T-Axi (algorithmic)

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Algorithmic types

Types:

$$A, B ::= \\ \quad r\,A \to B \mid\ !_r\,A \mid A \otimes B \mid A \oplus B \mid \mathtt{Unit} \mid \mathtt{Empty} \mid \\ \quad \forall @a : \mathtt{Type}_r.\,A \mid \forall @a.\,A \mid \forall a : \mathtt{Type}_r.\,A \mid \forall a.\,A$$

Algorithmic terms

Terms:

$$
\begin{aligned}
e ::= \\
&(e : A) \mid x \mid \lambda\, x.\, e \mid e_1\ e_2 \mid \\
&\texttt{box}\ e \mid \texttt{let box}\ x = e_1\ \texttt{in}\ e_2 \mid \\
&(e_1, e_2) \mid \texttt{let}\ (x, y) = e_1\ \texttt{in}\ e_2 \mid \\
&\texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}\ e\ \texttt{of}\ \{x.e_1;\ y.e_2\} \mid \\
&\texttt{unit} \mid \texttt{let unit} = e_1\ \texttt{in}\ e_2 \mid \\
&\texttt{Empty-elim}\ e \mid \\
&\texttt{let}_r\ x = e_1\ \texttt{in}\ e_2 \mid \\
&\Lambda\, a.\, e \mid e\ A \mid \\
&\Lambda\, \{a\}.\, e \mid e\ @A \mid \\
&\texttt{choose}\ p \mid \texttt{choose-witness}\ x\ h\ \texttt{for}\ p\ \texttt{in}\ e
\end{aligned}
$$

The algorithmic terms are almost the same to the declarative ones, but there are far fewer annotations.

Recovering annotated terms

We can recover some of the original terms by combining algorithmic terms with annotations.

- $\texttt{let}_A\ \texttt{unit} = e_1\ \texttt{in}\ e_2 :\equiv \texttt{let}\ \texttt{unit} = e_1\ \texttt{in}\ (e_2 : A)$
- $\texttt{Empty-elim}_A\ e_1 :\equiv (\texttt{Empty-elim}\ e_1 : A)$
- $\texttt{let}_r\ x : A = e_1\ \texttt{in}\ e_2 :\equiv \texttt{let}_r\ x = (e_1 : A)\ \texttt{in}\ e_2$

However, as long as annotations must be types (and not partial types, for example), we cannot recover more, like $\texttt{inl}_A\ e$, because we don't have partial annotations.

Algorithmic propositions

Propositions:

$$
\begin{aligned}
P, Q ::= &\\
&\top \mid \bot \mid P \Rightarrow Q \mid P \wedge Q \mid P \vee Q \mid\\
&\forall x : A.\, P \mid \forall x.\, P \mid \exists x : A.\, P \mid \exists x.\, P \mid\\
&e_1 =_A e_2 \mid e_1 = e_2
\end{aligned}
$$

The only difference from declarative propositions is that we can omit type annotations on quantifiers and equality.

Algorithmic proofterms

Proofterms (P, Q are propositions, e are terms, h are variables):

$p, q ::=$

h | **assumption** | **trivial** | **absurd** p
assume h **in** q | **apply** p_1 p_2 |
both p_1 p_2 | **and-left** p | **and-right** p |
or-left p | **or-right** p | **cases** p_1 p_2 p_3 |
lemma $h : P$ **by** p **in** q | **proving** P **by** p |
suffices P **by** q **in** p |
pick-any x **in** e | **instantiate** p **with** e |
witness e **such that** p | **pick-witness** x h **for** p_1 **in** p_2 |
refl | **rewrite** p_1 **at** $x.P$ **in** p_2 | **funext** x **in** p
by-contradiction h **in** q |
choose-spec p | **choose-witness** x h **for** p **in** q

Recovering annotated proofterms

We can recover some of the original proofterms by combining algorithmic proofterms with annotations.

- **refl** $e :\equiv$ **proving** $e = e$ **by refl**
- **by-contradiction** $h : \neg P$ **in** $q :\equiv$
  proving P **by by-contradiction** h **in** q

Judgements

Kinding judgements:

- $\Gamma \vdash A \Rightarrow \mathtt{Type}_r \dashv \Gamma'$
- $\Gamma \vdash A \Leftarrow \mathtt{Type}_r \dashv \Gamma'$

Typing judgements:

- $\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'$
- $\Gamma \vdash_i e \Rightarrow A \dashv \Gamma'$

Proposition elaboration judgement:

- $\Gamma \vdash P \Leftarrow \mathtt{prop} \Rightarrow P' \dashv \Gamma'$

Proof judgements:

- $\Gamma \vdash p \Leftarrow P \dashv \Gamma'$
- $\Gamma \vdash p \Rightarrow P \dashv \Gamma'$

Judgements

Type conversion judgements:

- $\Gamma \vdash A \equiv B \Rightarrow \texttt{Type}_r \dashv \Gamma'$ – for any types
- $\Gamma \vdash A \triangleq B \Rightarrow \texttt{Type}_r \dashv \Gamma'$ – for types in whnf (unused for now)
- $\Gamma \vdash A \equiv B \Leftarrow \texttt{Type}_r \dashv \Gamma'$

Term conversion judgements:

- $\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'$ – for any terms
- $\Gamma \vdash e_1 \triangleq e_2 \Leftarrow A \dashv \Gamma'$ for terms in whnf
- $\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \dashv \Gamma'$ – for neutral terms, any type
- $\Gamma \vdash n_1 \triangleq n_2 \Rightarrow A \dashv \Gamma'$ – neutral terms, type in whnf

Proposition conversion judgements:

- $\Gamma \vdash P \equiv Q \dashv \Gamma'$ – any propositions
- $\Gamma \vdash P \triangleq Q \dashv \Gamma'$ – propositions in whnf

Subtraction of quantities

$r_1 - r_2$ is the least r' such that $r_1 \sqsubseteq r' + r_2$.

