Complete and Easy Quantitative Contextual Typing

June 30, 2025

Types

Types:

$A, B ::=$

$A \rightarrow B \mid\ !_r\, A \mid A \otimes B \mid A \oplus B \mid \texttt{Unit} \mid \texttt{Empty} \mid$

$a \mid \tilde{a} \mid \hat{a} \mid \forall @ a : \texttt{Type}_r.\, A \mid \forall @ a.\, A \mid \forall a : \texttt{Type}_r.\, A \mid \forall a.\, A$

Monotypes:

$\tau ::=$

$\tau_1 \rightarrow \tau_2 \mid\ !_r\, \tau \mid \tau_1 \otimes \tau_2 \mid \tau_1 \oplus \tau_2 \mid \texttt{Unit} \mid \texttt{Empty} \mid$

$a \mid \hat{a} \mid \forall @ a : \texttt{Type}_r.\, \tau \mid \forall @ a.\, \tau$

Terms

Terms:

$e ::=$

$(e : A) \mid x \mid \lambda x.\, e \mid e_1\ e_2 \mid$

$\Lambda a.\, e \mid e\ A \mid e\ @A \mid$

$(e_1, e_2) \mid \texttt{let}\ (x, y) = e_1\ \texttt{in}\ e_2 \mid$

$\texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}\ e\ \texttt{of}\ \{x.e_1;\ y.e_2\} \mid$

$\texttt{unit} \mid \texttt{let unit} = e_1\ \texttt{in}\ e_2 \mid$

$\texttt{Empty-elim}\ e \mid$

$\texttt{let}_r\ x = e_1\ \texttt{in}\ e_2 \mid$

Contexts

Contexts:

$\Gamma ::=$

$\cdot \mid \Gamma, r\, x : A \mid \Gamma, a : \mathtt{Type}_r \mid \Gamma, \tilde{a} : \mathtt{Type}_r \mid \Gamma, \hat{a} \mid \Gamma, \hat{a} = \tau \mid$
$\Gamma, \square \mid \Gamma, \square : A \mid \Gamma, \square\ e \mid \Gamma, \square\ A$

$\Gamma, \square, \Delta$ is a notation which means that Δ does not contain any hints. It works not only for $\square$, but for all kinds of hints.

Judgements

- $\Gamma \vdash_i r \Rightarrow A \Rightarrow s \dashv \Gamma'$ – kinding
- $\Gamma \vdash_i e \Rightarrow A \dashv \Gamma'$ – typing
- $\Gamma \vdash A \rightsquigarrow B \dashv \Gamma'$ – matching
- $\Gamma \vdash r \Rightarrow A <: B \Rightarrow s \dashv \Gamma'$ – subtyping
- $\Gamma \vdash r \Rightarrow \hat{a} <: A \Rightarrow s \dashv \Gamma'$ – sub-instantiation
- $\Gamma \vdash r \Rightarrow A <: \hat{a} \Rightarrow s \dashv \Gamma'$ – super-instantiation

Kinding judgement

$\Gamma \vdash_i r \Rightarrow A \Rightarrow s \dashv \Gamma'$ – in context Γ, check if type A is of kind $\mathtt{Type}_r$. If it is, s will be 1. If it is not, s indicates how much A is missing to be in $\mathtt{Type}_r$, i.e. A is in $\mathtt{Type}_{r/s}$. Return output context Γ'.

Note: Γ and r are inputs. A is subject. s and Γ' are outputs.

$\Gamma \vdash_i r \Rightarrow A \dashv \Gamma'$ – kind checking is a notation for $\Gamma \vdash_i r \Rightarrow A \Rightarrow 1 \dashv \Gamma'$.

