Unit: 9 Alternating Current (AC) Circuits

9.1 AC Sources

An **Alternating Current, AC**, is current whose magnitude alternates (or oscillates) between a positive maximum and a negative minimum like that of a sinusoidal wave. In the previous section we learned that changing magnetic flux can induce an emf according to Faradays law of induction. In particular, if a coil rotates in the presence of a magnetic field, the induced emf varies **sinusoidally** with time and leads to an alternating current (AC), and provides a source of AC power. The symbol for an AC voltage source is shown in Figure 9.1.

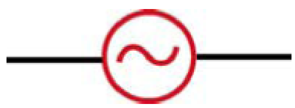

Figure 9.1 Symbol for an AC voltage source

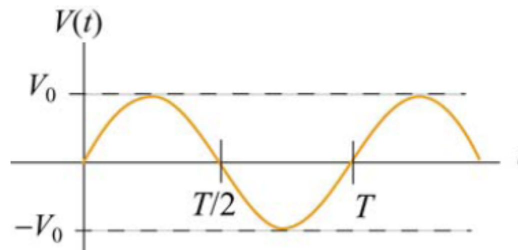

Figure 9.2 Sinusoidal Voltage Source

The sinusoidal voltage which drives the AC varies with time as

$$V = V_o \sin \omega t ....(1)$$

where V_o is called the **maximum output voltage** of the AC source, or the voltage amplitude. The voltage varies between V_o and $-V_o$ since a sine function varies between $+1$ and -1. A graph of voltage as a function of time is shown in Figure 9.2. There are various possibilities for A.C sources, including generators,and electrical oscillators. In a home, each electrical outlet serves as an AC source.

The sine function is periodic in time. This means that the value of the voltage at time t will be exactly the same at a later time $t^{'} = t + T$ where T is the **period**. The **frequency**, f, defined as $f = \frac{1}{T}$, has the unit of inverse seconds (s^{-1}), or **hertz** (Hz). The angular frequency is defined to be $\omega = 2\pi f$.

When a voltage source is connected to an RLC circuit, energy is provided to compensate the energy dissipation in the resistor, and the oscillation will no longer damp out. The oscillations of charge, current and potential difference are called driven or forced oscillations.

After an initial "transient time," an AC current will flow in the circuit as a response to the driving voltage source. The current, written as

$$I(t) = I_o \sin(\omega t - \phi)...................(2)$$

will oscillate with the same frequency as the voltage source, with an amplitude I_o and phase ϕ that depends on the driving frequency.

9.2 Resistors in an AC Circuit

Consider a simple AC circuit consisting of a resistor and an AC source , as shown in Figure 9.3. At any instant, the algebraic sum of the voltages around a closed loop in a circuit must be zero.

$$V(t) - V_R(t) = V(t) - I(t)R = 0 \text{..................}(3)$$

where $V_R(t) = I_R R$ is the instantaneous voltage drop across the resistor.

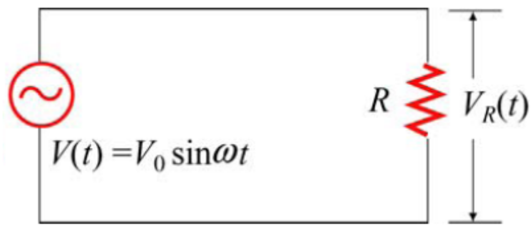

Fig. 9.3 A purely resistive circuit

The instantaneous current in the resistor is given by

$$I_R = \frac{V_R(t)}{R} = \frac{V_{RO}\sin\omega t}{R} = I_{RO}\sin\omega t \text{...................}(4)$$

where $V_{RO} = V_O$, and $I_{RO} = V_{RO}/R$ is the maximum current. Comparing Eq. (4) with Eq. (2), we find $\phi = 0$, which means that $V_R(t)$ and $I_R(t)$ are **in phase with each other, meaning that they reach their maximum or minimum values at the same time**. The time dependence of the current and the voltage across the resistor is depicted in Figure 9. 4(a).