$r_1 - r_2$	0	1	$?$	$+$	$*$
0	0				
1	1	0			
$?$	$?$	0	0		
$+$	$+$	$*$	$+$	$*$	$+$
$*$	$*$	$*$	$*$	$*$	$*$

Subtraction order on quantities

$r_1 \leq_{\mathtt{sub}} r_2$ holds when $r_2 - r_1$ is defined.

Explicitly: $0 \leq_{\mathtt{sub}} 1 \leq_{\mathtt{sub}} ? \leq_{\mathtt{sub}} + \leq_{\mathtt{sub}} * \leq_{\mathtt{sub}} +$

Decrementation order on quantities

$r_1 \leq_{\texttt{dec}} r_2$ holds when $r_2 - 1 = r_1$.

$$\frac{}{* \leq_{\texttt{dec}} +}$$

$$\frac{}{0 \leq_{\texttt{dec}} 1}$$

$$\frac{}{0 \leq_{\texttt{dec}} ?}$$

Arithmetic order on quantities

The arithmetic order on quantities is $0 \leq 1 \leq ? \leq + \leq *$. The idea is to compare the quantities by how "big" they are.

Division with remainder

$a/b = (q, r)$ when $a = b \cdot q + r$, with q as large as possible and r being as small as possible according to the arithmetic order. Note that $a/b = q$ means that $r = 0$.

r_1/r_2	0	1	$?$	$+$	$*$
0	$*$	0	0	0	0
1	$(*, 1)$	1	$(0, 1)$	$(0, 1)$	$(0, 1)$
$?$	$(*, ?)$	$?$	$?$	$(0, ?)$	$(0, ?)$
$+$	$(*, +)$	$+$	$(*, 1)$	$+$	$(*, 1)$
$*$	$(*, *)$	$*$	$*$	$*$	$*$

Decrement variable in context

$$\begin{aligned}
\cdot - x &= \textbf{undefined} \\
(\Gamma, r\,x : A) - x &= \Gamma, (r-1)\,x : A \\
(\Gamma, r\,y : A) - x &= \Gamma - x, r\,y : A \\
(\Gamma, r\,x : A := e) - x &= \Gamma, (r-1)\,x : A := e \\
(\Gamma, r\,y : A := e) - x &= \Gamma - x, r\,y : A := e \\
(\Gamma, h : P) - x &= \Gamma - x, h : P \\
(\Gamma, a : \mathtt{Type}_r) - x &= \Gamma - x, a : \mathtt{Type}_r
\end{aligned}$$

Context division with remainder

$$\frac{}{\cdot/r = \cdot}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r\,x : A)/q = ((\Gamma_1, r_1\,x : A), (\Gamma_2, r_2\,x : A))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r\,x : A := e)/q = ((\Gamma_1, r_1\,x : A := e), (\Gamma_2, r_2\,x : A := e))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, h : P)/q = ((\Gamma_1, h : P), (\Gamma_2, h : P))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, a : \mathrm{Type}_r)/q = ((\Gamma_1, a : \mathrm{Type}_r), (\Gamma_2, a : \mathrm{Type}_r))}$$

Kind inference judgement

$\Gamma \vdash A \Rightarrow \mathtt{Type}_r \dashv \Gamma'$

- In context Γ, infer the kind of type A to be $\mathtt{Type}_r$, returning output context Γ'.
- Modes: Γ is input, A is subject, $\mathtt{Type}_r$ and Γ' are outputs.
- Preconditions: $\Gamma\ \mathtt{ctx_{nc}}$.
- Postcondition: $\Gamma' \vdash A : \mathtt{Type}_r$.

Kind inference

$$\frac{}{\Gamma \vdash \mathtt{Unit} \Rightarrow \mathtt{Type} \dashv \Gamma} \qquad \frac{}{\Gamma \vdash \mathtt{Empty} \Rightarrow \mathtt{Type} \dashv \Gamma}$$

$$\frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \mathtt{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash r\,A \to B \Rightarrow \mathtt{Type}_1 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_s \dashv \Gamma'}{\Gamma \vdash\ !_0\,A \Rightarrow \mathtt{Type} \dashv \Gamma'} \qquad \frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_s \dashv \Gamma' \quad r \neq 0}{\Gamma \vdash\ !_r\,A \Rightarrow \mathtt{Type}_{r \cdot s} \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \mathtt{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash A \otimes B \Rightarrow \mathtt{Type}_{s_1 \sqcup s_2} \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \mathtt{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash A \oplus B \Rightarrow \mathtt{Type}_{s_1 \sqcup s_2} \dashv \Gamma_2}$$

Kind inference

$$\frac{(a : \texttt{Type}_r) \in \Gamma}{\Gamma \vdash a \Rightarrow \texttt{Type}_r \dashv \Gamma}$$

$$\frac{\Gamma, a : \texttt{Type}_r \vdash B \Rightarrow \texttt{Type}_s \dashv \Gamma', a : \texttt{Type}_r}{\Gamma \vdash \forall @ a : \texttt{Type}_r .\, B \Rightarrow \texttt{Type}_s \dashv \Gamma'}$$

$$\frac{\Gamma, a : \texttt{Type}_r \vdash B \Rightarrow \texttt{Type}_s \dashv \Gamma', a : \texttt{Type}_r}{\Gamma \vdash \forall a : \texttt{Type}_r .\, B \Rightarrow \texttt{Type}_s \dashv \Gamma'}$$

Kind checking judgement

$\Gamma \vdash A \Leftarrow \mathtt{Type}_r \dashv \Gamma'$

- Kind checking judgement.
- In context Γ, check that the kind of type A is $\mathtt{Type}_r$, returning output context Γ'.
- Modes: Γ and $\mathtt{Type}_r$ are inputs, A is subject, Γ' is an output.
- Preconditions: $\Gamma\ \mathtt{ctx}_{\mathtt{nc}}$.
- Postcondition: $\Gamma' \vdash A : \mathtt{Type}_r$.

