{Type}_+ \mid \mathtt{Type}_*$$

$\mathtt{Type}$ is a notation for $\mathtt{Type}_*$

Types

Types:

$A, B ::=$

$$r\, A \to B \mid \mathtt{!}_r\, A \mid A \otimes B \mid A \oplus B \mid \mathtt{Unit} \mid \mathtt{Empty} \mid$$
$$\forall \mathtt{@} a : \mathtt{Type}_r .\, A \mid \forall a : \mathtt{Type}_r .\, A$$

Terms

Terms:

$e ::=$

$$
\begin{aligned}
&x \mid \lambda_r\, x : A.\, e \mid e_1\ e_2 \mid \\
&\texttt{box}_r\ e \mid \texttt{let}_A\ \texttt{box}\ x = e_1\ \texttt{in}\ e_2 \mid \\
&(e_1, e_2) \mid \texttt{let}_A\ (x, y) = e_1\ \texttt{in}\ e_2 \mid \\
&\texttt{inl}_A\ e \mid \texttt{inr}_A\ e \mid \texttt{case}_A\ e\ \texttt{of}\ \{x.e_1;\ y.e_2\} \mid \\
&\texttt{unit} \mid \texttt{let}_A\ \texttt{unit} = e_1\ \texttt{in}\ e_2 \mid \\
&\texttt{Empty-elim}_A\ e \mid \\
&\texttt{let}_r\ x : A = e_1\ \texttt{in}\ e_2 \mid \\
&\Lambda\, a : \texttt{Type}_r.\, e \mid e\ A \mid \\
&\Lambda\, \{a : \texttt{Type}_r\}.\, e \mid e\ @A \mid \\
&\texttt{choose}\ p \mid \texttt{choose-witness}\ x\ h\ \texttt{for}\ p\ \texttt{in}\ e
\end{aligned}
$$

`choose` p and `choose-witness` x h `for` p `in` e are noncomputable terms, whereas all others are computable.

Propositions

Propositions:

$$
\begin{aligned}
P, Q ::= \; & \\
& \top \mid \bot \mid P \Rightarrow Q \mid P \wedge Q \mid P \vee Q \mid \\
& \forall x : A.\, P \mid \exists x : A.\, P \mid \\
& e_1 =_A e_2
\end{aligned}
$$

Notations:

$\neg P$ stands for $P \Rightarrow \bot$

$P \Leftrightarrow Q$ stands for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Proofterms

Proofterms (P, Q are propositions, e are terms, h are variables):

$p, q ::=$

h | **assumption** | **trivial** | **absurd** p
assume $h : P$ **in** q | **apply** p_1 p_2 |
both p_1 p_2 | **and-left** p | **and-right** p |
or-left p | **or-right** p | **cases** p_1 p_2 p_3 |
lemma $h : P$ **by** p **in** q | **proving** P **by** p |
suffices P **by** q **in** p |
pick-any $x : A$ **in** e | **instantiate** p **with** e |
witness e **such that** p | **pick-witness** x h **for** p_1 **in** p_2 |
refl e | **rewrite** p_1 **in** p_2 | **funext** $x : A$ **in** p
by-contradiction $h : \neg P$ **in** q |
choose-spec p | **choose-witness** x h **for** p **in** q

Contexts

Contexts:

$$\Gamma ::= \cdot \mid \Gamma, r\, x : A \mid \Gamma, r\, x : A := e \mid \Gamma, h : P \mid \Gamma, a : \texttt{Type}_r$$

Judgements

Well-formed context judgement: $\Gamma\ \mathtt{ctx}_i$, where i is either $\mathtt{c}$ or $\mathtt{nc}$.

Well-formed type judgement: $\Gamma \vdash A : \mathtt{Type}_r$
Type conversion judgement: $\Gamma \vdash A \equiv B : \mathtt{Type}_r$

Typing judgement: $\Gamma \vdash_i e : A$, where i is either $\mathtt{c}$ or $\mathtt{nc}$
Conversion judgement: $\Gamma \vdash e_1 \equiv e_2 : A$

Well-formed proposition judgement: $\Gamma \vdash P\ \mathtt{prop}$
Proposition conversion judgement: $\Gamma \vdash P \equiv Q\ \mathtt{prop}$

Proof judgement: $\Gamma \vdash p : P$

Sanity checks

We'll set up the system so that:

- If $\Gamma \vdash e_1 \equiv e_2 : A$, then $\Gamma \vdash_{\mathtt{nc}} e_1 : A$ and $\Gamma \vdash_{\mathtt{nc}} e_2 : A$.
- If $\Gamma \vdash_i e : A$, then $\Gamma\ \mathtt{ctx}_i$ and $|\Gamma| \vdash A : \mathtt{Type}_{\mathtt{qty}(A)}$.
- If $\Gamma \vdash A \equiv B : \mathtt{Type}_r$, then $\Gamma \vdash A : \mathtt{Type}_r$ and $\Gamma \vdash B : \mathtt{Type}_r$.
- If $\Gamma \vdash A : \mathtt{Type}_r$, then $\Gamma\ \mathtt{ctx}_{\mathtt{nc}}$.
- If $\Gamma \vdash p : P$, then $\Gamma \vdash P\ \mathtt{prop}$.
- If $\Gamma \vdash P \equiv Q\ \mathtt{prop}$, then $\Gamma \vdash P\ \mathtt{prop}$ and $\Gamma \vdash Q\ \mathtt{prop}$.
- If $\Gamma \vdash P\ \mathtt{prop}$, then $\Gamma\ \mathtt{ctx}_{\mathtt{nc}}$.
- If $\Gamma\ \mathtt{ctx}_{\mathtt{nc}}$, then $\Gamma\ \mathtt{ctx}_{\mathtt{c}}$.