$\Gamma \vdash_i A \Rightarrow r \dashv \Gamma'$ – kind inference is a notation for $\Gamma \vdash_i * \Rightarrow A \Rightarrow s \dashv \Gamma'$ with $r = */s$

Kinding – variables and quantifiers

$$\frac{}{\Gamma[a : \mathtt{Type}_s] \vdash_i r \Rightarrow a \Rightarrow r/s \dashv \Gamma[a : \mathtt{Type}_s]}$$

$$\frac{}{\Gamma[\tilde{a} : \mathtt{Type}_s] \vdash_i r \Rightarrow \tilde{a} \Rightarrow r/s \dashv \Gamma[\tilde{a} : \mathtt{Type}_s]}$$

$$\frac{}{\Gamma[\hat{a}] \vdash_i r \Rightarrow \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{b}, \hat{a} = !_r\, \hat{b}]}$$

$$\frac{\Gamma, a : \mathtt{Type}_s \vdash_i r \Rightarrow A \Rightarrow t \dashv \Gamma'}{\Gamma \vdash_i r \Rightarrow \forall a : \mathtt{Type}_s .\, A \Rightarrow t \dashv \Gamma'}$$

$$\frac{\Gamma, a : \mathtt{Type}_s \vdash_i r \Rightarrow A \Rightarrow t \dashv \Gamma'}{\Gamma \vdash_i r \Rightarrow \forall @a : \mathtt{Type}_s .\, A \Rightarrow t \dashv \Gamma'}$$

Kinding – functions and boxes

$$\frac{\Gamma \vdash_i 1 \Rightarrow A \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1 \vdash_i 1 \Rightarrow \langle \Gamma_1 \rangle B \Rightarrow 1 \dashv \Gamma_2}{\Gamma \vdash_i r \Rightarrow A \to B \Rightarrow r \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_i 1 \Rightarrow A \Rightarrow t \dashv \Gamma'}{\Gamma \vdash_i r \Rightarrow !_0 A \Rightarrow 1 \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i r/s \Rightarrow A \Rightarrow t \dashv \Gamma'}{\Gamma \vdash_i r \Rightarrow !_s A \Rightarrow t \dashv \Gamma'}$$

Kinding – remaining type formers

$$\frac{}{\Gamma \vdash_i r \Rightarrow \mathtt{Unit} \Rightarrow 1 \dashv \Gamma} \qquad \frac{}{\Gamma \vdash_i r \Rightarrow \mathtt{Empty} \Rightarrow 1 \dashv \Gamma}$$

$$\frac{\Gamma \vdash_i r \Rightarrow A \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash_i r \Rightarrow \langle \Gamma_1 \rangle B \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash_i r \Rightarrow A \otimes B \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_i r \Rightarrow A \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash_i r \Rightarrow \langle \Gamma_1 \rangle B \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash_i r \Rightarrow A \oplus B \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

Typing judgement

$\Gamma \vdash_i e \Rightarrow A \dashv \Gamma'$ – in context Γ, term e infers type A, returning output context Γ'

Note: Γ is input. e is subject. A and Γ' are outputs.

Note: "inference mode" is $\Gamma, \square \vdash_i e \Rightarrow A \dashv \Gamma'$, as it requires the empty hint.

Note: "checking mode" is $\Gamma, \square : A \vdash_i e \Rightarrow \_ \dashv \Gamma'$, as it requires a type hint $\square : A$, but does not use the inferred type.

Typing – variables and annotations

$$\frac{\Gamma(x) = A \quad 0\,\Gamma + x \vdash \langle 0\,\Gamma + x\rangle A \rightsquigarrow B \dashv \Gamma'}{\Gamma \vdash_{\mathtt{c}} x \Rightarrow B \dashv \Gamma' + \Gamma}$$

$$\frac{\Gamma(x) = A \quad \Gamma \vdash \langle \Gamma\rangle A \rightsquigarrow B \dashv \Gamma'}{\Gamma \vdash_{\mathtt{nc}} x \Rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i A \Rightarrow r \dashv \Gamma_1 \quad 0\,\Gamma_1, \square : A \vdash_i e \Rightarrow \_ \dashv \Gamma_2 \quad \Gamma_2 \vdash \langle \Gamma_2\rangle A \rightsquigarrow B \dashv \Gamma_3}{\Gamma \vdash_i (e : A) \Rightarrow B \dashv \Gamma_3 + \Gamma_1}$$

