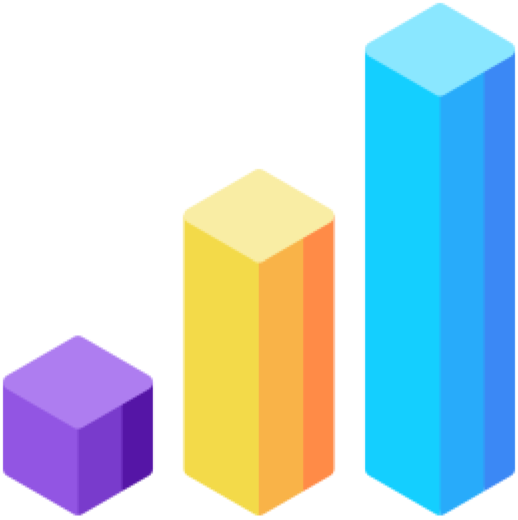


SIT787 Mathematics for AI

Assessment 1

Michael Rideout
Student Id: 225065259

Question 1

Given the function:

$$f(x) = \begin{cases} \sqrt{x^2 - 2x + 1}, & |x| \leq 9 \\ ax^2 + bx + 2, & |x| > 9 \end{cases}$$

The equivalent equation with absolute values and square root signs removed can be derived using the following steps:

<u>Square Root Removal</u>

The first component of the piecewise function $\sqrt{x^2 - 2x + 1}$ is a perfect square which can be re-written as $\sqrt{(x-1)^2}$

Given the rule $\sqrt{x^2} = |x|$ we can rewrite $\sqrt{(x-1)^2}$ as $|x-1|$

Therefore the function with square roots removed becomes:

$$f(x) = \begin{cases} |x-1|, & |x| \leq 9 \\ ax^2 + bx + 2, & |x| > 9 \end{cases}$$

<u>Absolute Values Removal</u>

The first component of the piecewise function $|x-1|$ can itself be defined as a piecewise function:

$$f(x) = |x-1| = \begin{cases} x-1, & |x| \geq 1 \\ 1-x, & |x| < 1 \end{cases}$$

And the constraint for $|x-1|$ is $|x| \leq 9$. Based on the rule $|x| \leq a \equiv -a \leq x \leq a$ the constraint for $|x-1|$ becomes $-9 \leq x \leq a$. Combining this constraint with the decomposed absolute for $|x-1|$ we get:

$$f(x) = |x-1| = \begin{cases} x-1, & 1 \leq x \leq 9 \\ 1-x, & -9 \leq x < 1 \end{cases}$$

Similarly for the component $ax^2 + bx + 2, |x| > 9$ the constraint can be rewritten based on the rule $|x| > a \equiv x < -a \text{ or } x > a$ there for $|x| > 9 \equiv x < -9 \text{ or } x > 9$

The resulting piecewise function with both square roots and absolute values removed is:

$$f(x) = \begin{cases} x-1 & 1 \leq x \leq 9 \\ 1-x, & -9 \leq x < 1 \\ ax^2 + bx + 2, & x < -9 \text{ or } x > 9 \end{cases}$$

Question 2

To determine if a function is differentiable, as described in the assignment specification sheet, the following conditions must be met:

1. Its constituent functions are differentiable on the corresponding open intervals
2. At the points where two subintervals touch, the corresponding derivatives of the two neighbouring subintervals should match.

The derivative of the function:

$$f(x) = \begin{cases} x-1 & 1 \leq x \leq 9 \\ 1-x, & -9 \leq x < 1 \\ ax^2+bx+2, & x < -9 \text{ or } x > 9 \end{cases}$$

is:

$$f'(x) = \begin{cases} 1 & 1 \leq x \leq 9 \\ -1, & -9 \leq x < 1 \\ 2ax+b, & x < -9 \text{ or } x > 9 \end{cases}$$

Each of the constituent functions of f(x) were differentiable on open intervals due to the power rule therefore the first condition of a differentiable function has been met.

