Unit 6
Calculation of Magnetic Fields

6.1 Magnetic Field of a point charge

A moving charge can generate a magnetic field $\vec{B}$ and it can also experience a magnetic force as it moves inside a magnetic field.The magnetic field produced by a moving point charge is calculated as

$$\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{q\vec{v}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi} \cdot \frac{q\vec{v}\times\vec{r}}{r^3} \tag{1}$$

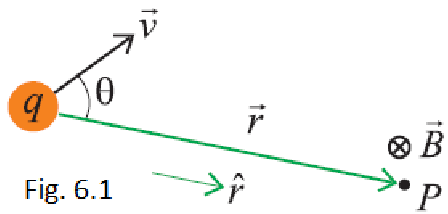


Fig. 6.1

where μ_0 is a constant called the permeability of free space(Magnetic constant):

$$\mu_0 = 4\pi \times 10^{-7} T \cdot m/A \tag{2}$$

The magnitude of the magnetic field due to the point charge is

$$B = \frac{\mu_0}{4\pi} \cdot \frac{qv\sin\theta}{r^2} \tag{3}$$

The field is maximum when $\theta = 90^0$ and minimum when $\theta = 0^0/180^0$

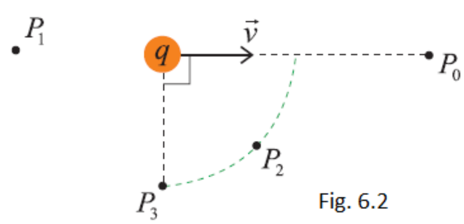


Fig. 6.2

$\vec{B}$ at $P_0 = 0 = \vec{B}$ at P_1
$\vec{B}$ at $P_2 < \vec{B}$ at P_3

♣ However, a single moving charge will NOT generate a steady magnetic field.

♣ stationary charges generate steady Electric field.

♣ Steady currents generate steady magnetic field.

6.2 Biot-Savart Law

Currents which arise due to the motion of charges are the source of magnetic fields. When charges move in a conducting wire and produce a current I, the magnetic field at any point P due to the current can be calculated by adding up the magnetic field contributions,$d\vec{B}$,from small segments of the wire $d\vec{s}$, (Figure 6.3).

These segments can be thought of as a vector quantity having a magnitude of the length of the segment $d\vec{s}$ and pointing in the direction of the current flow(I). The infinitesimal current source can then be written as $Id\vec{s}$.

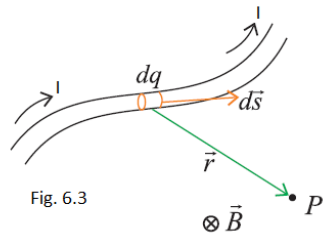


Fig. 6.3

Magnetic field at point P can be obtained by *integrating* the contribution from individual current segments. (**Principle of Superposition**)

From equation(1), the magnetic field due to an infinitesimal charge (which can be considered as a point charge) at distance r, with $\hat{r}$ as its corresponding unit vector, is given by

$$\therefore d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{dq\ \vec{v}\times\hat{r}}{r^2}$$

Notice: $dq\ \vec{v} = dq \cdot \frac{d\vec{s}}{dt} = Id\vec{s}$

$$\therefore d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I\ d\vec{s}\times\hat{r}}{r^2} \tag{4}$$

Equation(4) is called **Biot-Savart Law**.

The Biot-Savart law gives an expression for the magnetic field contribution, $d\vec{B}$,from the current source, $Id\vec{s}$, Notice that the expression is remarkably similar to the Coulombs law for the electric field due to a charge element dq:

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r} \tag{5}$$

Adding up these contributions to find the magnetic field at the point P requires integrating over the current source,

$$\vec{B} = \int_{wire} d\vec{B} = \frac{\mu_0 I}{4\pi}\int_{wire}\frac{d\vec{s}\times\hat{r}}{r^2} \tag{6}$$

The integral is a vector integral, which means that the expression for $\vec{B}$ is really three integrals, one for each component of $\vec{B}$. The vector nature of this integral appears in the cross product $Id\vec{s}\times\hat{r}$. Understanding how to evaluate this cross product and then perform the integral will be the key to learning how to use the Biot-Savart law.

♠**Biot-Savart Law** is to magnetic field as **Coulomb's Law** is to electric field.

♠ Basic element of $\vec{E}$-field: Electric charges dq.

♠ Basic element of $\vec{B}$-field: Current element $Id\vec{s}$.

