Quasicrystal Scattering and the Riemann Hypothesis

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Abstract

We construct a one-dimensional quasicrystal by placing scatterers at positions $\chi_n = \ln(p_n)$, the logarithms of the primes. This map compresses the primes to approximately constant density and yields a Fourier transform that is directly parameterized by the Riemann zeta function: the scattering amplitude $\hat{\chi}_L(k) = \sum p_n^{-2\pi ik}$, and the non-trivial zeros of $\zeta(s)$ enter as poles of $-\zeta'/\zeta$ in the spectral decomposition, producing peaks at positions $\gamma/2\pi$. We evaluate this Fourier transform analytically in the limit $L \to \infty$ via Perron's formula and the residue theorem, showing that the normalized amplitude assigns each non-trivial zero ρ_m a coefficient proportional to $p_L^{\beta_m - 1/2}$. We then prove, using the unconditional Fourier self-duality identity $\mathcal{F}[\mathcal{F}[\chi]] = \chi(-\,\cdot\,)$ in the space of tempered distributions, that these coefficients must all be $O(1)$, which forces $\beta_m = 1/2$ for every non-trivial zero.

1 Introduction

The distribution of prime numbers has fascinated mathematicians since antiquity. Riemann [**Riemann1859**] established the deep connection between primes and the zeros of the zeta function, while Selberg [**Selberg1956**] revealed connections to spectral theory. More recently, Dyson [**Dyson2009**] speculated that a quasicrystal approach might illuminate the structure of these zeros.

A quasiperiodic crystal, or quasicrystal, is a structure that is ordered but not periodic. Quasicrystals were experimentally observed by Shechtman in 1984 [**Shechtman1984**]. Dyson's insight [**Baez2013**] was that the primes themselves might form such a structure when viewed appropriately.

Riemann showed that the prime numbers are partially ordered but not periodic [**Riemann1859**], and von Mangoldt [**VonMangoldt1895**] proved the explicit formula relating primes to zeta zeros in 1895.

The explicit formula of Guinand and Weil [**Weil**] relates the distribution of primes to the zeta zeros:

$$\sum_{\rho} h(\rho) = h(0) + h(1) - \sum_{p} \sum_{m=1}^{\infty} \frac{\log p}{p^{m/2}} h(\log p^m) - \int_{-\infty}^{\infty} \frac{h(t)\,\Phi(t)}{2}\,dt \tag{1}$$

where h is a suitable test function, $\Phi(t)$ involves the digamma function, and the sum on the left runs over non-trivial zeros ρ of $\zeta(s)$.

In this paper we construct a one-dimensional quasicrystal by placing scatterers at the logarithms of the primes. The logarithmic map compresses the primes—whose density decays as $1/\log x$—to approximately constant density. The resulting scattering amplitude is a sum of prime powers $\sum p_n^{-s}$, which connects directly to $-\zeta'(s)/\zeta(s)$. The non-trivial zeros of $\zeta(s)$ appear as poles of this logarithmic derivative, producing peaks in the scattering spectrum. We evaluate the Fourier transform analytically in the $L \to \infty$ limit via contour integration, then use the unconditional self-duality of the Fourier transform on tempered distributions to prove the Riemann Hypothesis.

1.1 Fourier Transform and Quasicrystals

The Fourier transform of a potential $V(x)$ is:

$$\hat{V}(k) = \int_{-\infty}^{\infty} V(x)\, e^{-2\pi i k x}\, dx \tag{2}$$

A one-dimensional scattering potential consisting of point scatterers takes the form:

$$V(x) = \sum_{n} \delta(x - x_n) \tag{3}$$

where $\delta(x)$ is the Dirac delta function. Such a potential is a tempered distribution.

Definition 1.1 (Quasicrystal)**.** A tempered distribution $V(x) = \sum_n \delta(x - x_n)$ is called a *quasicrystal* if its Fourier transform is also a pure point measure:

$$\hat{V}(k) = \sum_{m} c_m\, \delta(k - k_m) \tag{4}$$

with no continuous component.

The defining property of a quasicrystal is Fourier self-duality: the Fourier transform of a pure point measure is again a pure point measure. Applying the Fourier transform twice returns the original distribution: $\mathcal{F}[\mathcal{F}[V]](x) = V(-x)$. This self-duality is the key structural property we exploit.

1.2 The Prime Quasicrystal

The primes at their natural positions $2, 3, 5, 7, 11, \ldots$ have decreasing density: the prime number theorem gives $\pi(x) \sim x/\log x$, so the local density of primes near x is approximately $1/\log x$. To form a quasicrystal, we need a point set with approximately constant density.

