UNIT 3: CAPACITANCE AND DIELECTRICS

CAPACITANCE

1. A capacitor is a device used in electric circuits that can store charge for a short period of time.

   a) Usually consists of 2 parallel conducting plates separated by a small distance.
   b) One plate is connected to positive voltage, the other to negative voltage.
   c) Electrons are pulled off of one of the plates ($+plate$) and are deposited onto the other plate ($-plate$) through (typically) a battery.
   d) The charge transfer stops when the potential difference across the plates equals the potential difference of the battery.
   e) A charged capacitor acts as a storehouse of charge and energy.

2. The capacitance C of a capacitor is the ratio of the magnitude of the charge on either conductor (e.g., plate) to the magnitude of the potential difference between the conductors:

$$C = \frac{Q}{\Delta V}$$

   a) Capacitance is measured in farads (F) in the SI system.

$$1F = \frac{1C}{V}$$

   b) One farad is a very large unit of capacitance. Capacitors usually range from 1 picofarad ($1pF = 10^{-12}F$) to 1 microfarad ($1\mu F = 10^{-6}F$).

3. We also can describe capacitance based on the geometry of the capacitor.
   Parallel-plate capacitor
   Two parallel metallic plates of equal area A are separated by a distance d, as shown in Fig.3.1. One plate carries a charge $+Q$, and the other carries a charge $-Q$.

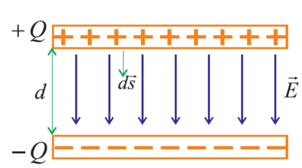


Fig. 3.1

The electric field between these plates is uniform and directed from the positive plate to the negative plate. You recall from unit one that the electric field due to a uniformly charged plate any where has magnitude $E = \frac{\sigma}{2\epsilon_0}$. Since between the two plates the fields produced by the two charged plates of the parallel plate capacitor have same directions, the magnitude of the resultant field anywhere between these plates has a magnitude twice of the magnitude of the field produced by one of the two plates. Hence,

$$E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Again from unit two, you recall that the electric potential difference between these two plates of the parallel plate capacitor can be calculated as

$$\Delta V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{s}$$

Again, notice that this integral is independent of the path taken.
$\therefore$ We can take the path that is <u>parallel</u> to the $\vec{E}$-field.

$$\begin{aligned} \therefore \quad \Delta V &= \int_+^- \vec{E} \cdot d\vec{s} \\ &= \int_+^- E \cdot ds \\ &= \frac{Q}{\epsilon_0 A} \underbrace{\int_+^- ds}_{\text{Length of path taken}} \\ &= \frac{Q}{\epsilon_0 A} \cdot d \end{aligned}$$

$$C = \epsilon_0 \frac{A}{d}$$

where A is the area of one of the plates (both plates have equal areas here), d is the separation distance of the plates, and ϵ_0 is the permittivity of free space (a constant).

4. In circuit diagrams, a capacitor is labeled with

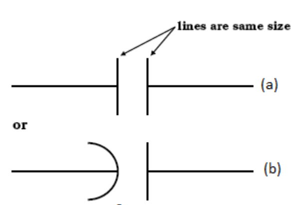


Fig 3.2

E. **Combination of Capacitors**

1. In circuits with multicomponents, always try to reduce the circuit to single components. a) Combine all capacitors to one capacitor.
   b) Combine all resistors (as shown in the next section of the notes) to one resistor.
   c) Hence, we can reduce most circuits to a simple circuit of an equivalent capacitor C_{eq} and an equivalent resistor R_{eq}.
   In this section, we will work only with capacitors.

2. **Capacitors in Parallel.**

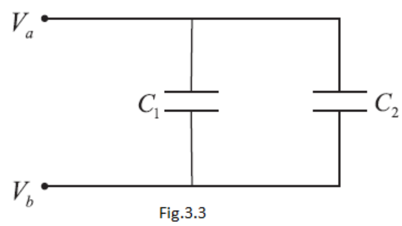


Fig.3.3

In this case, it's the *potential difference* $V = V_a - V_b$ that is the same across the capacitor.

BUT: *Charge* on each capacitor different

$$\begin{aligned} \text{Total charge} \quad Q &= Q_1 + Q_2 \\ &= C_1V + C_2V \\ Q &= \underbrace{(C_1 + C_2)}_{\text{Equivalent capacitance}}V \end{aligned}$$

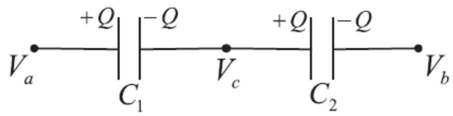
$\therefore$ For capacitors in parallel: $\boxed{C = C_1 + C_2}$

If we extend this treatment to three or more capacitors connected in parallel, we find the equivalent capacitance to be

$C_{eq} = C_1 + C_2 + C_3 + ... = \sum_i C_i$ (parallel combination)

Thus,**the equivalent capacitance of a parallel combination of capacitors is the algebraic sum of the individual capacitances and is greater than any of the individual capacitances.**

3. **Capacitors in Series.**



The *charge across capacitors* are the same.