Quantities

- 0 means a resource has been used up and is no longer available.
- 1 means a resource must be used exactly once.
- ? (pronounced "few") means a resource must be used at most once.
- $+$ (pronounced "many") means a resource must be used at least once.
- $*$ (pronounced "any") means no restrictions on usage.

Subusage ordering

$r_1 \sqsubseteq r_2$ means that a resource with quantity r_1 may be used when quantity r_2 is expected. This ordering is called sub-usaging. The definition below is just the skeleton, the full ordering is the reflexive-transitive closure of it.

$$
\begin{gathered}
* \sqsubseteq ? \\
* \sqsubseteq + \\
+ \sqsubseteq 1 \\
? \sqsubseteq 1 \\
? \sqsubseteq 0
\end{gathered}
$$

We will denote the greatest lower bound in this order with $r_1 \sqcap r_2$ and the least upper bound (if it exists) with $r_1 \sqcup r_2$.

Addition of quantities

When we have two quantities of the same resource, we can sum the quantities.

$$
\begin{aligned}
0 + r &= r \\
r + 0 &= r \\
?+? &= * \\
? + * &= * \\
*+? &= * \\
* + * &= * \\
\_ + \_ &= +
\end{aligned}
$$

Multiplication of quantities

When we have a quantity r_1 of resource A that contains quantitty r_2 of resource B, then we in fact have quantity $r_1 \cdot r_2$ of resource B.

$$
\begin{aligned}
0 \cdot r &= 0 \\
r \cdot 0 &= 0 \\
1 \cdot r &= r \\
r \cdot 1 &= r \\
? \cdot ? &= ? \\
+ \cdot + &= + \\
\_ \cdot \_ &= *
\end{aligned}
$$

The algebra of quantities

Quantities $\mathcal{Q}$ form a positive ordered commutative semiring with no zero divisors, i.e.:

- $(\mathcal{Q}, +, 0)$ is a commutative monoid.
- $(\mathcal{Q}, \cdot, 1)$ is a commutative monoid.
- 0 annihilates multiplication.
- Multiplication distributes over addition.
- Addition and multiplication preserve the subusage ordering in both arguments.
- If $r_1 + r_2 = 0$, then $r_1 = 0$ and $r_2 = 0$.
- If $r_1 \cdot r_2 = 0$, then $r_1 = 0$ or $r_2 = 0$.

Operations on contexts

Operations on contexts:

$\Gamma_1 \sqsubseteq \Gamma_2$ – context subusaging

$\Gamma_1 + \Gamma_2$ – context addition

$r\,\Gamma$ – context scaling

$|\Gamma|$ – cartesianization

$|\Gamma|_x$ – spotlight x

Context subusaging

$$\frac{}{\cdot \sqsubseteq \cdot}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \quad r_1 \sqsubseteq r_2}{\Gamma_1, r_1\, x : A \sqsubseteq \Gamma_2, r_2\, x : A}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \quad r_1 \sqsubseteq r_2}{\Gamma_1, r_1\, x : A := e \sqsubseteq \Gamma_2, r_2\, x : A := e}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2}{\Gamma_1, h : P \sqsubseteq \Gamma_2, h : P}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \quad r_1 \sqsubseteq r_2}{\Gamma_1, a : \mathtt{Type}_{r_1} \sqsubseteq \Gamma_2, a : \mathtt{Type}_{r_2}}$$

Context addition

$$
\begin{gathered}
\cdot + \cdot = \cdot \\
(\Gamma_1, r_1\, x : A) + (\Gamma_2, r_2\, x : A) = (\Gamma_1 + \Gamma_2), (r_1 + r_2)\, x : A \\
(\Gamma_1, r_1\, x : A := e) + (\Gamma_2, r_2\, x : A := e) = (\Gamma_1 + \Gamma_2), (r_1 + r_2)\, x : \\
A := e \\
(\Gamma_1, h : P) + (\Gamma_2, h : P) = (\Gamma_1 + \Gamma_2), h : P \\
(\Gamma_1, a : \mathtt{Type}_r) + (\Gamma_2, a : \mathtt{Type}_r) = (\Gamma_1 + \Gamma_2), a : \mathtt{Type}_r
\end{gathered}
$$

Context scaling

$$
\begin{aligned}
s \cdot &= \cdot \\
s\,(\Gamma, r\,x : A) &= s\,\Gamma, (s \cdot r)\,x : A \\
s\,(\Gamma, r\,x : A := e) &= s\,\Gamma, (s \cdot r)\,x : A := e \\
s\,(\Gamma, h : P) &= s\,\Gamma, h : P \\
s\,(\Gamma, a : \mathtt{Type}_r) &= s\,\Gamma, a : \mathtt{Type}_r
\end{aligned}
$$

Spotlight

$$
\begin{aligned}
|\cdot|_x &= \textbf{undefined} \\
|\Gamma, r\,x : A|_x &= 0\,\Gamma, 1\,x : A \\
|\Gamma, r\,y : A|_x &= |\Gamma|_x, 0\,y : A \\
|\Gamma, r\,x : A := e|_x &= 0\,\Gamma, 1\,x : A := e \\
|\Gamma, r\,y : A := e|_x &= |\Gamma|_x, 0\,y : A := e \\
|\Gamma, h : P|_x &= |\Gamma|_x, h : P \\
|\Gamma, a : \mathtt{Type}_r|_x &= |\Gamma|_x, a : \mathtt{Type}_r
\end{aligned}
$$