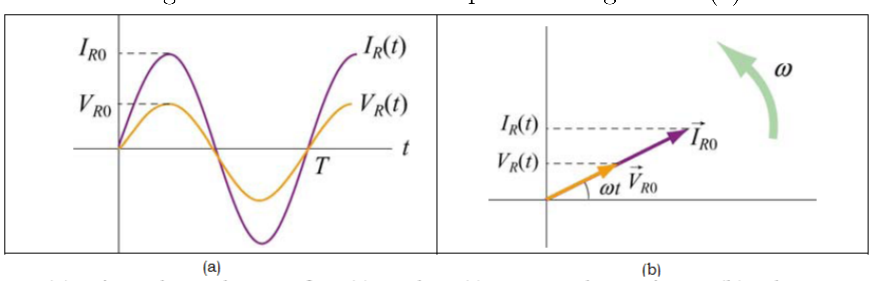

Figure 9.4 (a) Time dependence of $I_R(t)$ and $V_R(t)$ across the resistor. (b) Phasor diagram for the resistive circuit.

The behavior of $V_R(t)$ and $I_R(t)$ can also be represented with a **phasor** diagram, as shown in Figure 9.4(b). A **phasor** is a rotating vector having the following properties:

(i) **length**: the length corresponds to the amplitude.

(ii) **angular speed**: the vector rotates counterclockwise with an angular speed ω.

(iii) **projection**: the projection of the vector along the vertical axis corresponds to the value of the alternating quantity at time t.

We shall denote a **phasor** with an arrow above it. The phasor $\vec{V_{RO}}$ has a constant magnitude of V_{RO}. Its projection along the vertical direction is $V_{RO}\sin\omega t$, which is equal to $V_R(t)$, the voltage drop across the resistor at time t. A similar interpretation applies to $\vec{I_{RO}}$ for the current passing through the resistor. From the phasor diagram, we readily see that both the current and the voltage are in phase with each other.

The average value of current over one period vanishes because average value of a sine function over one period is zero.

It is convenient to define the **root-mean-square (rms)** current as the ratio of the maximum (peak) current to $\sqrt{2}$.

$$I_{rms} = \frac{I_{RO}}{\sqrt{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

In a similar manner, the rms voltage can be defined as

$$V_{rms} = \frac{V_{RO}}{\sqrt{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

power dissipated in the resistor is

$$P_R(t) = I_R(t)V_R(t) = I_R^2(t)R \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

from which the average over one period is obtained as:

$$P_{av}(t) = \frac{1}{2}I_{RO}^2 R = I_{rms}^2 R = I_{rms}V_{rms} = \frac{V_{rms}^2}{R} \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

9.3 Capacitor in an AC Circuit

Figure 9. 5 shows an AC circuit consisting of a capacitor connected across the terminals of an AC source. Kirchhoff's loop rule applied to this circuit gives

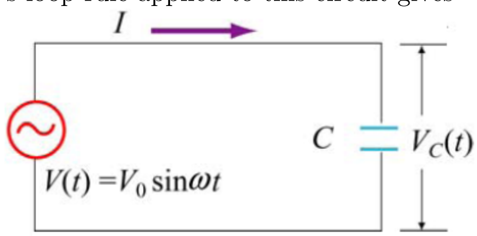

Fig. 9. 5 A purely capacitive circuit

$$V(t) - V_C(t) = V(t) - \frac{Q(t)}{C} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

which yields

$$Q(t) = CV(t) = CV_C(t) = CV_{CO}\sin\omega t \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

where $V_{CO} = V_O$ On the other hand, the current is

$$I_C(t) = \frac{dQ}{dt} = \omega CV_{CO}\cos\omega t = \omega CV_{CO}\sin(\omega t + \frac{\pi}{2}) \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

where we have used the trigonometric identity $\cos\omega t = \sin(\omega t + \frac{\pi}{2})$

The above equation indicates that the maximum value of the current is

$$I_{CO} = \omega CV_{CO} = \frac{V_{CO}}{X_C} \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

where

$$X_C = \frac{1}{\omega C} \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

is called the **capacitance reactance**. It also has SI units of ohms and represents the **effective resistance for a purely capacitive circuit**. Note that X_C is inversely proportional to both C and ω , and diverges as ω approaches zero.

By comparing Eq. (11) to Eq. (2), the phase constant is given by

$$\phi = -\frac{\pi}{2}$$

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 9. 6.