BUT: *Potential difference* (P.D.) across capacitors different

$$\Delta V_1 = V_a - V_c = \frac{Q}{C_1} \qquad \text{P.D. across } C_1$$
$$\Delta V_2 = V_c - V_b = \frac{Q}{C_2} \qquad \text{P.D. across } C_2$$

∴ Potential difference

$$\begin{aligned}\Delta V &= V_a - V_b \\ &= \Delta V_1 + \Delta V_2 \\ \Delta V &= Q\,(\frac{1}{C_1} + \frac{1}{C_2}) = \frac{Q}{C}\end{aligned}$$

where C is the **Equivalent Capacitance**

$$\therefore \quad \boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}}$$

When this analysis is applied to three or more capacitors connected in series, the relationship for the equivalent capacitance is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \qquad \text{(series combination)}$$

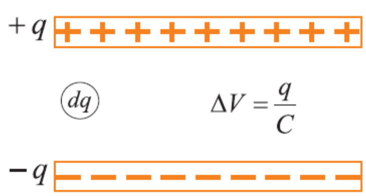
This shows that the inverse of the equivalent capacitance is the algebraic sum of the inverses of the individual capacitances and the equivalent capacitance of a series combination is always less than any individual capacitance in the combination.

PROBLEM-SOLVING HINTS

Capacitors

- Be careful with units. When you calculate capacitance in farads, make sure that distances are expressed in meters. When checking consistency of units, remember that the unit for electric fields can be either N/C or V/m.
- When two or more capacitors are connected in parallel, the potential difference across each is the same. The charge on each capacitor is proportional to its capacitance; hence, the capacitances can be added directly to give the equivalent capacitance of the parallel combination. The equivalent capacitance is always larger than the individual capacitances.
- When two or more capacitors are connected in series, they carry the same charge, and the sum of the potential differences equals the total potential difference applied to the combination. The sum of the reciprocals of the capacitances equals the reciprocal of the equivalent capacitance, which is always less than the capacitance of the smallest individual capacitor.

Energy stored in a capacitor



In charging a capacitor, *positive charge* is being moved from the *negative plate* to the *positive plate*.
$\Rightarrow$ NEEDS WORK DONE!

Suppose we move charge dq from $-ve$ to $+ve$ plate, *change in potential energy*

$$dU = \Delta V \cdot dq = \frac{q}{C}\, dq$$

Suppose we keep putting in a total charge Q to the capacitor, the *total potential energy*

$$U = \int dU = \int_0^Q \frac{q}{C}\, dq$$

$$\therefore \quad \boxed{U = \frac{Q^2}{2C} = \frac{1}{2} C \Delta V^2} \qquad (\because Q = C\Delta V)$$

The energy stored in the capacitor is stored in the **electric field** between the plates.

Note : In a parallel-plate capacitor, the *E-field is constant between the plates.*

$\therefore$ We can consider the E-field energy

$$\text{density u} = \frac{\text{Total energy stored}}{\text{Total volume with E-field}}$$

$$\therefore \quad u = \frac{U}{\underbrace{Ad}_{\text{Rectangular volume}}}$$

Recall

$$\begin{cases} C &= \dfrac{\epsilon_0 A}{d} \\ E &= \dfrac{\Delta V}{d} \quad \Rightarrow \quad \Delta V = Ed \end{cases}$$

$$\therefore \quad u = \frac{1}{2}(\overbrace{\frac{\epsilon_0 A}{d}}^{C}) \cdot (\overbrace{Ed}^{(\Delta V)^2})^2 \cdot \overbrace{\frac{1}{Ad}}^{\frac{1}{Volume}}$$

$$\boxed{u = \frac{1}{2}\epsilon_0 E^2}$$

Energy per unit volume of the electrostatic field

$\uparrow$ can be generally applied

Although the above Equation was derived for a parallel-plate capacitor, the expression is generally valid, regardless of the source of the electric field. **That is, the energy density in any electric field is proportional to the square of the magnitude of the electric field at a given point.**

