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Digital Experiences in Mathematics Education

ISSN 2199-3246
Volume 7
Number 1

Digit Exp Math Educ (2021) 7:167-191
DOI 10.1007/s40751-020-00073-5

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https://doi.org/10.1007/s40751-020-00073-5

Conversations with Materials and Diagrams about some of the Intricacies of Oscillatory Motion

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Published online: 18 September 2020

Abstract
This article relates a case study on how a conversation with materials and diagrams – the actual use of materials and diagrams to think, imagine, explain, collaborate, design and build – featured a certain kind of interplay between material and digital components. The physical components present in this setting included a water wheel, which is a wheel driven by flow of water whose rotational motion is a classic example of chaotic dynamics regulated by Lorenz equations. Digital components allowed for real-time graphical displays corresponding to the turning of the water wheel. We selected for this article a sequence of episodes from an interview with Jake, an undergraduate student majoring in engineering. Through a micro-ethnographic analysis, we reflect on how Jake combined the responsiveness of the digital displays with the tangibility of the water wheel to gain insight into some of the intricacies of oscillatory motion.

Keywords Diagrams · Materials · Micro-ethnography · Oscillatory motion · Plural speech · Polyvalent events · Tool use · Motion sensors · Embodied cognition

In this article, we discuss mathematical activities such as using tools and creating diagrams as particular instances of conversations with materials and diagrams. We start by theoretically framing this statement, articulating the seemingly contradictory idea of "conversation" in such contexts. We are particularly interested in highlighting the

Electronic supplementary material The online version of this article (https://doi.org/10.1007/s40751-020-00073-5) contains supplementary material, which is available to authorized users.

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complexity of these conversations, pointing at the irreducible significance of physicality and feelings in the doing of mathematics. To do so, we present and analyze episodes from an interview in which an undergraduate student, Jake, makes sense of a particular device we call the *water wheel*, and investigates certain graphical configurations expressing its movement. The functioning of the water wheel will be described in the methods section, which also contains an explanation of the kind of graphs that are digitally produced by means of photogate sensors and a software interface.

We present six episodes selected from the second interview with Jake, where he investigates some of these motions by using hand-produced diagrams, digitally produced ones and the actual manipulation of the wheel. As he encounters an unexpected diagram, we observe how the conversation with the water wheel and the graphs evolves along the expressive, performative and temporal dimensions of activity with diagrams. We characterize the ways in which qualitative differences emerge through bodily interaction with devices allowing for the development of new kinds of sensitivities, and we highlight the role of materiality-feeling as crucially involved in the mathematical investigations with the water wheel.

In this article, we discuss two kinds of diagrams: 1) inscriptions on a white board, and 2) computer-generated graphs displaying data collected by means of electronic sensors. The latter allows us to explore a particular type of interplay between physical and digital tools; namely, the behavior and responsiveness of a complex physical artifact intertwined with the generation of graphical displays reflecting temporal events that may not be easily perceivable in a direct fashion. The main example of this interplay that has been investigated in mathematics education is the use of motion detectors (e.g. Nemirovsky, 1994; Nemirovsky, Tierney & Wright, 1998) in which case materials are physical bodies in movement and digital diagrams are real-time depictions of kinesthesia. This is a setting in which diagrams become responsive to physical movement and bodies strive through moving to produce intended diagrammatic shapes.

With the water wheel, the body movement in question centers on the rotation of the wheel that can be driven by hand or autonomously turned by means of water flow. Something that is primarily distinctive in the use of motion detectors is that graphical displays tend to be endowed with feelings arising from bodily engagements with proprioception and kinesthesia. The power of these felt bodily engagements for mathematics learning is that they evoke sense-making resources that are not clearly otherwise at play (e.g. Ferrari, 2019).

Theoretical Background

In this section, we elaborate on two themes: 1) prior and contemporary conceptualizations of diagrams and materials in mathematics education, and 2) conversations with materials and diagrams and plural speech. We overview ideas within mathematics education in relation to the use of materials and diagrams to situate better the contribution of this article. Plural speech is a type of conversation that helped us conceive how interactions with materials and diagrams can be properly described as conversations.

Prior and Contemporary Conceptualizations of Diagrams and Materials in Mathematics Education

During most of the last quarter of the twentieth century, representationalism has been a prevalent perspective in mathematics education to characterize the use of diagrams (e.g. graphs) and materials (e.g. manipulatives) – diagrams and materials as representing mathematical ideas in ways that are more or less direct, misleading or partial (Post, Wachsmuth, Lesh & Behr, 1983). Given the biases of any representation in featuring some aspects and blurring others, the necessity of using multiple representations (Lesh, Post & Behr, 1987) seemed evident, such that the common core of the represented idea would emerge through their mutual translation.

Theorists of representationalism have often invoked the distinction between internal (i.e. mental) and external (i.e. physical) representations and the ways they transformed into each other via externalization and internalization (Goldin & Kaput, 1996). The prevalence of representationalism has been disrupted by the massive socio-technological presence of calculators and computers. It is clear that even a calculator does something more than representing ideas: it seems somehow to process and transform them, or at least assist in these pursuits.

New theoretical perspectives arose with headings such as "distributed cognition", "situated cognition" and "extended mind" (Clark & Chalmers, 1998; Greeno, 1998; Nunes, Schliemann & Carraher, 1993; Salomon & Perkins, 1989). Most of the work of distributed cognition theorists focused on information processing and ranged from the thesis that cognitive work "off-loads" onto the environment to the one that the environment is part of the cognitive system (Wilson, 2002). Papert (1980/1993) developed an influential perspective arguing that, beyond the processing of information, materials and computers are tools "to think with".

The conception of having conversations with materials and diagrams advanced in this article moves towards a view according to which they co-think with living beings. It resonates with post-humanistic perspectives de-centering thought from human individuals (de Freitas & Sinclair, 2014). In this view, thoughts circulate or flow across living beings, materials and diagrams. While diagrams have a material existence, such as inked inscriptions on paper or pixelated marks on a computer screen, we distinguish diagrams from materials because, in many cases, diagrams themselves are oblivious to forces or mechanisms that tend to be prominent in causal relations among materials, such as gravitational forces or chemical mechanisms of combustion. Within limits, there is a sense in which diagrams can be physically immutable (Latour, 1990), whereas materials cannot. In addition, diagrams do not exist apart from actual or imaginary performances they participate in (Nemirovsky, Kelton & Rhodehamel, 2013). Devoid of performative enactments, diagrams vanish leaving, at most, a faint leftover of marks or traces.

Conversations with Materials and Diagrams and Plural Speech

This article elaborates on the idea of conversations with materials and diagrams – a phrase that we have borrowed from Bamberger & Schön (1983) to which we added "diagrams – referring to the actual use of materials and diagrams to think, imagine, explain, design and build. Referring to conversations with materials and diagrams seems to be a misnomer. How is it possible to converse with things that do not speak, do not wait for their turn to respond and do not argue for or adopt thoroughgoing commitments? If the notion of conversation with materials and diagrams is to have any sustenance, it must refer to a type of conversation that can also happen among people. We propose that this kind of conversation appears to be well characterized by Blanchot's (1992) term "plural speech", namely one that, as initially characterized by Blanchot and elaborated upon by Bojesen (2019), helped us to conceive how interactions with materials and diagrams can be properly described as conversations.