The points where the subintervals touch are $x = -9, x = 1, x = 9$
We test either side of each point to determine if they match

<u>Boundary Point x = 1</u>

$\lim_{x \to 1^-} f'(x) = -1$

$\lim_{x \to 1^+} f'(x) = 1$

Since $\lim_{x \to 1^-} f'(x) \neq \lim_{x \to 1^+} f'(x)$ the derivatives of these two neighbouring subintervals do not match therefore the function f(x) is not differentiable.

<u>Boundary Point x = -9</u>

$\lim_{x \to -9^-} f'(x) = 2a(-9) + b$

$\lim_{x \to -9^+} f'(x) = -1$

This boundary point is differentiable when $-18a + b = -1$. Otherwise it is not.

<u>Boundary Point x = 9</u>

$\lim_{x \to 9^-} f'(x) = 2a(9) + b$

$\lim_{x \to 9^+} f'(x) = 1$

This boundary point is differentiable when $18a + b = 1$. Otherwise it is not.

As boundary point x = 1 is always not differentiable, the function f(x) is not differentiable.

Question 3

Given the function

$$f(x) = \begin{cases} x-1 & 1 \le x \le 9 \\ 1-x, & -9 \le x < 1 \\ \frac{1}{9}x^2 + \frac{-1}{3}x + 2, & x < -9 \text{ or } x > 9 \end{cases}$$

To find all the intercepts of $y = f(x)$ we must find the x-intercept and the y-intercept

<u>X-intercept</u>

The x-intercept occurs when $f(x) = 0$. We must test each sub interval for when this occurs.

- For the interval $1 \le x \le 9$:
  - $f(x) = x - 1$
  - Therefore $f(x) = 0 \equiv x - 1 = 0 \Rightarrow x = 1$
  - A x-intercept occurs at $x = 1$ within this interval
- For the interval $-9 \le x \le 1$
  - $f(x) = 1 - x$
  - Therefore $f(x) = 0 \equiv 1 - x = 0 \Rightarrow x = 1$
  - A x-intercept occurs at $x = 1$ within this interval
- For the interval $x < 9 \text{ or } x > 9$
  - $f(x) = \dfrac{1}{9}x^2 - \dfrac{1}{3}x + 2$
  - Therefore $f(x) = 0 \equiv \dfrac{1}{9}x^2 - \dfrac{1}{3}x + 2 = 0$
  - Multiply by 9 when then get $x^2 - 3x + 18 = 0$ and we use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to determine the discriminant $x = \dfrac{3 \pm \sqrt{-3^2 - 4.1.18}}{2} = \dfrac{3 \pm \sqrt{-63}}{2}$ there is no solution for $\sqrt{-63}$ in the real number systems so there is no x-intercept for this interval
- Therefore there is only one x-intercept (1,0)

<u>Y-intercept</u>

The y-intercept occurs at $x = 0$. Therefore $f(0) = 1 - x = 1 - 0 = 1$
The y-intercept is at (0, 1)

Question 4

As defined in the assignment specification sheet, the critical points for a given function $y = f(x)$ are:

- The points $x = c$ where $f'(c) = 0$
- The points $x = c$ where $f'(c)$ does not exist

Given:

$$f(x) = \begin{cases} x - 1 & 1 \leq x \leq 9 \\ 1 - x, & -9 \leq x < 1 \\ \frac{1}{9}x^2 + \frac{-1}{3}x + 2, & x < -9 \text{ or } x > 9 \end{cases}$$

The first derivative is:

$$f'(x) = \begin{cases} 1 & 1 \leq x \leq 9 \\ -1, & -9 \leq x < 1 \\ \frac{2}{9}x - \frac{1}{3}, & x < -9 \text{ or } x > 9 \end{cases}$$

Points where $f'(c) = 0$

We check each interval to determine if or when $f'(c) = 0$ for that interval.