Cartesianization

Cartesianization turns a context into a context with the same shape but unlimited resources.

$$
\begin{aligned}
|\cdot| &= \cdot \\
|\Gamma, r\,x : A| &= |\Gamma|, x : A \\
|\Gamma, r\,x : A := e| &= |\Gamma|, x : A := e \\
|\Gamma, h : P| &= |\Gamma|, h : P \\
|\Gamma, a : \mathtt{Type}_r| &= |\Gamma|, a : \mathtt{Type}_r
\end{aligned}
$$

Well-formed contexts

$$\frac{}{\cdot\ \mathtt{ctx}_{\mathtt{c}}}$$

$$\frac{\Gamma\ \mathtt{ctx}_{\mathtt{c}} \quad |\Gamma| \vdash A : \mathtt{Type}_s \quad x \notin \Gamma}{\Gamma, r\,x : A\ \mathtt{ctx}_{\mathtt{c}}}$$

$$\frac{\Gamma\ \mathtt{ctx}_{\mathtt{c}} \quad |\Gamma| \vdash_{\mathtt{nc}} e : A \quad x \notin \Gamma}{\Gamma, r\,x : A := e\ \mathtt{ctx}_{\mathtt{c}}}$$

$$\frac{\Gamma\ \mathtt{ctx}_{\mathtt{c}} \quad |\Gamma| \vdash P\ \mathtt{prop} \quad h \notin \Gamma}{\Gamma, h : P\ \mathtt{ctx}_{\mathtt{c}}}$$

$$\frac{\Gamma\ \mathtt{ctx}_{\mathtt{c}} \quad a \notin \Gamma \quad r \neq 0}{\Gamma, a : \mathtt{Type}_r\ \mathtt{ctx}_{\mathtt{c}}}$$

Well-formed cartesian contexts

$$\frac{\Gamma\ \mathtt{ctx_c} \quad \Gamma = |\Gamma|}{\Gamma\ \mathtt{ctx_{nc}}}$$

Well-formed types

$$\frac{\Gamma\ \mathtt{ctx}_{\mathtt{nc}}}{\Gamma \vdash \mathtt{Unit} : \mathtt{Type}} \qquad \frac{\Gamma\ \mathtt{ctx}_{\mathtt{nc}}}{\Gamma \vdash \mathtt{Empty} : \mathtt{Type}}$$

$$\frac{\Gamma \vdash A : \mathtt{Type}_{s_1} \qquad \Gamma \vdash B : \mathtt{Type}_{s_2}}{\Gamma \vdash r\,A \to B : \mathtt{Type}_1}$$

$$\frac{\Gamma \vdash A : \mathtt{Type}_s}{\Gamma \vdash\ !_0\,A : \mathtt{Type}} \qquad \frac{\Gamma \vdash A : \mathtt{Type}_s \qquad r \neq 0}{\Gamma \vdash\ !_r\,A : \mathtt{Type}_{r \cdot s}}$$

$$\frac{\Gamma \vdash A : \mathtt{Type}_{s_1} \qquad \Gamma \vdash B : \mathtt{Type}_{s_2}}{\Gamma \vdash A \otimes B : \mathtt{Type}_{s_1 \sqcup s_2}} \qquad \frac{\Gamma \vdash A : \mathtt{Type}_{s_1} \qquad \Gamma \vdash B : \mathtt{Type}_{s_2}}{\Gamma \vdash A \oplus B : \mathtt{Type}_{s_1 \sqcup s_2}}$$

Well-formed types

$$\frac{\Gamma\ \mathtt{ctx}_{\mathtt{nc}} \qquad (a : \mathtt{Type}_r) \in \Gamma}{\Gamma \vdash a : \mathtt{Type}_r}$$

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash B : \mathtt{Type}_s}{\Gamma \vdash \forall @ a : \mathtt{Type}_r .\, B : \mathtt{Type}_s}$$

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash B : \mathtt{Type}_s}{\Gamma \vdash \forall a : \mathtt{Type}_r .\, B : \mathtt{Type}_s}$$

The inherent quantity of a type

$$\begin{aligned}
\mathtt{qty}(\mathtt{Unit}) &= * \\
\mathtt{qty}(\mathtt{Empty}) &= * \\
\mathtt{qty}(!_0\, A) &= * \\
\mathtt{qty}(!_r\, A) &= r \cdot \mathtt{qty}(A) \\
\mathtt{qty}(A \otimes B) &= \mathtt{qty}(A) \sqcup \mathtt{qty}(B) \\
\mathtt{qty}(A \oplus B) &= \mathtt{qty}(A) \sqcup \mathtt{qty}(B) \\
\mathtt{qty}(r\, A \to B) &= 1 \\
\mathtt{qty}(\forall @a : \mathtt{Type}_r.\, B) &= \mathtt{qty}(B) \\
\mathtt{qty}(\forall a : \mathtt{Type}_r.\, B) &= \mathtt{qty}(B)
\end{aligned}$$

Total quantity added to context

$$r_A^{\mathtt{nc}} = *$$
$$r_A^{\mathtt{c}} = r \cdot \mathtt{qty}(A)$$

Using variables

$$\frac{\Gamma\ \mathtt{ctx}_i \quad (x : A) \in \Gamma \quad \Gamma \sqsubseteq |\Gamma|_x}{\Gamma \vdash_i x : A}$$