In order to work with an example of a conversation with materials, as a reference for the ensuing introduction of plural speech, we will describe the events analyzed by Bamberger and Schön (1983). Their article describes Dora and Ann's exploratory arrangements of Montessori bells. Dora and Ann are two adults without musical training who participated in this working session with these bells, which are a set of five, labeled X, Y, Z, P and Q, that look identical, each generating a sound with a distinct pitch. The main instruction had been to use the bells to generate tunes that they like. Over her first four moves, Dora physically separated two groups of bells, X–Y–Z, and P–Q, sensing these two groups do not belong together. Dora named the sound of P–Q as one of a doorbell: "Dora unintentionally discovers within the materials a surprising new object [i.e. a doorbell]" (p. 72). Shortly thereafter, Ann remarked that P and Q "belong to a different set" (p. 72). This comment seemed to prompt Dora to look for a different arrangement, one that included Q separated from P. After several attempts, Dora wondered what could be done with Q. Then Ann tried X–Q and then X–Q–Y. Listening to the sound that the latter sequence generated, Dora found that she liked the tune: "Oh! That sounds nice!" (p. 72). Ann and Dora's conversation with the Montessori bells followed a meandering course along which they discovered, among others, a doorbell and a nice tune. The role of the Montessori bells in these transactions was neither passive nor marginal: that the sound Q–P constituted a doorbell, for example, was fully embedded in their materiality.

In contrast to dialogical conversations, Blanchot worked to articulate an alternative type, one which would not be "a matter of teaching something or of extracting the truth by going from one interlocutor to another, as did Socrates in order to keep seeking the true through the vicissitudes of an unyielding conversation" (1992, p. 213). This alternative conversation type – plural speech – would encompass those that, "unlike dialogue and dialectic, [do] not imply that contradicting and contrasting thoughts should be brought to shared consensus or internal resolution" (Bojesen 2019, p. 650). Blanchot argued that plural speech must be one in which, rather than developing general points or arguments, the conversants proceed through separate affirmations. Instead of striving to explicate "an ordered set of words, experiences, and principles", speakers avoid development, so that it is, "a matter of thinking by separate affirmations. Someone says something and goes no further. Without proof, reasoning and logical consequence." (1992, p. 339). In plural speech, he added, the point is not so much to say what is thought, but to think what is said. As it can be illustrated by the P–Q bells sounding a doorbell, the aim of plural speech is not to reach consensus or convince, but to mobilize thought, allowing for the interruption of speech "by which the unknown announces itself" (p. 78).

Plural speech is a kind of conversation for which it makes sense that materials and diagrams can participate as legitimate interlocutors. They affirm things, such as that in certain conditions a material structure reaches a state of instability or a certain diagram transforms itself into a desired form, without necessarily advocating for encompassing viewpoints or lines of action, and without asserting the truth or the existence of something, which means that rather than reaching final closure, puzzling or perplexing questions often mobilize other supervening puzzling or perplexing questions. By participating in plural speech, materials and diagrams, like the Montessori bells in the work of Ann and Dora, co-think: they contribute to emerging ways of thinking which can be equally or symmetrically attributed to brains, bodies, tasks, materials and their socio-individual histories.

The Water Wheel

The water wheel is a fascinating example of a physical device that can exhibit a variety of different rotational motions, including pendulum-like behavior (with both small and large amplitudes), more complicated periodic behavior and chaotic behavior. The dynamics of the water wheel can be described with a system of three differential equations (see Nemirovsky & Tinker, 1993, for a derivation of these equations). Those that model the dynamics of the water wheel bear a striking resemblance to the famous system of three differential equations that Edward Lorenz created to study air circulation and which ultimately led to the discovery of sensitive dependence on initial conditions, which is one of the main properties of chaos (Gleick, 1987).

In the analysis presented here, we focus on the behavior of the water wheel that resembles pendulum-like behavior. The students we worked with were all mathematics, science and engineering majors and, hence, were familiar with oscillatory motion, especially that of a pendulum with small displacements. The water wheel, however, also affords an opportunity to examine oscillatory motion where the back-and-forth rotations range between 180 and 360 degrees.[1] In the interviews, the students created and explored a variety of different graphical means to depict such oscillatory motion, including time-series graphs of angular velocity and angular acceleration, and phase-plane graphs in the angular velocity-angular acceleration plane. We next give an overview of the physical device and how it operates.

As shown in Fig. 1, the water wheel consists of a circular plexiglass plate with 32 one-inch diameter plastic tubes around its edge. Each tube has a small hole at the bottom. The plate turns on an axle and is free to rotate. The tilt of the axle can be adjusted between 0 and 45 degrees from the vertical. Water showers into the eight uppermost tubes from a curved pipe with holes along its underside. A submersible pump sends water to the pipe, with a valve to regulate the flow. An oil bath between nested cylinders at the center of the wheel provides dynamic friction for the axis of rotation. Raising or lowering an oil reservoir varies the oil level in between these cylinders. (See the supplementary video clip "Jake and the water wheel" from the first interview.)

[1] A pendulum under small displacements reaches its maximum angular acceleration at the point of maximum displacements, which is also where velocity is zero. If the displacements are large, however, the maximum angular acceleration is obtained when the weight is at the same height as the center of rotation and does not correspond to the velocity being zero.

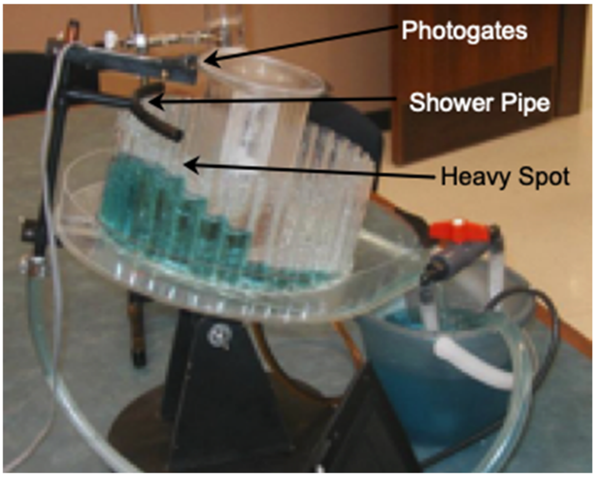


Fig. 1 The water wheel

The angular velocity of the water wheel is measured by two photo-electronic sensors that detect the motion of a pattern of black lines on the wheel top. A computer interface permits users to graph angular velocity versus time, angular acceleration versus time, and angular velocity versus angular acceleration while the wheel is turning (Nemirovsky & Tinker, 1993). Water showers into the tubes when they are carried underneath the shower pipe. As the wheel turns, the water gathered in each tube provides a torque around the axis of the wheel. Because each tube leaks water from the bottom, the amount of water in each tube decreases over time, until that tube again swings upward to be under the shower pipe to receive more water. During periodic motion, water tends to accumulate in a bell-shaped distribution in the tubes, which students often call "the heavy spot" (see Fig. 1 or the video in the supplementary material entitled "Water Wheel"). With different choices of tilt angle, flow rate, bearing friction and initial water distribution, the motion of the wheel exhibits a variety of periodic, almost periodic and chaotic motion, as well as period doubling and transition into chaos.