Example: **Magnetic field due to straight current segment**

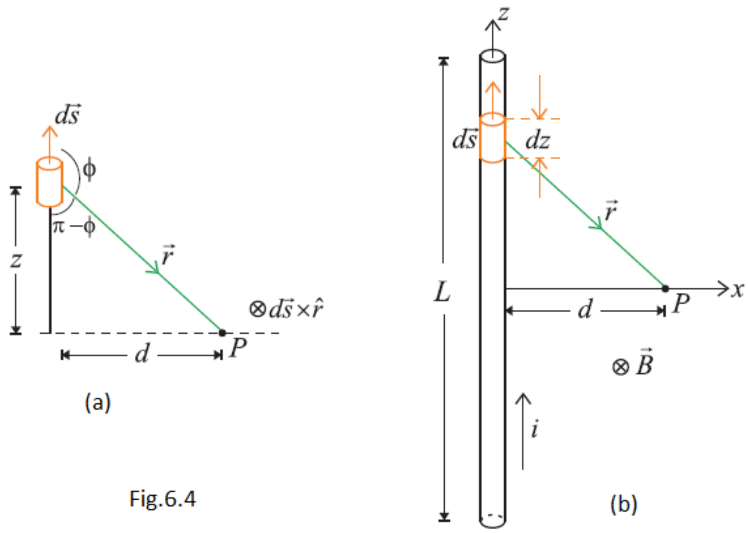


Fig.6.4

$\therefore \quad |\, d\vec{s}\times\hat{r}\,| = dz\sin\phi = dz\sin(\pi-\phi) = dz\cdot\frac{d}{r} = \frac{d\cdot dz}{\sqrt{d^2+z^2}}$
$d\vec{B} = \frac{\mu_0}{4\pi}\cdot\frac{Idz}{r^2}\cdot\frac{d}{r} = \frac{\mu_0 I}{4\pi}\cdot\frac{d}{(d^2+z^2)^{3/2}}dz$
$\therefore \quad B = \int_{-L/2}^{L/2} dB = \frac{\mu_0 Id}{4\pi}\int_{-L/2}^{L/2}\frac{dz}{(d^2+z^2)^{3/2}}$
$B = \frac{\mu_0 I}{4\pi d}\cdot\frac{z}{(d^2+z^2)^{1/2}}\Big|_{-L/2}^{+L/2}$
$B = \frac{\mu_0 I}{4\pi d}\cdot\frac{L}{(d^2+\frac{L^2}{4})^{1/2}}$
Limiting Case: When $L \gg d$, the magnetic field due to a long straight wire will have the following form:

$$(\frac{L^2}{4}+d^2)^{-1/2} \approx (\frac{L^2}{4})^{-1/2} = \frac{2}{L}$$

$$\therefore \quad B = \frac{\mu_0 I}{2\pi d} \qquad (7)$$

The direction of the magnetic field is determined from right-hand rule.
Recall: $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 d}$ for an infinite long line of charge.

6.3 Force Between Two Parallel Wires

We have already seen that a current-carrying wire produces a magnetic field. In addition, when placed in a magnetic field, a wire carrying a current will experience a net force. Thus, we expect two current-carrying wires to exert force on each other.
Consider two parallel wires separated by a distance r and carrying currents I_1 and I_2 in the $+x$-direction, as shown in Figure 6.5.

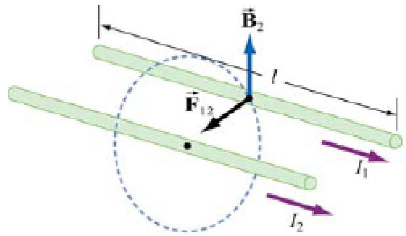


Figure 6.5 Force between two parallel wires

The magnetic force, $\vec{F}_{12}$, exerted on wire 1 by wire 2 may be computed as follows: Using the result from the previous example, the magnetic field lines due to I_2 going in the $+x$-direction are circles concentric with wire 2, with the field $\vec{B}_2$ pointing in the tangential direction. Thus, at an arbitrary point P on wire 1, we have $\vec{B}_2 = -\frac{\mu_0 I_2}{2\pi r}\hat{j}$,which points in the direction perpendicular to wire 1, as depicted in Figure 6.5. Therefore,

$$\vec{F}_{12} = I_1\vec{l} \times \vec{B}_2 = I_1(l\hat{i}) \times (\frac{-\mu_0 I_2}{2\pi r}\hat{j}) = -\frac{\mu_0 I_1 I_2 l}{2\pi r}\hat{k} \tag{8}$$

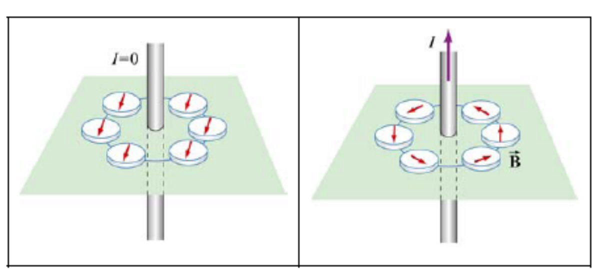
Clearly $\vec{F}_{12}$ points toward wire 2. The conclusion we can draw from this simple calculation is that two parallel wires carrying currents in the same direction will attract each other. On the other hand, if the currents flow in opposite directions, the resultant force will be repulsive.