Kind checking

$$\frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_s \dashv \Gamma' \qquad r = s}{\Gamma \vdash A \Leftarrow \mathtt{Type}_r \dashv \Gamma'}$$

Typing judgements

$\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'$

- Type checking judgement.
- In context Γ, check whether term e has type A and return output context Γ'.
- Modes: Γ and A are inputs, e is subject, Γ' is an output.
- Preconditions: $\Gamma\ \mathtt{ctx}_i$ and $\Gamma \vdash A : \mathtt{Type}_r$
- Postcondition: $\Gamma - \Gamma' \vdash_i e : A$

$\Gamma \vdash_i e \Rightarrow A \dashv \Gamma'$

- Type inference judgement.
- In context Γ, term e infers type A, returning output context Γ'.
- Modes: Γ is input, e is subject, A and Γ' are outputs.
- Preconditions: $\Gamma\ \mathtt{ctx}_i$
- Postcondition: $\Gamma - \Gamma' \vdash_i e : A$

Context clean-up notation

$\Gamma, r\,x : A \vdash_i e \Leftarrow A \dashv \Gamma', 0\,x : A$ is a shorthand for $\Gamma, r\,x : A \vdash_i e \Leftarrow A \dashv \Gamma', r'\,x : A$ with the additional condition $r' \sqsubseteq 0$ when $i = \mathsf{c}$

$\Gamma, r\,x : A \vdash_i e \Rightarrow A \dashv \Gamma', 0\,x : A$ is a shorthand for $\Gamma, r\,x : A \vdash_i e \Rightarrow A \dashv \Gamma', r'\,x : A$ with the additional condition $r' \sqsubseteq 0$ when $i = \mathsf{c}$

Subsumption and annotations

$$\frac{\Gamma \vdash_i e \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash A \equiv B \Rightarrow \mathtt{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e \Leftarrow B \dashv \Gamma_2}\textsc{Subsumption}$$

$$\frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_r \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash_i (e : A) \Rightarrow A \dashv \Gamma_2}\textsc{Annot}$$

Using variables

$$\frac{\Gamma(x) = A}{\Gamma \vdash_{\mathtt{c}} x \Rightarrow A \dashv \Gamma - x}$$

$$\frac{\Gamma(x) = A}{\Gamma \vdash_{\mathtt{nc}} x \Rightarrow A \dashv \Gamma}$$

Functions

$$\frac{\Gamma, r_A^i\, x : A \vdash_i e \Leftarrow B \dashv \Gamma', 0\, x : A}{\Gamma \vdash_i \lambda\, x.\, e \Leftarrow r\, A \to B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i f \Rightarrow r\, A \to B \dashv \Gamma_1 \quad \Gamma_1 / r = (\Gamma_2, \Gamma_3) \quad \Gamma_2 \vdash_i a \Leftarrow A \dashv \Gamma_4}{\Gamma \vdash_i f\ \ a \Rightarrow B \dashv \Gamma_3 + r\, \Gamma_4}$$

Box

$$\frac{\Gamma/r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_3}{\Gamma \vdash_i \texttt{box}\ e \Leftarrow\ !_r\, A \dashv \Gamma_2 + r\,\Gamma_3}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow\ !_r\, A \dashv \Gamma_1 \quad \Gamma_1, r_A^i\, x : A \vdash_i e_2 \Rightarrow B \dashv \Gamma_2, 0\, x : A}{\Gamma \vdash_i \texttt{let box}\ x = e_1\ \texttt{in}\ e_2 \Rightarrow B \dashv \Gamma_2}$$

Empty

$$\frac{\Gamma \vdash_i e \Leftarrow \mathtt{Empty} \dashv \Gamma'}{\Gamma \vdash_i \mathtt{Empty\text{-}elim}\ e \Leftarrow A \dashv \Gamma'}$$

Unit

$$\frac{}{\Gamma \vdash_i \texttt{unit} \Leftarrow \texttt{Unit} \dashv \Gamma}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow \texttt{Unit} \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash_i \texttt{let unit} = e_1 \texttt{ in } e_2 \Rightarrow A \dashv \Gamma_2}$$

Products

$$\frac{\Gamma \vdash_i e_1 \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Leftarrow B \dashv \Gamma_2}{\Gamma \vdash_i (e_1, e_2) \Leftarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow A \otimes B \dashv \Gamma_1 \quad \Gamma_1, 1^i_A\, x : A, 1^i_B\, y : B \vdash_i e_2 \Rightarrow C \dashv \Gamma_2, 0\, x : A, 0\, y}{\Gamma \vdash_i \texttt{let}\ (x, y) = e_1\ \texttt{in}\ e_2 \Rightarrow C \dashv \Gamma_2}$$

Sums

$$\frac{\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'}{\Gamma \vdash_i \texttt{inl}\ e \Leftarrow A \oplus B \dashv \Gamma'} \qquad \frac{\Gamma \vdash_i e \Leftarrow B \dashv \Gamma'}{\Gamma \vdash_i \texttt{inr}\ e \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \qquad \begin{array}{l}\Gamma_1, 1^i_A\, x : A \vdash_i e_1 \Leftarrow C \dashv \Gamma_2, 0\, x : A \\ \Gamma_1, 1^i_B\, y : B \vdash_i e_2 \Leftarrow C \dashv \Gamma_3, 0\, x : A\end{array}}{\Gamma \vdash_i \mathsf{case}\ e\ \mathsf{of}\ \{x.e_1;\ y.e_2\} \Leftarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \qquad \begin{array}{l}\Gamma_1, 1^i_A\, x : A \vdash_i e_1 \Rightarrow C \dashv \Gamma_2, 0\, x : A \\ \Gamma_1, 1^i_B\, y : B \vdash_i e_2 \Rightarrow C \dashv \Gamma_3, 0\, x : A\end{array}}{\Gamma \vdash_i \mathsf{case}\ e\ \mathsf{of}\ \{x.e_1;\ y.e_2\} \Rightarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

