A Short and Simple Proof of the Riemann's Hypothesis and its generalized version

Charaf ECH-CHATBI *

Tuesday 26 January 2021

Abstract

We present a short proof of the Riemann's Hypothesis (RH) and the Generalized Riemann's Hypothesis (GRH) where only undergraduate mathematics is needed.

Keywords: (Generalized) Riemann Hypothesis; Zeta function; Dirichlet L-functions; Prime Numbers; Millennium Problems.

1 The Riemann Hypothesis

1.1 The importance of the Riemann Hypothesis

The prime number theorem gives us the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper[1], it asserts that all the 'non-trivial' zeros of the zeta function are complex numbers with real part 1/2.

1.2 Riemann Zeta Function

For a complex number s where $\Re(s) > 1$, the Zeta function is defined as the sum of the following series:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \tag{1}$$

In his 1859 paper[1], Riemann went further and extended the zeta function $\zeta(s)$, by analytical continuation, to an absolutely convergent function in the half plane $\Re(s) > 0$, minus a simple pole at s = 1:

$$\zeta(s) = \frac{s}{s-1} - s\int_{1}^{+\infty} \frac{\{x\}}{x^{s+1}} dx \tag{2}$$

*One Raffles Quay, North Tower Level 35. 048583 Singapore. Email: charaf.chatbi@gmail.com. The opinions of this article are those of the author and do not reflect in any way the views or business of his employer.

Where $\{x\} = x - [x]$ and $[x]$ is the integer part of x. Riemann also obtained the analytic continuation of the zeta function to the whole complex plane.

Riemann[1] has shown that Zeta has a functional equation[1]

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{4}$$

Where $\Gamma(s)$ is the Gamma function. Using the above functional equation, Riemann has shown that the non-trivial zeros of ζ are located symmetrically with respect to the line $\Re(s) = 1/2$, inside the critical strip $0 < \Re(s) < 1$. Riemann has conjectured that all the non trivial-zeros are located on the critical line $\Re(s) = 1/2$. In 1921, Hardy & Littlewood[3] showed that there are infinitely many zeros on the critical line. In 1896, Hadamard[3] and De la Vallée Poussin[3] independently proved that $\zeta(s)$ has no zeros of the form $s = 1 + it$ for $t \in \mathbb{R}$. Some of the known results[3] of $\zeta(s)$ are as follows:

- $\zeta(s)$ has no zero for $\Re(s) > 1$.
- $\zeta(s)$ has no zero of the form $s = 1 + i\tau$. i.e. $\zeta(1 + i\tau) \neq 0$, $\forall\, \tau$.
- $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.
- $\zeta(s)$ has all the trivial zeros at the negative even integers $s = -2k$, $k \in \mathbb{N}^*$.
- The non-trivial zeros are inside the critical strip: i.e. $0 < \Re(s) < 1$.
- If $\zeta(s) = 0$, then $1 - s$, $\bar{s}$ and $1 - \bar{s}$ are also zeros of ζ: i.e. $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$.

Therefore, to prove the "Riemann Hypothesis" (RH), it is sufficient to prove that ζ has no zero on the right hand side $1/2 < \Re(s) < 1$ of the critical strip.

1.3 Proof of the Riemann Hypothesis

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $\zeta(s) = 0$.

We have from the Riemann's integral above:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{+\infty} \frac{\{x\}}{x^{s+1}}dx \tag{5}$$

We have $s \neq 1$, $s \neq 0$ and $\zeta(s) = 0$, therefore:

$$\frac{1}{s-1} = \int_1^{+\infty} \frac{\{x\}}{x^{s+1}}dx \tag{6}$$

[1] This is slightly different from the functional equation presented in Riemann's paper[1]. This is a variation that is found everywhere in the litterature[2,3,4]. Another variant using the cos:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s) \tag{3}$$

Therefore:

$$\frac{1}{\sigma + i\tau - 1} = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma+i\tau+1}}dx \tag{7}$$

And

$$\frac{\sigma - 1 - i\tau}{(\sigma-1)^2+\tau^2} = \int_1^{+\infty} \frac{\Big(\cos(\tau \ln(x)) - i\sin(\tau\ln(x))\Big)\{x\}}{x^{\sigma+1}}dx \tag{8}$$

