Unit 8: Inductance

8.1 Self Inductance

Consider a coil consisting of N turns and carrying current I as shown in Figure 1. If the current is steady, then the magnetic flux through the loop will remain constant. However, if the current changes this change in current induces an e.m.f. or voltage in that coil (or circuit) in such a direction to oppose what is causing it. If the current in the circuit is increasing, the induced e.m.f will be in the direction to oppose the increase in the current. The production of an e.m.f in a circuit by a change in current in that circuit is called **self induction**. According to Faraday's law of induction, the induced e.m.f. depends on the rate at which the current in that circuit is changing.

$$\varepsilon = -L\frac{dI}{dt}$$

Negative sign is in accordance with Lenz's law.

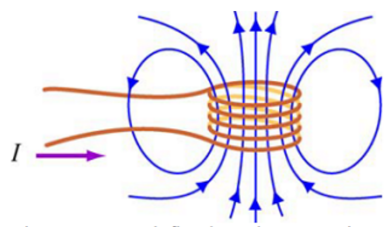

Figure 1. Magnetic flux through a current loop

Combining this equation with the previous Faraday's law of induction, $\varepsilon = -N\frac{d\phi_B}{dt}$, yields

$$L = N\frac{\phi_B}{I}$$

Inductance is flux per unit current. Physically, the inductance L is a measure of an inductor's "resistance" to the change of current; the larger the value of L, the lower the rate of change of current. The unit of inductance is called henry (H).

Inductance of a Solenoid

Consider a subsection of length l of the solenoid:

Flux linkage $= N\,\Phi_B = nl \cdot BA$ where A is cross-sectional area

$$\therefore \boxed{\begin{aligned} L &= \frac{N\Phi_B}{\mathrm{I}} = \mu_0 n^2 l A \\ \frac{L}{l} &= \mu_0 n^2 A = \mathit{Inductance\ per\ unit\ length} \end{aligned}}$$

Notice : (i) $L \propto n^2$

(ii) The inductance, like the capacitance, depends only on geometric factors, not on I.

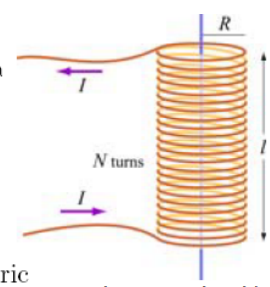

Figure 2. Solenoid

8.2 Mutual Inductance

Suppose two coils are placed near each other, as shown in Figure 3. The first coil has N_1 turns and carries a current I_1 which gives rise to a magnetic field $\vec{B_1}$. Since the two coils are close to each other, some of the magnetic field lines through coil 1 will also pass through coil 2. Let ϕ_{21} denote the magnetic flux through one turn of coil 2 due to I_1 . Now, by varying I_1 with time, there will be an induced emf associated with the changing magnetic flux in the second coil:

$$\varepsilon_2 = -N_2 \frac{d\phi_{21}}{dt}$$

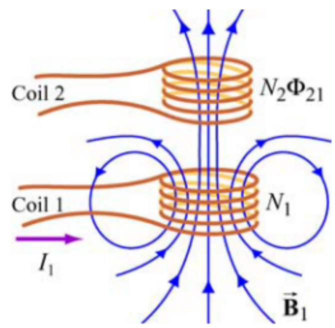

Figure 3. Changing current in coil 1 produces changing magnetic flux in coil 2.

Since the time rate of change of magnetic flux ϕ_{21} in coil 2 is proportional to the time rate of change of the current in coil 1, the induced e.m.f can be written as

$$\varepsilon_2 = -M \frac{dI_1}{dt}$$

where the proportionality constant M is called the **mutual inductance**. It has same SI unit as that of self inductance which is the henry (H). Similarly, e.m.f. induced in the first coil by the change in current in the second coil is written as

$$\varepsilon_1 = -M \frac{dI_2}{dt}$$

The production of an e.m.f. or voltage in one circuit due to change in current in the nearby second circuit is called **mutual inductance.**

8.3 RL Circuits

Consider the RL circuit, a circuit containing a resistor and an inductor in series, shown in Figure 4. At $t = 0$ the switch is closed. We find that the current does not rise immediately to its maximum value $I = \varepsilon/R$. This is due to the presence of the self-induced emf in the inductor.

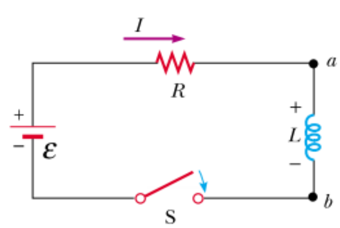

Figure 4. A series RL circuit. As the current increases toward its maximum value, an emf that opposes the increasing current is induced in the inductor.