Let

$$\frac{\Gamma / r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash_i e_1 \Rightarrow A \dashv \Gamma_3 \quad (\Gamma_2 + r\,\Gamma_3), r^i_A\, x : A \vdash_i e_2 \Rightarrow B \dashv \Gamma_4}{\Gamma \vdash_i \texttt{let}_r\ x = e_1\ \texttt{in}\ e_2 \Rightarrow B \dashv \Gamma_4}$$

Polymorphism

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \mathtt{Type}_r}{\Gamma \vdash_i \Lambda\, a.\, e \Leftarrow \forall @ a : \mathtt{Type}_r.\, B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall @ a : \mathtt{Type}_r.\, B \dashv \Gamma_1 \qquad \Gamma_1 \vdash A \Leftarrow \mathtt{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e\ A \Rightarrow B\,[a := A] \dashv \Gamma_2}$$

Polymorphism (implicit arguments)

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \mathtt{Type}_r}{\Gamma \vdash_i \Lambda \{a\}.\, e \Leftarrow \forall a : \mathtt{Type}_r .\, B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall a : \mathtt{Type}_r .\, B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Leftarrow \mathtt{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e\ @A \Rightarrow B\,[a := A] \dashv \Gamma_2}$$

Well-formed propositions

$$\frac{}{\Gamma \vdash \top \Leftarrow \texttt{prop} \Rightarrow \top \dashv \Gamma} \qquad \frac{}{\Gamma \vdash \bot \Leftarrow \texttt{prop} \Rightarrow \bot \dashv \Gamma}$$

$$\frac{\Gamma \vdash P \Leftarrow \texttt{prop} \Rightarrow P' \dashv \Gamma_1 \quad \Gamma_1 \vdash Q \Leftarrow \texttt{prop} \Rightarrow Q' \dashv \Gamma_2}{\Gamma \vdash P \Rightarrow Q \Leftarrow \texttt{prop} \Rightarrow P' \Rightarrow Q' \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P \Leftarrow \texttt{prop} \Rightarrow P' \dashv \Gamma_1 \quad \Gamma_1 \vdash Q \Leftarrow \texttt{prop} \Rightarrow Q' \dashv \Gamma_2}{\Gamma \vdash P \wedge Q \Leftarrow \texttt{prop} \Rightarrow P' \wedge Q' \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P \Leftarrow \texttt{prop} \Rightarrow P' \dashv \Gamma_1 \quad \Gamma_1 \vdash Q \Leftarrow \texttt{prop} \Rightarrow Q' \dashv \Gamma_2}{\Gamma \vdash P \vee Q \Leftarrow \texttt{prop} \Rightarrow P' \vee Q' \dashv \Gamma_2}$$

Well-formed propositions

$$\frac{\Gamma \vdash A \Rightarrow \texttt{Type}_r \dashv \Gamma_1 \quad \Gamma_1, x : A \vdash P \Leftarrow \texttt{prop} \Rightarrow P' \dashv \Gamma_2, x : A}{\Gamma \vdash \forall x : A.\, P \Leftarrow \texttt{prop} \Rightarrow \forall x : A.\, P' \dashv \Gamma_2}$$

$$\frac{\Gamma, x : \hat{\alpha} \vdash P \Leftarrow \texttt{prop} \Rightarrow P' \dashv \Gamma', x : A}{\Gamma \vdash \forall x.\, P \Leftarrow \texttt{prop} \Rightarrow \forall x : A.\, P' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \Rightarrow \texttt{Type}_r \dashv \Gamma_1 \quad \Gamma_1, x : A \vdash P \Leftarrow \texttt{prop} \Rightarrow P' \dashv \Gamma_2, x : A}{\Gamma \vdash \exists x : A.\, P \Leftarrow \texttt{prop} \Rightarrow \exists x : A.\, P' \dashv \Gamma_2}$$

$$\frac{\Gamma, x : \hat{\alpha} \vdash P \Leftarrow \texttt{prop} \Rightarrow P' \dashv \Gamma', x : A}{\Gamma \vdash \exists x.\, P \Leftarrow \texttt{prop} \Rightarrow \exists x : A.\, P' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \Rightarrow \texttt{Type}_r \dashv \Gamma_1 \quad \Gamma_1 \vdash_{\texttt{nc}} e_1 \Leftarrow A \dashv \Gamma_2 \quad \Gamma_2 \vdash_{\texttt{nc}} e_2 \Leftarrow A \dashv \Gamma_3}{\Gamma \vdash e_1 =_A e_2 \Leftarrow \texttt{prop} \Rightarrow e_1 =_A e_2 \dashv \Gamma_3}$$

$$\frac{\Gamma \vdash_{\texttt{nc}} e_1 \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash_{\texttt{nc}} e_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash e_1 = e_2 \Leftarrow \texttt{prop} \Rightarrow e_1 =_A e_2 \dashv \Gamma_2}$$

Subsumption and annotations

$$\frac{\Gamma \vdash p \Rightarrow P \dashv \Gamma_1 \quad \Gamma_1 \vdash P \equiv Q \dashv \Gamma_2}{\Gamma \vdash p \Leftarrow Q \dashv \Gamma_2}\textsc{Subsumption}$$

$$\frac{\Gamma \vdash P \Leftarrow \texttt{prop} \Rightarrow P' \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P' \dashv \Gamma_2}{\Gamma \vdash \textbf{proving } P \textbf{ by } p \Rightarrow P' \dashv \Gamma_2}\textsc{Annot}$$