The magnitude of the force per unit length that one of the wires exert on the other is given by:

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r} \tag{9}$$

6.4 Ampere's law and its application

We have seen that moving charges or currents are the source of magnetism. This can be readily demonstrated by placing compass needles near a wire. As shown in Figure 6.6a, all compass needles point in the same direction in the absence of current. However, when $I \neq 0$ the needles will be deflected along the tangential direction of the circular path (Figure 6.6b).



(a) (b)

Figure 6.6 Deflection of compass needles near a current-carrying wire

Let us now divide a circular path of radius r into a large number of small length vectors $\Delta\vec{s} = \Delta s\hat{\varphi}$, that point along the tangential direction with magnitude Δs (Figure 6.7).

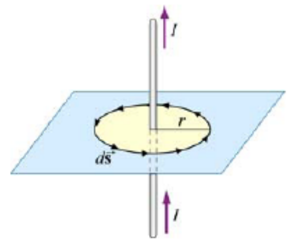


Figure 6.7 Amperian loop

In the limit $\Delta\vec{s} \to 0$, we obtain

$$\oint \vec{B} \cdot d\vec{s} = B \oint ds = (\frac{\mu_0 I}{2\pi r})(2\pi r) = \mu_0 I \qquad (10)$$

The result above is obtained by choosing a closed path, or an "Amperian loop" that follows one particular magnetic field line. Let's consider a slightly more complicated Amperian loop, as that shown in Figure 6.8.

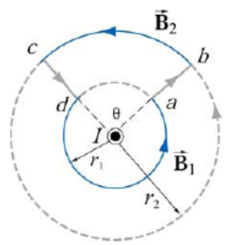


Figure 6.8 An Amperian loop involving two field lines

The generalization to any closed loop of arbitrary shape that involves many magnetic field lines is known as Amperes law:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc} \qquad (11)$$

Ampere's law in magnetism is analogous to Gauss's law in electrostatics. In order to apply them, the system must possess certain symmetry. In the case of an infinite wire, the system possesses cylindrical symmetry and Amperes law can be readily applied. However, when the length of the wire is finite, Biot-Savart law must be used instead.

Amperes law is applicable to the following current configurations:

1. Infinitely long straight wires carrying a steady current I
2. Infinitely large sheet of thickness b with a current density J
3. Infinite solenoid
4. Toroid

~~Example:~~ **Field Inside and Outside a Current-Carrying Wire**

Consider a long straight wire of radius R carrying a current I of uniform current density, as shown in Figure 6.9. Find the magnetic field everywhere.

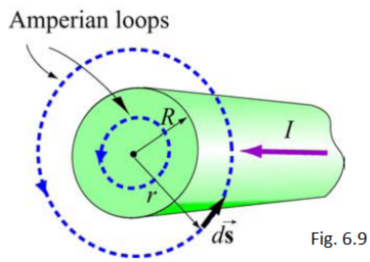


Figure 6.9. Amperian loops for calculating the magnetic field of a conducting wire of radius R.

Solution:

(i) Outside the wire where $r \geq R$,the Amperian loop (circle 1) completely encircles the current, i.e.,$I_{enc} = I$. Applying Ampere's law yields

$$\oint \vec{B} \cdot d\vec{s} = B \oint ds = B(2\pi r) = \mu_0 I$$

which implies

$$B = \frac{\mu_0 I}{2\pi r}$$

(ii) Inside the wire where $r < R$, the amount of current encircled by the Amperian loop (circle 2) is proportional to the area enclosed, i.e.,

$$I_{enc} = (\frac{\pi r^2}{\pi R^2})I$$

Thus, we have

$$\oint \vec{B} \cdot d\vec{s} = B(2\pi r) = \mu_0 I(\frac{\pi r^2}{\pi R^2})$$

$$\Rightarrow B = \frac{\mu_0 I r}{2\pi R^2}$$

We see that the magnetic field is zero at the center of the wire and increases linearly with r until r=R. Outside the wire, the field falls off as 1/r. The qualitative behavior of the field is depicted in Figure 6.10 below.

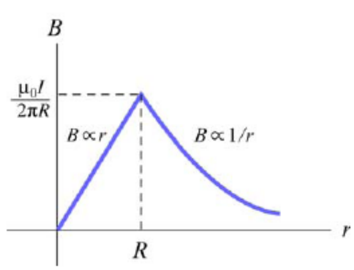


Figure 6.10 Magnetic field of a conducting wire of radius R carrying a steady current I .