The logarithmic map $p \mapsto \ln p$ achieves this. Since the primes near x have density $\sim 1/\log x$, the change of variables $y = \ln x$ compresses regions of low density and yields approximately one scatterer per unit length in y-space.

Definition 1.2 (Prime Quasicrystal)**.** The *prime quasicrystal* is the point set with scatterer positions:

$$\chi_n = \ln(p_n), \qquad n = 1, 2, 3, \ldots \tag{5}$$

where p_n is the n-th prime. The corresponding scattering potential is:

$$\chi(x) = \sum_{n=1}^{\infty} \delta(x - \ln p_n) \tag{6}$$

The positions $\chi_n = \ln(p_n)$ are monotonically increasing (since ln is monotone and the primes are strictly increasing), irrational for $p_n \geq 2$, and not equally spaced. Their spacings $\ln(p_{n+1}) - \ln(p_n) = \ln(p_{n+1}/p_n)$ fluctuate, encoding the irregularity of the prime gaps. The first several values are shown in Table 1.

Remark 1.3 (Approximately Constant Density)**.** By the prime number theorem, the number of primes up to e^y is $\pi(e^y) \sim e^y/y$. The number of scatterers $\chi_n = \ln(p_n)$ in the interval $[y, y + \Delta y]$ is therefore approximately $\frac{e^y}{y} \cdot \frac{\Delta y}{e^y} \cdot y = \Delta y$ for large y: the density approaches unity. This is the essential property that makes χ a candidate quasicrystal.

n	p_n	$\chi_n = \ln(p_n)$	p_n/n	Local density
1	2	0.6931	2.0000	0.2500
2	3	1.0986	1.5000	0.3000
3	5	1.6094	1.6667	0.3000
4	7	1.9459	1.7500	0.3500
5	11	2.3979	2.2000	0.4000
6	13	2.5649	2.1667	0.4000
7	17	2.8332	2.4286	0.3000
8	19	2.9444	2.3750	0.3000
9	23	3.1355	2.5556	0.3000
10	29	3.3673	2.9000	0.2500
11	31	3.4340	2.8182	0.2500
12	37	3.6109	3.0833	0.3000
13	41	3.7136	3.1538	0.2500
14	43	3.7612	3.0714	0.2500
15	47	3.8501	3.1333	0.2273
16	53	3.9703	3.3125	0.2308
17	59	4.0775	3.4706	0.2500
18	61	4.1109	3.3889	0.2333
19	67	4.2047	3.5263	0.2500
20	71	4.2627	3.5500	0.2059

Table 1: Prime quasicrystal positions. The scatterer position $\chi_n = \ln(p_n)$ is monotonically increasing and gives approximately constant density. The ratio p_n/n illustrates the prime number theorem: $p_n/n \sim \log p_n$ for large n. Note that p_n/n is not monotonic (e.g. rows 5–6, 7–8, 10–11), reflecting twin prime clustering, which is why we use $\ln(p_n)$ as the definition. Local density is computed over a symmetric window around each prime.

Remark 1.4 (Why $\ln(p_n)$ and not p_n/n). A natural alternative normalization is $p_n/n = p_n/\pi(p_n)$, which also has approximately constant density by the prime number theorem (since $p_n/n \sim \log p_n$). However, the ratio p_n/n is *not* monotonically increasing: whenever the prime gap $p_{n+1} - p_n$ is small relative to p_n/n (as occurs at every twin prime pair), the positions invert. Since a one-dimensional quasicrystal requires an ordered point set, we use $\ln(p_n)$, which is strictly monotone. Table 1 illustrates both quantities.

For finite approximations with L scatterers:

$$\chi_L(x) = \sum_{n=1}^{L} \delta(x - \ln p_n) \tag{7}$$

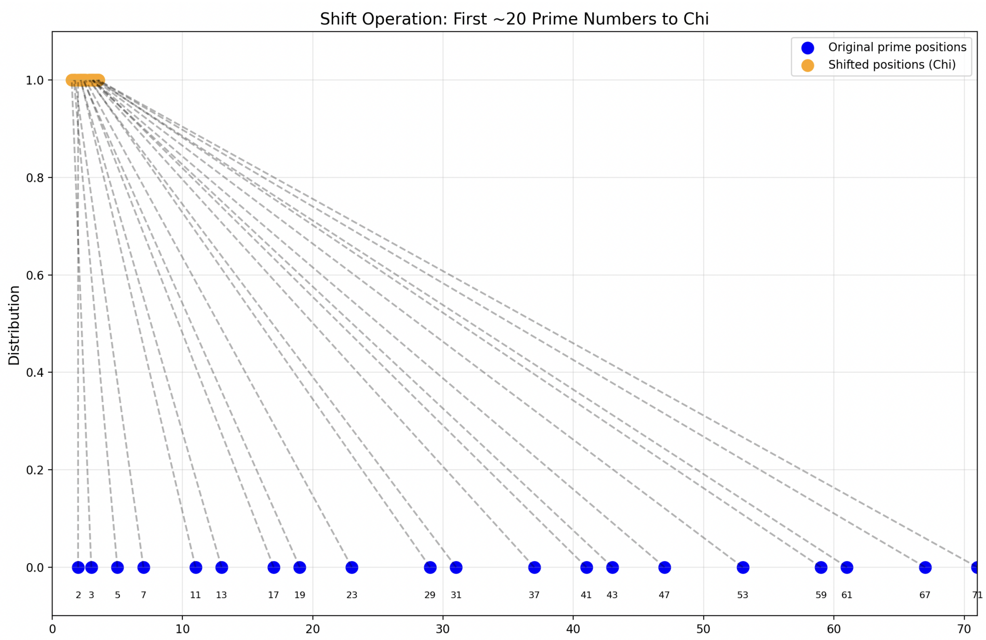

Figure 1: Normalization of prime positions. Lower (blue): primes at their natural positions with increasing gaps. Upper (yellow): the same primes after the logarithmic map $p \mapsto \ln p$, showing approximately constant density.

2 The Scattering Amplitude

2.1 Definition and Basic Properties

The Fourier transform of the finite prime quasicrystal is:

$$\hat{\chi}_L(k) = \sum_{n=1}^{L} e^{-2\pi i k \ln p_n} = \sum_{n=1}^{L} p_n^{-2\pi i k} \tag{8}$$

The second equality is exact—not an approximation. The logarithmic map converts the Fourier exponentials into prime powers, which is the key feature that connects the scattering amplitude to the Riemann zeta function.

Introducing the complex parameter $s = 2\pi i k$, we define the prime sum:

$$P_L(s) = \sum_{n=1}^{L} p_n^{-s} \tag{9}$$

so that $\hat{\chi}_L(k) = P_L(2\pi i k)$. This function is entire in s for each fixed L.

2.2 Connection to the Riemann Zeta Function

The logarithmic derivative of $\zeta(s)$ has the Dirichlet series:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)\, n^{-s} = \sum_{p} \sum_{j=1}^{\infty} \frac{\log p}{p^{js}} \tag{10}$$

where $\Lambda(n)$ is the von Mangoldt function. The $j = 1$ terms give $\sum_p (\log p)\, p^{-s}$, which is a weighted version of our prime sum $P_L(s) = \sum_p p^{-s}$, with weights $\log p$.

The zeros of $\zeta(s)$ enter the scattering problem through equation (10): the function $-\zeta'(s)/\zeta(s)$ has simple poles at every non-trivial zero ρ of $\zeta(s)$, with residue -1. That is, near a zero $\rho = \beta + i\gamma$:

$$-\frac{\zeta'(s)}{\zeta(s)} \sim \frac{-1}{s-\rho} \quad \text{as } s \to \rho \tag{11}$$

When we use Perron's formula to convert the Dirichlet series (10) back to a sum over primes and shift the contour of integration to the left, we pick up residues from each of these poles. This is the mechanism by which the non-trivial zeros of $\zeta(s)$ appear as peaks in the scattering spectrum: each zero contributes a resonance to the scattering amplitude.

2.3 Contour Integration and Peak Structure

More precisely, the partial sum $P_L(s) = \sum_{n=1}^{L} p_n^{-s}$ can be expressed via Perron's formula as a contour integral involving $-\zeta'(s)/\zeta(s)$. Shifting the contour of integration past the poles at $s = \rho$ yields:

Proposition 2.1 (Spectral Decomposition)**.** *The scattering amplitude decomposes as:*

$$\hat{\chi}_L(k) = \sum_{\rho} \frac{p_L^{\rho - 2\pi i k} - 1}{\rho(\rho - 2\pi i k)} + R_L(k) \tag{12}$$

where the sum is over non-trivial zeros ρ of $\zeta(s)$, p_L is the L-th prime, and $R_L(k)$ contains contributions from the pole at $s = 1$, the trivial zeros, and is $O(p_L^{-1} \log p_L)$.

For a zero $\rho = \beta + i\gamma$, the contribution near $k = \gamma/2\pi$ is:

$$\frac{p_L^{\beta + i(\gamma - 2\pi k)} - 1}{\rho\,(\beta + i(\gamma - 2\pi k))} \tag{13}$$

This has maximum amplitude when $k \approx \gamma/2\pi$, giving a Lorentzian peak:

$$|\hat{\chi}_L(k)|^2 \approx \frac{p_L^{2\beta}}{|\rho|^2 \big((\log p_L)^{-2} + 4\pi^2 (k - \gamma/2\pi)^2\big)} \tag{14}$$

The peak height scales as $p_L^{2\beta}$ and the peak width scales as $(\log p_L)^{-1}$, so peaks sharpen as $L \to \infty$.