In the episodes that we have selected for this article, the wheel was used without water flow allowing for dynamic exploration with physical interaction, either by turning it by hand or by adding weights – marbles inserted in some of the tubes (see Fig. 2a). In the latter case, the water wheel was mathematically equivalent to a pendulum that could swing all around its center of rotation, even making complete revolutions, as illustrated by Fig. 2b.

Methods

We conducted a total of eight, 90- to 120-minute, open-ended, individual interviews with three undergraduate students. Each student we interviewed had completed differential, integral and multi-variable calculus and had taken or was currently taking differential equations. The interviews used a set of pre-planned tasks as a springboard for exploration of mathematical ideas that proved of interest to the student, rather than as a strict progression of problems to complete. We did not know ahead of time what would be of interest to students, what they would find problematic, or what they would find challenging. We therefore used the prepared tasks as a resource rather than as a strict progression of problems to complete.

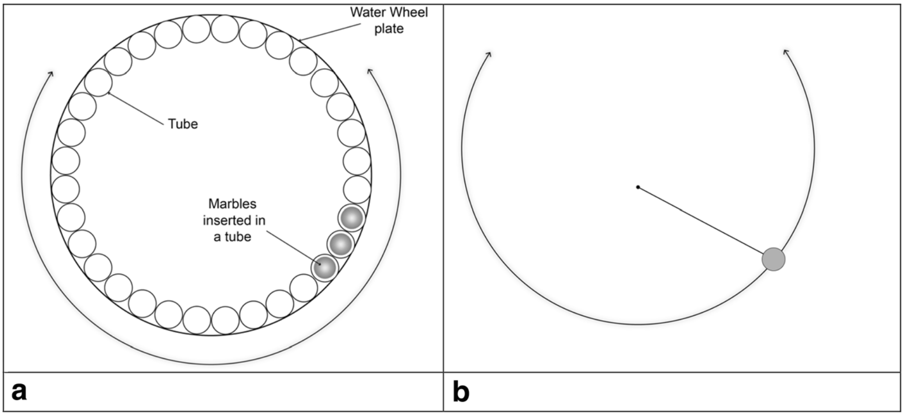


Fig. 2 **a** Marbles inserted as weights in some of the tubes (top view); **b** equivalent pendulum able to turn all the way around

As a result, each interview took on its own character, even though students engaged in similar problems. This was not surprising for us because the mathematics that students encounter is not in the tasks nor is it in the water wheel, but rather develops through their interactions and dialogue with the tasks, the interviewer, the available tools (e.g. the water wheel, the computer software), and the norms and expectations that emerge during the interviews. Our perspective on this issue is influenced by the theory of symbolic interactionism as developed by Blumer (1969) and a theoretical orientation that views learning as both a social and psychological accomplishment (Cobb & Bauersfeld, 1995).

In this article, we attend to the experiences of one student, Jake, in his second of three interviews. We focus on this because it was in this session that he created and justified graphs and diagrams, and also explored digitally produced graphs of the water wheel's motion. As such, this interview was most helpful in our understanding the nature of conversations with materials and diagrams. Jake encountered a perplexing computer-generated graph that was qualitatively different from what he had predicted. Moreover, he eventually reconciled the graphs, in large part due to touching physically and sensing the forces at play in the motion of the water wheel.

All interviews were videotaped and transcribed. Copies of any written work produced by the student and interviewer were also retained. Since the second interview integrated a computer interface with photogates (see Fig. 1), which enabled students to generate real-time graphs of angular velocity and angular acceleration, we used two video cameras during this interview – one focusing on the participants and one on the computer screen. We systematically reviewed all videotapes and transcripts, first breakingdown each interview into approximately ten-minute episodes, wherever there seemed to be a natural break or change in focus. We then developed written summaries for each of these episodes. Systematic review of these summaries helped us identify several key episodes that were then analyzed in detail. Analyses of these episodes were discussed by all four authors in order to develop a shared sense of interpretation and to minimize careless interpretations.

Our analytic methods are grounded in micro-ethnography (Erickson, 1996, 2004; Goodwin, 2003). Micro-ethnography encompasses a collection of techniques and means of analysis tracing the moment-by-moment bodily and situated activity of subjects engaged in certain events and interactions. Talk, gesture, facial expression, body posture, drawing of symbols, manipulation of tools, pointing, pace and gaze are all instances of modalities to be traced. This approach to analyzing the data enabled us to trace body activity over short periods of time in complex multiplicity (i.e. talk, gaze, gesture, posture, facial expression, tone of voice, etc.). These fine-grained, documented events helped us shape and extend our interpretations of the conversations Jake engaged in with the material resources at hand, the diagrams and graphs he produced, and the computer-generated graphs that spoke to him.

We avoid interpretive commentaries reducible to saying that a certain event is an example of some phenomenon or process previously identified in the literature. While the event is likely to be related to some of them, it must inspire original contributions. Micro-ethnographers document and pay close attention to potentially countless nuances, because any event can prove a reflection of a complex mesh of material, social, psychological, technological, symbolic, historical and biological aspects, including those regarding the worlds of the analysts themselves. Literature from philosophy, mathematics and the social sciences play crucial roles in expanding our interpretive horizons and not in prescribing what we would see. If all we could see was a faithful repetition of that which has been reported in the literature, it would mean that we had stopped practicing close listening.

Selected Episodes

We present our analysis of conversations with diagrams via six contiguous episodes during the second session. Except for Episodes 3 and 5, which are only briefly summarized, each includes one or more tables with transcript excerpts and annotated figures highlighting relevant gestures and bodily engagement. Tables are followed by reflective analysis. The use of italics inside the numbered transcripts marks words or expressions that are uttered in an emphasized manner (e.g. slightly prolonged).

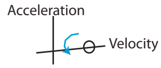
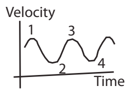
Episode 1: Temporalizing Diagrams on a Whiteboard

Informed by his work in the previous interview, Jake starts his analysis by re-creating acceleration versus velocity graphs based on two graphs of velocity versus time (a sinusoidal graph centered on the t-axis and a sinusoidal graph above the t-axis).

Commentaries about Table 1

Invoking his recollection, Jake draws a circle. A prominent aspect of our analysis is that a conversation with diagrams, which is what Jake develops, is a performance. The diagrams – graphs in this case – enter into an animated interaction among themselves and with the diagram user which is bodily enacted by means of gestures, talk and drawings. The most central contribution of a performance with diagrams, we propose, is the occurrence of polyvalent events. The qualifier "polyvalent" conveys the sense that these events combine diverse temporalities, such as the temporality of the performative acts (e.g. the series of utterances by Jake in Table 1) and the temporality of the diagrammed events (e.g. imaginary turning of the water wheel at varying speed).