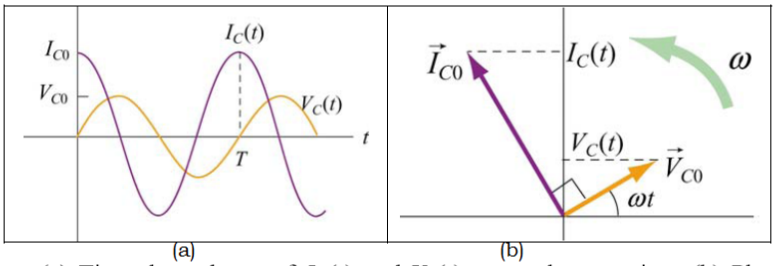

Figure 9.6 (a) Time dependence of $I_c(t)$ and $V_c(t)$ across the capacitor. (b) Phasor diagram for the capacitive c

Notice that at $t = 0$, the voltage across the capacitor is zero while the current in the circuit is at a maximum. In fact, $I_C(t)$ reaches its maximum before $V_C(t)$ by one quarter of a cycle ($\phi = \frac{\pi}{2}$). Thus, we say that **the current leads the voltage by $\frac{\pi}{2}$ in a capacitive circuit.**

9.4 Inductor in an AC circuit

Figure 9.7 shows a purely inductive circuit with an inductor connected to an AC generator.

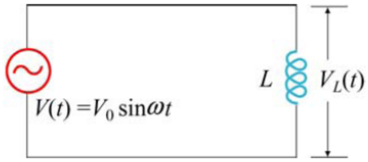

Figure 9.7 A purely inductive circuit

Kirchhoff's loop rule applied to this circuit gives

$$V(t) - V_L(t) = V(t) - L\frac{dI_L}{dt} = 0 \text{...................}(14)$$

which implies

$$\frac{dI_L}{dt} = \frac{V(t)}{L} = \frac{V_{LO}\sin\omega t}{L} \text{................}(15)$$

where $V_{LO} = V_O$. Integrating over the above equation, we find

$$I_L(t) = \int dI_L = \frac{V_{LO}}{L}\int \sin\omega t\ dt = -(\frac{V_{LO}}{\omega L})\cos\omega t = (\frac{V_{LO}}{\omega L})\sin(\omega t - \frac{\pi}{2}) \text{..................}(16)$$

where we have used the trigonometric identity $-\cos\omega t = \sin(\omega t - \frac{\pi}{2})$ for rewriting the last expression. Comparing Eq. (16) with Eq. (2), we see that the amplitude of the current through the inductor is

$$I_{LO} = \frac{V_{LO}}{\omega L} = \frac{V_{LO}}{X_L} \text{...................}(17)$$

where

$$X_L = \omega L \text{...........................}(18)$$

is called the **inductive reactance**. It has SI units of ohms (Ω), just like resistance. However, unlike resistance, X_L depends linearly on the angular frequency ω. Thus, the resistance to current flow increases with frequency. This is due to the fact that at higher frequencies the current changes more rapidly than it does at lower frequencies. On the other hand, the inductive reactance vanishes as ω approaches zero.

By comparing Eq.(16) to Eq.(2), we also find the phase constant to be

$$\phi = +\frac{\pi}{2} \text{................................}(19)$$

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 9. 8

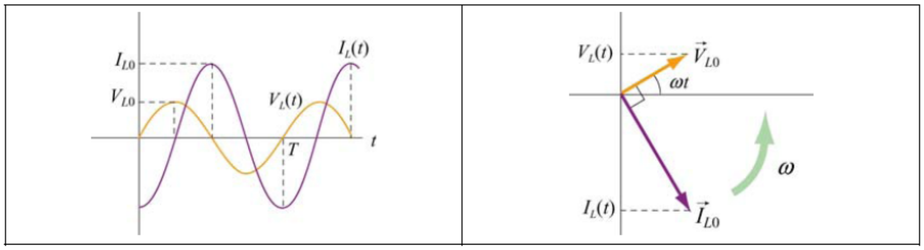

Figure 9. 8 (a) Time dependence of $I_L(t)$ and $V_L(t)$ across the inductor. (b) Phasor diagram for the inductive circuit.

As can be seen from the figures, the current $I_L(t)$ is out of phase with $V_L(t)$ by $\phi = \pi/2$; it reaches its maximum value after $V_L(t)$ does by one quarter of a cycle. Thus, we say that: **the current lags behind the voltage by $\phi = \pi/2$ in a purely inductive circuit**

9.5 The RLC Series Circuit

Consider now the driven series RLC circuit shown in Figure 9.9.

