1) Let’s prove it by induction.

Let $f(x)=5^{2x+2}-24x-25$ then $\forall n\in\mathbb{N}: 576|f(n)\Rightarrow f(n)\,mod\,576=0$

I. <u>Base Case:</u> $f(1)=5^{2+2}-24(1)-25=625-25-24=576$

II. <u>Induction step</u>: assuming the assertion holds to n, prove it to n+1

$$f(n+1)=5^{2(n+1)+2}-24(n+1)-25=25(5^{2n+2})-24n-49$$
$$=25(f(n)+24n+25)-24n-49=25f(n)+600n-24n+625-49$$
$$=25f(n)+576n+576$$

therefore:

$$f(n+1)\ mod\ 576=(25f(n)+576n+576)\ mod\ 576$$
$$=25f(n)\ mod\ 576+(576n+576)\ mod\ 576$$
$$=25f(n)\ mod\ 576+576(n+1)\ mod\ 576=25f(n)\ mod\ 576$$
$$=0$$

2) let’s prove it by induction.

I. <u>Base Case</u>: $A^1=A=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}=\begin{pmatrix}a^1 & 0\\ 0 & b^1\end{pmatrix}$

<u>II. Induction step</u>: assuming the assertion holds to n, prove it to n+1

$$A^{n+1}=A\cdot A^n=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\cdot\begin{pmatrix}a^n & 0\\ 0 & b^n\end{pmatrix}$$
$$=\begin{pmatrix}a\cdot a^n+0\cdot 0 & a\cdot 0+0\cdot b^n\\ 0\cdot a^n+b\cdot 0 & 0\cdot 0+b\cdot b^n\end{pmatrix}$$
$$=\begin{pmatrix}a^{n+1} & 0\\ 0 & b^{n+1}\end{pmatrix}$$

3) the sequence is $a_n=a_{n-1}+a_{n-2}\ \ n\geq 2$ $\qquad a_0=a_1=1$

solving it using generating functions

$$\sum_{n=2}^{\infty} a_n-a_{n-1}-a_{n-2}=0$$
$$\sum_{n=2}^{\infty} a_n x^n-\sum_{n=2}^{\infty} a_{n-1} x^n-\sum_{n=2}^{\infty} a_{n-1} x^n=0$$
$$((\sum_{n=0}^{\infty} a_n x^n)-a_1x-a_0)-(\sum_{n=1}^{\infty} a_n x^n x)-(\sum_{n=0}^{\infty} a_n x^n x^2)$$
$$(a(x)-a_1x-a_0)-x(a(x)-a_0)-x^2a(x)=0$$
$$a(x)-x-1-a(x)x+x-x^2a(x)=0$$
$$a(x)-xa(x)-x^2a(x)=1$$
$$a(x)=\frac{1}{1-x-x^2}$$

Factorize a(x)   First find root of the denominator

$$1-x-x^2=0 \Rightarrow x=\frac{-(-1)\pm\sqrt{(-1)^2-4(-1)(1)}}{2\cdot -1}$$

$$x=\frac{1\pm\sqrt{5}}{-2}=\frac{-1\mp\sqrt{5}}{2}$$

$$Let\, r_1=\frac{-1+\sqrt{5}}{2}, r_2=\frac{-1-\sqrt{5}}{2}$$

therefore $1-x-x^2 = -(x-r_1)(x-r_2)$

$$a(x)=\frac{-1}{(x-r_1)(x-r_2)}=\frac{A}{x-r_1}+\frac{B}{x-r_2}$$

solve for A and B

$$-1=A(x-r_2)+B(x-r_1)$$
$$-1=(A+B)x-(r_2\cdot A+r_1\cdot B)$$

$$A+B=0\cdots\cdots\cdots\cdots\cdots(1)$$
$$r_2 A+r_1 B=1\cdots\cdots\cdots\cdots\cdots(1)$$

solving (1) and substituting to (2)

$$A=-B$$
$$B(r_1-r_2)=1 \;\Rightarrow B=\frac{1}{r_1-r_2}$$
$$A=-B=\frac{1}{r_2-r_1}$$

so

$$a(x)=-\frac{A}{(r_1-x)}-\frac{B}{(r_2-x)}$$
$$=-(\frac{1/r_1}{r_2-r_1})(\frac{1}{1-x/r_1})-(\frac{1/r_2}{r_1-r_2})(\frac{1}{1-x/r_2})$$
$$=(\frac{r_1^{-1}}{r_1-r_2})\sum_{n=0}^{\infty} r_1^{-n}\cdot x^n-(\frac{r_2^{-1}}{r_1-r_2})\sum_{n=0}^{\infty} r_2^{-n} x^n$$
$$=\sum_{n=0}^{\infty}(\frac{r_1^{-n-1}-r_2^{-n-1}}{r_1-r_2})x^n$$

therefore :

$$a_n=\frac{r_1^{-n-1}-r_2^{-n-1}}{r_1-r_2} \quad =\frac{\frac{1}{r_1^{n+1}}-\frac{1}{r_2^{n+1}}}{r_1-r_2} \quad =\frac{1}{\sqrt{5}}\cdot\frac{r_2^{n+1}-r_1^{n+1}}{(r_1\cdot r_2)^{n+1}}$$
$$=\frac{\sqrt{5}}{5}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

4)

$$a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0 \ \forall n \geq 3 \qquad a_0 = 0, a_1 = 1, a_2 = 10$$

$$\sum_{n=3}^{\infty} (a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3}) x^n = 0$$

$$\sum_{n=3}^{\infty} a_n x^n - 9\sum_{n=3}^{\infty} a_{n-1} x^n + 26\sum_{n=3}^{\infty} a_{n-2} x^n - 24\sum_{n=3}^{\infty} a_{n-3} x^n = 0$$

$$(a(x) - a_2 x^2 - a_1 x - a_0) - 9x(a(x) - a_1 x - a_0) + 26x^2(a(x) - a_0) - 24x^3 a(x) = 0$$

$$a(x) - 10x^2 - x - 9x\,a(x) + 9x^2 + 26x^2 a(x) - 24x^3 a(x) = 0$$

$$a(x)(1 - 9x + 26x^2 - 24x^3) = x^2 - x$$

$$a(x) = \frac{x^2 - x}{-24x^3 + 26x^2 - 9x + 1} = \frac{x^2 - x}{(4x-1)(3x-1)(2x-1)}$$

$$= \frac{2.5}{1-4x} - \frac{4}{1-3x} + \frac{1.5}{1-2x}$$

$$= 2.5\sum_{n=0}^{\infty} 4^n x^n - 4\sum_{n=0}^{\infty} 3^n x^n + 1.5\sum_{n=0}^{\infty} 2^n x^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{5}{2} \cdot 4^n + 4 \cdot 3^n + -\frac{3}{2} \cdot 2^n\right) x^n$$

$$\therefore a_n = \frac{5}{2} \cdot 4^n + 4 \cdot 3^n + -\frac{3}{2} \cdot 2^n$$