Table 1 Re-creating an acceleration versus velocity graph for the case of the sinusoidal velocity–time graph above the *t*-axis

#	Text	Gesture	Image
1	Jake: "If I recall correctly, was somewhere, like, around here.	Draws a circle counterclockwise from the 3 o'clock position on the *x*-axis	
2	Because, as the, uh…	Points with left hand to the start of the velocity versus time graph	
3	These are the points where the acceleration is zero. So, that's, these two,	Points to first local max (1), first local min (2)	
4	these points where the acceleration is zero.	Points to second local max (3), second local min (4)	
5	Because velocity is zero. Wait, wait, wait."	Crosses arm and bring hand to chin	

The time of diagrammed events by the water wheel and the time of the performance itself are two temporal trajectories that can merge and bifurcate: aspects that occur simultaneously in one may succeed each other in the other. For example, a "zero acceleration" condition on the motion of the water wheel can be indicated by a series of drawing acts, such as marking the circle (Table 1, row 1) as it crosses the horizontal axis and a local max or min (Table 1, rows 3–4), and by uttering twice, "These are the points where the acceleration is zero" (Table 1, rows 3–4). An event would be univalent if it expressed itself along a single temporal trajectory.

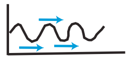
Note how any act of drawing introduces its own performative temporality: he draws the circle counterclockwise, from a certain point to another (Table 1, row 1). The arrow of time in a drawing act can be more or less relevant; perhaps the only thing that mattered to Jake here was to obtain a circular shape, whether it were drawn counterclockwise or not. However, the rotational direction will soon become significant as he strives to show how the velocity versus time graph creates necessary constraints on how the circle gets to be drawn.

As he continues, Jake says, "These are the points where the acceleration is zero", probably indicating the points where the circle crosses the horizontal axis. Then he "deactivates" his left hand in order to move his right one to indicate the points on the velocity versus time graph where the acceleration is zero (Table 1, rows 3–4). Note that this kind of choices, such as moving up the right hand and deactivating the left one, are not deliberate decisions but rather spontaneous bodily acts that fall smoothly into place; they are part of what the body already knows as it engages in a conversation with diagrams. But then Jake says something that pulls him back (Table 1, row 5): "Because velocity is zero. Wait, wait, wait."

Commentaries About Table 2

At the end of Table 1, Jake says, "Because velocity is zero" at a time in which the diagram tells that it is not. The mismatch calls for a "stepping back" ("Wait, wait, wait"). Upon sensing a contradictory attribution of acceleration and velocity being both simultaneously zero at a point in time in which velocity is not zero, Jake moved to reconfigure the overall relationship between the two graphs: "acceleration is the derivative of velocity", "the slope [i.e. not the function] is zero when

Table 2 Locating where acceleration is zero.

#	Text	Gesture	Image
1	Jake: "Acceleration is the,	Points toward the acceleration versus velocity graph	
2	uh, derivative of velocity	Points toward the velocity versus time graph	
3	So, yes, the, this, uh, the slope is zero	Draws twice horizontal tangent line at first local max	
4	That's where the acceleration is zero."	Draws horizontal tangent line at (2), (3), (4)	

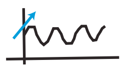
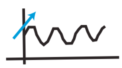
the acceleration is zero". This kind of reconfiguration of relationships triggered by local contradictions is, in all likelihood, a common occurrence in conversations with diagrams: a momentary sense of contradiction pulls the diagram-maker back to reassess overarching relationships across diagrams.

Jake ends by marking all the instances of an "acceleration zero" event (Table 2, row 4). Marking the successive occurrences of an event is a discrete temporalizing that "jumps" from one occurrence to the other. Our next commentary will highlight a continuous temporalizing of diagrams.

Commentaries about Table 3

In Table 3, Jake temporalizes the velocity versus time graph in a manner different from how he had done just moments ago: it is now about a continuous going from the origin to the first local maximum, instead of marking discrete sequential moments. From the origin to the first max "velocity increased", "acceleration's positive" and, along the way, acceleration reaches a maximum after which it goes toward zero. This kind of temporalizing "to become" ("acceleration becomes zero. But, the velocity becomes maximum.") indicates the nature of the change: it is a continuous becoming. Jake spontaneously moves his finger from–to instead of jumping the marker here-and-then-here as he did in Table 2.

Table 3 Noting the interplay between velocity and acceleration.

#	Text	Gesture	Image
1	Jake: "So, um, and this velocity increased	Points to top-most velocity versus time graph and traces from origin to first local max	
2	So, the acceleration . . . OK, now, over here, acceleration's positive.	Points to y-axis and the velocity graph	
3	So, it must be here right where the maximum acceleration	Traces from y-axis to first local max	
4	And, acceleration becomes zero. But, the velocity becomes maximum. And, as velocity starts to"		

Commentaries about Table 4

Jake discusses going from the origin to the first max of the velocity graph, but this time we can see that he is also tracing the corresponding arc in the acceleration versus velocity graph (Table 4, row 3). Jake follows a becoming, by co-tracing, with his left and right hands, the goings of both graphs. Note again the verbs that populate his utterances, some of them in present continuous tense: “acceleration is going”, “velocity is also increasing”, “velocity becomes maximum” and “acceleration becomes zero”. Perhaps the clearest example of a continuous change corresponding to a kind of “going through” is the last one: he taps the crossing of the horizontal axis and, rather than describing it as acceleration “being” zero, he voices its “becoming negative”.

Episode 2: Creating Digital Diagrams by Driving the Wheel

In this episode, Jake transitions to the water wheel where he pushes and moves the wheel by hand to create digitally produced graphs. This is the first time that he has used the sensor capability to produce graphs in real time.

Commentaries about Table 5

As Jake works to have the water wheel turning in an anticlockwise direction and going in a cyclic sequence of “fast and slow”, he uses both hands to drive the wheel accordingly, while keeping one or the other hand in touch with the wheel. Afterwards, following his first glimpse at the computer screen, he notices that parts of the graph are invisible, as they occupy regions falling outside the screen’s graphic space. Jake’s assessment is immediate: “I made it too fast. Too drastic…” From this experience of drawing digital diagrams by driving materials –the water wheel itself in this case– a key question comes to us: how is this way of generating diagrams different from doing so with markers on a whiteboard, as discussed in Episode 1?

One first discernible difference is that the speed with which he moved the marker on the whiteboard was not a matter of concern. We are so used to drawing with a marker

Table 4 Co-ordinating the velocity versus time graph with the acceleration versus velocity graph.

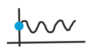
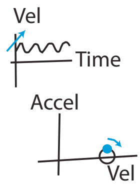
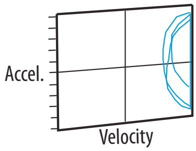
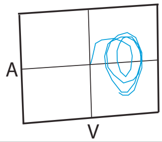
#	Text	Gesture	Image
1	Jake: “OK. So, as we start here,	Points to the start of the velocity graph on the y-axis	
2	acceleration is going, is at, at, its max.	Points to 12 o’clock position on circle with right hand	
3	And, velocity is also increasing, too.	Traces along sine graph towards first local max (1) and moves clockwise on circle in 12 to 3 position	
4	And, over here, velocity becomes maximum.	Points to first local max (1) of sine and 3 o’clock position on circle	
5	But, the, uh, acceleration becomes zero. And, acceleration becomes negative here.”	Lifts marker from and taps on circle in 3 o’clock position	

Table 5 Creating digital diagrams.