The integral is absolutely convergent. We take the real part and the imaginary part of both sides of the above equation and define the functions F and G as following:

$$\begin{aligned} F(\sigma,\tau) &= \frac{\sigma-1}{(\sigma-1)^2+\tau^2} && (9)\\ &= \int_1^{+\infty} \frac{\Big(\cos(\tau\ln(x))\Big)\{x\}}{x^{\sigma+1}}dx && (10) \end{aligned}$$

And

$$\begin{aligned} G(\sigma,\tau) &= \frac{\tau}{(\sigma-1)^2+\tau^2} && (11)\\ &= \int_1^{+\infty} \frac{\Big(\sin(\tau\ln(x))\Big)\{x\}}{x^{\sigma+1}}dx && (12) \end{aligned}$$

We also have $1-\bar{s} = 1-\sigma+i\tau = \sigma_1 + i\tau_1$ a zero for ζ with a real part σ_1 such that $0 < \sigma_1 = 1-\sigma < 1$ and an imaginary part τ_1 such that $\tau_1 = \tau$. Therefore

$$\begin{aligned} F(1-\sigma,\tau) &= \frac{\sigma_1-1}{(\sigma_1-1)^2+\tau_1^2} && (13)\\ &= \frac{-\sigma}{\sigma^2+\tau^2} && (14)\\ &= \int_1^{+\infty} \frac{\Big(\cos(\tau\ln(x))\Big)\{x\}}{x^{2-\sigma}}dx && (15) \end{aligned}$$

And

$$\begin{aligned} G(1-\sigma,\tau) &= \frac{\tau_1}{(\sigma_1-1)^2+\tau_1^2} && (16)\\ &= \frac{\tau}{\sigma^2+\tau^2} && (17)\\ &= \int_1^{+\infty} \frac{\Big(\sin(\tau\ln(x))\Big)\{x\}}{x^{2-\sigma}}dx && (18) \end{aligned}$$

Therefore:

$$\frac{F(\sigma,\tau)}{F(1-\sigma,\tau)} = \frac{1-\sigma}{\sigma}\frac{\sigma^2+\tau^2}{(1-\sigma)^2+\tau^2} \tag{19}$$

And

$$\frac{G(\sigma,\tau)}{\sigma^2+\tau^2} = \frac{G(1-\sigma,\tau)}{(1-\sigma)^2+\tau^2} \tag{20}$$

By combining the equations (19) and (20) we get:

$$\boxed{\frac{1-\sigma}{\sigma} = \frac{F(\sigma,\tau)}{G(\sigma,\tau)}\frac{G(1-\sigma,\tau)}{F(1-\sigma,\tau)}} \tag{21}$$

We also, note that the expression of G corresponds to the Laplace transform of the function *Sinus* and we write the following:

$$G(\sigma,\tau) = \frac{\tau}{(\sigma-1)^2+\tau^2} \tag{22}$$

$$= \int_0^{+\infty} \sin(\tau x)e^{-(1-\sigma)x}dx \tag{23}$$

And

$$G(1-\sigma,\tau) = \frac{\tau}{\sigma^2+\tau^2} \tag{24}$$

$$= \int_0^{+\infty} \sin(\tau x)e^{-\sigma x}dx \tag{25}$$

And after a change of variables we get the new expression of G:

$$G(\sigma,\tau) = \int_0^{+\infty} \sin(\tau x)e^{-(1-\sigma)x}dx \tag{26}$$

$$= \int_1^{+\infty} \frac{\sin(\tau\ln(x))}{x^{2-\sigma}}dx \tag{27}$$

And

$$G(1-\sigma,\tau) = \int_0^{+\infty} \sin(\tau x)e^{-\sigma x}dx \tag{28}$$

$$= \int_1^{+\infty} \frac{\sin(\tau\ln(x))}{x^{\sigma+1}}dx \tag{29}$$

Also, the expression of F corresponds to the Laplace transform of the function *Cosinus* and we write the following:

$$F(\sigma,\tau) = \frac{-(1-\sigma)}{(\sigma-1)^2+\tau^2} \tag{30}$$

$$= -\int_0^{+\infty} \cos(\tau x)e^{-(1-\sigma)x}dx \tag{31}$$

And

$$F(1-\sigma,\tau) = \frac{-\sigma}{\sigma^2+\tau^2} \tag{32}$$

$$= -\int_0^{+\infty} \cos(\tau x)e^{-\sigma x}dx \tag{33}$$

And after a change of variables we get the new expression of F:

$$F(\sigma,\tau) = -\int_0^{+\infty} \cos(\tau x)e^{-(1-\sigma)x}dx \tag{34}$$
$$= -\int_1^{+\infty} \frac{\cos(\tau \ln(x))}{x^{2-\sigma}}dx \tag{35}$$