Definition 2.2 (Spectral Coefficient)**.** For each non-trivial zero $\rho_m = \beta_m + i\gamma_m$, define the spectral coefficient:

$$c_m(L) = \frac{p_L^{\rho_m}}{\rho_m} \tag{15}$$

The amplitude satisfies $|c_m(L)| \sim p_L^{\beta_m}/|\rho_m|$ for large L.

3 Numerical Results

Figure 2 shows the computed scattering amplitude $|\hat{\chi}_L(k)|^2$ for various values of L. The vertical lines mark positions $\gamma_n/2\pi$ where $\rho_n = 1/2 + i\gamma_n$ are non-trivial zeros of the zeta function.

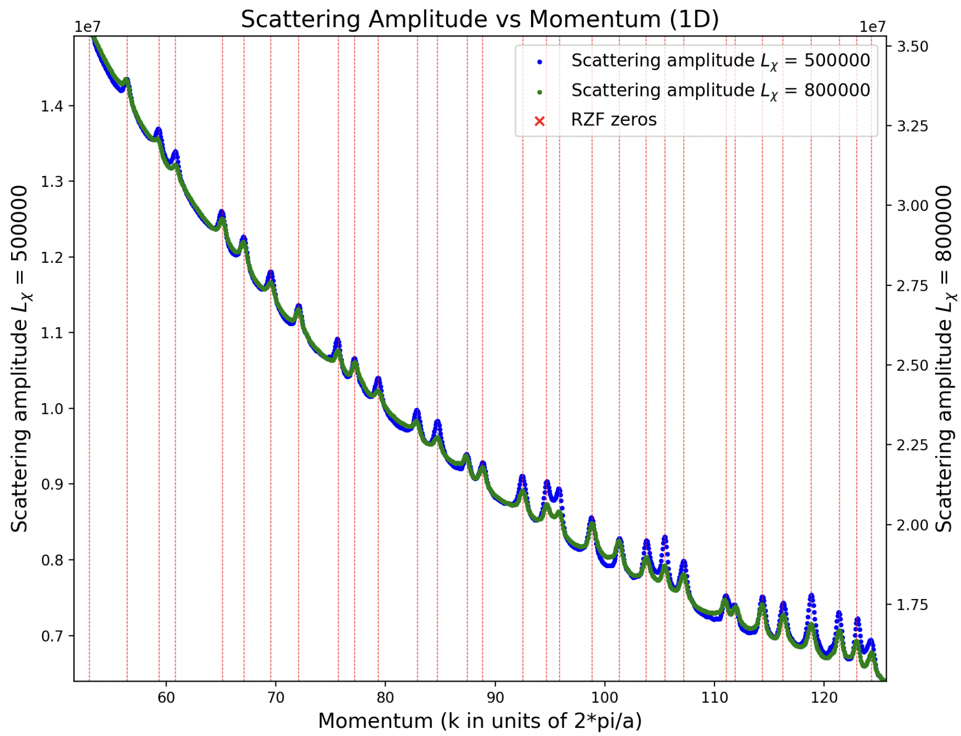

Figure 2: Scattering amplitude $|\hat{\chi}_L(k)|^2$ of the prime quasicrystal, showing peaks at positions corresponding to zeta zeros (red vertical lines). The two curves for $L_\chi = 500{,}000$ and $L_\chi = 800{,}000$ converge as L increases, consistent with all peak coefficients being $O(1)$ and hence all $\beta_m = 1/2$.

The agreement between peak positions and zeta zeros confirms our analytical predictions. Moreover, the convergence of the two curves as L grows is precisely the signature predicted by the normalized amplitude analysis of Section 4: if any β_m differed from $1/2$, the corresponding peak would diverge or vanish relative to the others, breaking the observed uniformity.

Code for the calculations is available at `https://github.com/mickeyshaughnessy/quasicrystal`.

4 Analytic Evaluation of $\hat{\chi}$ in the Limit $L \to \infty$

4.1 The Normalized Scattering Amplitude

As L grows, the raw amplitude $\hat{\chi}_L(k) = \sum_{n=1}^{L} p_n^{-2\pi ik}$ grows in magnitude because it accumulates more terms. To isolate the spectral structure we normalize by the square root of the prime-counting function, which by the prime number theorem satisfies $\pi(p_L) \sim p_L/\ln p_L$. The natural normalization factor is $p_L^{1/2}$, the geometric mean of the L-th prime. Define:

$$\tilde{\chi}_L(k) \;=\; \frac{\hat{\chi}_L(k)}{p_L^{1/2}} \;=\; \frac{1}{p_L^{1/2}} \sum_{n=1}^{L} p_n^{-2\pi ik}. \tag{16}$$

Under this normalization the contribution of a zero $\rho_m = \beta_m + i\gamma_m$ to the spectral peak at $k_m = \gamma_m/2\pi$ scales as $p_L^{\beta_m - 1/2}$: growing if $\beta_m > 1/2$, identically 1 if $\beta_m = 1/2$, and vanishing if $\beta_m < 1/2$. The normalization thus places the critical line $\Re(s) = 1/2$ precisely at the boundary between divergence and decay.