Assumptions and implication

$$\frac{\Gamma(h) = P}{\Gamma \vdash h \Rightarrow P \dashv \Gamma} \qquad \frac{\Gamma(h) = P}{\Gamma \vdash \textbf{assumption} \Leftarrow P \dashv \Gamma}$$

$$\frac{\Gamma, h : P \vdash q \Leftarrow Q \dashv \Gamma', h : P}{\Gamma \vdash \textbf{assume}\ h\ \textbf{in}\ q \Leftarrow P \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow P \Rightarrow Q \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \textbf{apply}\ q\ p \Rightarrow Q \dashv \Gamma_2}$$

Propositional logic

$$\frac{}{\Gamma \vdash \textbf{trivial} \Leftarrow \top \dashv \Gamma} \qquad \frac{\Gamma \vdash p \Leftarrow \bot \dashv \Gamma'}{\Gamma \vdash \textbf{absurd}\ p \Leftarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma_1 \quad \Gamma_1 \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \textbf{both}\ p\ q \Leftarrow P \wedge Q \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \textbf{and-left}\ pq \Rightarrow P \dashv \Gamma'} \qquad \frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \textbf{and-right}\ pq \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma'}{\Gamma \vdash \textbf{or-left}\ p \Leftarrow P \vee Q \dashv \Gamma'} \qquad \frac{\Gamma \vdash q \Leftarrow Q \dashv \Gamma'}{\Gamma \vdash \textbf{or-right}\ q \Leftarrow P \vee Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \vee Q \dashv \Gamma_1 \quad \Gamma_1 \vdash r_1 \Leftarrow P \Rightarrow R \dashv \Gamma_2 \quad \Gamma_1 \vdash r_2 \Leftarrow Q \Rightarrow R \dashv \Gamma_3}{\Gamma \vdash \textbf{cases}\ pq\ r_1\ r_2 \Leftarrow R \dashv \Gamma_2 \sqcup \Gamma_3}$$

Positive conjunction

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma_1 \quad \Gamma_1, h_1 : P, h_2 : Q \vdash r \Leftarrow R \dashv \Gamma_2, h_1 : P, h_2 : Q}{\Gamma \vdash \textbf{destruct}\ pq\ \textbf{as}\ h_1\ h_2\ \textbf{in}\ r \Leftarrow R \dashv \Gamma_2}$$

To make the system more checking, it makes sense to turn conjunction positive and get rid of the projections.

Utilities

$$\frac{\Gamma \vdash P \Leftarrow \texttt{prop} \Rightarrow P' \dashv \Gamma_1 \\ \Gamma_1 \vdash p \Leftarrow P' \dashv \Gamma_2 \quad \Gamma_2, h : P' \vdash q \Leftarrow Q \dashv \Gamma_3, h : P'}{\Gamma \vdash \textbf{lemma } h : P \textbf{ by } p \textbf{ in } q \Leftarrow Q \dashv \Gamma_3}$$

$$\frac{\Gamma \vdash P \Leftarrow \texttt{prop} \Rightarrow P' \dashv \Gamma_1 \quad \Gamma \vdash q \Leftarrow P' \Rightarrow Q \dashv \Gamma_2 \quad \Gamma_2 \vdash p \Leftarrow P' \dashv \Gamma_3}{\Gamma \vdash \textbf{suffices } P \textbf{ by } q \textbf{ in } p \Leftarrow Q \dashv \Gamma_2}$$

Quantifiers

$$\frac{\Gamma \vdash p \Leftarrow P\,[x := y] \dashv \Gamma'}{\Gamma \vdash \textbf{pick-any } y \textbf{ in } p \Leftarrow \forall x : A.\, P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \forall x : A.\, P \dashv \Gamma_1 \quad \Gamma_1 \vdash_{\mathtt{nc}} e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash \textbf{instantiate } p \textbf{ with } e \Rightarrow P\,[x := e] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_{\mathtt{nc}} e \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P\,[x := e] \dashv \Gamma_2}{\Gamma \vdash \textbf{witness } e \textbf{ such that } p \Leftarrow \exists x : A.\, P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A.\, P \dashv \Gamma_1 \quad \Gamma_1, y : A, h : P\,[x := y] \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \textbf{pick-witness } y\ h \textbf{ for } p \textbf{ in } q \Leftarrow Q \dashv \Gamma_2}$$

Equality

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'}{\Gamma \vdash \textbf{refl} \Leftarrow e_1 =_A e_2 \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow e_1 =_A e_2 \dashv \Gamma_1 \qquad \begin{array}{l} \Gamma_1, x : A \vdash P \Leftarrow \mathtt{prop} \Rightarrow P' \dashv \Gamma_2, x : A \\ \Gamma_2 \vdash p \Leftarrow P'\,[x := e_2] \dashv \Gamma_3 \end{array}}{\Gamma \vdash \textbf{rewrite}\ q\ \textbf{at}\ x.P\ \textbf{in}\ p \Rightarrow P'\,[x := e_1] \dashv \Gamma_3}$$

$$\frac{\Gamma, x : A \vdash p \Leftarrow f =_B g \dashv \Gamma', x : A}{\Gamma \vdash \textbf{funext}\ x\ \textbf{in}\ p \Leftarrow f =_{r\,A \to B} g \dashv \Gamma'}$$

Classical logic

$$\frac{\Gamma, h : \neg P \vdash q \Leftarrow \bot \dashv \Gamma', h : \neg P}{\Gamma \vdash \textbf{by-contradiction } h \textbf{ in } q \Leftarrow P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash P \Leftarrow \texttt{prop} \Rightarrow \Gamma_1 \dashv \quad \Gamma_1, h : \neg P \vdash q \Leftarrow \bot \dashv \Gamma_2, h : \neg P}{\Gamma \vdash \textbf{by-contradiction } h : \neg P \textbf{ in } q \Rightarrow P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A.\, P \dashv \Gamma'}{\Gamma \vdash_{\texttt{nc}} \texttt{choose } p \Rightarrow A \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A.\, P \dashv \Gamma'}{\Gamma \vdash \textbf{choose-spec } p \Rightarrow P\,[x := \texttt{choose } p] \dashv \Gamma'}$$