And

$$F(1-\sigma,\tau) = -\int_0^{+\infty} \cos(\tau x)e^{-\sigma x}dx \tag{36}$$
$$= -\int_1^{+\infty} \frac{\cos(\tau \ln(x))}{x^{\sigma+1}}dx \tag{37}$$

We do another change of variables and get the new expression of F and G. Also we define new functions f and g for each $\lambda > 0$ and $\mu > 0$ as follows:

$$F(\sigma,\tau) = -\lambda^{1-\sigma}\int_\lambda^{+\infty} \frac{\cos(\tau \ln(x/\lambda))}{x^{2-\sigma}}dx \tag{38}$$
$$= -\lambda^{1-\sigma} f(\lambda, 1-\sigma) \tag{39}$$

And

$$F(1-\sigma,\tau) = -\mu^{\sigma}\int_\mu^{+\infty} \frac{\cos(\tau \ln(x/\mu))}{x^{\sigma+1}}dx \tag{40}$$
$$= -\mu^{\sigma} f(\mu, \sigma) \tag{41}$$

And

$$G(\sigma,\tau) = \lambda^{1-\sigma}\int_\lambda^{+\infty} \frac{\sin(\tau \ln(x/\lambda))}{x^{2-\sigma}}dx \tag{42}$$
$$= \lambda^{1-\sigma} g(\lambda, 1-\sigma) \tag{43}$$

And

$$G(1-\sigma,\tau) = \mu^{\sigma}\int_\mu^{+\infty} \frac{\sin(\tau \ln(x/\mu))}{x^{\sigma+1}}dx \tag{44}$$
$$= \mu^{\sigma} g(\mu, \sigma) \tag{45}$$

And we inject these new expressions into the equation (21) to get for each $\lambda > 0$ and $\mu > 0$:

$$\boxed{(1-\sigma)g(\lambda,1-\sigma)f(\mu,\sigma) = \sigma f(\lambda,1-\sigma)g(\mu,\sigma)} \tag{46}$$

The functions $\lambda \to f(\lambda,\sigma)$ (res. $\lambda \to f(\lambda, 1-\sigma)$) and $\lambda \to g(\lambda,\sigma)$ (resp. $\lambda \to g(\lambda, 1-\sigma)$), thanks to the fact that $1+\sigma > 1$ and $2-\sigma > 1$, are continuous and infinitely differentiable, and of class C^{∞}. Their first derivatives are as follows:

$$f\prime(\lambda,\sigma) = -\frac{1}{\lambda^{(1+\sigma)}} + \frac{\tau}{\lambda^{(1+\sigma)}} \int_{1}^{+\infty} \frac{\sin(\tau \ln(x))}{x^{\sigma+1}} dx \quad (47)$$

$$= -\frac{1}{\lambda^{(1+\sigma)}} + \frac{\tau}{\lambda^{(1+\sigma)}} G(1-\sigma,\tau) \quad (48)$$

$$g\prime(\lambda,\sigma) = -\frac{\tau}{\lambda^{(1+\sigma)}} \int_{1}^{+\infty} \frac{\cos(\tau \ln(x))}{x^{\sigma+1}} dx \quad (49)$$

$$= \frac{\tau}{\lambda^{(1+\sigma)}} F(1-\sigma,\tau) \quad (50)$$

And

$$f\prime(\lambda,1-\sigma) = -\frac{1}{\lambda^{(2-\sigma)}} + \frac{\tau}{\lambda^{(2-\sigma)}} G(\sigma,\tau) \quad (51)$$

$$g\prime(\lambda,1-\sigma) = \frac{\tau}{\lambda^{(2-\sigma)}} F(\sigma,\tau) \quad (52)$$

And we calculate the derivatives of both sides of the equation (46) in regard to λ and μ to get:

$$\boxed{(1-\sigma) g\prime(\lambda,1-\sigma) f\prime(\mu,\sigma) = \sigma f\prime(\lambda,1-\sigma) g\prime(\mu,\sigma)} \quad (53)$$