- For the interval $1 \leq x \leq 9$
  - $\forall x \text{ such that } 1 \leq x \leq 9, \quad f'(c) = 1$
  - For this interval f'(c) can never be zero therefore it is not a critical point
- For the interval $-9 \leq x < 1$
  - $\forall x \text{ such that } -9 \leq x < 1, \quad f'(c) = -1$
  - For this interval f'(c) can never be zero therefore it is not a critical point
- For the interval $x < -9 \text{ or } x > 9$
  - $f'(c) = 0 \Rightarrow \dfrac{2}{9}x - \dfrac{1}{3} = 0 \Rightarrow x = \dfrac{3}{2}$
  - $x = \dfrac{3}{2}$ is outside this interval therefore it is not a critical point

Points where $f'(c)$ does not exist

We check each boundary point between the intervals to determine if the derivative of the function is continuous. It is a critical point if it is not continuous. The boundary points for the derivative function are $x = -9, x = 1, x = 9$

- For $x = -9$
  - $\lim_{x \to -9^-} f'(x) = -2\dfrac{1}{3}$

    - $\lim_{x \to -9^+} f'(x) = -1$
    - $-2\frac{1}{3} \neq -1$ therefore $x = -9$ is a critical point
- For $x = 1$
    - $\lim_{x \to 1^-} f'(x) = -1$
    - $\lim_{x \to 1^+} f'(x) = 1$
    - $1 \neq -1$ therefore $x = 1$ is a critical point
- For $x = 9$
    - $\lim_{x \to 9^-} f'(x) = 1$
    - $\lim_{x \to 9^+} f'(x) = 1\frac{2}{3}$
    - $1 \neq 1\frac{2}{3}$ therefore $x = 9$ is a critical point
- The critical points are at $x = -9, x = 1, x = 9$

Question 5

The classification of all critical points of $y = f(x)$ entails testing each critical point for changes in direction before and after the point with regards to the first derivative. If f'(x) changes from positive to negative at x=c then it is a local maximum point. If f'(x) changes from negative to positive at x=c then it is a local minimum point.

- For $x = -9$
  - $\lim_{x \to -9^-} f'(x) = -2\frac{1}{3} \text{ negative slope}$
  - $\lim_{x \to -9^+} f'(x) = -1 \text{ negative slope}$
  - $x = -9$ is neither a local maximum or minimum as there is no change in slope direction
- For $x = 1$
  - $\lim_{x \to 1^-} f'(x) = -1 \text{ negative slope}$
  - $\lim_{x \to 1^+} f'(x) = 1 \text{ positive slope}$
  - The slope changes from negative to positive therefore $x = 1$ is a local minimum
- For $x = 9$
  - $\lim_{x \to 9^-} f'(x) = 1 \text{ positive slope}$
  - $\lim_{x \to 9^+} f'(x) = 1\frac{2}{3} \text{ positive slope}$
  - $x = 9$ is neither a local maximum or minimum as there is no change in slope direction

There is a local minimum at $x = 1$

Question 6

A function $f(x)$ is increasing if $f'(x)$ is positive and decreasing if $f'(x)$ is negative.

- For $x = -9$
  - $\lim_{x \to -9^-} f'(x) = -2\frac{1}{3}$ negative slope
  - $\lim_{x \to -9^+} f'(x) = -1$ negative slope
- For $x = 1$
  - $\lim_{x \to 1^-} f'(x) = -1$ negative slope
  - $x = -9 \lim_{x \to 1^+} f'(x) = 1$ positive slope
  - The slope is negative. Before the critical point $x = 1$ the slope is negative (including both sides of the critical point $x = -9$) but changes to positive after $x = 1$. Therefore the function $f(x)$ is decreasing for the interval $(-\infty, 1]$
- For $x = 9$
  - $\lim_{x \to 9^-} f'(x) = 1$ positive slope
  - $\lim_{x \to 9^+} f'(x) = 1\frac{2}{3}$ positive slope
  - $x = 9$ is neither a local maximum or minimum as there is no change in slope direction
- Before and after the critical point $x = -9$,the slope is negative. Before the critical point $x = 1$ the slope is negative but changes to positive after $x = 1$. Therefore the function f(x) is decreasing for the interval $(-\infty, 1]$
- After the critical point $x = 1$ and before and after the critical point $x = 9$ the slope is positive therefore the function f(x) is increasing for the interval $[1, \infty)$