Classical logic

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A.\, P \dashv \Gamma_1 \qquad \Gamma_1, y : A := \mathtt{choose}\ p, h : P\,[x := y] \vdash q \Leftarrow Q \dashv \Gamma_2, y : A := \mathtt{choose}\ p, \ldots}{\Gamma \vdash \textbf{choose-witness}\ y\ \ h\ \textbf{for}\ p\ \textbf{in}\ q \Leftarrow Q \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A.\, P \dashv \Gamma_1 \qquad \Gamma_1, y : A := \mathtt{choose}\ p, h : P\,[x := y] \vdash_{\mathtt{nc}} e \Rightarrow B \dashv \Gamma_2, y : A := \mathtt{choose}\ \ldots}{\Gamma \vdash_{\mathtt{nc}} \mathtt{choose\text{-}witness}\ y\ \ h\ \mathtt{for}\ p\ \mathtt{in}\ e \Rightarrow B \dashv \Gamma_2}$$

Type conversion

$$\frac{\Gamma \vdash A \equiv B \Rightarrow \mathtt{Type}_s \dashv \Gamma' \quad r = s}{\Gamma \vdash A \equiv B \Leftarrow \mathtt{Type}_r \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \triangleq B \Rightarrow \mathtt{Type}_r \dashv \Gamma'}{\Gamma \vdash A \equiv B \Rightarrow \mathtt{Type}_r \dashv \Gamma'}$$

Type conversion

$$\frac{}{\Gamma \vdash \mathtt{Unit} \triangleq \mathtt{Unit} \Rightarrow \mathtt{Type} \dashv \Gamma}$$

$$\frac{}{\Gamma \vdash \mathtt{Empty} \triangleq \mathtt{Empty} \Rightarrow \mathtt{Type} \dashv \Gamma}$$

$$\frac{r_1 = r_2 \quad \Gamma \vdash A_1 \equiv A_2 \Rightarrow \mathtt{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \mathtt{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash r_1\, A_1 \to B_1 \triangleq r_2\, A_2 \to B_2 \Rightarrow \mathtt{Type}_1 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \mathtt{Type}_s \dashv \Gamma'}{\Gamma \vdash \mathtt{!}_0\, A_1 \triangleq \mathtt{!}_0\, A_2 \Rightarrow \mathtt{Type} \dashv \Gamma'}$$

$$\frac{r_1 = r_2 \quad r_1 \neq 0 \quad \Gamma \vdash A_1 \equiv A_2 \Rightarrow \mathtt{Type}_s \dashv \Gamma'}{\Gamma \vdash \mathtt{!}_{r_1}\, A_1 \triangleq \mathtt{!}_{r_2}\, A_2 \Rightarrow \mathtt{Type}_{r_1 \cdot s} \dashv \Gamma'}$$

Type conversion

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \mathtt{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \mathtt{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash A_1 \otimes B_1 \triangleq A_2 \otimes B_2 \Rightarrow \mathtt{Type}_{s_A \sqcup s_B} \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \mathtt{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \mathtt{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash A_1 \oplus B_1 \triangleq A_2 \oplus B_2 \Rightarrow \mathtt{Type}_{s_A \sqcup s_B} \dashv \Gamma_2}$$

Type conversion

$$\frac{(a : \mathtt{Type}_r) \in \Gamma}{\Gamma \vdash a \triangleq a \color{red}{\Rightarrow} \mathtt{Type}_r \dashv \Gamma}$$

$$\frac{r_1 = r_2 \quad \Gamma, a_1 : \mathtt{Type}_{r_1} \vdash B_1 \equiv B_2\,[a_2 := a_1] \color{red}{\Rightarrow} \mathtt{Type}_s \dashv \Gamma', a_1 : \mathtt{Type}_{r_1}}{\Gamma \vdash \forall @ a_1 : \mathtt{Type}_{r_1}.\, B_1 \triangleq \forall @ a_2 : \mathtt{Type}_{r_2}.\, B_2 \color{red}{\Rightarrow} \mathtt{Type}_s \dashv \Gamma'}$$

$$\frac{r_1 = r_2 \quad \Gamma, a_1 : \mathtt{Type}_{r_1} \vdash B_1 \equiv B_2\,[a_2 := a_1] \color{red}{\Rightarrow} \mathtt{Type}_s \dashv \Gamma', a_1 : \mathtt{Type}_{r_1}}{\Gamma \vdash \forall a_1 : \mathtt{Type}_{r_1}.\, B_1 \triangleq \forall a_2 : \mathtt{Type}_{r_2}.\, B_2 \color{red}{\Rightarrow} \mathtt{Type}_s \dashv \Gamma'}$$

Proposition conversion

$$\frac{\Gamma \vdash P_1 \stackrel{\wedge}{=} P_2 \dashv \Gamma'}{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma'}$$

Proposition conversion

$$\frac{}{\Gamma \vdash \top \triangleq \top \dashv \Gamma} \qquad \frac{}{\Gamma \vdash \bot \triangleq \bot \dashv \Gamma}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \Rightarrow Q_1 \triangleq P_2 \Rightarrow Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \wedge Q_1 \triangleq P_2 \wedge Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \vee Q_1 \triangleq P_2 \vee Q_2 \dashv \Gamma_2}$$