And we inject the first derivatives of f and g into the last equation:

$$(1-\sigma)\left(\frac{\tau}{\lambda^{(2-\sigma)}} F(\sigma,\tau)\right)\left(-\frac{1}{\mu^{(1+\sigma)}} + \frac{\tau}{\mu^{(1+\sigma)}} G(1-\sigma,\tau)\right) \quad (54)$$

$$= \sigma\left(-\frac{1}{\lambda^{(2-\sigma)}} + \frac{\tau}{\lambda^{(2-\sigma)}} G(\sigma,\tau)\right)\left(\frac{\tau}{\mu^{(1+\sigma)}} F(1-\sigma,\tau)\right) \quad (55)$$

Thanks to the equation (21) we can simplify the last equation and get the following:

$$(1-\sigma) F(\sigma,\tau) = \sigma F(1-\sigma,\tau) \quad (56)$$

We use the expressions of $F(1-\sigma,\tau)$ and $F(\sigma,\tau)$ from the equations (9) and (14) to get that:

$$\frac{(1-\sigma)^2}{(1-\sigma)^2+\tau^2} = \frac{\sigma^2}{\sigma^2+\tau^2} \quad (57)$$

Therefore

$$\sigma = \frac{1}{2} \quad (58)$$

Hence, the Riemann's Hypothesis is true. ■

Remark. *The main idea here was to not work with the original defintions of F and G that come from the Zeta function definition. But instead work with their simpler versions from the Laplace transform of the functions Sinus and Consinus.*

1.4 Conclusion

We saw that if s is a zeta zero, then real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Riemann's Hypothesis is true: *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$*. In the next section, we will apply the same method to prove the Generalized Riemann Hypothesis (GRH).

2 The Generalized Riemann Hypothesis

2.1 Dirichlet $L-$functions

Let's $(z_n)_{n\geq 1}$ be a sequence of complex numbers. A Dirichlet series[3] is a series of the form $\displaystyle\sum_{n=1}^{\infty}\frac{z_n}{n^s}$, where $s=\sigma+i\tau$ is complex. The zeta function is a Dirichlet series. Let's define the function $L(s)$ of the complex s: $L(s)=\displaystyle\sum_{n=1}^{\infty}\frac{z_n}{n^s}$.

- If $(z_n)_{n\geq 1}$ is a bounded, then the corresponding Dirichlet series converges absolutely on the open half-plane where $\Re(s)>1$.
- If the set of sums $z_n+z_{n+1}+...+z_{n+k}$ for each n and $k\geq 0$ is bounded, then the corresponding Dirichlet series converges on the open half-plane where $\Re(s)>0$.
- In general, if $z_n=O(n^k)$, the corresponding Dirichlet series converges absolutely in the half plane where $\Re(s)>k+1$.

The function $L(s)$ is analytic on the corresponding open half plane[2-3].

To define Dirichlet $L-$functions we need to define Dirichlet characters. A function $\chi:\mathbb{Z}\longmapsto\mathbb{C}$ is a Dirichlet character modulo q if it satisfies the following criteria:

- (i) $\chi(n)\neq 0$ if $(n,q)=1$.
- (ii) $\chi(n)=0$ if $(n,q)>1$.
- (iii) χ is periodic with period q :$\chi(n+q)=\chi(n)$ for all n.
- (iv) χ is multiplicative :$\chi(mn)=\chi(m)\chi(n)$ for all integers m and n.

The trivial character is the one with $\chi_0(n)=1$ whenever $(n,q)=1$.

Here are some known results for a Dirichlet character modulo q. For any integer n we have $\chi(1)=1$. Also if $(n,q)=1$, we have $(\chi(n))^{\phi(q)}=1$ with ϕ is Euler's totient function. $\chi(n)$ is a $\phi(q)-$th root of unity. Therefore, $|\chi(n)|=1$ if $(n,q)=1$, and $|\chi(n)|=0$ if $(n,q)>1$.
Also, we recall the cancellation property for Dirichlet characters modulo q: For any n integer

$$\sum_{i=1}^{q}\chi(i+n)=\begin{cases}\phi(q), & \text{if } \chi=\chi_0 \text{ the trivial character}\\ 0, & \text{if otherwise, } \chi\neq\chi_0\end{cases}\tag{59}$$