Functions

$$\frac{\Gamma, r^i_A\, x : A \vdash_i e : B}{\Gamma \vdash_i \lambda_r\, x : A.\, e : r\, A \to B}$$

$$\frac{\Gamma \sqsubseteq \Gamma_1 + r\,\Gamma_2 \quad \Gamma_1 \vdash_i e_1 : r\, A \to B \quad \Gamma_2 \vdash_i e_2 : A}{\Gamma \vdash_i e_1\ e_2 : B}$$

Box

$$\frac{\Gamma \sqsubseteq r\,\Gamma' \quad \Gamma' \vdash_i e : A}{\Gamma \vdash_i \texttt{box}_r\ e : \texttt{!}_r\,A}$$

$$\frac{\Gamma = \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash_i e_1 : \texttt{!}_r\,A \quad \Gamma_2, r^i_A\,x : A \vdash_i e_2 : B}{\Gamma \vdash_i \texttt{let}_B\ \texttt{box}\ x = e_1\ \texttt{in}\ e_2 : B}$$

Empty

$$\frac{|\Gamma| \vdash A : \texttt{Type}_r \quad \Gamma \vdash_i e : \texttt{Empty}}{\Gamma \vdash_i \texttt{Empty-elim}_A\ e : A}$$

Unit

$$\frac{\Gamma\ \mathtt{ctx}_i \quad \Gamma \sqsubseteq 0\,\Gamma}{\Gamma \vdash_i \mathtt{unit} : \mathtt{Unit}}$$

$$\frac{\Gamma - \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash_i e_1 : \mathtt{Unit} \quad \Gamma_2 \vdash_i e_2 : A}{\Gamma \vdash_i \mathtt{let}_A\ \mathtt{unit} = e_1\ \mathtt{in}\ e_2 : A}$$

Products

$$\frac{\Gamma - \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash_i a : A \quad \Gamma_2 \vdash_i b : B}{\Gamma \vdash_i (a, b) : A \otimes B}$$

$$\frac{\Gamma - \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash_i e_1 : A \otimes B \quad \Gamma_2, 1^i_A\, x : A, 1^i_B\, y : B \vdash_i e_2 : C}{\Gamma \vdash_i \mathtt{let}_C\ (x, y) = e_1\ \mathtt{in}\ e_2 : C}$$

Sums

$$\frac{\Gamma \vdash_i e : A \quad |\Gamma| \vdash B : \mathtt{Type}_r}{\Gamma \vdash_i \mathtt{inl}_B\ e : A \oplus B} \qquad \frac{|\Gamma| \vdash A : \mathtt{Type}_r \quad \Gamma \vdash_i e : B}{\Gamma \vdash_i \mathtt{inr}_A\ e : A \oplus B}$$

$$\frac{\Gamma - \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash_i e : A \oplus B \quad \begin{array}{c} \Gamma_2, 1^i_A\, x : A \vdash_i e_1 : C \\ \Gamma_2, 1^i_B\, y : B \vdash_i e_2 : C \end{array}}{\Gamma \vdash_i \mathtt{case}_C\ e\ \mathtt{of}\ \{x.e_1;\ y.e_2\} : C}$$

Q: Do we want first-order representation of the branches?
Probably yes.

Let

$$\frac{\Gamma \sqsubseteq r\,\Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash_i e_1 : A \quad \Gamma_2, r^i_A\, x : A := e_1 \vdash_i e_2 : B}{\Gamma \vdash_i \mathtt{let}_r\; x : A = e_1 \;\mathtt{in}\; e_2 : B}$$

Polymorphism

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash_i e : B}{\Gamma \vdash_i \Lambda\, a : \mathtt{Type}_r .\, e : \forall @ a : \mathtt{Type}_r .\, B}$$

$$\frac{\Gamma \vdash_i e : \forall @ a : \mathtt{Type}_r .\, B \qquad |\Gamma| \vdash A : \mathtt{Type}_r}{\Gamma \vdash_i e\ A : B\,[a := A]}$$

Polymorphism (implicit arguments)

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash_i e : B}{\Gamma \vdash_i \Lambda \{a : \mathtt{Type}_r\}.\, e : \forall a : \mathtt{Type}_r.\, B}$$

$$\frac{\Gamma \vdash_i e : \forall a : \mathtt{Type}_r.\, B \qquad |\Gamma| \vdash A : \mathtt{Type}_r}{\Gamma \vdash_i e\ \mathtt{@}A : B\,[a := A]}$$

Well-formed propositions

$$\frac{\Gamma\ \mathtt{ctx}_{\mathtt{nc}}}{\Gamma \vdash \top\ \mathtt{prop}} \qquad \frac{\Gamma\ \mathtt{ctx}_{\mathtt{nc}}}{\Gamma \vdash \bot\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash P\ \mathtt{prop} \quad \Gamma \vdash Q\ \mathtt{prop}}{\Gamma \vdash P \Rightarrow Q\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash P\ \mathtt{prop} \quad \Gamma \vdash Q\ \mathtt{prop}}{\Gamma \vdash P \wedge Q\ \mathtt{prop}} \qquad \frac{\Gamma \vdash P\ \mathtt{prop} \quad \Gamma \vdash Q\ \mathtt{prop}}{\Gamma \vdash P \vee Q\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash A : \mathtt{Type}_r \quad \Gamma, x : A \vdash P\ \mathtt{prop}}{\Gamma \vdash \forall x : A.\, P\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash A : \mathtt{Type}_r \quad \Gamma, x : A \vdash P\ \mathtt{prop}}{\Gamma \vdash \exists x : A.\, P\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash A : \mathtt{Type}_r \quad \Gamma \vdash_{\mathtt{nc}} e_1 : A \quad \Gamma \vdash_{\mathtt{nc}} e_2 : A}{\Gamma \vdash e_1 =_A e_2\ \mathtt{prop}}$$

Substitution

The notation is $P\,[x := e]$ for substitution in propositions.