#	Text	Gesture	Image
1	Chris: "So, you are trying to make it go…?"	While recording a graph of acceleration versus velocity, Jake turns the wheel in an anti-clockwise direction for fifteen seconds; he is standing up and silently looking at the water wheel	
2	Jake: "Fast and slow. Yes. And, I think I made it too fast. Too drastic."	Looking at the computer screen and noticing that parts of the graph step outside the visible range of the axes	
3	Chris helps Jake to change the scale of the axes so that the graph can display a wider range of velocities. Jake turns the wheel in an anti-clockwise direction for eighteen seconds; he is standing up and silently looking at the water wheel. As compared with the previous run, he turns the wheel more slowly, with slight touches of both hands: the computer screen displays the graph shown.		

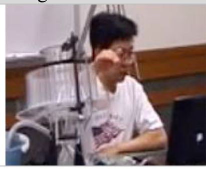
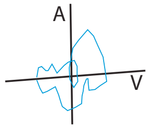
or pen that whatever speed is "natural" to the drawing body is to be enacted. On the other hand, Jake acknowledges that there is something like going too fast or too drastic with the turning of the water wheel. Exceeding desirable boundaries in the motion of the water wheel is a matter relative to the graphical display.

In addition to changing the velocity scale to register a wider range of velocities, during Jake's next attempt to record a graph (Table 5, row 3), he moves the wheel much slower. Each fast/slow cycle gets displayed as successive vertical ovals shifting right and left, all of them within the displayed region of the graph (Table 5, row 3). While the change of scale involves a choice of new quantitative limits for the velocity axis, the overall slowing of the turning wheel, as driven by his hands, requires a qualitative difference in the kinesthesia of Jake's hands and arms, and of the water wheel: the wheel is to go slower in a way that is not quantified but felt. Jake's subsequent examination of the computer screen validates the new less-drastic movement of the wheel.

Commentaries on Table 6

If, compared with Table 5, one thinks of the utterance of Table 6 as a distinct way of generating a diagram, namely moving the wheel while looking at the computer screen, the difference is clear: imagine drawing a considerably complex diagram or graph with a marker that does not leave traces on the whiteboard until it is finished, when it becomes visible all at once. Jake, who in Table 5 had been making the wheel oscillate by slowly dragging it back and forth with both hands while standing over it and looking at it, now completely changes his way of handling the wheel, by regularly pushing it back and forth from the same point of contact (Table 6). He is now sitting and keeps his focus on the computer screen. We observe a change in his overall pattern of action (from standing to sitting; from using both hands to pushing with one hand; from focusing just on the wheel to gazing continuously at the screen while moving the wheel), which entailed a new mode of visuo-tactile performance.Following Chris's

Table 6 Attending to the graph production in real time.

#	Text	Gesture	Image
1	Chris encouraged Jake to proceed differently by "doing the wheel and watching the graph...in real time [i.e. simultaneously]"	Jake proceeds accordingly, starting at around the 9 o'clock position, rotating the wheel counterclockwise to around 3 o'clock position	
2		Jake then rotates clockwise back to 9 o'clock position, all the while looking at the computer screen	

advice, he calibrates and feels the movement of his hand/wheel *while* ascertaining visuo-graphical responsiveness on the computer screen.

Episode 3 (Brief Summary): Creating Digital Diagrams with a Heavy Spot

The "heavy spot" was a term coined by Jake in the previous interview to describe bell-shaped accumulation of water that happens with back-and-forth, pendulum-like motion. To generate a heavy spot without water flowing through the tubes, Chris gives Jake several stacks of marbles taped together, which they insert inside several contiguous tubes of the water wheel. Jake then creates digital diagrams using the marble-made heavy spot. In particular, he creates a series of slightly superimposed circles centered on the origin by setting the wheel in motion, so that the heavy spot does not move far from its equilibrium position at the bottom of the wheel. He seems unsurprised with the circular-shaped graph, since it appears sufficiently in line with his initial drawing on the blackboard and his initial recollections and expectations. He observes the circle gradually reducing in size and the graph collapsing to the origin point, as both acceleration and velocity tend to zero when the wheel slows down and, eventually, stops at the equilibrium point. This sets the stage for the next episode in which Jake encountered a surprising digital graph.

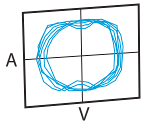
Episode 4: Expecting a Circle but Finding an Apple

This episode begins with Chris proposing a new situation to be explored with the water wheel, one for which itsmovement starts with the heavy spot at the top position, then it is released to oscillate freely.

Commentaries about Table 7

This new situation (Table 7) might be seen, in principle, as simply a variation on the case Jake was investigating just before, as it still involves back-and-forth movement of the wheel, but this time starting with the heavy spot close to the top height. Nevertheless, it actually reconfigures the conversation. Think of any conversation between two people: a little variation, for example a rewording of a previous utterance or a slight

Table 7 Finding a surprising, digitally-produced, acceleration versus velocity graph.

#	Text	Gesture	Image
1	Chris: "So, what if we take our heavy spot and started farther, like up here?"	Chris rotates the wheel such that marbles are at 11 o'clock position	
2	Jake: "Oh, OK." Chris: "Just when it's about to. Ready?"	Jake projects his body towards the wheel with eyes wide open, then moves to the computer screen and clicks to restart the software	
3	Jake: "Yeah."	Chris releases the wheel, while Jake stares at the screen where the following graph is originating	
4		Jake sits back and looks puzzled, first at the wheel, then at the computer screen, folding his arms	
5	Chris: "Huh. Why do you sit back and fold your arms, Jake?"	Chris folds his arms, too, and laughs	

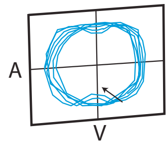
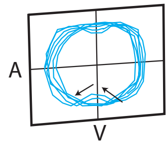
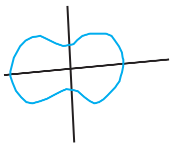
change in the tone, can completely change the course of the discourse, e.g. creating discomfort or distance in one of the interlocutors. This is so because each event is permeated by affective tones, which in turn characterize the ways in which we are responsive to events.

Commentaries about Table 8

In this case (Table 7 and 8), the movement of the water wheel creates a diagram on the screen that is unexpected for Jake. It is as if the wheel was now proposing a new diagram, different from the expected "more or less circular" one discussed by Jake at the whiteboard in Episode 1. Now, the overall shape is described similar to "an apple" as soon as Chris recreated it at the whiteboard (Table 8, rows 4–5). Jake interrogates himself about the strange behavior of the graph, "why the acceleration went down momentarily" (Table 8, rows 1–2). This is a bit like a conversation in which one of the conversants says something puzzling, even nonsensical, prompting others to enquire, questions of the sort of: Why? What do you mean? How come?

While in a regular conversation another utterance follows on, in this conversation with materials it reverberates primarily in the ways in which Jake is moving. Jake steps back, laughs and raises his eyebrows, then he brings the left hand to his chin appearing puzzled about the apple-shaped graph. By looking at these ways of moving, even if we had not seen the graph on the screen, we would acknowledge a rupture or discontinuity in the conversation.

Table 8 Puzzling over the unexpected graph.