The Dirichlet $L-$functions are simply the sum of the Dirichlet series. Let's χ be a Dirichlet character modulo q, The Dirichlet $L-$function

$L(s,\chi)$ is defined for $\Re(s) > 1$ as the following:

$$L(s,\chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} \tag{60}$$

They are a natural generalization of the Riemann zeta-function $\zeta(s)$ to an arithmetic progression and are a powerful tool in analytic number theory. The Dirichlet series, converges absolutely and uniformly in any bounded domain in the complex s-plane for which $\Re(s) \geq 1+\gamma$, $\gamma > 0$. In the particular case of the trivial character χ_0, $L(s,\chi_0)$ extends to a meromorphic function[5-8] in $\Re(s) > 0$ with the only pole at $s = 1$.

If χ is a non-trivial character, we have

$$L(s,\chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \tag{61}$$

Since the sum in the integrand is bounded, this formula gives an analytic continuation of $L(s,\chi)$ to a regular function in the half-plane $\Re(s) > 0$. Also, like the zeta function, the Dirichlet $L-$functions have their Euler product[2,5, 9, 10]. For $\Re(s) > 1$:

$$L(s,\chi) = \prod_{p \, \mathrm{Prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \tag{62}$$

Therefore, if $\chi = \chi_0$ is a trivial character $\mod q$, we have, for $q = 1$,

$$L(s,\chi_0) = \zeta(s) \tag{63}$$

And for $q > 1$, we have,

$$L(s,\chi_0) = \zeta(s) \prod_{p/q} \left(1 - \frac{1}{p^s}\right) \tag{64}$$

For this reason the properties of $L(s,\chi_0)$ in the entire complex plane are determined by the properties of $\zeta(s)$.

Let's now $q\prime$ be the smallest divisor (prime) of q. Let's $\chi\prime$ be the Dirichlet character $\chi\prime \mod q\prime$. For any integer n such that $(n,q) = 1$, we have also $(n,q\prime) = 1$ and $\chi(n) = \chi\prime(n)$. $\chi\prime$ is called primitive and $L(s,\chi)$ and $L(s,\chi\prime)$ are related analytically such that if $\chi \neq \chi_0$:

$$L(s,\chi) = L(s,\chi\prime) \prod_{p/q} \left(1 - \frac{\chi\prime(p)}{p^s}\right) \tag{65}$$

$L(s,\chi)$ and $L(s,\chi\prime)$ have the same zeros in the critical strip $0 < \Re(s) < 1$.

Also, for a primitive character χ, $(i.e.\chi = \chi\prime)$ $L(s,\chi)$ has the following functional equation:

$$\tau(\chi)\Gamma(\frac{1-s+a}{2})\, L(1-s,\chi) = \sqrt{\pi}(\frac{q}{\pi})^s i^a q^{\frac{1}{2}}\, \Gamma(\frac{s+a}{2})\, L(s,\overline{\chi}) \tag{66}$$

Where Γ is the Gamma function and $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$, and $\tau(\chi) = \sum_{k=1}^{q} \chi(k) \exp(\frac{2\pi k i}{q})$.

When $\Re(s) > 1$ there is no zero for $L(s,\chi)$. When $\Re(s) < 0$, for a primitive character χ, we have the trivial zeros of $L(s,\chi)$: $s = a - 2k$, where k is a positive integer and a is defined above. For more details, please refer to the references[2-11].

2.2 The GRH Statement

The Generalized Riemann Hypothesis states that the Dirichlet $L-$functions have all their non-trivial zeros on the critical line $\Re(s) = \frac{1}{2}$.

We know that for any character χ modulo q, all non-trivial zeros of $L(s, \chi)$ lies in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. From the functional equation above we have that if:

- s_0 is a non-trivial zero of $L(s, \chi)$, then $1 - s_0$ is a zero of $L(s, \overline{\chi})$.
- s_0 is a non-trivial zero of $L(s, \overline{\chi})$, then $1 - s_0$ is a zero of $L(s, \chi)$.

Therefore, we just need to prove that for any character[2] χ modulo q, there is no non-trivial zeros of $L(s, \chi)$ in the right hand side of the critical strip $\{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$.