Assumptions and implication

$$\frac{\Gamma\ \texttt{ctx}_{\texttt{nc}} \quad (h : P) \in \Gamma}{\Gamma \vdash h : P} \qquad \frac{\Gamma\ \texttt{ctx}_{\texttt{nc}} \quad (h : P) \in \Gamma}{\Gamma \vdash \textbf{assumption} : P}$$

$$\frac{\Gamma, h : P \vdash q : Q}{\Gamma \vdash \textbf{assume}\ h : P\ \textbf{in}\ q : P \Rightarrow Q}$$

$$\frac{\Gamma \vdash q : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \textbf{apply}\ q\ p : Q}$$

Propositional logic

$$\frac{\Gamma\ \mathtt{ctx}_{\mathtt{nc}}}{\Gamma \vdash \textbf{trivial} : \top} \qquad \frac{\Gamma \vdash Q\ \mathtt{prop} \quad \Gamma \vdash p : \bot}{\Gamma \vdash \textbf{absurd}\ p : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash q : Q}{\Gamma \vdash \textbf{both}\ p\ q : P \wedge Q}$$

$$\frac{\Gamma \vdash pq : P \wedge Q}{\Gamma \vdash \textbf{and-left}\ pq : P} \qquad \frac{\Gamma \vdash pq : P \wedge Q}{\Gamma \vdash \textbf{and-right}\ pq : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash Q\ \mathtt{prop}}{\Gamma \vdash \textbf{or-left}\ p : P \vee Q} \qquad \frac{\Gamma \vdash P\ \mathtt{prop} \quad \Gamma \vdash q : Q}{\Gamma \vdash \textbf{or-right}\ q : P \vee Q}$$

$$\frac{\Gamma \vdash pq : P \vee Q \quad \Gamma \vdash r_1 : P \Rightarrow R \quad \Gamma \vdash r_2 : Q \Rightarrow R}{\Gamma \vdash \textbf{cases}\ pq\ r_1\ r_2 : R}$$

Utilities

$$\frac{\Gamma \vdash p : P \quad \Gamma, h : P \vdash q : Q}{\Gamma \vdash \textbf{lemma}\ h : P\ \textbf{by}\ p\ \textbf{in}\ q : Q}$$

$$\frac{\Gamma \vdash p : P}{\Gamma \vdash \textbf{proving}\ P\ \textbf{by}\ p : P}$$

$$\frac{\Gamma \vdash pq : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \textbf{suffices}\ P\ \textbf{by}\ pq\ \textbf{in}\ p : Q}$$

Quantifiers

$$\frac{\Gamma, x : A \vdash p : P}{\Gamma \vdash \textbf{pick-any } x : A \textbf{ in } p : \forall x : A.\, P}$$

$$\frac{\Gamma \vdash p : \forall x : A.\, P \quad \Gamma \vdash_{\mathtt{nc}} e : A}{\Gamma \vdash \textbf{instantiate } p \textbf{ with } e : P\,[x := e]}$$

$$\frac{\Gamma \vdash_{\mathtt{nc}} e : A \quad \Gamma, x : A \vdash P \ \mathtt{prop} \quad \Gamma \vdash p : P\,[x := e]}{\Gamma \vdash \textbf{witness } e \textbf{ such that } p : \exists x : A.\, P}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P \quad \Gamma \vdash Q \ \mathtt{prop} \quad \Gamma, x : A, h : P \vdash q : Q}{\Gamma \vdash \textbf{pick-witness } x\ h \textbf{ for } p \textbf{ in } q : Q}$$

Equality

$$\frac{\Gamma \vdash_{\mathtt{nc}} e : A}{\Gamma \vdash \textbf{refl } e : e =_A e}$$

$$\frac{\Gamma \vdash q : e_1 =_A e_2 \quad \Gamma, x : A \vdash P \ \mathtt{prop} \quad \Gamma \vdash p : P\,[x := e_2]}{\Gamma \vdash \textbf{rewrite } q \textbf{ in } p : P\,[x := e_1]}$$

$$\frac{\Gamma, x : A \vdash p : f \ x =_B g \ x}{\Gamma \vdash \textbf{funext } x : A \textbf{ in } p : f =_{r\,A \to B} g}$$

Classical Logic

$$\frac{\Gamma, h : \neg P \vdash q : \bot}{\Gamma \vdash \textbf{by-contradiction}\ h : \neg P\ \textbf{in}\ q : P}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P}{\Gamma \vdash_{\mathtt{nc}} \mathtt{choose}\ p : A}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P}{\Gamma \vdash \textbf{choose-spec}\ p : P\,[x := \mathtt{choose}\ p]}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P \quad \Gamma \vdash Q\ \mathtt{prop} \quad \Gamma, x : A := \mathtt{choose}\ p, h : P \vdash q : Q}{\Gamma \vdash \textbf{choose-witness}\ x\ h\ \textbf{for}\ p\ \textbf{in}\ q : Q}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P \quad \Gamma \vdash B : \mathtt{Type}_r \quad \Gamma, x : A := \mathtt{choose}\ p, h : P \vdash_{\mathtt{nc}} e : B}{\Gamma \vdash_{\mathtt{nc}} \mathtt{choose\text{-}witness}\ x\ h\ \mathtt{for}\ p\ \mathtt{in}\ e : B}$$