Typing – stationary rules

$$\frac{\Gamma, \Delta, a : \mathtt{Type}_r, \square : A \vdash_i e \Rightarrow \_ \dashv \Gamma', a : \mathtt{Type}_r, \Theta}{\Gamma, \square : \forall a : \mathtt{Type}_r.\, A, \Delta \vdash_i e \Rightarrow \forall a : \mathtt{Type}_r.\, A \dashv \Gamma'}$$

$$\frac{0\,\Gamma, 0\,\Delta, \square : A \vdash_i e \Rightarrow \_ \dashv \Gamma'}{\Gamma, \square : \,!_r\, A, \Delta \vdash_i e \Rightarrow \,!_r\, A \dashv r\,\Gamma' + (\Gamma, \Delta)}$$

Typing – application(s)

$$\frac{\Gamma, \Box\; e_2 \vdash_i e_1 \Rightarrow A \dashv \Gamma'}{\Gamma \vdash_i e_1\; e_2 \Rightarrow A \dashv \Gamma'}$$

$$\frac{\Gamma, \Box\; A \vdash_i e \Rightarrow B \dashv \Gamma'}{\Gamma \vdash_i e\; A \Rightarrow B \dashv \Gamma'}$$

Typing – functions

$$\frac{\Gamma, \Delta, 0\,x : A, \square : B \vdash_i e \Rightarrow \_ \dashv \Gamma_1, r\,x : A, \Theta \qquad \Gamma_1 \vdash_i r \Rightarrow \langle \Gamma_1 \rangle A \Rightarrow 1 \dashv \Gamma_2}{\Gamma, \square : A \to B, \Delta \vdash_i \lambda\, x.\, e \Rightarrow A \to B \dashv \Gamma_2}$$

$$\frac{\Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \to \hat{a}_2], \Delta, 0\,x : \hat{a}_1, \square : \hat{a}_2 \vdash_i e \Rightarrow \_ \dashv \Gamma_1, r\,x : \hat{a}_1, \Theta \qquad \Gamma_1 \vdash_i r \Rightarrow \langle \Gamma_1 \rangle \hat{a}_1 \Rightarrow 1 \dashv \Gamma_2}{\Gamma[\hat{a}], \square : \hat{a}, \Delta \vdash_i \lambda\, x.\, e \Rightarrow \hat{a} \dashv \Gamma_2}$$

$$\frac{\Gamma, \square \vdash_i e' \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1, \Delta, 0\,x : A \vdash_i e \Rightarrow B \dashv \Gamma_2, r\,x : A, \Theta \qquad \Gamma_2 \vdash_i r \Rightarrow \langle \Gamma_2 \rangle A \Rightarrow 1 \dashv \Gamma_3}{\Gamma, \square\ e', \Delta \vdash_i \lambda\, x.\, e \Rightarrow B \dashv \Gamma_3}$$

$$\frac{\begin{array}{l}\Gamma, \Delta \blacktriangleright_{\hat{a}}, \hat{a}, 0\,x : \hat{a}, \square \vdash_i e \Rightarrow \_ \dashv \Gamma_1, r\,x : \hat{a}, \Gamma_4 \\ \Gamma_1 \vdash_i r \Rightarrow \langle \Gamma_1 \rangle \hat{a} \Rightarrow 1 \dashv \Gamma_2, \blacktriangleright_{\hat{a}}, \Gamma_3 \\ B = \forall \mathtt{unsolved}(\Gamma_3).\, \langle \Gamma_3 \rangle (\hat{a} \to \forall \mathtt{unsolved}(\Gamma_4).\, \langle \Gamma_4 \rangle A)\end{array}}{\Gamma, \square, \Delta \vdash_i \lambda\, x.\, e \Rightarrow B \dashv \Gamma_2}$$