Question 7

Given:

$$f(x) = \begin{cases} x-1 & 1 \le x \le 9 \\ 1-x, & -9 \le x < 1 \\ \frac{1}{9}x^2 + \frac{-1}{3}x + 2, & x < -9 \text{ or } x > 9 \end{cases}$$

The first derivative is:

$$f'(x) = \begin{cases} 1 & 1 \le x \le 9 \\ -1, & -9 \le x < 1 \\ \frac{2}{9}x - \frac{1}{3}, & x < -9 \text{ or } x > 9 \end{cases}$$

The second derivative is:

$$f''(x) = \begin{cases} 0 & 1 \le x \le 9 \\ 0, & -9 \le x < 1 \\ \frac{2}{9}, & x < -9 \text{ or } x > 9 \end{cases}$$

Question 8

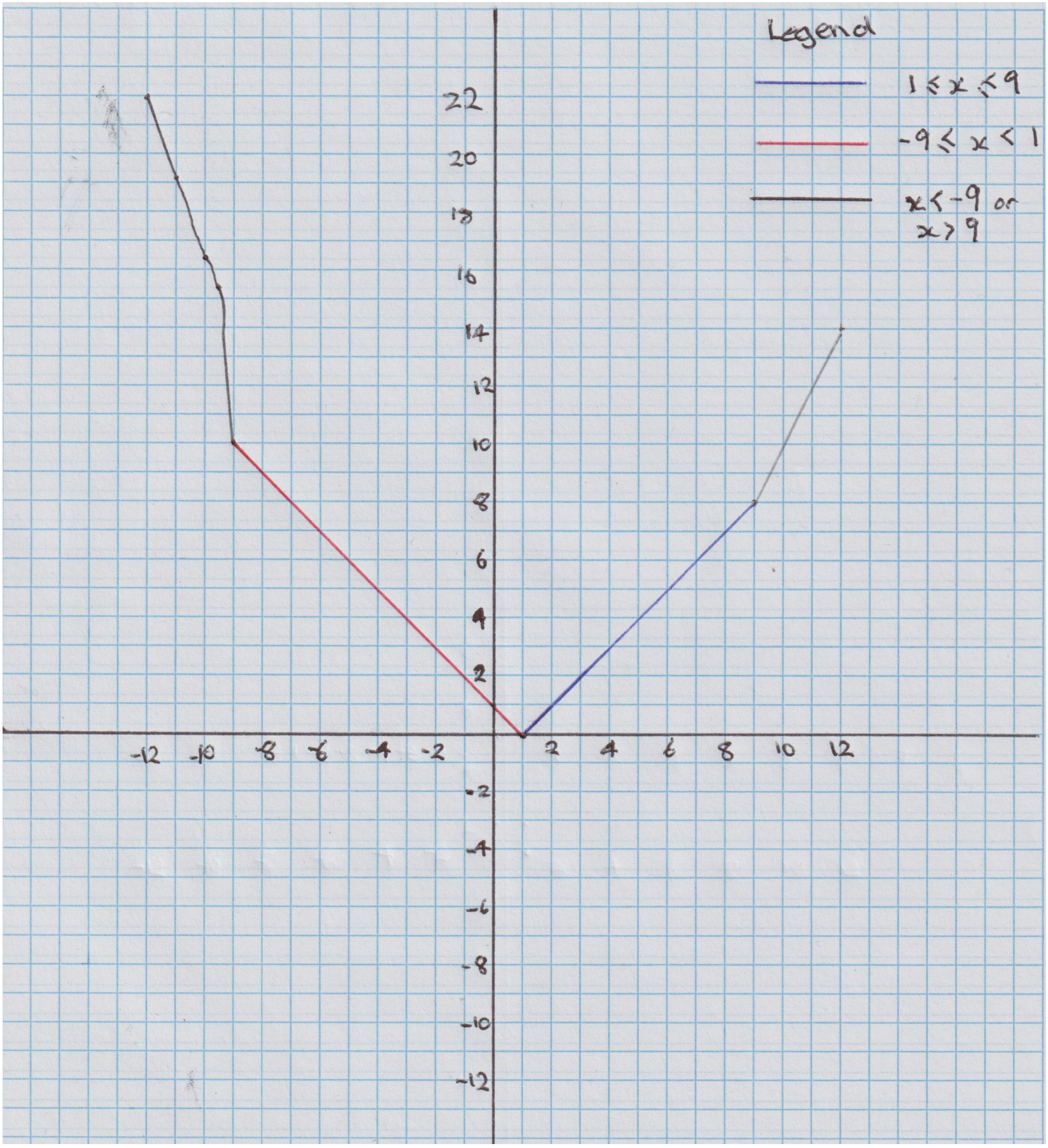
From the assignment specification sheet it states that “A point of inflection occurs at a point where f’’(x) = 0 AND the second derivative changes sign”

