WILKINSON (2005) CHAPTER 13 NOTES BY MICK MCQUAID

SPACE

Wilkinson defines a graphics frame as a set of tuples ranging over all possible values in the domain of a p-dimensional varset. Already we are defining terms using other terms previously defined in the book, and defined in a casual way. A tuple, (a, b) is a pair and an n-tuple is a set, $(a_1, a_2, \ldots, a_n)$.

But a pair of what? A set of what? What can you depict in a graphic? It seems that only correspondences can be depicted. There can't be a single object; there must be at least two. Is that true?

Wilkinson defines *graph* in two different ways: (1) a subset of the tuples in a graphics frame and (2) as a set of vertices and edges (which is the graph-theoretic definition of a graph). This can be confusing as the chapter develops, since he explores different ways to think about space, including the graph-theoretic approach.

Wilkinson defines a graphic as the perceptual realization of a graph (first sense) and claims we use two spaces to make a graphic: the underlying space with a mathematical definition and a display space which is always $2D$ or $3D$ Euclidean space.

He gives an example of a graphical depiction of beadlet anemones on a rock, showing a photograph of them and two minimum spanning trees describing them. He actually defines minimum spanning trees later in the chapter but suffice it for now to say that more than one is possible and that one might be preferred over another for some purposes. By the way, Wikipedia gives what I consider a better definition of a minimum spanning tree as a subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight.

In Wilkinson's example, the two minimum spanning trees arise from two different sets of information about the beadlet anemones. One is the (x, y) coordinates of the anemones on the rock (as if it were a plane) while the other is a list of anemone name pairs and weights for the implied edges between them. In the first case, we generate the weights from the (x, y) coordinates and in the other case, the weights are given. In both cases, the output is a display of a tree in two-dimensional space on the page of the book.

The two trees look different from each other but we have no way of knowing whether the difference matters unless we know something about the underlying space which, in this case, means knowing the underlying given data.

Thus concludes Wilkinson's introduction to space. Next he considers the mathematical concept of space, the psychological concept of space, and the graph-theoretic concept of space.

13.1 MATHEMATICAL SPACE

The next section of the book is devoted to a description of popular mathematical spaces for graphics. All of these are topological and are subsets of topological spaces. These spaces obey some mathematical axioms which are listed and some of them are metric, defined by another set of axioms.

Some of these spaces are connected in the sense that they can't be partitioned and some are totally disconnected in that each point is in its own space as is a graphic of entirely categorical variables. Continuous variables are embedded in a connected space. If you have continuous and categorical variables in the same graphics frame, you have a set of connected spaces, which lead to empty regions in graphics. The result is the opportunity to use the empty space to encode additional sources of variation using aesthetics (things we can see, such as color and position).

Metric spaces can be problematic. We can't encode information that violates the three axioms I mentioned earlier, identity, symmetry, and the triangle inequality in a metric space.

13.1.4 Maps. Wilkinson then defines maps (not the geographic kind, the topological kind) that can be injective, surjective, or bijective. The map, $f : X \rightarrow Y$ is injective if no element in Y has more than one element of X mapped to it. It is surjective if every element in Y has at least one element of X mapped to it, and it is bijective if it is both injective and surjective. Why you need to know this is unclear at the moment. It leads into the following definition, though, and may be helpful there.

13.1.5 Embeddings. A map $f : S \rightarrow P$ from one topological space to another is an embedding if S and its image $f(S)$ in P are homeomorphic. A homeomorphism is a continuous one-to-one function whose inverse is also continuous. Continuous means that points that are arbitrarily close in S are also arbitrarily close in P.

It would be wonderful if the relationship between our mathematical space and physical space were an embedding, but it often isn't and this explains, in a mathematical sense, why many visualizations can't work or are intrinsically hard to comprehend. There is a famous case where Milton Friedman, the economist, used dots that were too large on a graphic and misinterpreted some major economic phenomenon as a result: two objects that were arbitrarily close in P were actually far apart in S. On the other hand, there are plenty of cases, like planar geographic maps, that are not embeddings but are useful representations.