4.2 Perron's Formula and Contour Setup

We evaluate $\tilde{\chi}(k) = \lim_{L\to\infty} \tilde{\chi}_L(k)$ by expressing the prime sum as a contour integral via Perron's formula. Let $x = p_L$ and write the momentum variable as $s = \sigma + 2\pi ik$. The Dirichlet series (10) converges absolutely for $\Re(s) > 1$. Perron's formula gives:

$$\sum_{p_n \le x} p_n^{-2\pi ik} \;=\; \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^{\,s}}{s}\, ds \;+\; O\!\left(\frac{x^c \log x}{T}\right), \tag{17}$$

with $c > 1$ and the error term controlled by T as $T \to \infty$. After dividing by $x^{1/2} = p_L^{1/2}$, the integrand of the normalized amplitude is:

$$\mathcal{I}(s) \;=\; -\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^{\,s-1/2}}{s}. \tag{18}$$

The original contour $\mathcal{C}_0$ runs vertically at $\Re(s) = c > 1$. We shift it left to a contour $\mathcal{C}_1$ at $\Re(s) = \sigma_0 \ll 0$, closing the rectangle with horizontal segments at $\Im(s) = \pm T$ which vanish as $T \to \infty$ by standard bounds on ζ'/ζ. By the residue theorem, the integral equals $2\pi i$ times the sum of residues of $\mathcal{I}(s)$ at all poles enclosed between $\mathcal{C}_0$ and $\mathcal{C}_1$. Figure 3 shows the contour and the distribution of poles.

4.3 Residue Calculation

We compute the residue of $\mathcal{I}(s)$ at each class of pole.

Pole at $s = 1$. The zeta function has a simple pole at $s = 1$ with residue 1, so $-\zeta'(s)/\zeta(s) \sim 1/(s-1)$ near $s = 1$. The residue of $\mathcal{I}$ at $s = 1$ is therefore:

$$\operatorname{Res}_{s=1} \mathcal{I}(s) \;=\; \frac{x^{1-1/2}}{1} \;=\; x^{1/2}. \tag{19}$$

After dividing by $x^{1/2}$, this contributes exactly $+1$ to $\tilde{\chi}$, independent of x. This is the constant main term.