#	Text	Gesture	Image
1	Jake: “Well, I was thinking why the acceleration...went down	Jake traces from the lowest point in quadrant four to the y-intercept, which elevates slightly, and then emphatically raises his eyebrows	
2	momentarily.	He traces from the lowest point in the fourth quadrantto the y-intercept, which elevates slightly, past to the lowest point in the third quadrant	
3	Ohhhh.”	He brings left hand to chin and looks again puzzled at the screen	
4	Chris: “So, it did something like,”	Chris starts drawing the curve on the blackboard, stops when the marker touches the vertical axes for the first time and looks at Jake	
5	Jake: “Yes, yes, like, an apple.” Chris: “Like an apple.” Jake: “Yeah.”	Chris completes the curve and Jake stares at the whiteboard where Chris is drawing	

Note the central role of the digital display based on the measures from specialized sensors. Normally, the apple shape is not visible during the water wheel’s motion; without the sensing instrument, it would have just been a skipped-over phenomenon. Like a doctor examining a patient, the conversation is not only with the patient: it is also with a panoply of observation techniques and instruments. Here we get to one of the core ideas we want to contribute for this special issue: the centrality of the interplay between the physical and the digital when the latter plays the role of an instrument, helping to make tangible and visible myriad aspects that otherwise would simply remain unnoticeable.

Episode 5 (Brief Summary): Developing a Theory for the Apple Shape

Between the end of Episode 4 and the beginning of this one, Jake spent two and a half minutes observing the wheel in motion and tracing it on the concurrent graphing, with a particular emphasis on the acceleration as the heavy spot neared the top-side. He identified that the upper and lower dimples on the apple graph corresponded to zero velocity: they had to have happened when the heavy spot was close to the top, coincident with the wheel changing direction. After a close visuo-tactile examination of the wheel in motion and the corresponding graphs of acceleration versus velocity, Jake remarked, smiling, that he was entertaining an explanation for the apple shape.

The focus of his investigation had been to identify the specific moments of the wheel's motion during which the creation of the dimples took place. Something must occur at that time, originating a dimple. Attending to transient qualities accompanying a puzzling event, we think, is a critical process in a conversation with materials. In striving to understand what would be happening to the water wheel as the heavy spot reached the highest levels, Jake had observed a certain "jiggle" of the marbles, which coupled the wheel's change of direction and the free room that the marbles had within the tube holding them at the heavy spot. The marbles seemed to be going up at the time in which the wheel started to push them back down, activating them as in a moment of hesitation. Jake wanted to show this happening to Chris. As he releases the wheel, he followed the heavy spot with his head, trying to observe minute movements of the marbles. As the "jiggle" of the marbles was not visually obvious, Jake resorts to an alternative: he leaves the room to come back with some paper tissues, which were subsequently used by Chris and Jake to stuff the tubes containing the marbles so that they could not move inside.

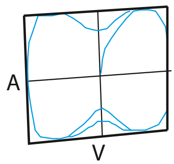
Episode 6: Touching and Feeling Variations in Weight

Between the end of Episode 5 and the beginning of this one, Jake and Chris stuff pieces of paper tissues into the tubes holding the marbles, so that the marbles would be tight inside, preventing them from moving. We pick up where Jake tests out his theory (Table 9).

Commentaries about Table 9

Upon releasing the water wheel, the wheel's response was immediate: the apple-shape movement is insensitive to little movements of the marbles. Against Jake's expectation for the validity of the theory he had developed in Episode 5, he felt shifted back to ground zero. He stayed holding his forehead while looking at the water wheel for a long time. What is the significance of such silent staring from

Table 9 Realizing that his theory was invalid

#	Text	Gesture	Image
1	Chris: "Tell me when to let go."	Chris holds the heavy spot near the top	
2	Jake: "OK."	Chris lets the wheel move	
3	Jake: "Huh! So, it's, it's not."	The software starts creating an 'apple-shaped' graph, Jake laughs	
4	Chris: "It's not the movement?" Jake: "It's not the movement."	Jake folds his arms and laughs	
5	Jake: "OK. Well. It's not the movement. Huh."	Jake laughs nervously then holds his forehead with his left hand for fifteen seconds	

slightly afar, away from touching and moving things? It is clear that thoughts are in intense motion, but what kind of thoughts? How can we grasp this thought-movement from a conversational point of view?

In fact, we noticed that this kind of silent staring happens very often in conversations with materials and diagrams: the material/diagram makers step back and quietly look at them. Unless a possible sense-making image of things quickly comes up to one of the conversants, they are likely to "step back" and inquire about the situation as a whole. It is like probing the place and its history from a rather holistic point of view, trying out this angle and that angle, until finding one that seems to be promising, triggering a more in-depth focused seeking.

Commentaries about Table 10

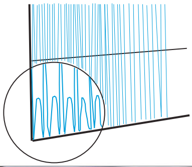
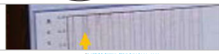
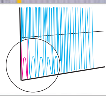
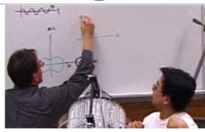
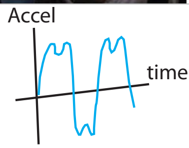
As the episode continues (Table 10), Chris suggests looking at alternative digitally created graphs. This prompts a radical change through which Jake ceases staring and gets into the active mode of touching and moving the water wheel.

Jake abandons his first theory, discussed in Episode 5, due to the experiment in which he uses the tissues to prevent the marbles from moving inside the tubes. Following Chris's suggestion, he looks for another type of graph to investigate the motion of the water wheel, or another "angle" from which to look at the strange event. These graphs are available within the system as different modes to capture digitally the phenomena modelled by the water wheel, enlarging the ways in which we can explore what is going on from different angles.

Concerning this, it is of interest for our contribution to the special issue to highlight two aspects. First, the digital sensors offer a widening perception of an event. It is hard to observe acceleration and changes in acceleration by direct sight while the movement of the wheel (and any movement) is happening, but these are unfolded through the sensor which is able to sense and display them graphically. Second, different graphs for the same event have the potential to broaden its polyvalence. The software connected to the sensor displays a number of graphs, which are different ways of unpacking the same "thing" from different points of views and ways of relating qualitative aspects of movement (as velocity and acceleration). Each of them speaks differently about the event and, at the same time, the event is more than their sum. As we see in this segment, selecting a new type of graph can trigger a new line of investigation and/or shed new light on a previous digital diagram.

Jake's choice of the acceleration versus time graph immediately surprises him ("whoa!", [Table 10, row 4]). The appearance of a "wicked move" (Table 10, row 10), namely a dimple that characterizes the periodic graph the software shows (Table 10, rows 4–6), does not explain the *why* of the situation, which is what Jake is investigating, but rather creates a different feeling for the same event, as if he has heard it differently uttered or explained, or seen it under a new light. That intuition might emerge because of the ways in which the event has been told, or because it reminds the conversant of a similar situation, creating a sense for which that aspect could be of interest in understanding the present situation.

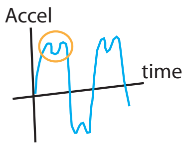
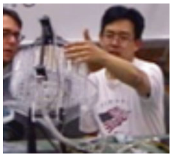
Table 10 Exploring acceleration versus time graphs.