Capacitors with Dielectrics

A dielectric is a nonconducting material, such as rubber, glass, or waxed paper. When a dielectric is inserted between the plates of a capacitor, the capacitance increases. If the dielectric completely fills the space between the plates, the capacitance increases by a dimensionless factor κ, which is called the **dielectric constant** of the material. The dielectric constant varies from one material to another.
a) For a capacitor with no dielectric, the voltage drop across the capacitor is

$$\Delta V_0 = \frac{Q_0}{C_0}$$

If a dielectric is inserted between the plates of a capacitor, the voltage drop is reduced by a scale factor κ (note that $\kappa > 1$):

$$\Delta V = \frac{\Delta V_\circ}{\kappa}$$

b) Because the charge on the capacitor will not change when a dielectric is introduced, the capacitance in the presence of a dielectric must change to the value

$$C = \frac{Q_\circ}{\Delta V} = \frac{Q_\circ}{\Delta V_\circ/\kappa} = \frac{\kappa Q_\circ}{\Delta V_\circ}$$

or the capacitance increases by the amount
$C = \kappa C_0$
That is, the capacitance increases by the factor κ when the dielectric completely fills the region between the plates. For a parallel-plate capacitor, where

$$C_0 = \epsilon_0 \frac{A}{d}$$

we can express the capacitance when the capacitor is filled with a dielectric as

$$C = \kappa \frac{\epsilon_0 A}{d}$$

For any given plate separation, there is a maximum electric field that can be produced in the dielectric before it breaks down and begins to conduct$\Rightarrow$ this maximum electric field is called the dielectric strength.
When designing circuits, one always needs to insure that the electric field generated by the stored charge in the capacitor does not exceed the dielectric strength of the dielectric material. If this occurs, the capacitor will short circuit (and sometimes blow up!).

Electric Dipole

An **electric dipole** consists of two charges of equal magnitude and opposite sign separated by a distance 2a, as shown in Figure 3.6. **The electric dipole moment** of this configuration is defined as the vector **p** directed from $-q$ toward $+q$ along the line joining the charges and having magnitude $2aq$:

$$p = 2aq$$

Now suppose that an electric dipole is placed in a uniform electric field **E**,established by some other charge distribution, as shown in Figure 3.7.Let us imagine that the dipole moment makes an angle θ with the field.
The electric forces acting on the two charges are equal in magnitude ($F = qE$) and opposite in direction as shown in Figure 3.7. Thus, the net force on the dipole is zero. However, the two forces produce a net torque on the dipole; as a result, the dipole rotates in the direction that brings the dipole moment vector into greater alignment with the field. The torque due to the force on the positive charge about an axis through O in Figure 3.7 has magnitude $Fa\sin\theta$, where $a\sin\theta$ is the moment arm of F about O. This force tends to produce a clockwise rotation. The torque about O on the negative charge is also of magnitude $Fa\sin\theta$; here again, the force tends to produce a clockwise rotation. Thus, the magnitude of the net torque about O is

$$\tau = 2Fa\sin\theta$$

Because $F = qE$ and $p = 2aq$, we can express τ as

$$\tau = 2aqE\sin\theta = pE\sin\theta$$

It is convenient to express the torque in vector form as the cross product of the vectors $\vec{p}$ and $\vec{E}$:

$$\vec{\tau} = \vec{p} \times \vec{E}$$

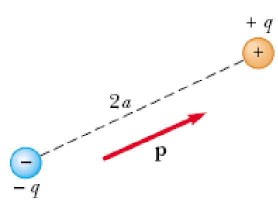


Fig.3.6

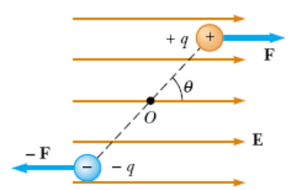


Fig.3.7

We can determine the potential energy of the systeman electric dipole in an external electric fieldas a function of the orientation of the dipole with respect to the field. To do this, we recognize that work must be done by an external agent to rotate the dipole through an angle so as to cause the dipole moment vector to become less aligned with the field. The work done is then stored as potential energy in the system. The work dW required to rotate the dipole through an angle $d\theta$ is $dW = \tau d\theta$. Because $\tau = pE\sin\theta$ and because the work results in an increase in the potential energy U, we find that for a rotation from θ_i to θ_f the change in potential energy of the system is

$$U_f - U_i = \int_{\theta_i}^{\theta_f} \tau \, d\theta = \int_{\theta_i}^{\theta_f} pE \sin \theta \, d\theta = pE \int_{\theta_i}^{\theta_f} \sin \theta \, d\theta$$
$$= pE[-\cos \theta]_{\theta_i}^{\theta_f} = pE(\cos \theta_i - \cos \theta_f)$$

The term that contains $\cos\theta_i$ is a constant that depends on the initial orientation of the dipole. It is convenient for us to choose a reference angle of $\theta_i = 90^0$, so that $\cos\theta_i = \cos 90^0$. Furthermore, let us choose $U_i = 0$ at $\theta_i = 90^0$ as our reference of potential energy. Hence, we can express a general value of $U = U_f$ as

$$U = -pE\cos\theta$$

We can write this expression for the potential energy of a dipole in an electric field as the dot product of the vectors $\vec{p}$ and $\vec{E}$:

$$U = -\vec{p} \bullet \vec{E}$$