Conversion rules

$$\frac{\Gamma \vdash_i e : A \quad |\Gamma| \vdash A \equiv B : \mathtt{Type}_r}{\Gamma \vdash_i e : B}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : A \quad \Gamma \vdash A \equiv B : \mathtt{Type}_r}{\Gamma \vdash e_1 \equiv e_2 : B}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash P \equiv Q\ \mathtt{prop}}{\Gamma \vdash p : Q}$$

Type conversion

$$\frac{\Gamma\ \mathtt{ctx}_{\mathtt{nc}}}{\Gamma \vdash \mathtt{Unit} \equiv \mathtt{Unit} : \mathtt{Type}} \qquad \frac{\Gamma\ \mathtt{ctx}_{\mathtt{nc}}}{\Gamma \vdash \mathtt{Empty} \equiv \mathtt{Empty} : \mathtt{Type}}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : \mathtt{Type}_{s_1} \qquad \Gamma \vdash B_1 \equiv B_2 : \mathtt{Type}_{s_2}}{\Gamma \vdash r\,A_1 \to B_1 \equiv r\,A_2 \to B_2 : \mathtt{Type}_1}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : \mathtt{Type}_s}{\Gamma \vdash\ !_0\,A_1 \equiv\ !_0\,A_2 : \mathtt{Type}} \qquad \frac{\Gamma \vdash A_1 \equiv A_2 : \mathtt{Type}_s \qquad r \neq 0}{\Gamma \vdash\ !_r\,A_1 \equiv\ !_r\,A_2 : \mathtt{Type}_{r \cdot s}}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : \mathtt{Type}_{s_1} \qquad \Gamma \vdash B_1 \equiv B_2 : \mathtt{Type}_{s_2}}{\Gamma \vdash A_1 \otimes B_1 \equiv A_2 \otimes B_2 : \mathtt{Type}_{s_1 \sqcup s_2}}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : \mathtt{Type}_{s_1} \qquad \Gamma \vdash B_1 \equiv B_2 : \mathtt{Type}_{s_2}}{\Gamma \vdash A_1 \oplus B_1 \equiv A_2 \oplus B_2 : \mathtt{Type}_{s_1 \sqcup s_2}}$$

Type conversion

$$\frac{\Gamma\ \mathtt{ctx_{nc}} \qquad (a : \mathtt{Type}_r) \in \Gamma}{\Gamma \vdash a \equiv a : \mathtt{Type}_r}$$

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash B_1 \equiv B_2 : \mathtt{Type}_s}{\Gamma \vdash \forall @ a : \mathtt{Type}_r .\, B_1 \equiv \forall @ a : \mathtt{Type}_r .\, B_2 : \mathtt{Type}_s}$$

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash B_1 \equiv B_2 : \mathtt{Type}_s}{\Gamma \vdash \forall a : \mathtt{Type}_r .\, B_1 \equiv \forall a : \mathtt{Type}_r .\, B_2 : \mathtt{Type}_s}$$

Type conversion – properties

The rules above are the complete definition of type conversion. We don't need to take any closures – type conversion already is an equivalence relation.

- $\Gamma \vdash A \equiv A : \mathtt{Type}$
- If $\Gamma \vdash A \equiv B : \mathtt{Type}$ then $\Gamma \vdash B \equiv A : \mathtt{Type}$
- If $\Gamma \vdash A \equiv B : \mathtt{Type}$ and $\Gamma \vdash B \equiv C : \mathtt{Type}$ then $\Gamma \vdash A \equiv C : \mathtt{Type}$

Proposition conversion

$$\frac{\Gamma\ \mathtt{ctx}_{\mathtt{nc}}}{\Gamma \vdash \top \equiv \top\ \mathtt{prop}} \qquad \frac{\Gamma\ \mathtt{ctx}_{\mathtt{nc}}}{\Gamma \vdash \bot \equiv \bot\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2\ \mathtt{prop} \quad \Gamma \vdash Q_1 \equiv Q_2\ \mathtt{prop}}{\Gamma \vdash P_1 \Rightarrow Q_1 \equiv P_2 \Rightarrow Q_2\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2\ \mathtt{prop} \quad \Gamma \vdash Q_1 \equiv Q_2\ \mathtt{prop}}{\Gamma \vdash P_1 \wedge Q_1 \equiv P_2 \wedge Q_2\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2\ \mathtt{prop} \quad \Gamma \vdash Q_1 \equiv Q_2\ \mathtt{prop}}{\Gamma \vdash P_1 \vee Q_1 \equiv P_2 \vee Q_2\ \mathtt{prop}}$$

Proposition conversion

$$\frac{\Gamma \vdash A_1 \equiv A_2 : \mathtt{Type}_r \quad \Gamma, x : A_1 \vdash P_1 \equiv P_2 \ \mathtt{prop}}{\Gamma \vdash \forall x : A_1 .\, P_1 \equiv \forall x : A_2 .\, P_2 \ \mathtt{prop}}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : \mathtt{Type}_r \quad \Gamma, x : A_1 \vdash P_1 \equiv P_2 \ \mathtt{prop}}{\Gamma \vdash \exists x : A_1 .\, P_1 \equiv \exists x : A_2 .\, P_2 \ \mathtt{prop}}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : \mathtt{Type}_r \quad \Gamma \vdash e_1 \equiv e_2 : A_1 \quad \Gamma \vdash e_1' \equiv e_2' : A_1}{\Gamma \vdash e_1 =_{A_1} e_1' \equiv e_2 =_{A_2} e_2' \ \mathtt{prop}}$$

Note that we don't worry about α-conversion and assume the variable is the same on both sides.