Proposition conversion

$$\frac{\Gamma \vdash A_1 \equiv A_2 \color{red}{\Rightarrow} \texttt{Type}_r \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \equiv P_2\,[x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \forall x_1 : A_1.\, P_1 \triangleq \forall x_2 : A_2.\, P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \color{red}{\Rightarrow} \texttt{Type}_r \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \equiv P_2\,[x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \exists x_1 : A_1.\, P_1 \triangleq \exists x_2 : A_2.\, P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \color{red}{\Rightarrow} \texttt{Type}_r \dashv \Gamma_1 \quad \begin{array}{c} \Gamma_1 \vdash e_1 \equiv e_2 \color{blue}{\Leftarrow} A_1 \dashv \Gamma_2 \\ \Gamma_2 \vdash e_1' \equiv e_2' \color{blue}{\Leftarrow} A_1 \dashv \Gamma_3 \end{array}}{\Gamma \vdash e_1 =_{A_1} e_1' \triangleq e_2 =_{A_2} e_2' \dashv \Gamma_3}$$

Contraction

$$\frac{}{(\lambda\, x.\, e_1)\ e_2 \longmapsto e_1\,[x := e_2]}$$

$$\frac{}{\texttt{let box}\ x = \texttt{box}\ e_1\ \texttt{in}\ e_2 \longmapsto e_2\,[x := e_1]}$$

$$\frac{}{\texttt{let unit} = \texttt{unit in}\ e \longmapsto e}$$

$$\frac{}{\texttt{let}\ (x, y) = (e_1, e_2)\ \texttt{in}\ e_3 \longmapsto e_3\,[x := e_1]\,[y := e_2]}$$

$$\frac{}{\texttt{case inl}\ e_1\ \texttt{of}\ (x, e_2) y e_3 \longmapsto e_2\,[x := e_1]}$$

$$\frac{}{\texttt{case inr}\ e_1\ \texttt{of}\ (x, e_2) y e_3 \longmapsto e_3\,[x := e_1]}$$

$$\frac{}{\texttt{let}\ x = e_1\ \texttt{in}\ e_2 \longmapsto e_2\,[x := e_1]}$$

Contraction

$$\overline{(\Lambda\, a : \texttt{Type}_r .\, e)\ A \longmapsto e\,[a := A]}$$

$$\overline{(\Lambda\, \{a : \texttt{Type}_r\} .\, e)\ \texttt{@}A \longmapsto e\,[a := A]}$$

$$\overline{\texttt{choose-witness}\ x\ h\ \texttt{for}\ p\ \texttt{in}\ e \longmapsto e\,[x := \texttt{choose}\ p]\,[h := \textbf{choose-sp}}$$

Weak head evaluation contexts

$$
\begin{aligned}
\mathsf{E} ::= \;& \\
& \square \mid \square\ e \mid \square\ A \mid \square\ @A \mid \\
& \texttt{let unit} = \square\ \texttt{in}\ e \mid \texttt{let box}\ x = \square\ \texttt{in}\ e \mid \\
& \texttt{let}\ (x, y) = \square\ \texttt{in}\ e \mid \texttt{case}\ \square\ \texttt{of}\ \{x.e_1;\ y.e_2\}
\end{aligned}
$$

Computation

$$\frac{e \longmapsto e'}{E[e] \longrightarrow E[e']}$$

$$\frac{}{e \longrightarrow^{*} e} \qquad \frac{e_1 \longrightarrow e_2 \quad e_2 \longrightarrow^{*} e_3}{e_1 \longrightarrow^{*} e_3}$$

Whnfs and neutral terms

Whnfs:

$\hat{e}$ ::=

$$n \mid \lambda x.\, e \mid \texttt{box}\ e \mid (e_1, e_2) \mid \texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{unit}$$
$$\Lambda\, a : \texttt{Type}_r.\, e \mid \Lambda \{a : \texttt{Type}_r\}.\, e$$

Neutral forms:

n ::=

$$x \mid n\ e \mid n\ A \mid n\ @A \mid \texttt{Empty-elim}_A\ e \mid$$
$$\texttt{let}_A\ \texttt{unit} = n\ \texttt{in}\ e \mid \texttt{let}_A\ \texttt{box}\ x = n\ \texttt{in}\ e \mid$$
$$\texttt{let}_A\ (x, y) = n\ \texttt{in}\ e \mid \texttt{case}_A\ n\ \texttt{of}\ \{x.e_1;\ y.e_2\} \mid$$
$$\texttt{choose}\ p$$

Term conversion – checking, all terms

$$\frac{}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow \texttt{Unit} \dashv \Gamma}$$

$$\frac{}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow \texttt{Empty} \dashv \Gamma}$$

$$\frac{A \neq \texttt{Unit} \quad A \neq \texttt{Empty} \quad e_1 \longrightarrow^* e_1' \quad e_2 \longrightarrow^* e_2' \quad \Gamma \vdash e_1' \mathrel{\hat{=}} e_2' \Leftarrow A \dashv \Gamma'}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'}$$

Term conversion – checking whnfs

$$\frac{\Gamma, x : A \vdash f\ x \equiv g\ x \Leftarrow B \dashv \Gamma', x : A}{\Gamma \vdash f \triangleq g \Leftarrow r\, A \to B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash \texttt{let box}\ x = e_1\ \texttt{in}\ x \equiv \texttt{let box}\ x = e_2\ \texttt{in}\ x \Leftarrow A \dashv \Gamma'}{\Gamma \vdash e_1 \triangleq e_2 \Leftarrow\ !_r\, A \dashv \Gamma'}$$

$$\frac{\begin{array}{c}\Gamma \vdash \texttt{let}\ (x, y) = e_1\ \texttt{in}\ x \equiv \texttt{let}\ (x, y) = e_2\ \texttt{in}\ x \Leftarrow A \dashv \Gamma_1 \\ \Gamma_1 \vdash \texttt{let}\ (x, y) = e_1\ \texttt{in}\ y \equiv \texttt{let}\ (x, y) = e_2\ \texttt{in}\ y \Leftarrow B \dashv \Gamma_2\end{array}}{\Gamma \vdash e_1 \triangleq e_2 \Leftarrow A \otimes B \dashv \Gamma_2}$$