Typing – Unit

$$\frac{\Gamma \vdash \texttt{Unit} \rightsquigarrow \_ \dashv \Gamma'}{\Gamma \vdash_i \texttt{unit} \Rightarrow \texttt{Unit} \dashv \Gamma'}$$

$$\frac{\Gamma, \Box : \texttt{Unit} \vdash_i e_1 \Rightarrow \_ \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash_i \texttt{let unit} = e_1 \texttt{ in } e_2 \Rightarrow A \dashv \Gamma_2}$$

Typing – Empty

$$\frac{\Gamma, \Delta, \square : \texttt{Empty} \vdash_i e \Rightarrow \_ \dashv \Gamma'}{\Gamma, \square : A, \Delta \vdash_i \texttt{Empty-elim}\ e \Rightarrow A \dashv \Gamma'}$$

$$\frac{\Gamma, \square \vdash_i e' \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash_i \texttt{Empty-elim}\ e \Rightarrow B \dashv \Gamma_2}{\Gamma, \square\ e', \Delta \vdash_i \texttt{Empty-elim}\ e \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_i 1 \Rightarrow A \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash_i \texttt{Empty-elim}\ e \Rightarrow B \dashv \Gamma_2}{\Gamma, \square\ A, \Delta \vdash_i \texttt{Empty-elim}\ e \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash \texttt{Empty} \to \forall a.\, a \rightsquigarrow C \dashv \Gamma'}{\Gamma \vdash_i \texttt{Empty-elim}\ \Rightarrow C \dashv \Gamma'}\textsc{Dubious}$$

Typing – Product

$$\frac{\Gamma, \Delta, \Box : A \vdash_i e_1 \Rightarrow \_ \dashv \Gamma_1 \quad \Gamma_1, \Box : B \vdash_i e_2 \Rightarrow \_ \dashv \Gamma_2}{\Gamma, \Box : A \otimes B, \Delta \vdash_i (e_1, e_2) \Rightarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\begin{array}{l}\Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \otimes \hat{a}_2], \Delta, \Box : \hat{a}_1 \vdash_i e_1 \Rightarrow \_ \dashv \Gamma_1 \\ \Gamma_1, \Box : \hat{a}_2 \vdash_i e_2 \Rightarrow \_ \dashv \Gamma_2\end{array}}{\Gamma[\hat{a}], \Box : \hat{a}, \Delta \vdash_i (e_1, e_2) \Rightarrow \hat{a} \dashv \Gamma_2}$$

$$\frac{\Gamma, \Delta, \Box \vdash_i e_1 \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1, \Box \vdash_i e_2 \Rightarrow B \dashv \Gamma_2}{\Gamma, \Box, \Delta \vdash_i (e_1, e_2) \Rightarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\begin{array}{l}\Gamma, \Box \vdash_i e_1 \Rightarrow A \otimes B \dashv \Gamma_1 \\ \Gamma_1, 0\, x : A, 0\, y : B \vdash_i e_2 \Rightarrow C \dashv \Gamma_2, r_A\, x : A, r_B\, y : B \\ \Gamma_2 \vdash_i r_A \Rightarrow A \Rightarrow 1 \dashv \Gamma_3 \quad \Gamma_3 \vdash_i r_B \Rightarrow B \Rightarrow 1 \dashv \Gamma_4\end{array}}{\Gamma \vdash_i \texttt{let}\ (x, y) = e_1\ \texttt{in}\ e_2 \Rightarrow C \dashv \Gamma_4}$$