13.1.6 Multidimensional scaling. The opposite of representing the relationship between points in a known space is to create space based on known relationships. This is called multidimensional scaling and begins with dissimilarities between points.

13.1.7 Geodesics. Geodesics are locally length-minimizing paths. In Euclidean $2D$ space they are lines, while on a sphere, they are arcs of great circles. Geodesics need algorithms for determining shortest paths, so they depend on the definition of a spatial metric.

13.1.8 Dimensions. How do you define the word dimension? A naive definition is the number of coordinates needed to represent an instance of an object in a Euclidean space. For example, a point is one, a circle is two, and a sphere is three.

On the other hand, you only need two coordinates to locate a point on a sphere. This exemplifies a more sophisticated definition as the number of coordinates needed to specify a point on an object.

People assign many different meanings to the word dimension. For example, the term $2\frac{1}{2}D$ is used by computer scientists, vision scientists, and geographers to refer to $2D$ displays with the half dimension depicting metadata that lets us construct a $3D$ scene in our minds from the $2D$ representation. As another example, dotplots are really one-dimensional but take up two dimensions so that there is someplace to put the dots.

13.1.9 Connected spaces. These are mostly metric spaces. The most common form of metric spaces is defined by Minkowski distance between two points x_j and x_k as follows.

$$d(x_j, x_k) = \left(\sum_{i=1}^{n} |x_{ij} - x_{ik}|^p \right)^{1/p}$$

This is a general formula that can be specified for different numbers of dimensions, p. If $p = 2$ this is Euclidean distance. If $p = 1$ this is city-block or Manhattan distance. Different distance metrics distort visualizations in different, sometimes surprising ways.

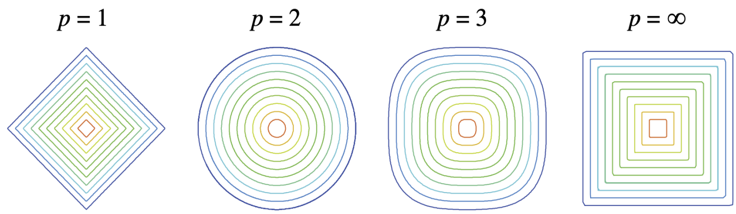


Figure 13.4 *Isodistance contours for power metrics in two dimensions*

The textbook's Figure 13.4 shows isodistance contours for different dimensions, using the above formula. If $p < 1$ as below, the contours take on an emaciated look that diminishes until we get to the circle at $p = 2$. Beyond $p = 2$, the contours become more and more square until at $p = \infty$ they are perfectly square.

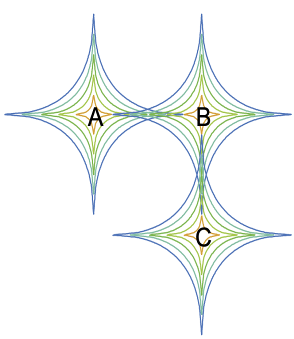


Figure 13.5 *Triangle inequality violation*

The textbook's Figure 13.5 shows a violation of the triangle inequality because the distance from A to C is so much greater (given the contours) than the sum of the distances AB and BC.

Wilkinson next describes assymetric distance maps, where the distance from point x_j to x_k is not the same as the distance from x_k to x_j. Distance is directional and can be shown by isocontour lines with an asymmetrical shape.

Visualizations often depict shortest paths. Of particular interest is the shortest path in $3D$ space, with the associated concept of the gradient. The gradient here can be thought of as the path a ball would take rolling down a smooth hill.

13.1.10 Fractals. Wilkinson discusses fractals, which are shapes where each part is similar to the whole and where the shape's Hausdorff dimension differs from its topological dimension. The definition of the Hausdorff dimension is beyond the scope of the book and is used only to justify the exclusion of fractals from the set of objects characterized by their topological dimensions.