#	Text	Gesture	Image
1	Chris: "We could try some other graphs…I don't know if that helps or not. But there is a whole palette of graphs. You know, feel free to choose whatever you like."	Chris and Jake look at each other	
2	Jake: "Yeah. OK, Yeah, yeah, OK. OK.	Jake nodding, grabs the mouse and looks for the different options of the software	
3	Yes, yes, OK. Actually then, I'm going to try. Well, velocity,	Jake rapidly gazes at the wheel, Chris stops it from swinging	
4	I know, won't help too much. So, I'll try acceleration versus time. Whoa!	He clicks on acceleration versus time, the software displays the graph; Jake is surprised by the little dimples at each min and max	
5	OK. See, look at this	He points to the first minimum in the graph	
6	I mean, uh, kind of goes down. But, minimizes, but, and, it comes up a little bit. And, then it goes down again, comes up"	He follows with index finger the piece of graph he is describing	
7	Chris: "So, let's write it on the board." Jake: "OK."	Jake looks at Chris and smiles	
8	Chris: "So, this is acceleration versus time?" Jake: "Right, right."	Chris grabs the marker and draws two orthogonal axes on the whiteboard	
9	Chris: "And, it does?" Jake: "It, uh." Chris: "Do you want to graph it?"	Chris and Jake both look at the computer screen and Chris passes the marker to Jake	
10	Jake: "Yeah. It's going to kind of go like this. This kind of wicked move."	Jake draws the graph, then stares at the graph with folded arms, then at the computer screen, then back at the whiteboard	

Commentaries about Table 11

Addressing Chris's question (Table 11, row 3), which focuses on the location of the heavy spot that corresponds to the "wicked move" in the graph, Jake starts to point out variations of acceleration as the wheel moves. He starts by re-stating that the maximum is near the top, when the velocity changes direction and localizing zero acceleration at the bottom position of the wheel (Table 11, rows 4–5). Note that, in the portion of the graph that is now of interest for Jake, acceleration undergoes a slight but very significant variation that creates the maximum-relative minimum–maximum sequence we observe in the dimple. Jake orients himself inside the conversation by localizing first two extremes of acceleration (zero and maximum) along the wheel.

Table 11 Focusing in on the ‘wicked move’.

#	Text	Gesture	Image
1	Chris: “Yeah. So, what is this wicked move up here?”	Chris points to the part of the graph that is in-between the two first high peaks	
2	Jake: “Yeah, that’s, I don’t know. Um.”	Jake looks at the water wheel	
3	Chris: “So, What’s. Where is it happening? Does it happen when it’s, like this?”	Chris places the heavy spot at a certain angle near the bottom	
4	Jake: “OK. When it’s up here, that’s when the acceleration is the maximum.	Jake indicates the heavy spot near the top on the left side	
5	And here, the acceleration is zero.”	He holds the heavy spot near the bottom	
6	Chris: “How do you know where the acceleration is maximum?”	He continues to move the heavy spot around to different locations	
7	Jake: “Um. Well. One thing that I just noticed is that, uh.	He continues to slowly move the wheel around with his right hand on the heavy spot	
8	It seems to, uh, uh, it, it seems to be proportional to the, well, see, as this thing goes up.	He moves the heavy spot with left hand, clockwise, to the top	
9	It seems to…What?!?”	He releases the wheel, smiles, puzzled, then holds his chin with left hand	

A new question by Chris (Table 11, row 6) asks Jake to focus on the point of maximum acceleration. Jake continues to move the wheel slowly around, and the heavy spot to different positions, controlling the oscillations with one hand or the other. He slows down his utterance, following the back-and-forth movement with his head (Table 11, rows 4–6). But then, while driving the oscillations with his hand, he starts to feel how heavy the heavy spot is at different angles and, contrary to his expectation, he feels that it is not heaviest at the top (Table 11, rows 7–8). He interrupts his own sentence and says “What?!?” (Table 11, row 9), looking again puzzled and stepping back from the wheel. We next examine the relevance of feeling weight and force in moving the wheel as a way of elucidating the role of touching and sensing variations and changes in conversation with materials.

Commentaries about Table 12

In Table 12, Jake articulates his insights about force gleaned through actual feeling the heavy spot on the wheel as it moves. At the bottom position, he says that, “the force is zero” (Table 12, row 7) while at the side he feels greater weight: “I feel more weight right now” (Table 12, row 2). Concerning the top position, he says, “the weight is almost zero up here” (Table 12, row 3). These utterances bring forth ways in which Jake is now adjusting his own mapping of acceleration around the wheel, creating a new sense for acceleration relative to the force felt at the side position. In Episodes 1 and 2, we commented on the performative aspects temporalizing diagrams on the

Table 12 Feeling variations in weight.

#	Text	Gesture	Image
1	Jake: "One, one thing I'm noticing is that uh.	He moves the wheel around with right hand, looks at the computer screen, left hand still holds his chin	
2	Well, see, as I move this, I feel more weight right now you know.	He moves heavy spot so that it is in the left most position (3 o'clock) –he is sensing the weight of the heavy spot with his index finger slightly inside one of the tubes	
3	But, the, uh, weight is almost zero up here.	He moves the heavy spot to the top	
4	Um. So, in other words, you know, up here [top], I don't have to apply, uh, as much force.	He moves slightly the wheel around the top position	
5	As a matter of fact, it pretty much stays there.	He removes his hand from the wheel	
6	But, if it's in, you know, the middle. But, over here is, the force is like, the, uh, like a maximum.	He brings the heavy spot at the side	
7	Um, um. And, uh, over, over here, of course, you know, the force is pretty much zero.	He brings the heavy spot at the bottom, opens his left hand	
8	So, uh, I don't know, force due to gravity, I think, it's, uh, like, like, almost zero.	He brings the heavy spot to the upper position, then lets it go down	
9	And, over here it, it seems to be most, the force seems to be most strongest. And, over here [at the bottom], of course, uh, the, the, uh, force is zero."	He brings the heavy spot to the left side (3 o'clock) and then releases the wheel	

whiteboard. Temporalizing the variation of acceleration as the wheel turns is now involved in a crucial performative aspect: sensing how heavy the heavy spot feels. Feeling the water wheel (the heavy spot) is hard work. It entails developing a new sensitivity to the movement of the wheel and a qualitative refinement to localize acceleration.

The previous comments bring forth a crucial point for our contribution: materiality-feeling gets to be significant through physical involvement with the wheel. Getting a feeling for acceleration by moving the device is something which is not mediated nor apparent, and is largely not replaceable by software-simulated behavior. It is true that one could find the point of maximum weight – that is the point of maximum acceleration – just by "applying" Newton's laws and the force of gravity, which is something that Jake had developed during the previous interview session. However, Jake is here illustrating another path, more direct, bodily and sensuous. This path is available through materiality-feeling and is too-often absent in the doing of mathematics.

Accounting for the Origins of the Wicked Move

To conclude, we summarize Jake's subsequent conversation with the water wheel and, given the space constraints, forego annotated transcripts and figures. As Jake continues his touching and sensing, he undergoes a key insight. Referring to the point in which the wheel changes direction in its oscillation, he observes that, "over here, it passes the maximum acceleration point. But, yet, over here [at the point where it changes direction, near the top], it's, *there's still acceleration left*." He is revealing that, at the top, when the velocity changes direction, "there's still acceleration left", as opposed to being maximum acceleration. This is a crucial realization to account for the wicked move, which encompasses making sense of a quick and subtle variation of acceleration as time passes.