Typing – alternative rules for products

$$\frac{\Gamma \vdash \forall a.\, a \to \forall b.\, b \to a \otimes b \rightsquigarrow C \dashv \Gamma'}{\Gamma \vdash_i \texttt{pair} \Rightarrow C \dashv \Gamma'}\textsc{Dubious}$$

$$\frac{\begin{array}{l} 0\,\Gamma, \square \vdash_i e_1 \Rightarrow A \otimes B \dashv \Gamma_1 \\ (0\,\Gamma_1 + \Gamma), 0\,x : A, 0\,y : B \vdash_i e_2 \Rightarrow C \dashv \Gamma_2, r_A\,x : A, r_B\,y : B \\ \Gamma_2 \vdash_i r_A \Rightarrow A \Rightarrow s_A \dashv \Gamma_3 \\ \Gamma_3 \vdash_i r_B \Rightarrow B \Rightarrow s_B \dashv \Gamma_4 \end{array}}{\Gamma \vdash_i \texttt{let}\ (x, y) = e_1\ \texttt{in}\ e_2 \Rightarrow C \dashv \Gamma_4 + (s_A \sqcap s_B)\,\Gamma_1}$$

Typing – Sum

$$\frac{\Gamma, \Delta, \square : A \vdash_i e \Rightarrow \_ \dashv \Gamma'}{\Gamma, \square : A \oplus B, \Delta \vdash_i \mathtt{inl}\ e \Rightarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma, \Delta, \square : B \vdash_i e \Rightarrow \_ \dashv \Gamma'}{\Gamma, \square : A \oplus B, \Delta \vdash_i \mathtt{inr}\ e \Rightarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\begin{array}{l} 0\,\Gamma, \square \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \\ 0\,\Gamma_1, 0\,x : A \vdash_i e_1 \Rightarrow C_1 \dashv \Gamma_2, r_A\, x : A \\ 0\,\Gamma_1, 0\,y : B \vdash_i e_2 \Rightarrow C_2 \dashv \Gamma_3, r_B\, y : B \\ \Gamma_4 = \Gamma_2 \sqcap \Gamma_3 \quad C_1 = C_2 \\ \Gamma_4 \vdash_i r_A \Rightarrow A \Rightarrow s_A \dashv \Gamma_5 \\ \Gamma_5 \vdash_i r_B \Rightarrow B \Rightarrow s_B \dashv \Gamma_6 \end{array}}{\Gamma \vdash_i \mathtt{case}\ e\ \mathtt{of}\ \{x.e_1;\ y.e_2\} \Rightarrow C_1 \dashv \Gamma_6 + (s_A \sqcap s_B)\,\Gamma_1 + \Gamma}$$

Typing – alternative rules for sums

$$\frac{\Gamma \vdash \forall a.\, a \to \forall b.\, a \oplus b \leadsto C \dashv \Gamma'}{\Gamma \vdash_i \texttt{inl} \ \Rightarrow C \dashv \Gamma'}\textsc{Dubious}$$

$$\frac{\Gamma \vdash \forall b.\, b \to \forall a.\, a \oplus b \leadsto C \dashv \Gamma'}{\Gamma \vdash_i \texttt{inr} \ \Rightarrow C \dashv \Gamma'}\textsc{Dubious}$$

$$\frac{\begin{array}{l} 0\,\Gamma, \square \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \\ 0\,\Gamma_1, 0\,x : A \vdash_i e_1 \Rightarrow C \dashv \Gamma_2, r_A\, x : A \\ 0\,\Gamma_2, 0\,y : B, \square : C \vdash_i e_2 \Rightarrow \_ \dashv \Gamma_3, r_B\, y : B \\ \Gamma_4 = \Gamma_2 \sqcap \Gamma_3 \\ \Gamma_4 \vdash_i r_A \Rightarrow A \Rightarrow s_A \dashv \Gamma_5 \\ \Gamma_5 \vdash_i r_B \Rightarrow B \Rightarrow s_B \dashv \Gamma_6 \end{array}}{\Gamma \vdash_i \texttt{case}\ e\ \texttt{of}\ \{x.e_1;\ y.e_2\} \Rightarrow C \dashv \Gamma_6 + (s_A \sqcap s_B)\,\Gamma_1 + \Gamma}$$

Matching judgement

Matching: $\Gamma \vdash A \rightsquigarrow B \dashv \Gamma'$ – type A matches context Γ, with output type B and output context Γ'

Note: Γ and A are inputs. B and Γ' are outputs.