In classifying fractals outside the realm of objects characterized by topological dimensions, Wilkinson seems to relegate them to a niche in visualization. However, he offers one example of a fractal being used to visualize another phenomenon (in other words, not just as a picture of a fractal for the sake of the fractal's beauty). This comes later in the chapter and is an example of gene sequences encoded as a fractal that can be observed from many angles.

13.1.12 Graph-theoretic space. This subsection covers some basic definitions of graph theory which are necessary to visualize social networks (or other networks, it's just that social networks are currently popular in the visualization world). I'm not going to repeat these definitions here and they are incomplete in any case.

This section begins with the observation that Pavlov believed that the brain is an associative network in which evocation of a response to a stimulus is likely to activate spatially associated cortical events. This idea, called spreading activation, remains popular over a hundred years later.

Wilkinson goes on to mention theories of perception and notes that most of them postulate at least two stages of processing: (1) pre-attentive stages, where we perceive a stimulus without effort, and (2) higher cognitive stages, in which we analyze aspects of the stimulus to make judgments. So the rest of this section discusses these two stages.

13.2.1 Spatial models of pre-attentive cognitive processes. Pre-attentive visual and auditory processes appear to be spatial, but complicated. Some research has suggested that perceived color space is metric. This is exemplified by the textbook's Figure 13.14, showing on the left a diagram of chromaticity used for color calibration, and on the right a picture of judgments about color similarity configured using multidimensional scaling.

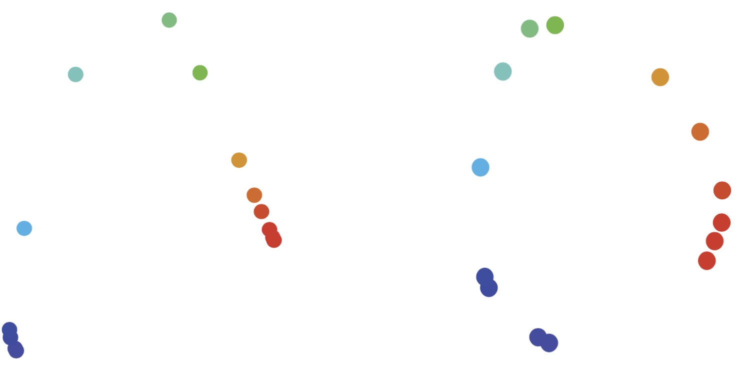


Figure 13.14 *1931 CIE (left), MDS of Ekman data (right)*

The important point about Figure 13.14 is how similar the two pictures are, suggesting that pre-attentive human perceptions of color are metric. Wilkinson gives another example, this time of the pre-attentive perception of form.

13.2.2 Spatial models of cognitive processes. Judgment is a different matter. There is plenty of evidence that human judgment is not metric, much of it supplied by Amos Tversky and Daniel Kahnemann in the 1970s and 80s and later summarized for the layperson in the popular (and controversial!) book *Thinking Fast and Slow* (2011).

Tversky and Kahnemann show that we can not assume that people will judge quantities and relationships the same way after a glance and after lengthy consideration.

A related issue is that we must consider biases introduced by cognitive processes when people judge spatial material. Another Tversky, Barbara Tversky, showed that adding a political boundary on a map changes the judgment of distances between towns in that map.

13.2.3 Spatial cognition. We need to also consider how we perceive and think about space itself. Evidence shows that the perceptual world is not Euclidean. In the outdoors we seem to see a flattened spherical world, with the sky perceived as closer at the zenith than at the horizon. The moon illusion, in which the moon is perceived as larger at the horizon than overhead has been cited as evidence of this perception. Indoors, spatial perception is influenced by the structure of the room. A famous artifact called the Ames room, with distorted walls, plays tricks on our perception. (I vividly remember visiting an Ames room in an amusement park as a child and being so disoriented that I staggered around!) The Ames room illusion makes people look gigantic or tiny depending on where they stand, and makes balls appear to roll uphill. I'm attaching a picture of one uploaded by the UK Royal Institution.