It would probably not be noticed or sensed if the digital graph were not available. Moreover, localizing it is achieved through direct manipulation of the wheel, and the development of a new sensitivity with respect to the dynamics of the wheel. Jake then locates the where/when of the wicked move: it is right above the 3 and 9 o'clock positions. He also makes a prediction for how to eliminate the apple shape: "in other words, if I start, like, up around here [at the 9 o'clock position] on the, it should go, *it shouldn't become like an apple shape*". Jake proceeds to test his new theory and finds that the digitally produced graph is as he predicted.

Discussion and Conclusion

This article details a case study on how a conversation with materials and diagrams –the actual use of materials and diagrams to think, imagine, explain, collaborate, design, and build– featured a certain kind of interplay between physical and digital components. The physical components present in this setting included the water wheel, while digital components allowed for real-time graphical displays corresponding to the turning of the water wheel. The article encompasses five episodes that have been transcribed and commented on. In this discussion, for each of the five annotated episodes, we distillthe foremost ideas we have learned about and hope to contribute.

Polyvalent Events

In Episode 1, we interpreted Jake's use of diagrams as temporalizing them by embedding them in a performative stream of activities meshed with the past and potential behavior of the water wheel. At the end of Table 1, the performative stream included a "stepping back", triggered by a hint of contradiction, aiming at reassessing overarching relationships among diverse diagrams. We distinguished two types of temporalizing – discrete and continuous – with the former characterizing the passage of time as a sequence of separate events, with the latter focusing on the present continuous mode of event-becoming. We characterized these events as polyvalent: they merge and bifurcate diverse temporalities.

Emerging Intensities

In Episode 2, we discussed differences between drawing diagrams on a whiteboard and generating digital diagrams by moving the water wheel. We pointed to a major difference in that, while drawing diagrams rendered many aspects of the drawing acts as inconsequential or to be taken for granted (e.g. how fast we trace a circle), the genesis of digital diagrams involved lively encounters with momentous features (e.g. how fast to turn the water wheel), which engaged Jake in the development of a sense or a feel for them. For our purposes, it is useful to think of such sense or feel as varying along an “intensity”:

> We are thus led to define the intensity of a superficial effort in the same way as that of a deep-seated psychic feeling. In both cases there is a qualitative progress and an increasing complexity, indistinctly perceived. (Bergson, 1913/1950, p. 26)

We interpret an intensity as a gradation of *qualitative* differences going from “less” to “more”, or vice versa (Bergson 1913/1950, Chapter 1), differing from an extensity such as the axis of velocity in the graph displayed in Table 1, row 1, which is ordered from left to right according to quantitative differences amenable to addition and multiplication, in that intensity is a source of *felt* differences unsuitable to arithmetic operations. Jake’s movement of the wheel, as he worked to appreciate its graphical effects, enabled him gradually to develop a kinesthetic sense for what might be “too fast” or “slow enough” attuned to contextual demands, which is what we customarily refer to as “getting a feel for it”.

Note that Jake’s development of a lived intensity for how fast to turn the water wheel was based on body motion, the exertion of subtle forces and the on-going graphical responsiveness on the computer screen. While it is possible to experience intensities with simulated mechanisms, such as, say, a digitally simulated water wheel with sliders continuously varying its rotational speed, the physicality of the water wheel offers an extraordinarily broad range of sensorial and kinesthetic engagements that are largely unavailable with a simulated one. Kinesthesia, muscular enactments, touch and tangibility are currently difficult to involve with pure simulationsproperly. The materiality of the water wheel can be thought of contributing a radical broadening and richness of emerging intensities, subsequently available to the collective imagination by means of gestures, diagrams and wordings.

Digital Sensors and the Emergence of New Forms of Sensitivity and Responsiveness

In Episode 4, the digital graphs display what appears to be an unmediated relationship with the motion of the water wheel. They are responsive, in real time, to minute changes in the angular speed and acceleration. This sensitivity to subtle changes as they occur opens up a new window into the behavior of the water wheel, and prompts sudden shifts in the conversation with materials and diagrams, such as the evident disruption occasioned by the unexpected dimples. The apple shape that momentarily follows the initial movement of the water wheel is not directly visible in its physical motion. In this sense, digital graphs act as micro- or macro-temporal lenses. But, even more significantly, their being responsive in real time enabled Jake to broaden the conversation, by making it inclusive of additional observational and handling techniques. Note that this real-time responsiveness is equally important in the use of body-motion detectors, as well as in dragging within a dynamic geometry application. Sensitivity and responsiveness help make a tangible myriad of kinesthetic and sensuous aspects that otherwise remain unnoticeable.

Conversation with Materials to Test Theories

In Episode 5, we described how Jake concluded that the dimples occurred during the first few cycles, when the heavy spot was close to the top. Something must have been happening then: the dimple was an event whose moment of occurrence Jake had identified. Then he turned to the water wheel for a close physical examination of the circumstances of the puzzling event. As he followed the motion of the heavy spot, he perceived the marbles were undergoing a subtle jiggle when they changed the angular direction of their movement. It was as if they hesitated,momentarily. However, when he tried to show the jiggle to Chris, it was not clearly occurring, which prompted Jake to think of indirect means of revealing the jiggle: materially disallowing it would eliminate the dimples.

Materiality-Feeling

In Episode 6, the failure of the jiggle to explain the dimples was followed by fifteen seconds during which Jake silently stared at the water wheel while holding his forehead. This suggested to us another kind of “stepping back”: probing the materials and diagrams holistically, trying out this angle and that one, until finding out a point of view that *seems to be promising*. This happened when Jake switched to graphs of velocity versus time: the “wicked move” became apparent. This new way of scrutinizing a puzzling event called for a novel examination of its circumstances: how the weight of the heavy spot felt at different angles. He underwent a surprising sensing: that maximum lightness occurred when the heavy spot was at the top, not at the lateral sides. This was the beginning of re-assessing when the wicked move started and ended, over the course of which Jake exercised the power of materiality-feeling. We concluded that a contribution of the physicality of the water wheel, not replaceable by ordinary simulated behavior, is bringing materiality-feeling into play.

With this article, we have strived to illustrate and advance theoretical approaches, hopefully inspirational, on how conversations with materials and diagrams can interweave physical and digital synergies, animating bodily ways of mathematics learning. The particulars of a selected set of short episodes necessarily restrict of what the study is a case. Ours is a case of an exploration of a complex dynamical system – a water wheel furnished with electronic sensors and real-time digital displays – led by an undergraduate student majoring in engineering who was keen to share his ways of doing things. Further investigating how some of the core ideas we have learned about (e.g. polyvalent events, emerging intensities, digital sensors and the emergence of new forms of sensitivity and responsiveness, conversations with materials to test theories and materiality-feeling) play out in other settings is an indispensable step to developing, tuning and critiquing these ideas further.

Acknowledgments On behalf of all authors, the corresponding author states that there is no conflict of interest.

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