Applying Kirchhoff's loop rule to this circuit, traversing the circuit in the clockwise direction, gives

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

The above equation can be rewritten as

$$\frac{dI}{I - \varepsilon/R} = -\frac{dt}{L/R}$$

Integrating over both sides and imposing the condition $I(t = 0) = 0$, the solution to the differential equation is

$$I(t) = \frac{\varepsilon}{R}(1 - e^{-Rt/L})$$

This expression shows how the inductor effects the current. The current does not increase instantly to its final equilibrium value when the switch is closed but instead increases according to an exponential function. If we remove the inductance in the circuit, which we can do by letting L approach zero, the exponential term becomes zero and we see that there is no time dependence of the current in this case, the current increases instantaneously to its final equilibrium value in the absence of the inductance. We can also write this expression as

$$I(t) = \frac{\varepsilon}{R}(1 - e^{-t/\tau})$$

where the constant τ is the **time constant** of the RL circuit:

$$\tau = \frac{L}{R}$$

Physically, τ is the time interval required for the current in the circuit to reach 63.2% of its final value ε/R. The time constant is a useful parameter for comparing the time responses of various circuits.

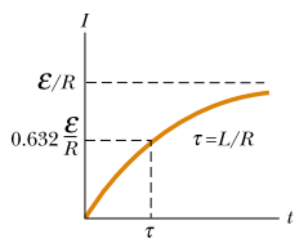

Figure 5. Plot of the current versus time for the RL circuit shown in Figure 4.

8.4 Energy stored in an Inductor

Consider once again the RL - circuit shown in Figure 4. which contain a battery that has negligible internal resistance. Suppose that the switch S is open for $t < 0$ and then closed at $t = 0$. The current in the circuit begins to increase, and a back emf that opposes the increasing current is induced in the inductor. Because the current is increasing, $\frac{dI}{dt}$ in Faraday's law is positive; thus, ε induced in inductor is negative. Applying Kirchhoff's loop rule to this circuit, in the clockwise direction, yields

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

After multiplying each term by I rearrangement yields

$$I\varepsilon = I^2R + LI\frac{dI}{dt}$$

Recognizing $I\varepsilon$ as the rate at which energy is supplied by the battery and I^2R as the rate at which energy is delivered to the resistor, we see that $LI(\frac{dI}{dt})$ must represent the rate at which energy is being stored in the inductor. If we let U denote the energy stored in the inductor at any time, then we can write the rate $\frac{dU}{dt}$ at which energy is stored as

$$\frac{dU}{dt} = LI\frac{dI}{dt}$$

To find the total energy stored in the inductor, we can rewrite this expression as $dU = LIdI$ and integrate:

$$U = \int dU = \int_0^I LIdI = L\int_0^I IdI$$
$$U = \frac{1}{2}LI^2$$

This expression represents the energy stored in the magnetic field of the inductor when the current is I and has the same form as that of the energy stored in the electric field of a capacitor, $U = \frac{1}{2}CV^2$. In either case, we see that energy is required to establish a field.

8.5 Magnetic Energy density

It is the energy stored in the magnetic field per unit volume, i.e.,$u_B = \frac{energy}{volume}$. For simplicity, consider a solenoid whose inductance is given by the Equation $L = \mu_o n^2 Al$. The magnetic field of a solenoid is given by Equation $B = \mu_o nI$.

Substituting the expression for L and $I = B/\mu_o n$ into Equation for energy stored in an inductor gives

$$U = \frac{1}{2}LI^2 = \frac{1}{2}\mu_o n^2 Al(B/\mu_o n)^2 = \frac{B^2}{2\mu_o}Al$$

Because Al is the volume of the solenoid, the **magnetic energy density, or the energy stored per unit volume in the magnetic field of the inductor** is

$$u_B = \frac{U}{Al} = \frac{B^2}{2\mu_o}$$

Although this expression was derived for the special case of a solenoid, it is **valid for any region of space in which a magnetic field exists**. Note that this Equation is similar in form to the Equation for the energy per unit volume stored in an electric field, $u_E = \frac{1}{2}\epsilon_o E^2$. **In both cases, the energy density is proportional to the square of the field magnitude**.

8.6 Oscillations in an LC circuits

Consider an LC circuit in which a capacitor is connected to an inductor, as shown in Figure 6. Suppose the capacitor initially has charge Q_o. When the switch is closed, the capacitor begins to discharge and the electric energy is decreased. On the other hand, the current created from the discharging process generates magnetic energy which then gets stored in the inductor. In the absence of resistance, the total energy is transformed back and forth between the electric energy in the capacitor and the magnetic energy in the inductor. This phenomenon is called electromagnetic oscillation.

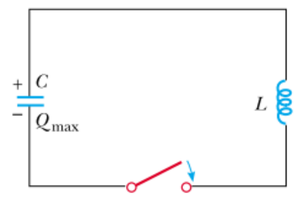

Figure 6

The total energy in the LC circuit at some instant after closing the switch is

$$U = U_C + U_L = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2$$

The fact that U remains constant implies that

$$\frac{dU}{dt} = \frac{d}{dt}(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0$$

or

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0..........(*)$$

where $I = -dQ/dt$ (and $dI/dt = -d^2Q/dt^2$). Notice the sign convention we have adopted here. The negative sign implies that the current I is equal to the rate of decrease of charge in the capacitor plate immediately after the switch has been closed. The same equation can be obtained by applying the modified Kirchhoff's loop rule clockwise:

$$\frac{Q}{C} - L\frac{dI}{dt} = 0$$

followed by our definition of current.