Perception is always influenced by by depth cues such as texture and color that govern our perception of 3*D* space. No matter how we try to flatten it, the 3*D* world is inescapable, says Wilkinson.

13.3 GRAPHING SPACE

This section is about mapping an underlying mathematical space to a 2*D* or 3*D* Euclidean space, which is where we see graphics. There are four possibilities for the underlying space that Wilkinson considers. These four are a connected space, a discrete space, a graph (in the graph-theoretic sense), and a collection of nested spaces.

13.3.1 Mapping connected space to Euclidean space. An example of mapping a non-Euclidean connected space to a connected space is given as the conversion of a table of city block distances between Manhattan landmarks and a 2*D* map of those landmarks. This is a felicitous example because of the way city blocks are laid out in mid-Manhattan as a grid (except for Broadway and Central Park). An interesting illusion occurs when we look at the geodesics in this example. A zigzag pattern appears to be shorter than a straighter path of equal length.

13.3.1.2 Mapping affine space to Euclidean space. Euclidean space is isotropic (from a Greek word meaning *the same in any direction*). An affine transformation of Euclidean space results in an anisotropic space. We can modify the formula for Euclidean distance by adding a weight, w, that differs in each direction i.

$$d(x_j, x_k) = \left(\sum_{i=1}^{n} w_i (x_{ij} - x_{ik})^2 \right)^{1/2}$$

This stretches or shrinks each dimension separately. An application of this principle is Mahalanobis distance, which can give us a rotated ellipse instead of a circle of distances as shown in textbook Figure 13.16.

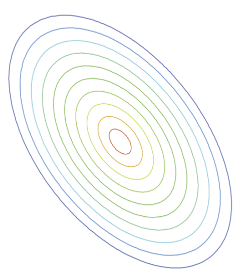


Figure 13.16 *Mahalanobic anisotropic isodistance contours*

An example of this with the famous iris dataset is shown in textbook Figure 13.17, where we see two species of iris, Versicolor and Virginica. In the left panel, the star represents a new iris that we would like to classify. At a glance it appears to belong to the red ellipse, Versicolor. But extending the contours in the right panel shows that it is slightly more like to belong to the blue ellipse, Virginica. Thus if we're trying to evaluate anisotropic distances, we may need multiple contours.

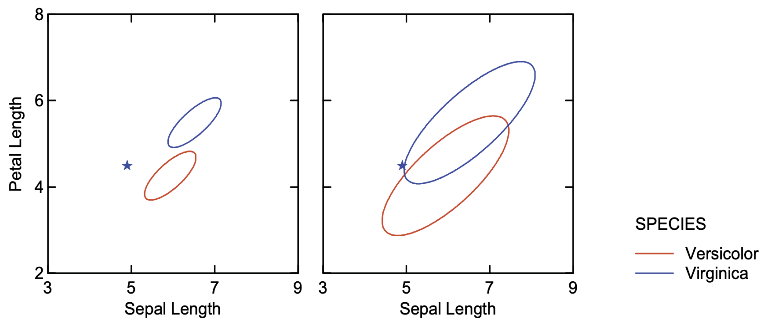


Figure 13.17 *Anisotropic classification*

13.3.1.4 Sharing space. Symbols on scatterplots may represent points, but they have a nonzero size. They occupy real space and often overlap as we saw in the Tableau tutorial when we created a bubble plot of markets and reduced the opacity of the symbols so they could be seen through each other. Wilkinson returns to the beadlet anemones to show an example of a bubble plot but in this case there is no sharing or stealing of space since the size of the bubbles represents the size of the anemones. This is atypical of bubble plots.

13.3.1.5 Label layouts. Graphics programs use algorithms called force-directed graph layouts to move labels into positions such that they don't occlude each other. You don't need to know much about these algorithms except that one might work better than another in a given situation and some graphics software lets you choose algorithms.