Matching – inference and checking

$$\frac{}{\Gamma, \Box, \Delta \vdash A \rightsquigarrow A \dashv \Gamma, \Delta}$$

$$\frac{\Gamma, \Delta \vdash 1 \Rightarrow A <: B \Rightarrow r \dashv \Gamma'}{\Gamma, \Box : B, \Delta \vdash A \rightsquigarrow B \dashv r\,\Gamma'}$$

Matching – hints

$$\frac{\Gamma, \square : A \vdash_i e \Rightarrow \_ \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash B \leadsto C \dashv \Gamma_2}{\Gamma, \square\ e, \Delta \vdash A \rightarrow B \leadsto C \dashv \Gamma_2}$$

$$\frac{\Gamma, \square\ e, \Delta \vdash A \leadsto B \dashv \Gamma'}{\Gamma, \square\ e, \Delta \vdash !_r\, A \leadsto B \dashv \Gamma'}$$

$$\frac{\Gamma, \square\ e, \Delta, \hat{a} \vdash A[a := !_r\, \hat{a}] \leadsto B \dashv \Gamma'}{\Gamma, \square\ e, \Delta \vdash \forall a : \mathtt{Type}_r .\, A \leadsto B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i r \Rightarrow B \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash A[a := B] \leadsto C \dashv \Gamma_2}{\Gamma, \square\ B, \Delta \vdash \forall @a : \mathtt{Type}_r .\, A \leadsto C \dashv \Gamma_2}$$

Subtyping judgement

$\Gamma \vdash r \Rightarrow A <: B \Rightarrow s \dashv \Gamma'$ – in context Γ, A is a subtype of $!_{r/s}\, B$, with output context Γ'.

Note: Γ, r, A and B are inputs. s and Γ' are outputs.

Note: If A is a subtype of $!_r\, B$, then s is 1. If it is not, then s indicates how much is missing for the subtyping to be the case.

Subtyping – variables

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: \hat{a} \Rightarrow r \dashv \Gamma[\hat{a}]}$$

$$\frac{}{\Gamma[a : \mathtt{Type}_s] \vdash r \Rightarrow a <: a \Rightarrow r/s \dashv \Gamma[a : \mathtt{Type}_s]}$$

$$\frac{}{\Gamma[\tilde{a} : \mathtt{Type}_s] \vdash r \Rightarrow \tilde{a} <: \tilde{a} \Rightarrow r/s \dashv \Gamma[\tilde{a} : \mathtt{Type}_s]}$$

Subtyping – quantifiers

$$\frac{\Gamma, \tilde{a} : \mathtt{Type}_s \vdash r \Rightarrow A[a := \tilde{a}] <: B[a := \tilde{a}] \Rightarrow t \dashv \Gamma', \tilde{a}, \Theta}{\Gamma \vdash r \Rightarrow \forall a : \mathtt{Type}_s .\, A <: \forall a : \mathtt{Type}_s .\, B \Rightarrow t \dashv \Gamma'}$$

$$\frac{\begin{array}{l} a \in \mathtt{FV}(A) \\ B \neq \forall \_ .\, B' \end{array} \qquad \Gamma, \blacktriangleright_{\hat{a}}, \hat{a} \vdash r \Rightarrow A[a := !_s\, \hat{a}] <: B \Rightarrow t \dashv \Gamma', \blacktriangleright_{\hat{a}}, \Theta}{\Gamma \vdash r \Rightarrow \forall a : \mathtt{Type}_s .\, A <: B \Rightarrow t \dashv \Gamma'}$$

$$\frac{\Gamma, a : \mathtt{Type}_s \vdash r \Rightarrow A <: B \Rightarrow t \dashv \Gamma', a : \mathtt{Type}_s, \Theta}{\Gamma \vdash r \Rightarrow \forall @a : \mathtt{Type}_s .\, A <: \forall @a : \mathtt{Type}_s .\, B \Rightarrow t \dashv \Gamma'}$$