Term conversion – closure

$$\frac{\Gamma \vdash_i e : A}{\Gamma \vdash e \equiv e : A}\textsc{Refl}$$

$$\frac{\Gamma \vdash e_2 \equiv e_1 : A}{\Gamma \vdash e_1 \equiv e_2 : A}\textsc{Sym}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : A \quad \Gamma \vdash e_2 \equiv e_3 : A}{\Gamma \vdash e_1 \equiv e_3 : A}\textsc{Trans}$$

Term conversion – Empty

$$\frac{\Gamma \vdash_{\mathtt{nc}} e_1 : \mathtt{Empty} \quad \Gamma \vdash_{\mathtt{nc}} e_2 : \mathtt{Empty}}{\Gamma \vdash e_1 \equiv e_2 : \mathtt{Empty}}$$

$$\frac{\Gamma \vdash A : \mathtt{Type}_r \quad \Gamma \vdash e_1 \equiv e_2 : \mathtt{Empty}}{\Gamma \vdash \mathtt{Empty\text{-}elim}_A\ e_1 \equiv \mathtt{Empty\text{-}elim}_A\ e_2 : A}$$

Term conversion – Unit

$$\frac{\Gamma \vdash_{\texttt{nc}} a : A}{\Gamma \vdash \texttt{let unit} = \texttt{unit in } a \equiv a : A}$$

$$\frac{\Gamma \vdash_{\texttt{nc}} u_1 : \texttt{Unit} \quad \Gamma \vdash_{\texttt{nc}} u_2 : \texttt{Unit}}{\Gamma \vdash u_1 \equiv u_2 : \texttt{Unit}}$$

$$\frac{\Gamma \vdash u_1 \equiv u_2 : \texttt{Unit} \quad \Gamma \vdash a_1 \equiv a_2 : A}{\Gamma \vdash \texttt{let}_A\ \texttt{unit} = u_1\ \texttt{in}\ a_1 \equiv \texttt{let}_A\ \texttt{unit} = u_2\ \texttt{in}\ a_2 : A}$$

Term conversion – functions

$$\frac{\Gamma, x : A \vdash_{\texttt{nc}} b : B \quad \Gamma \vdash_{\texttt{nc}} a : A}{\Gamma \vdash (\lambda_r\, x : A.\, b)\ a \equiv b\,[x := a] : B}$$

$$\frac{\Gamma, x : A \vdash f\ x \equiv g\ x : B}{\Gamma \vdash f \equiv g : r\, A \to B}$$

$$\frac{\Gamma \vdash f_1 \equiv f_2 : r\, A \to B \quad \Gamma \vdash a_1 \equiv a_2 : A}{\Gamma \vdash f_1\ a_1 \equiv f_2\ a_2 : B}$$

Term conversion – box

$$\frac{\Gamma \vdash_{\mathtt{nc}} a : A \quad \Gamma, x : A \vdash_{\mathtt{nc}} b : B}{\Gamma \vdash \mathtt{let}_B\ \mathtt{box}\ x = \mathtt{box}_r\ a\ \mathtt{in}\ b \equiv b\,[x := a] : B}$$

$$\frac{\Gamma \vdash \mathtt{let}_A\ \mathtt{box}\ x = e_1\ \mathtt{in}\ x \equiv \mathtt{let}_A\ \mathtt{box}\ x = e_2\ \mathtt{in}\ x : A}{\Gamma \vdash e_1 \equiv e_2 : !_r\, A}$$

$$\frac{\Gamma \vdash a_1 \equiv a_2 : !_r\, A \quad \Gamma, x : A_1 \vdash b_1 \equiv b_2 : B}{\Gamma \vdash \mathtt{let}_B\ \mathtt{box}\ x = a_1\ \mathtt{in}\ b_1 \equiv \mathtt{let}_B\ \mathtt{box}\ x = a_2\ \mathtt{in}\ b_2 : B}$$

Term conversion – product

$$\frac{\Gamma \vdash_{\mathtt{nc}} a : A \quad \Gamma \vdash_{\mathtt{nc}} b : B \quad \Gamma, x : A, y : B \vdash_{\mathtt{nc}} c : C}{\Gamma \vdash \mathtt{let}_C\ (x, y) = (a, b)\ \mathtt{in}\ c \equiv c\,[x := a]\,[y := b] : C}$$

$$\frac{\begin{array}{c}\Gamma \vdash \mathtt{let}_A\ (x, y) = e_1\ \mathtt{in}\ x \equiv \mathtt{let}_A\ (x, y) = e_2\ \mathtt{in}\ x : A \\ \Gamma \vdash \mathtt{let}_B\ (x, y) = e_1\ \mathtt{in}\ y \equiv \mathtt{let}_B\ (x, y) = e_2\ \mathtt{in}\ y : B\end{array}}{\Gamma \vdash e_1 \equiv e_2 : A \otimes B}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : A \otimes B \quad \Gamma, x : A, y : B \vdash c_1 \equiv c_2 : C}{\Gamma \vdash \mathtt{let}_C\ (x, y) = e_1\ \mathtt{in}\ c_1 \equiv \mathtt{let}_C\ (x, y) = e_2\ \mathtt{in}\ c_2 : C}$$