13.3.2 Mapping discrete space to Euclidean space. When we map discrete space to Euclidean space, there are areas inside the frame but outside the image of the mapping. That is, there is free space where no points go. We may use free space for shape and size aesthetics and for annotation.

Wilkinson provides a really good example of this in textbook Figure 13.22, where the middle row shows some egregious graphical practice and the bottom row has been arranged to make the relationships more perceivable. Note that both the ordering and spacing differs on the bottom row.

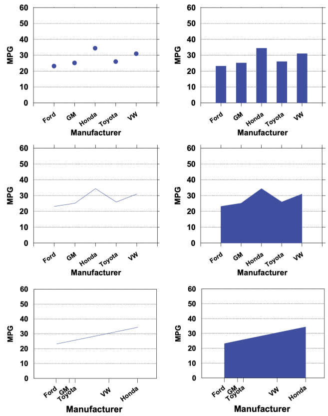


Figure 13.22 Bar, dot, line, and area charts of EPA data

13.3.3 Mapping graph-theoretic space to Euclidean space. Wilkinson covers spanning trees (including the minimum spanning tree mentioned earlier), ultrametric trees, and additive trees in this subsection. Of special note are dendrograms, which result from clustering algorithms. A number of dendrograms are depicted, although Wilkinson never calls them by name.

An ultrametric space is not explicitly defined by Wilkinson, but Wikipedia provides a concise definition where instead of the triangle inequality:

$$d(x, z) \leqslant d(x, y) + d(y, z)$$

distances in the space obey the ultrametric inequality:

$$d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$$

A dendrogram is an ultrametric tree. Wilkinson shows three dendrograms in Figure 13.26, each constructed using a different algorithm on the anemone data. In this case, average linkage works best to reveal the true clusters, but in practice, no one method (single, linkage, complete linkage, or average linkage) dominates all cases. Thus, for visualization purposes, it is often best to try all three. Software that creates dendrograms, such as several R packages, usually allows you to select the linkage algorithm.

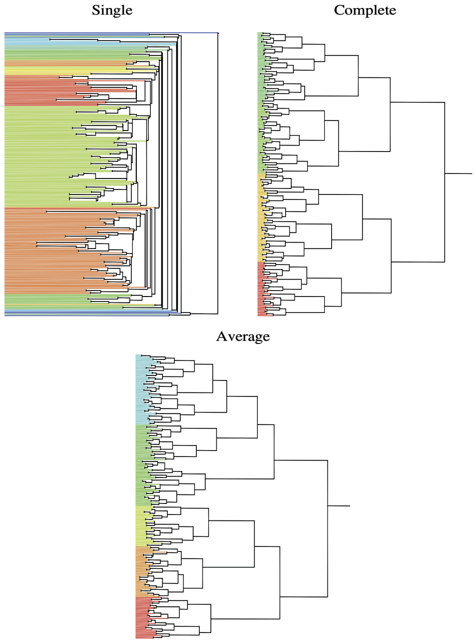


Figure 13.26 *Cluster trees on anemone data*

13.3.4 Mapping nested space to Euclidean space. Wilkinson describes treemaps, Temple MVV, and region trees in this subsection. All of these represent nested spaces in Euclidean spaces, but some are more successful than others. For example, treemaps have been popular in showing the contents of disk-based filesystems.

The procedure for constructing a treemap is given in this subsection, but it is usual for software to carry out the procedure, with the user simply specifying the variables to be used and a linkage algorithm. Textbook Figure 13.30 shows an example of a tree converted to a treemap, but more information is needed than is shown in the picture to understand how it is done. First there is some variable for splitting the tree, then there is some variable for determining the size of each tile.

