Quasicrystal Scattering and the Riemann Zeta Function

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Abstract

I carry out numerical scattering calculations against a family of one-dimensional point-like arrangements of atoms, $\chi(\mathbf{x})$, related to the distribution of prime numbers by a shift operation making the atomic density approximately constant. I show how the Riemann Zeta Function (RZF) naturally parameterizes the analytic structure of the scattering amplitude.

1 Introduction

The curious relationship between prime numbers and the non-trivial zeros of the Riemann Zeta Function (RZF) has intrigued mathematicians and physicists for decades [1–4]. Freeman Dyson's speculation [5] about using quasicrystals to explore this relationship particularly caught my attention.

Quasicrystals, structures that are ordered but not periodic, were first observed experimentally by Shechtman in 1984 [6]. Their unique properties make them an interesting lens through which to study number-theoretic concepts.

In this work, I explore the connection between quasicrystals and the RZF through numerical scattering calculations. I construct a family of one-dimensional, point-like atomic arrangements $\chi(\mathbf{x})$ related to prime number distribution and demonstrate how the RZF naturally parameterizes the resulting scattering amplitude. For a more detailed background on the RZF and related concepts, see Appendix **??**.

1.1 Fourier Transform and Scattering

The scattering of a wave from a potential is represented by the Fourier transform:

$$\hat{V}(k) = \int_{-\infty}^{\infty} V(x) e^{-i2\pi kx} dx \tag{1}$$

For a one-dimensional scattering potential of the form:

$$V(x) = \sum_{x_n \in X} \delta_D(x - x_n) \tag{2}$$

its Fourier transform may contain a tempered distribution:

$$\mathcal{F}\{V(x)\} = \hat{V}(k) = \hat{h}(k) + \sum_{k_m \in X^*} \hat{V_m} \delta_D(k - k_m) \tag{3}$$

When $\hat{h}(k) = 0$ everywhere, $V(x)$ is a quasicrystal. Further details on the properties of quasicrystals and their Fourier transforms can be found in Appendix **??**.

1.2 Wave Scattering and χ

Inspired by Varma's approach [9], I define a specific 1-dimensional atomic arrangement for scattering calculations. Unlike Varma, I apply the shift transformation directly to the real space atomic positions, not to the k-space positions of the RZF zeros.

I consider a scattering potential $V(x)$ with Dirac delta functions at each prime number. To normalize the atomic density, I apply a shift operation:

$$p(x_n) = x_n \cdot \frac{1}{\pi(x_n)} \approx \log(x_n) \tag{4}$$

where x_n is the nth prime and $\pi(x_n)$ is the prime counting function. The full derivation of this shift operation and its relation to the prime counting function is provided in Appendix **??**.

This normalization is crucial for understanding how the RZF zeros enter the scattering calculation. The RZF zeros create poles along the axis Re(z) = 1/2 in the complex plane, directly influencing the structure of the scattering amplitude.

2 Results and Discussion

Figure 2 shows the scattering amplitude for finite L_χ. Key observations include:

1. A series of peaks and troughs in the scattering amplitude, with overall amplitude decreasing as momentum increases.

2. Clear correspondence between RZF zero positions and scattering amplitude peaks.

3. Similar behavior across different L_χ values, suggesting robustness of the observed features.

4. A consistent pattern of peaks and troughs related to RZF zero distribution when viewed across a wider momentum range.

These results support my hypothesis about how RZF zeros enter the scattering calculation through the normalization process. The alignment of scattering amplitude peaks with RZF zeros likely manifests the poles created by these zeros in the complex plane. A more detailed analysis of this correspondence is provided in Appendix **??**.