The general solution to Equation $(*)$ is

$$Q(t) = Q_o\cos(\omega_o t + \phi)$$

where Q_o is the amplitude of the charge and ϕ is the phase. The angular frequency ω_o is given by

$$\omega_o = \frac{1}{\sqrt{LC}}$$

The corresponding current in the inductor is

$$I(t) = -\frac{dQ}{dt} = \omega_o Q_o \sin(\omega_o t + \phi) = I_o \sin(\omega_o t + \phi)$$

where $I_o = \omega_o Q_o$. From the initial conditions $Q(t=0) = Q_o$ and $I(t=0) = 0$, the phase ϕ can be determined to be $\phi = 0$. Thus, the solutions for the charge and the current in our LC circuit are $Q(t) = Q_o \cos\omega_o t$..........$(**)$ and $I(t) = I_o \sin\omega_o t$.........$(***)$
The time dependence of and $Q(t)$ and $I(t)$ are depicted in Figure 7.

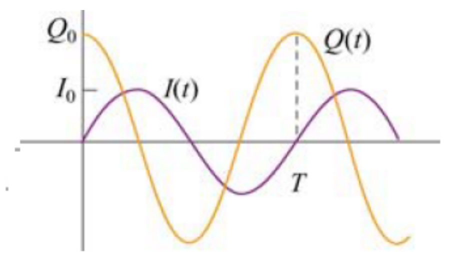

Figure 7. Charge and current in the LC circuit as a function of time.

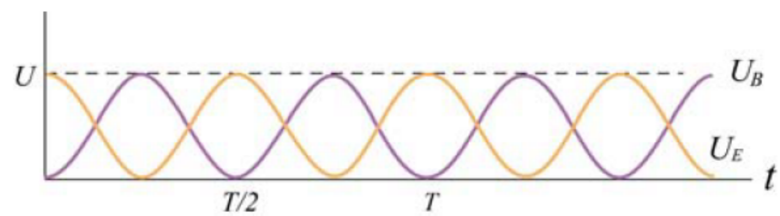

Figure 8. Electric and magnetic energy oscillations

Using Equations $(**)$ and $(***)$, we see that at any instant of time, the electric energy and the magnetic energies are given by

$$U_E = \frac{Q^2(t)}{2C} = (\frac{Q_o^2}{2C})\cos^2\omega_o t$$

$$U_B = \frac{1}{2}LI^2(t) = \frac{LI_o^2}{2}\sin^2\omega_o t = \frac{L(-\omega_o Q_o)^2}{2}\sin^2\omega_o t = (\frac{Q_o^2}{2C})\sin^2\omega_o t$$

respectively. One can easily show that the total energy remains constant:

$$U = U_E + U_B = (\frac{Q_o^2}{2C})\cos^2\omega_o t + (\frac{Q_o^2}{2C})\sin^2\omega_o t = \frac{Q_o^2}{2C}$$

The electric and magnetic energy oscillation is illustrated in Figure 8.

8. 7 The RLC Circuit

We now consider a series RLC circuit which contains a resistor, an inductor and a capacitor, as shown in Figure 9. Imagine that switch S_1 is closed and S_2 is open, so that the capacitor has an initial charge Q_{max}. Next, S_1 is opened and S_2 is closed. Once S_2 is closed and a current is established, the total energy stored in the capacitor and inductor at any time is given by

$$U = U_L + U_C = \frac{LI^2}{2} + \frac{Q^2}{2C}$$

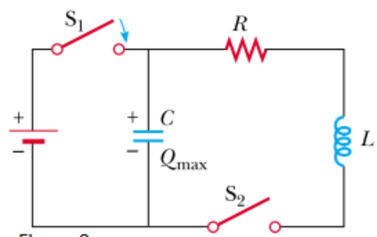

Figure 9

However, this total energy is no longer constant, as it was in the LC circuit, because the resistor causes transformation to internal energy. Because the rate of energy transformation to internal energy within a resistor is I^2R, we have

$$\frac{dU}{dt} = -I^2R$$

where the negative sign signifies that the energy U of the circuit is decreasing in time. Substituting the above expression for U in this result gives

$$\frac{d}{dt}(\frac{LI^2}{2} + \frac{Q^2}{2C}) = LI\frac{dI}{dt} + \frac{Q}{C}\frac{dQ}{dt} = -I^2R$$

To convert this equation into a form that allows us to compare the electrical oscillations with their mechanical analog, we first use the fact that $I = dQ/dt$ and move all terms to the left-hand side to obtain

$$LI\frac{d^2Q}{dt^2} + I^2R + \frac{Q}{C}I = 0$$

Now we divide through by I :

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

In simplest case the solution for the above second order differential equation is

$$Q = Q_{max}e^{-Rt/2L}\cos\omega_d t$$

where ω_d, the angular frequency at which the circuit oscillates, is given by

$$\omega_d = [\frac{1}{LC} - (\frac{R}{2L})^2]^{1/2}$$

That is, the value of the charge on the capacitor undergoes a damped harmonic oscillation in analogy with a blockspring system moving in a viscous medium.