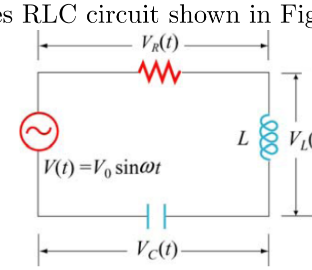

Figure 9. 9 Driven series *RLC* Circuit

Applying Kirchhoff's loop rule, we obtain

$$V(t) - VR(t) - V_L(t) - V_C(t) = V(t) - IR - L\frac{dI_L}{dt} - \frac{Q}{C} = 0 \text{.....................(21)}$$

which leads to the following differential equation:

$$L\frac{dI_L}{dt} + IR + \frac{Q}{C} = V_O \sin\omega t \text{.....................(22)}$$

Assuming that the capacitor is initially uncharged so that $I = dQ/dt$ is proportional to the **increase** of charge in the capacitor, the above equation can be rewritten as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_O \sin\omega t \text{.....................(23)}$$

One possible solution to Eq. (21) is

$$Q(t) = Q_O(t)\cos(\omega t - \phi) \text{.....................(24)}$$

The corresponding current is

$$I(t) = \frac{dQ}{dt} = I_O \sin(\omega t - \phi) \text{.....................(25)}$$

with an amplitude

$$I_O = -Q_O\omega = -\frac{V_O}{\sqrt{R^2 + (X_L - X_C)^2}} \text{.....................(26)}$$

and the phase angle is

$$\phi = \frac{1}{R}(\omega L - \frac{1}{\omega C}) = \frac{X_L - X_C}{R} \text{..........................(27)}$$

Notice that the current has the same amplitude and phase at all points in the series RLC circuit.

On the other hand, the instantaneous voltage across each of the three circuit elements R, L and C has a different amplitude and phase relationship with the current, as can be seen from the phasor diagrams shown in Figure 9.10.

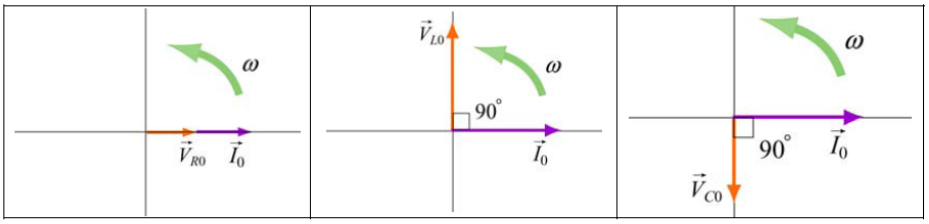

Figure 9.10 Phasor diagrams for the relationships between current and voltage in (a) the resistor, (b) the inductor, and (c) the capacitor, of a series *RLC* circuit.

From Figure 9. 10, the instantaneous voltages can be obtained as:

$$V_R(t) = I_O R \sin\omega t = V_{RO}\sin\omega t$$
$$V_L(t) = I_O X_L \sin(\omega t + \tfrac{\pi}{2}) = V_{LO}\cos\omega t$$
$$V_C(t) = I_O X_C \sin(\omega t - \tfrac{\pi}{2}) = -V_{CO}\cos\omega t......................(28)$$

where

$$V_{RO} = I_O R =, \quad V_{LO} = I_O X_L, \qquad V_{CO} = I_O X_C..........................(29)$$

are the amplitudes of the voltages across the circuit elements. The sum of all three voltages is equal to the instantaneous voltage supplied by the AC source:

$$V(t) = V_R(t) + V_L(t) + V_C(t).................(30)$$

Using the phasor representation, the above expression can also be written as

$$\vec{V}_O = \vec{V}_{RO} + \vec{V}_{LO}(t) + \vec{V}_{CO}.................(31)$$

as shown in Figure 9. 11 (a). Again we see that current phasor $\vec{I}_O$ leads the capacitive voltage phasor $\vec{V}_{CO}$ by $\pi/2$ but lags the inductive voltage phasor $\vec{V}_{LO}$ by $\pi/2$. The three voltage phasors rotate counterclockwise as time passes, with their relative positions fixed.

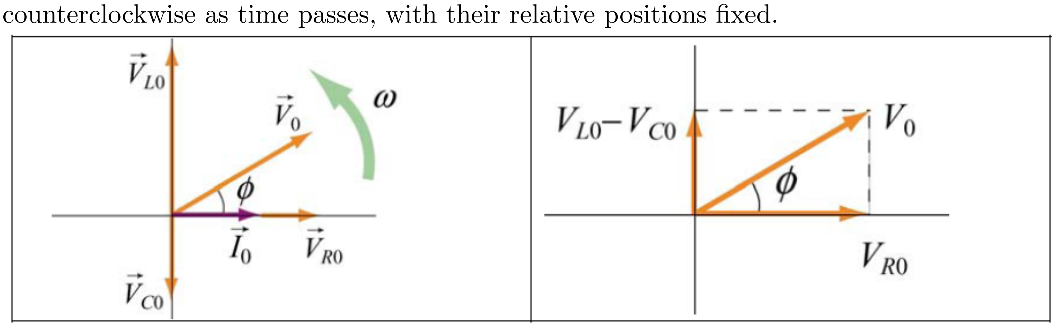

Figure 9.11 (a) Phasor diagram for the series *RLC* circuit. (b) voltage relationship