Term conversion – sum

$$\frac{\Gamma \vdash_{\mathtt{nc}} a : A \quad \Gamma, x : A \vdash_{\mathtt{nc}} c_1 : C \quad \Gamma, y : B \vdash_{\mathtt{nc}} c_2 : C}{\Gamma \vdash \mathtt{case}_C\ \mathtt{inl}\ a\ \mathtt{of}\ \{x.c_1;\ y.c_2\} \equiv c_1\,[x := a] : C}$$

$$\frac{\Gamma \vdash_{\mathtt{nc}} b : B \quad \Gamma, x : A \vdash_{\mathtt{nc}} c_1 : C \quad \Gamma, y : B \vdash_{\mathtt{nc}} c_2 : C}{\Gamma \vdash \mathtt{case}_C\ \mathtt{inr}\ b\ \mathtt{of}\ \{x.c_1;\ y.c_2\} \equiv c_2\,[y := b] : C}$$

$$\frac{\Gamma \vdash a_1 \equiv a_2 : A \quad \Gamma \vdash B : \mathtt{Type}}{\Gamma \vdash \mathtt{inl}\ a_1 \equiv \mathtt{inl}\ a_2 : A \oplus B} \qquad \frac{\Gamma \vdash A : \mathtt{Type} \quad \Gamma \vdash b_1 \equiv b_2 : B}{\Gamma \vdash \mathtt{inr}\ b_1 \equiv \mathtt{inr}\ b_2 : A \oplus B}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : A \oplus B \qquad \begin{array}{l}\Gamma, x : A \vdash c_1 \equiv c_1' : C \\ \Gamma, y : B \vdash c_2 \equiv c_2' : C\end{array}}{\Gamma \vdash \mathtt{case}_C\ e_1\ \mathtt{of}\ \{x.c_1;\ y.c_2\} \equiv \mathtt{case}_C\ e_1'\ \mathtt{of}\ \{x.c_1';\ y.c_2'\} : C}$$

Term conversion – let

$$\frac{\Gamma \vdash_{\mathtt{nc}} e_1 : A \quad \Gamma, x : A \vdash_{\mathtt{nc}} e_2 : B}{\Gamma \vdash \mathtt{let}\ x = e_1\ \mathtt{in}\ e_2 \equiv e_2\,[x := e_1] : B}$$

$$\frac{\Gamma \vdash a_1 \equiv a_2 : A \quad \Gamma, x : A \vdash b_1 \equiv b_2 : B}{\Gamma \vdash \mathtt{let}\ x = a_1\ \mathtt{in}\ b_1 \equiv \mathtt{let}\ x = a_2\ \mathtt{in}\ b_2 : B}$$

Q: Should we have both rules, or just the first one?

Term conversion – polymorphism

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash_{\mathtt{nc}} e : B \qquad |\Gamma| \vdash A : \mathtt{Type}_r}{\Gamma \vdash (\Lambda\, a : \mathtt{Type}_r.\, e)\ A \equiv e\,[a := A] : B\,[a := A]}$$

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash f\ \ a \equiv g\ \ a : B}{\Gamma \vdash f \equiv g : \forall @ a : \mathtt{Type}_r.\, B}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : \forall @ a : \mathtt{Type}_r.\, B \qquad \Gamma \vdash A_1 \equiv A_2 : \mathtt{Type}_r}{\Gamma \vdash e_1\ \ A_1 \equiv e_2\ \ A_2 : B\,[a := A_1]}$$

Term conversion – polymorphism (implicit arguments)

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash_{\mathtt{nc}} e : B \quad |\Gamma| \vdash A : \mathtt{Type}_r}{\Gamma \vdash (\Lambda\,\{a : \mathtt{Type}_r\}.\, e)\ @A \equiv e\,[a := A] : B\,[a := A]}$$

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash f\ @a \equiv g\ @a : B}{\Gamma \vdash f \equiv g : \forall a : \mathtt{Type}_r.\, B}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : \forall a : \mathtt{Type}_r.\, B \quad \Gamma \vdash A_1 \equiv A_2 : \mathtt{Type}_r}{\Gamma \vdash e_1\ @A_1 \equiv e_2\ @A_2 : B\,[a := A_1]}$$

Environments

Global environments:

$\Sigma ::=$

$$\emptyset \mid \Sigma, h : P := p \mid \Sigma, x : A := e \mid$$
$$\Sigma, \texttt{partial}\ x : A := e \mid \Sigma, \texttt{totality}\ x\ p$$

Well-formed environments

$$\frac{}{\emptyset\ \texttt{env}}$$

$$\frac{\Sigma\ \texttt{env} \quad h \notin \Sigma \quad \Sigma \mid \cdot \vdash p : P}{\Sigma, h : P := p\ \texttt{env}}$$

$$\frac{\Sigma\ \texttt{env} \quad x \notin \Sigma \quad \Sigma \mid \cdot \vdash_{\mathsf{c}} e : A}{\Sigma, x : A := e\ \texttt{env}}$$

$$\frac{\Sigma\ \texttt{env} \quad x \notin \Sigma \quad \Sigma \mid \cdot \vdash_{\mathsf{c}} e : A \quad e\ \textbf{fails syntactic check}}{\Sigma, \texttt{partial}\ x : A := e\ \texttt{env}}$$

$$\frac{\Sigma\ \texttt{env} \quad \Sigma = \Sigma_1, \texttt{partial}\ x : A := e, \Sigma_2 \quad \Sigma \mid \cdot \vdash p : \exists r : A.\, e =?= r}{\Sigma, \texttt{totality}\ x\ p\ \texttt{env}}$$