2.3 Proof of the GRH

2.3.1 Case of χ non-trivial

Let's take a complex number s such that $s = \sigma + i\tau$. Unless we explicitly mention otherwise, let's suppose that $0 < \sigma < 1$, $\tau > 0$ and $L(s, \chi) = 0$ where $L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$ and χ is a non-trivial Dirichlet character χ modulo q. When $q = 2$ there is only one Dirichlet character and it is trivial. So here we are going to assume that $q \geq 3$.

We have from the integral equation (59) above:

$$L(s, \chi) = s \int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx \tag{67}$$

We first develop the integral above as following:

$$\int_1^{+\infty} \frac{\sum_{n=1}^{n \leq x} \chi(n)}{x^{s+1}} dx = \int_1^{+\infty} \frac{\chi(1)}{x^{s+1}} dx + \int_1^{+\infty} \frac{\sum_{n=2}^{n \leq x} \chi(n)}{x^{s+1}} dx \tag{68}$$

We have $\chi(1) = 1$, $s \neq 0$ and $L(s, \chi) = 0$, therefore:

$$\int_1^{+\infty} \frac{1}{x^{s+1}} dx = -\int_1^{+\infty} \frac{\sum_{n=2}^{n \leq x} \chi(n)}{x^{s+1}} dx \tag{69}$$

Let's define the function $H(x)$ as following:

$$H(x) = -\sum_{n=2}^{n \leq x} \chi(n) = h(x)\, e^{-i\alpha_x} \tag{70}$$

Where the function $h(x)$ being the function norm of the function $H(x)$. i.e $h(x) = \|H(x)\|$. The functions $H(x)$ and α_x are piecewise constant as for each integer $n \geq 1$, for each real $x \in (n, n+1)$, we have $H(x) = H(n) = h(n)\, e^{-i\alpha_n}$ and $\alpha_x = \alpha_n \in [0, 2\pi]$. Also, the function H is bounded thanks

[2] In fact, we just need to prove it for the primitive characters.

to the cancellation property of the Dirichlet chararcters (5). Let's denote M such that

$$M = \sup_{x \geq 1} h(x) \tag{71}$$

Therefore:

$$\frac{1}{s} = \int_1^{+\infty} \frac{H(x)}{x^{s+1}} dx \tag{72}$$

Therefore:

$$\frac{1}{\sigma + i\tau} = \int_1^{+\infty} \frac{H(x)}{x^{\sigma+i\tau+1}} dx \tag{73}$$

And

$$\frac{\sigma - i\tau}{\sigma^2 + \tau^2} = \int_1^{+\infty} \frac{\Big(\cos(\tau \ln(x) + \alpha_x) - i \sin(\tau \ln(x) + \alpha_x)\Big) h(x)}{x^{\sigma+1}} dx \tag{74}$$

The integral is absolutely convergent. We take the real part of both sides of the above equation and define the functions F and G as following:

$$F(\sigma, \tau) = \frac{\sigma}{\sigma^2 + \tau^2} \tag{75}$$

$$= \int_1^{+\infty} \frac{\Big(\cos(\tau \ln(x) + \alpha_x)\Big) h(x)}{x^{\sigma+1}} dx \tag{76}$$

And

$$G(\sigma, \tau) = \frac{\tau}{\sigma^2 + \tau^2} \tag{77}$$

$$= \int_1^{+\infty} \frac{\Big(\sin(\tau \ln(x) + \alpha_x)\Big) h(x)}{x^{\sigma+1}} dx \tag{78}$$

We also have $1 - s = 1 - \sigma - i\tau = \sigma_1 + i\tau_1$ a zero for $L(s, \bar{\chi})$ with a real part σ_1 such that $0 < \sigma_1 = 1 - \sigma < 1$ and an imaginary part τ_1 such that $\tau_1 = -\tau$. Therefore

$$F(1 - \sigma, -\tau) = \frac{\sigma_1}{\sigma_1^2 + \tau_1^2} \tag{79}$$

$$= \frac{1 - \sigma}{(1 - \sigma)^2 + \tau^2} \tag{80}$$

$$= \int_1^{+\infty} \frac{\Big(\cos(-\tau \ln(x) - \alpha_x)\Big) h(x)}{x^{1-\sigma+1}} dx \tag{81}$$

$$= \int_1^{+\infty} \frac{\Big(\cos(\tau \ln(x) + \alpha_x)\Big) h(x)}{x^{2-\sigma}} dx \tag{82}$$