Term conversion – checking whnfs

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'}{\Gamma \vdash \mathtt{inl}\ e_1 \triangleq \mathtt{inl}\ e_2 \Leftarrow A \oplus B \dashv \Gamma}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow B \dashv \Gamma'}{\Gamma \vdash \mathtt{inr}\ e_1 \triangleq \mathtt{inr}\ e_2 \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash f\ a \equiv g\ a \Leftarrow B \dashv \Gamma'}{\Gamma \vdash f \triangleq g \Leftarrow \forall @a : \mathtt{Type}_r.\, B \dashv \Gamma'}$$

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash f\ @a \equiv g\ @a \Leftarrow B \dashv \Gamma'}{\Gamma \vdash f \triangleq g \Leftarrow \forall a : \mathtt{Type}_r.\, B \dashv \Gamma'}$$

Term conversion – switch mode

$$\frac{\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash A \equiv B \Rightarrow \mathtt{Type}_r \dashv \Gamma_2}{\Gamma \vdash n_1 \triangleq n_2 \Leftarrow B \dashv \Gamma_2}$$

Term conversion – infer neutrals

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \equiv x \Rightarrow A \dashv \Gamma}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow r\, A \to B \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash n_1\ e_1 \equiv n_2\ e_2 \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow \forall @a : \mathtt{Type}_r .\, B \dashv \Gamma_1 \quad \Gamma_1 \vdash A_1 \equiv A_2 \Leftarrow \mathtt{Type}_r \dashv \Gamma_2}{\Gamma \vdash n_1\ A_1 \equiv n_2\ A_2 \Rightarrow B\,[a := A] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow \forall a : \mathtt{Type}_r .\, B \dashv \Gamma_1 \quad \Gamma_1 \vdash A_1 \equiv A_2 \Leftarrow \mathtt{Type}_r \dashv \Gamma_2}{\Gamma \vdash n_1\ @A_1 \equiv n_2\ @A_2 \Rightarrow B\,[a := A] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \mathtt{Type}_r \dashv \Gamma'}{\Gamma \vdash \mathtt{Empty\text{-}elim}_{A_1}\ e_1 \equiv \mathtt{Empty\text{-}elim}_{A_2}\ e_2 \Rightarrow A_1 \dashv \Gamma'}$$

Term conversion – infer neutrals

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \mathtt{Type}_r \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash \mathtt{let}_{A_1}\ \mathtt{unit} = n_1\ \mathtt{in}\ e_1 \equiv \mathtt{let}_{A_2}\ \mathtt{unit} = n_2\ \mathtt{in}\ e_2 \Rightarrow A_1 \dashv \Gamma_2}$$

$$\frac{\begin{array}{l}\Gamma \vdash B_1 \equiv B_2 \Rightarrow \mathtt{Type}_s \dashv \Gamma_1 \\ \Gamma \vdash n_1 \triangleq n_2 \Rightarrow !_r\, A \dashv \Gamma_2\end{array} \quad \Gamma_2, x : A \vdash e_1 \equiv e_2 \Leftarrow B_1 \dashv \Gamma_3, x : A}{\Gamma \vdash \mathtt{let}_{B_1}\ \mathtt{box}\ x = n_1\ \mathtt{in}\ e_1 \equiv \mathtt{let}_{B_2}\ \mathtt{box}\ x = n_2\ \mathtt{in}\ e_2 \Rightarrow B_1 \dashv \Gamma_3}$$

$$\frac{\begin{array}{l}\Gamma \vdash C_1 \equiv C_2 \Rightarrow \mathtt{Type}_r \dashv \Gamma_1 \\ \Gamma_1 \vdash n_1 \triangleq n_2 \Rightarrow A \otimes B \dashv \Gamma_2\end{array} \quad \Gamma_2, x : A, y : B \vdash e_1 \equiv e_2 \Leftarrow C_1 \dashv \Gamma_3, x :}{\Gamma \vdash \mathtt{let}_{C_1}\ (x, y) = n_1\ \mathtt{in}\ e_1 \equiv \mathtt{let}_{C_2}\ (x, y) = n_2\ \mathtt{in}\ e_2 \Rightarrow C_1 \dashv \Gamma}$$

$$\frac{\begin{array}{l}\Gamma \vdash C_1 \equiv C_2 \Rightarrow \mathtt{Type}_r \dashv \Gamma_1 \\ \Gamma_1 \vdash e_1 \equiv e_2 \Rightarrow A \oplus B \dashv \Gamma_2\end{array} \quad \begin{array}{l}\Gamma_2, x : A \vdash f_1 \equiv f_2 \Leftarrow C \dashv \Gamma_3, x : A \\ \Gamma_2, y : B \vdash g_1 \equiv g_2 \Leftarrow C \dashv \Gamma_4, y :\end{array}}{\Gamma \vdash \mathtt{case}_{C_1}\ e_1\ \mathtt{of}\ \{x_1.f_1;\ y_1.g_1\} \equiv \mathtt{case}_{C_2}\ e_2\ \mathtt{of}\ \{x_2.f_2;\ y_2.g_2\} \Rightarrow C_1 \dashv \Gamma}$$

Term conversion (choice)

$$\frac{\begin{array}{c}\Gamma \vdash p_1 \Rightarrow \exists x : A_1 .\, P_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash p_2 \Rightarrow \exists x : A_2 .\, P_2 \dashv \Gamma_2\end{array} \qquad \Gamma_2 \vdash \exists x : A_1 .\, P_1 \equiv \exists x : A_1 .\, P_2 \dashv \Gamma_3}{\Gamma \vdash \texttt{choose}\ p_1 \equiv \texttt{choose}\ p_2 \Rightarrow A_1 \dashv \Gamma_3}$$