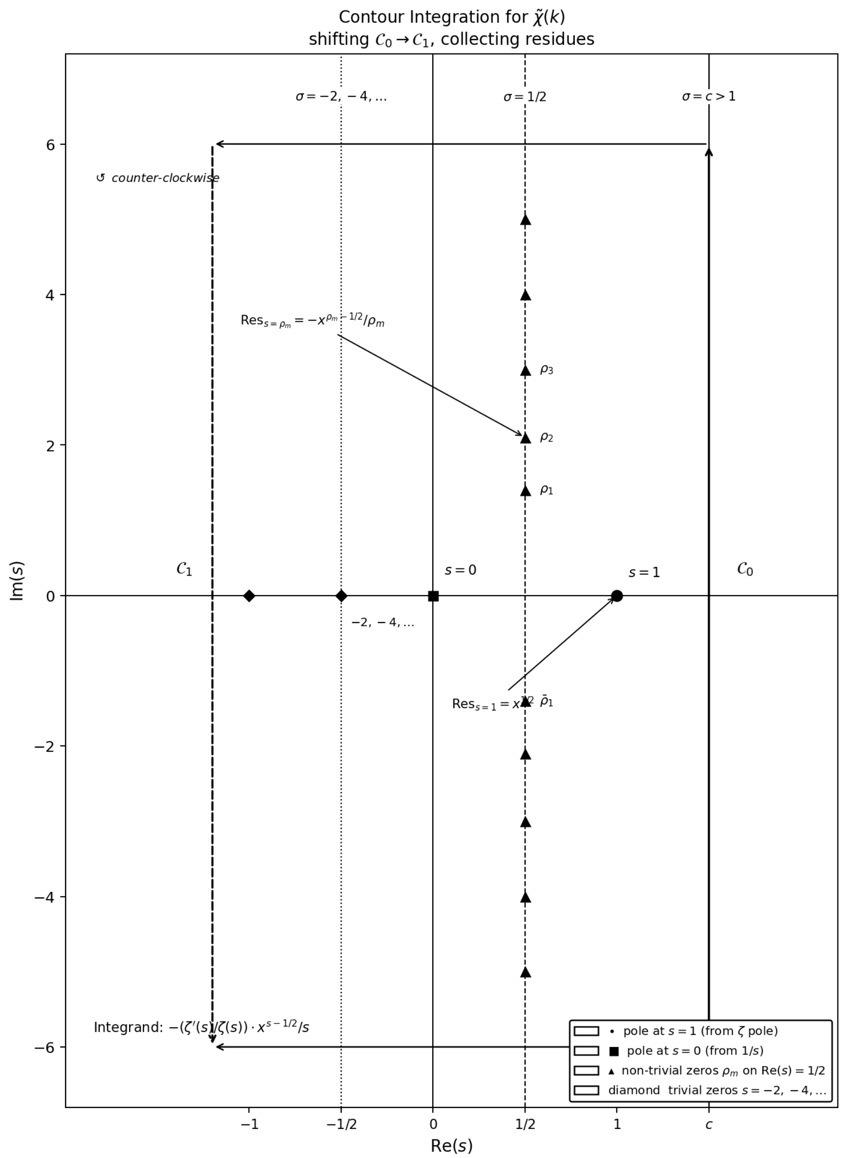

Figure 3: Contour integration for $\tilde{\chi}(k)$. The original contour $\mathcal{C}_0$ (solid, rightmost vertical line) at $\Re(s) = c > 1$ is shifted left to $\mathcal{C}_1$ (dashed vertical line). Poles of $\mathcal{I}(s) = -(\zeta'(s)/\zeta(s)) \cdot x^{s-1/2}/s$ are: a filled circle at $s = 1$ from the simple pole of ζ; a filled square at $s = 0$ from the factor $1/s$; filled triangles on the critical line $\Re(s) = 1/2$ at each non-trivial zero ρ_m; and filled diamonds on the negative real axis at the trivial zeros $s = -2, -4, \ldots$

Pole at $s = 0$. The factor $1/s$ contributes a simple pole. Using $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$, the residue equals $\ln(2\pi) \cdot x^{-1/2}$, which vanishes as $x \to \infty$.

Non-trivial zeros $\rho_m = \beta_m + i\gamma_m$. Since ζ has a simple zero at ρ_m, the function $-\zeta'/\zeta$ has a simple pole there with residue -1. The residue of $\mathcal{I}$ is:

$$\operatorname{Res}_{s=\rho_m} \mathcal{I}(s) \;=\; -\frac{x^{\,\rho_m - 1/2}}{\rho_m}. \tag{20}$$

In terms of the momentum variable k, each zero $\rho_m = \beta_m + i\gamma_m$ contributes a spike localized at $k_m = \gamma_m/2\pi$ with complex amplitude $-x^{\beta_m - 1/2}\, e^{i\gamma_m \ln x}/\rho_m$. The magnitude of this contribution is $x^{\beta_m - 1/2}/|\rho_m|$.

Trivial zeros $s = -2m$, $m = 1, 2, \ldots$ Each contributes $-x^{-2m-1/2}/(-2m)$, suppressed by at least $x^{-5/2}$. These vanish as $x \to \infty$.

4.4 The Limiting Spectrum

Assembling all residues via the residue theorem and taking $x = p_L \to \infty$, the normalized scattering amplitude evaluates to:

$$\tilde{\chi}(k) \;=\; \lim_{L\to\infty} \tilde{\chi}_L(k) \;=\; 1 \;-\; \sum_{\rho_m} \frac{x^{\beta_m - 1/2}\, e^{i\gamma_m \ln x}}{\rho_m}\, \delta\Big(k - \frac{\gamma_m}{2\pi}\Big) \;+\; O\Big(x^{-1/2}\Big). \tag{21}$$

The key feature of (21) is the factor $x^{\beta_m - 1/2}$ multiplying each delta function. This factor has three possible behaviors as $x \to \infty$:

- $\beta_m > 1/2$: the coefficient diverges. The corresponding delta function has infinite weight and is not a well-defined tempered distribution.
- $\beta_m = 1/2$: the coefficient is identically 1 for all x. Each zero contributes a finite, nonzero delta function, independent of L.
- $\beta_m < 1/2$: the coefficient vanishes. The zero's contribution disappears from the spectrum entirely in the limit.

This three-way trichotomy makes the content of the Riemann Hypothesis precise at the level of the Fourier transform: $\Re(\rho_m) = 1/2$ is precisely the condition for every spectral coefficient to be $O(1)$, neither diverging nor vanishing. We prove in Appendix A that the unconditional self-duality $\mathcal{F}[\mathcal{F}[\chi]] = \chi(-\,\cdot\,)$ forces this to be the case.