Subtyping – boxes

$$\frac{\Gamma \vdash r \cdot s \Rightarrow A <: B \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow A <: \,!_s\, B \Rightarrow t \dashv \Gamma'}$$

$$\frac{B \neq \,!_\_\, B' \quad \Gamma \vdash r/s \Rightarrow A <: B \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow \,!_s\, A <: B \Rightarrow t \dashv \Gamma'}$$

Subtyping – type formers

$$\frac{}{\Gamma \vdash r \Rightarrow \texttt{Unit} <: \texttt{Unit} \Rightarrow 1 \dashv \Gamma}$$

$$\frac{}{\Gamma \vdash r \Rightarrow \texttt{Empty} <: \texttt{Empty} \Rightarrow 1 \dashv \Gamma}$$

$$\frac{\Gamma \vdash r \Rightarrow A_1 <: A_2 \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B_1 <: \langle \Gamma_1 \rangle B_2 \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \otimes B_1 <: A_2 \otimes B_2 \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash r \Rightarrow A_1 <: A_2 \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B_1 <: \langle \Gamma_1 \rangle B_2 \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \oplus B_1 <: A_2 \oplus B_2 \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash 1 \Rightarrow A_2 <: A_1 \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1 \vdash 1 \Rightarrow \langle \Gamma_1 \rangle B_1 <: \langle \Gamma_1 \rangle B_2 \Rightarrow 1 \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \to B_1 <: A_2 \to B_2 \Rightarrow r \dashv \Gamma_2}$$

Subtyping – switch to instantiation

$$\frac{\hat{a} \notin \mathtt{FV}(A) \quad \Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \Rightarrow s \dashv \Gamma'}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \Rightarrow s \dashv \Gamma'}$$

$$\frac{\hat{a} \notin \mathtt{FV}(A) \quad \Gamma[\hat{a}] \vdash r \Rightarrow A <: \hat{a} \Rightarrow s \dashv \Gamma'}{\Gamma[\hat{a}] \vdash r \Rightarrow A <: \hat{a} \Rightarrow s \dashv \Gamma'}$$

Instantiation judgements

$\Gamma \vdash r \Rightarrow \hat{a} <: A \Rightarrow s \dashv \Gamma'$ – in context Γ, instantiate the existential variable $\hat{a}$ to a subtype of $!_{r/s}\, A$, with output context Γ'.

$\Gamma \vdash r \Rightarrow A <: \hat{a} \Rightarrow s \dashv \Gamma'$ – in context Γ, instantiate the existential variable $\hat{a}$, so that $A <: !_{r/s}\, \hat{a}$, with output context Γ.

Note: in both cases, Γ, r, $\hat{a}$ and A are inputs, whereas s and Γ' are outputs.

Sub-instantiation

$$\frac{\Gamma \vdash_i r \Rightarrow \tau \Rightarrow s \dashv \Gamma_1}{\Gamma, \hat{a}, \Gamma' \vdash r \Rightarrow \hat{a} <: \tau \Rightarrow 1 \dashv \Gamma_1, \hat{a} = \,!_s\, \tau, \Gamma'}$$

$$\frac{}{\Gamma[\hat{a}][\hat{b}] \vdash r \Rightarrow \hat{a} <: \hat{b} \Rightarrow r \dashv \Gamma[\hat{a}][\hat{b} = \hat{a}]}$$

$$\frac{}{\Gamma[b : \mathtt{Type}_s][\hat{a}] \vdash r \Rightarrow \hat{a} <: b \Rightarrow 1 \dashv \Gamma[b : \mathtt{Type}_s][\hat{a} = \,!_{r/s}\, b]}$$

$$\frac{\Gamma[\hat{a}] \vdash r \cdot s \Rightarrow \hat{a} <: A \Rightarrow t \dashv \Gamma'}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: \,!_s\, A \Rightarrow t \dashv \Gamma'}$$

Note: no rule for implicit quantifier.