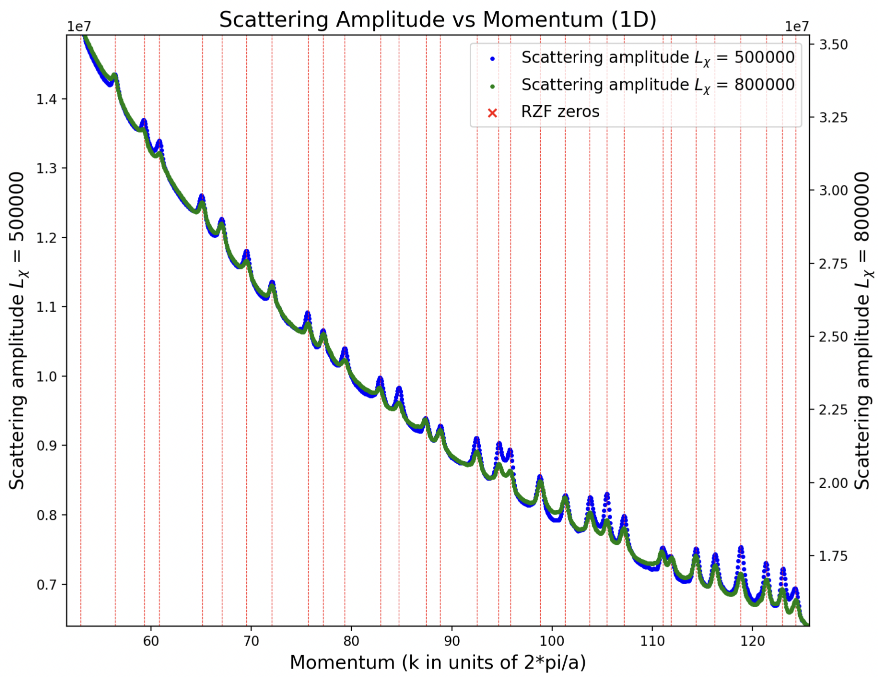

Figure 1: Scattering amplitude for finite L_χ. Vertical red lines indicate the positions of the imaginary part of the non-trivial RZF zeros.

3 Conclusion

This work provides a novel physical interpretation of RZF zeros in terms of wave scattering from a carefully constructed potential. It suggests that RZF zeros might be understood as resonances in this scattering system, with each zero corresponding to a particular scattering mode.

The robustness of these features across different L_χ values indicates that this relationship between RZF zeros and scattering amplitudes is a fundamental property of the system I've constructed, not an artifact of finite approximation.

Future work could explore extending this approach to higher dimensions, investigating different normalization schemes, or studying how potential perturbations affect the relationship between scattering amplitudes and RZF zeros.

In conclusion, I've demonstrated an intriguing connection between number theory, specifically the Riemann Zeta Function and its zeros, and wave scattering physics. This new perspective suggests potential approaches for studying these mathematical objects through physical analogies.

4 Acknowledgements

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A Background on RZF and Related Concepts

B. Riemann showed the prime numbers are partially ordered and are not periodic [1] in 1859, and H. Von Mangoldt [7] proved the explicit formula in 1895.

The explicit formula of Guinand and Weil [8] is a formula for the Fourier transform of the RZF zeros as a sum over prime powers, plus additional terms:

$$\sum_{\rho} h(\rho) = h(0) + h(1) - \sum_{p} \sum_{m=1}^{\infty} \frac{h(\log p^m)}{p^{m/2}} \log p - \int_{-\infty}^{\infty} \frac{h(t)\Phi(t)}{2} dt \tag{5}$$

The explicit formula of Guinand and Weil is the dual of the prime counting function expression of Von Mangoldt [7].

B Properties of Quasicrystals and Their Fourier Transforms

By applying the Fourier transform a second time to $\hat{V}(k)$, it is evident that $\hat{V}(k)$ is also a quasicrystal when $V(x)$ is a quasicrystal. When all the x_n lie along a line, the k_m must also lie along a line in the complex plane - applying the Fourier transform twice gives us back our original $V(x)$.

C Derivation of Shift Operation and Relation to Prime Counting Function

The exact expression [1] for $\pi(x)$ when $x > 1$ is:

$$\pi(x) = \pi_0(x) - \frac{1}{2} = R(x) - \sum_{\rho} R(x^{\rho}) - \frac{1}{2} \tag{6}$$

where

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} li(x^{\frac{1}{n}}) \tag{7}$$

where $\mu(x)$ is the Möbius function, $li(x)$ is the logarithmic integral function, and ρ runs over all the zeros of the RZF.

If the trivial zeros are collected and the sum is taken only over the non-trivial zeros, then:

$$\pi_0(x) \approx R(x) - \sum_{\rho} R(x^{\rho}) - \frac{1}{\log(x)} + \frac{1}{\pi}\arctan(\frac{1}{\log(x)}) \tag{8}$$

It is well known that $\pi(x) \sim \frac{x}{log(x)}$ in a rougher approximation.

The quantity $\pi(x)/x$ has units of density - it represents the density of atoms around x in the scattering potential defined by $V(x)$.

D Detailed Analysis of Scattering Amplitude and RZF Zero Correspondence

[This section would contain a more detailed quantitative analysis of the correspondence between scattering amplitude peaks and RZF zeros, which was mentioned as a TODO in the original document.]

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