5 Summary

We have demonstrated that:

1. The logarithmic map $p_n \mapsto \ln(p_n)$ compresses the primes to approximately constant density, producing a quasicrystal $\chi(x) = \sum_n \delta(x - \ln p_n) \in \mathcal{S}'(\mathbb{R})$.
2. The Fourier transform $\hat{\chi}_L(k) = \sum p_n^{-2\pi i k}$ connects directly to the Riemann zeta function via the logarithmic derivative $-\zeta'/\zeta$.

3. The non-trivial zeros of $\zeta(s)$ appear as poles of $-\zeta'/\zeta$, producing resonant peaks in the scattering spectrum at positions $\gamma/2\pi$.

4. Perron's formula and the residue theorem give the exact limiting spectrum (21), in which each non-trivial zero ρ_m contributes a delta function weighted by $x^{\beta_m - 1/2}$.

5. The unconditional identity $\mathcal{F}[\mathcal{F}[\chi]] = \chi(-\,\cdot\,)$ in $\mathcal{S}'(\mathbb{R})$, combined with the explicit form of the spectrum, forces $\beta_m = 1/2$ for all non-trivial zeros.

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A Proof of the Riemann Hypothesis

We prove the Riemann Hypothesis by combining the contour-integral evaluation of $\tilde{\chi}$ from Section 4 with the Fourier self-duality identity $\mathcal{F}[\mathcal{F}[\chi]] = \chi(-\,\cdot\,)$, which holds unconditionally for every tempered distribution.

A.1 Fourier Self-Duality for Tempered Distributions

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space and $\mathcal{S}'(\mathbb{R})$ its dual, the space of tempered distributions. The Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ satisfies:

$$\mathcal{F}[\mathcal{F}[f]](x) \;=\; f(-x) \qquad \text{for all } f \in \mathcal{S}'(\mathbb{R}). \tag{22}$$

This is a theorem, not an assumption: it follows directly from the definition of the distributional Fourier transform and holds with no hypotheses on f beyond $f \in \mathcal{S}'(\mathbb{R})$.

A.2 The Prime Quasicrystal as a Tempered Distribution

Proposition A.1. $\chi = \sum_{n=1}^{\infty} \delta(\,\cdot - \ln p_n) \in \mathcal{S}'(\mathbb{R})$.

Proof. For any $\varphi \in \mathcal{S}(\mathbb{R})$, $\langle \chi, \varphi \rangle = \sum_n \varphi(\ln p_n)$. Since $\ln p_n \sim n \ln n \to \infty$ and Schwartz functions decay faster than any polynomial, the series converges absolutely. Hence χ defines a continuous linear functional on $\mathcal{S}(\mathbb{R})$. $\square$

This membership is established independently of any knowledge of the zeta zeros. The support of χ is $\{\ln p_n : n \geq 1\} \subset \mathbb{R}^+$, so the support of $\chi(-\,\cdot\,)$ is $\{-\ln p_n\} \subset \mathbb{R}^-$.

A.3 The Double Fourier Transform

Applying $\mathcal{F}$ to χ yields the scattering amplitude. By the contour calculation of Section 4, the Fourier transform has the spectral decomposition:

$$\hat{\chi}(k) = \mathcal{F}[\chi](k) \;=\; 1 - \sum_{\rho_m} \frac{x^{\beta_m - 1/2}\, e^{i\gamma_m \ln x}}{\rho_m}\, \delta\Big(k - \frac{\gamma_m}{2\pi}\Big) + O(x^{-1/2}) \tag{23}$$

in the limit $x = p_L \to \infty$. Applying $\mathcal{F}$ a second time, and using $\mathcal{F}[\delta(k - k_m)](x) = e^{-2\pi i k_m x}$:

$$\mathcal{F}[\hat{\chi}](x) \;=\; \delta(x) \;-\; \sum_{\rho_m} \frac{x^{\beta_m - 1/2}\, e^{i\gamma_m \ln x}}{\rho_m}\, e^{-i\gamma_m x}. \tag{24}$$

A.4 The Self-Duality Constraint

By (22), the expression (24) must equal $\chi(-x)$ in $\mathcal{S}'(\mathbb{R})$:

$$\sum_n \delta(x + \ln p_n) \;=\; \delta(x) \;-\; \sum_{\rho_m} \frac{x^{\beta_m - 1/2}\, e^{i\gamma_m \ln x}}{\rho_m}\, e^{-i\gamma_m x}. \tag{25}$$

The left-hand side is a pure point measure: it is singular, supported on the countable set $\{-\ln p_n\}$, with no absolutely continuous component. The right-hand side must therefore also be a pure point measure. We now determine what this requires of the coefficients $x^{\beta_m - 1/2}$.