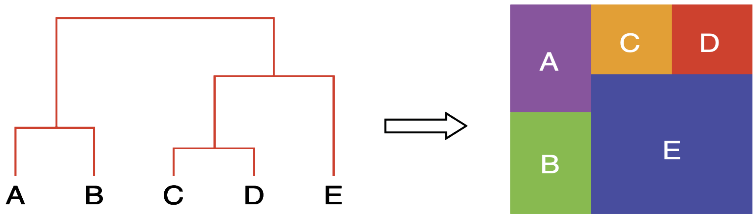
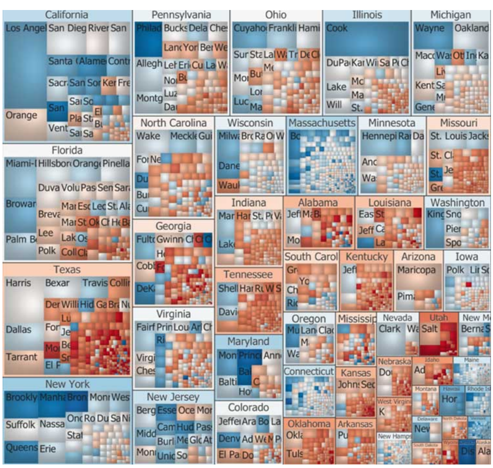


Figure 13.30 Mapping a tree to a treemap

To make this process clearer, consider the following treemap from Butler Analytics. The tiles are sized according to number of voters and the tiles are created by a two-level hierarchy of state and county. Notice that the tiles are colored by political affiliation.



Wilkinson strongly prefers region trees over Temple MVV and has much negative to say about OLAP (online analytical processing) cubes, which are the basis for Temple MVV. As an example, he shows two views of the Titanic data, with textbook Figure 13.33 supplying the Temple MVV view and textbook Figure 13.34 supplying the region tree view.

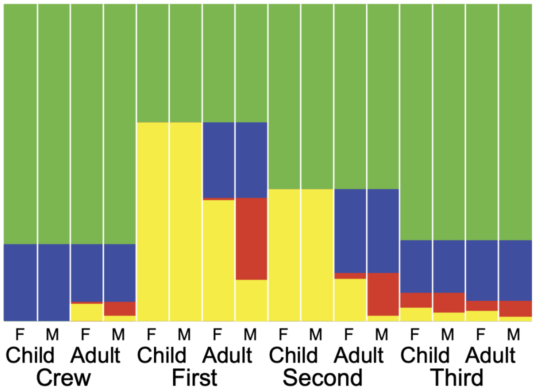


Figure 13.33 *Nested tiling of Titanic data*

This famous dataset lists the survivors of the Titanic, according to age, class, and gender. As you may know, the children and most of the women in first class survived, but the other groups suffered more fatalities according to class and gender. This is fairly easy to see in the region tree but much harder in the Temple MVV visualization.

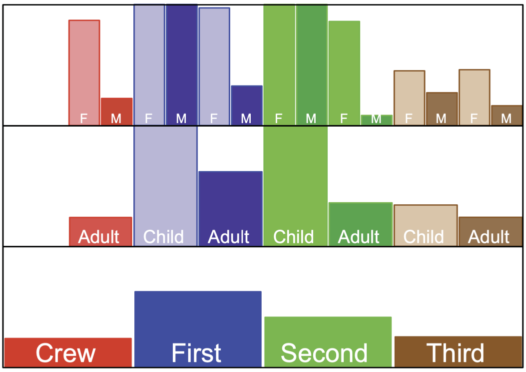


Figure 13.34 *Region tree of Titanic data*

The region tree exemplifies a faceted display, a very popular data visualization tool for dividing data into facets of interest. In this case we see three separate facets, gender at the top, age in the middle, and class at the bottom.

CONCLUSION

These notes just gloss over some of the important points of Wilkinson's chapter on space. It is worthwhile to read the chapter for more detail and for references to some of the relevant literature on the topic. Although the book was published in 2005, most of the material is relevant today and it is possible to update your knowledge of the other areas by following the citation networks of the references. The only deficiency is that he spells Shneiderman wrong, making it a little harder to search for information about treemaps, of which there are many varieties today.