And

$$G(1-\sigma,-\tau) = \frac{\tau_1}{\sigma_1^2+\tau_1^2} \tag{83}$$

$$= \frac{-\tau}{(1-\sigma)^2+\tau^2} \tag{84}$$

$$= \int_1^{+\infty} \frac{\Big(\sin(-\tau\ln(x)-\alpha_x)\Big)h(x)}{x^{2-\sigma}}dx \tag{85}$$

Therefore:

$$\frac{F(1-\sigma,-\tau)}{F(\sigma,\tau)} = \frac{1-\sigma}{\sigma}\frac{\sigma^2+\tau^2}{(1-\sigma)^2+\tau^2} \tag{86}$$

And

$$-\frac{G(1-\sigma,-\tau)}{G(\sigma,\tau)} = \frac{\sigma^2+\tau^2}{(1-\sigma)^2+\tau^2} \tag{87}$$

By combining the equations (84) and (85) we get an equation similar to the equation (21) of the case of Riemann Hypothesis:

$$\boxed{\frac{1-\sigma}{\sigma} = -\frac{F(1-\sigma,-\tau)}{G(1-\sigma,-\tau)}\frac{G(\sigma,\tau)}{F(\sigma,\tau)}} \tag{88}$$

Once here, we use exactly the same Laplace transform technique and we follow the same steps as in the section of the proof of the Riemann Hypothesis above to finish off. Everything else stays the same.

Therefore

$$\sigma = \frac{1}{2} \tag{89}$$

Hence, the Generalized Riemann's Hypothesis is true. ■

2.3.2 Case of χ trivial

For the trivial character χ_0 modulus q we have:

$$\zeta(s) = L(s,\chi_0)\prod_{p\,\mathrm{Prime},p/q}\Big(1-\frac{1}{p^s}\Big)^{-1} \tag{90}$$

Where ζ is the zeta function.

The product $\prod_{p\,\mathrm{Prime},p/q}\Big(1-\frac{1}{p^s}\Big)^{-1}$ is finite, bounded and nonzero ($s\neq 0$). s is a zero for $L(s,\chi_0)$, then thanks to (73), it is also a zero for the zeta $\zeta(s)$. We proved in the first section, that Riemann's Hypothesis is true, therefore $\sigma = \frac{1}{2}$.

2.4 Conclusion

We saw that if s is a zero of $L(s,\chi)$, then the real part $\Re(s)$ can only be $\frac{1}{2}$. Therefore the Generalized Riemann Hypothesis is true: *The non-trivial zeros of $L(s,\chi)$ have their real part equal to $\frac{1}{2}$.*

References

[1] Bernhard Riemann. *On the Number of Prime Numbers less than a Given Quantity*
https://www.claymath.org/sites/default/files/ezeta.pdf

[2] Aleksandar Ivic. *The Riemann Zeta-Function: Theory and Applications*

[3] Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*
http://wayback.cecm.sfu.ca/~pborwein/TEMP_PROTECTED/book.pdf

[4] Jørgen Veisdal. *The Riemann Hypothesis, explained*
https://medium.com/cantors-paradise/the-riemann-hypothesis-explained-fa01c1f75d3f

[5] Thai Pham. *Dirichlet's Theorem on Arithmetic Progressions*
https://web.stanford.edu/ thaipham/papers/MIT_18.104_Review_Paper.pdf

[6] G. H. Hardy. *The general theory of dirichlet series.*
https://archive.org/details/generaltheoryofd029816mbp/page/n9

[7] Garrett, Paul. *Primes in arithmetic progressions, 2011.*
http : //www.math.umn.edu/ garrett/m/mfms/notes_c/dirichlet.pdf

[8] A. Wintner. *The fundamental lemma in Dirichlet's theory of the arithmetical progressions. Amer. J. Math., 68:285-292, 1946.*

[9] G. Bhowmik, D. Essouabri and B. Lichtin. *Meromorphic continuation of multivariable Euler products, Forum Math. p. 1-29, DOI 10.1515, 2006.*

[10] E. Bombieri and D. A. Hejhal. *On the distribution of zeros of linear combinations of Euler products. Duke Math. J., 80(3):821-862, 1995.*

[11] H. Davenport. *Multiplicative Number Theory, 2nd edition, revised by H. Montgomery, Graduate Texts in Mathematics, Vol. 74, Springer-Verlag, New York, 1980.*