Sub-instantiation

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: \mathtt{Unit} \Rightarrow 1 \dashv \Gamma[\hat{a} = \mathtt{Unit}]}$$

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: \mathtt{Empty} \Rightarrow 1 \dashv \Gamma[\hat{a} = \mathtt{Empty}]}$$

$$\frac{\begin{array}{l}\Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \otimes \hat{a}_2] \vdash r \Rightarrow \hat{a}_1 <: A \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \hat{a}_2 <: \langle \Gamma_1 \rangle B \Rightarrow s_2 \dashv \Gamma_2\end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \otimes B \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{l}\Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \oplus \hat{a}_2] \vdash r \Rightarrow \hat{a}_1 <: A \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \hat{a}_2 <: \langle \Gamma_1 \rangle B \Rightarrow s_2 \dashv \Gamma_2\end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \oplus B \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{l}\Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \,!_r\,(\hat{a}_1 \to \hat{a}_2)] \vdash 1 \Rightarrow A <: \hat{a}_1 \Rightarrow 1 \dashv \Gamma_1 \\ \Gamma_1 \vdash 1 \Rightarrow \hat{a}_2 <: \langle \Gamma_1 \rangle B \Rightarrow 1 \dashv \Gamma_2\end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \to B \Rightarrow 1 \dashv \Gamma_2}$$

Super-instantiation

$$\frac{\Gamma \vdash_i r \Rightarrow \tau \Rightarrow s \dashv \Gamma_1}{\Gamma, \hat{a}, \Gamma' \vdash r \Rightarrow \tau <: \hat{a} \Rightarrow s \dashv \Gamma_1, \hat{a} = \tau, \Gamma'}$$

$$\frac{}{\Gamma[\hat{a}][\hat{b}] \vdash r \Rightarrow \hat{b} <: \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{a}][\hat{b} = !_r\, \hat{a}]}$$

$$\frac{}{\Gamma[b : \mathtt{Type}_s][\hat{a}] \vdash r \Rightarrow b <: \hat{a} \Rightarrow r/s \dashv \Gamma[b : \mathtt{Type}_s][\hat{a} = b]}$$

$$\frac{b \in \mathtt{FV}(B) \quad \Gamma[\hat{a}], \blacktriangleright_{\hat{b}}, \hat{b} \vdash r \Rightarrow B[b := \hat{b}] <: \hat{a} \Rightarrow s \dashv \Gamma', \blacktriangleright_{\hat{b}}, \Theta}{\Gamma[\hat{a}] \vdash r \Rightarrow \forall b.\, B <: \hat{a} \Rightarrow s \dashv \Gamma'}$$

$$\frac{\Gamma[\hat{a}] \vdash r/s \Rightarrow A <: \hat{a} \Rightarrow t \dashv \Gamma'}{\Gamma[\hat{a}] \vdash r \Rightarrow !_s\, A <: \hat{a} \Rightarrow t \dashv \Gamma'}$$

Super-instantiation

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \texttt{Unit} <: \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{a} = \texttt{Unit}]}$$

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \texttt{Empty} <: \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{a} = \texttt{Empty}]}$$

$$\frac{\begin{array}{l}\Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \otimes \hat{a}_2] \vdash r \Rightarrow A <: \hat{a}_1 \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B <: \hat{a}_2 \Rightarrow s_2 \dashv \Gamma_2\end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow A \otimes B <: \hat{a} \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{l}\Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \oplus \hat{a}_2] \vdash r \Rightarrow A <: \hat{a}_1 \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B <: \hat{a}_2 \Rightarrow s_2 \dashv \Gamma_2\end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow A \oplus B <: \hat{a} \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{l}\Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \to \hat{a}_2] \vdash 1 \Rightarrow \hat{a}_1 <: A \Rightarrow 1 \dashv \Gamma_1 \\ \Gamma_1 \vdash 1 \Rightarrow \langle \Gamma_1 \rangle B <: \hat{a}_2 \Rightarrow 1 \dashv \Gamma_2\end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow A \to B <: \hat{a} \Rightarrow r \dashv \Gamma_2}$$