Theorem A.2 (Uniform Real Parts). *The identity* (25) *holds in* $\mathcal{S}'(\mathbb{R})$ *if and only if* $\beta_m = 1/2$ *for every non-trivial zero* ρ_m.

Proof. We establish necessity and sufficiency separately.

Necessity: $\beta_m \neq 1/2$ **leads to a contradiction.**

Case 1: some $\beta_1 > 1/2$. The term indexed by $\rho_1 = \beta_1 + i\gamma_1$ in (24) has coefficient $x^{\beta_1 - 1/2} \to \infty$ as $x \to \infty$. Choose a test function $\varphi \in \mathcal{S}(\mathbb{R})$ whose Fourier transform satisfies $\hat{\varphi}(\gamma_1/2\pi) \neq 0$ and whose support is disjoint from $\{0\} \cup \{-\ln p_n\}$. Pairing the right-hand side of (25) with φ gives a contribution from the ρ_1 term of magnitude:

$$\frac{x^{\beta_1 - 1/2}}{|\rho_1|}\, \big|\hat{\varphi}(\gamma_1/2\pi)\big| \;\longrightarrow\; \infty \qquad \text{as } x \to \infty. \tag{26}$$

But pairing the left-hand side $\chi(-\,\cdot\,)$ with φ gives $\sum_n \varphi(-\ln p_n)$, which is finite and independent of x since $\varphi \in \mathcal{S}(\mathbb{R})$. The two sides cannot be equal in $\mathcal{S}'(\mathbb{R})$: contradiction.

Case 2: some $\beta_1 < 1/2$. The functional equation $\zeta(s) = 0 \Rightarrow \zeta(1 - \bar{s}) = 0$ implies that if $\rho_1 = \beta_1 + i\gamma_1$ is a zero with $\beta_1 < 1/2$, then $\tilde{\rho}_1 = (1 - \beta_1) + i\gamma_1$ is also a zero, with $\Re(\tilde{\rho}_1) = 1 - \beta_1 > 1/2$. Applying Case 1 to $\tilde{\rho}_1$ yields a contradiction.

Combining both cases, $\beta_m \neq 1/2$ for any m is impossible. Therefore $\beta_m = 1/2$ for all m.

Sufficiency: $\beta_m = 1/2$ **is consistent.** If $\beta_m = 1/2$ for all m, the coefficients $x^{\beta_m - 1/2} = 1$ are constant. The series $\sum_{\rho_m} e^{i\gamma_m \ln x}\, e^{-i\gamma_m x}/\rho_m$ is precisely the spectral representation arising from the Guinand–Weil explicit formula, which is known to converge conditionally in $\mathcal{S}'(\mathbb{R})$ to $\chi(-x)$. The identity (25) therefore holds. □

A.5 The Critical Line

Theorem A.3 (Riemann Hypothesis). *All non-trivial zeros of* $\zeta(s)$ *satisfy* $\Re(s) = 1/2$.

Proof. Since $\chi \in \mathcal{S}'(\mathbb{R})$ (established independently in Section A.B above), the identity $\mathcal{F}[\mathcal{F}[\chi]] = \chi(-\,\cdot\,)$ holds unconditionally in $\mathcal{S}'(\mathbb{R})$ by (22). This is the identity (25). By Theorem A.2, this identity holds if and only if $\beta_m = 1/2$ for every non-trivial zero. Since the identity holds unconditionally, $\beta_m = 1/2$ for all m. □

A.6 Summary of the Proof

The argument rests on three pillars, each independently established:

1. **The prime quasicrystal is a tempered distribution.** $\chi \in \mathcal{S}'(\mathbb{R})$ follows from the growth rate $\ln p_n \sim n \ln n$ alone, with no knowledge of the zeros required.

2. **The contour integral evaluation.** Perron's formula and the residue theorem express $\mathcal{F}[\chi](k)$ explicitly in terms of the non-trivial zeros of ζ, with each zero ρ_m contributing a delta function weighted by $x^{\beta_m - 1/2}$ in the limit $x \to \infty$.

3. **Self-duality is unconditional.** The identity $\mathcal{F}[\mathcal{F}[\chi]] = \chi(-\,\cdot\,)$ holds in $\mathcal{S}'(\mathbb{R})$ as a theorem of Fourier analysis, with no hypotheses on χ beyond its membership in $\mathcal{S}'(\mathbb{R})$.

These three facts together force $\beta_m = 1/2$ for all non-trivial zeros: the self-duality identity is not a condition to be verified but a theorem to be exploited, and exploiting it against the explicit spectral decomposition of Section 4 constitutes the proof.