The relationship between different voltage amplitudes is depicted in Figure 9. 11(b). From the Figure, we see that

$$V_O = |\vec{V}_O| = |\vec{V}_{RO} + \vec{V}_{LO} + \vec{V}_{CO}| = \sqrt{V_{RO}^2 + (V_{LO} - V_{CO})^2} = \sqrt{(I_O R)^2 + (I_O X_O - I_O X_C)^2} = I_O\sqrt{R^2 + (X_L - X_C)^2}.................(32)$$

which leads to the same expression for I_O as that obtained in Eq. (25).

It is crucial to note that the maximum amplitude of the AC voltage source is not equal to the sum of the maximum voltage amplitudes across the three circuit elements:

$$V_O \neq V_{RO} + V_{LO} + V_{CO} \text{.................(33)}$$

This is due to the fact that the voltages are not in phase with one another, and they reach their maxima at different times.

Impedance

We have already seen that the **inductive reactance** $X_L = \omega L$ and **capacitance reactance** $X_C = 1/\omega C$ play the role of an **effective resistance** in the purely inductive and capacitive circuits, respectively. In the series RLC circuit, the effective resistance is the **impedance**, defined as

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \text{..............................(34)}$$

The relationship between Z, X_L *and* X_C can be represented by the diagram shown in Figure 9.12.

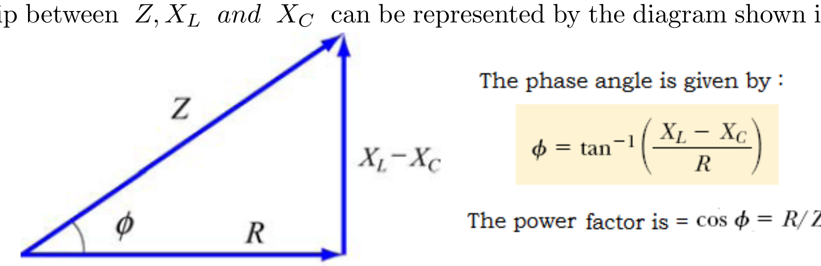

Figure 9. 12

The SI unit of impedance is ohm . In terms of Z, the current may be rewritten as

$$I(t) = \frac{V_O \sin(\omega t - \phi)}{Z} \text{............................(35)}$$

Notice that the impedance Z also depends on the angular frequency ω, as do X_L and X_C.

Table below gives impedance values and phase angles for various series circuits containing different combinations of elements.

Resonance

A series RLC circuit is said to be in **resonance** when the current has its maximum value. In general, the rms current can be written

$$I_{rms} = \frac{V_{rms}}{Z}$$

Substituting the expression for Z into the above equation gives

$$I_{rms} = \frac{V_{rms}}{\sqrt{R^2 + (X_O - X_C)^2}}$$

Because the impedance depends on the frequency of the source, the current in the RLC circuit also depends on the frequency. The frequency ω_O at which $X_L - X_C = 0$ is called the **resonance frequency** of the circuit. To find ω_O, we use the condition $X_L = X_C$, from which we obtain $\omega_O L = 1/\omega_O C$, or

$$\omega_O = \frac{1}{\sqrt{LC}} \text{.....................(36)}$$

Power in an AC circuits

Average Power in an AC Capacitive and Inductive circuits

In an electrical circuit, energy is supplied by the supply p.d.,stored by capacitive and inductive elements and dissipated by resistive elements.

The principle of conservation of energy means that, at any time t the rate at which energy is supplied by the supply p.d. must equal the sum of the rate at which it is stored in the capacitive and inductive elements and dissipated by the resistive elements (here we assume that ideal capacitors and inductors have no internal resistance).

If for an AC circuit the supply p.d and the current are written as in Eq. (1) and Eq. (2) respectively, the average power is calculated as

$$Average\ Power = V_{rms} I_{rms} \cos\phi \text{..........................(37)}$$

In a resistive circuit since the p.d and the current are in phase, $\phi = 0^0 \Rightarrow \cos 0^0 = 1$, so that $P_{av} = V_{rms} I_{rms}$. However, in a pure capacitive and inductive circuits the phase angle $\phi = 90^0 \Rightarrow \cos 90^0 = 0$ so that the average power is zero.