The intervals where $f''(x) = 0$ are $1 \leq x \leq 9$ and $-9 \leq x < 1$. Checking each interval:

- <u>For interval</u> $1 \leq x \leq 9$
  - $\lim_{x \to 1^-} f''(x) = 0 \text{ no sign}$
  - $\lim_{x \to 1^+} f''(x) = 0 \text{ no sign}$
  - $\lim_{x \to 9^-} f''(x) = 0 \text{ no sign}$
  - $\lim_{x \to 9^+} f''(x) = \frac{2}{3} \text{ positive sign}$
  - There are no sign changes for this interval
- <u>For interval</u> $-9 \leq x < 1$
  - $\lim_{x \to -9^-} f''(x) = \frac{2}{3} \text{ positive sign}$
  - $\lim_{x \to -9^+} f''(x) = 0 \text{ no sign}$
  - $\lim_{x \to 1^-} f''(x) = 0 \text{ no sign}$
  - $\lim_{x \to 1^+} f''(x) = \ 0 \text{ no sign}$
  - There are no sign changes for this interval

Since there are no sign changes for points where f’’(x) = 0, there are no inflection points for this function

Question 9



Legend
1 ≤ x ≤ 9
-9 ≤ x < 1
x < -9 or x > 9
22
20
18
16
14
12
10
8
6
4
2
-2
-4
-6
-8
-10
-12
-12
-10
-8
-6
-4
-2
2
4
6
8
10
12

Question D1

Given:

$$f(x) = \begin{cases} \sqrt{x^2 - 2x + 1} & \text{if } |x| \leq c \\ ax^2 + bx + 2 & \text{if } |x| > c \end{cases}$$

With square roots and absolute signs removed the function becomes:

$$f(x) = \begin{cases} x - 1 & 1 \leq x \leq c \\ 1 - x, & -c \leq x < 1 \\ ax^2 + bx + 2, & x < -c \text{ or } x > c \end{cases}$$

And for any $c \in \mathbb{N}$

To determine if the function $f(x)$ is continuous for the whole of $\mathbb{R}$ we check that the boundary points of the function are continuous.

A function f is continuous from the right at a number a if $\lim\limits_{x \to a^+} = f(a)$ and f is continuous from the left at a if $\lim\limits_{x \to a^-} = f(a)$ (Stewart 2013, p48)

The boundary points for the function occur at $x = -c, x = 1, x = c$

<u>For boundary point</u> $x = 1$

- $\lim\limits_{x \to 1^-} f(x) = 1 - 1 = 0$
- $\lim\limits_{x \to 1^+} f(x) = 1 - 1 = 0$
- $f(1) = 1 - 1 = 0$
- Since $\lim\limits_{x \to 1^-} f(x) = f(1) = \lim\limits_{x \to 1^+} f(x)$ the function is continuous at the boundary point $x = 1$

<u>For boundary point</u> $x = c$

- $\lim\limits_{x \to c^-} f(x) = c - 1$
- $\lim\limits_{x \to c^+} f(x) = ac^2 + bc + 2$
- $f(c) = c - 1$
- $ac^2 + bc + 2 = c - 1$ to be continuous

<u>For boundary point</u> $x = -c$

- $\lim\limits_{x \to c^-} f(x) = ac^2 - bc + 2$

- $\lim_{x \to c^+} f(x) = 1 - x$
- $f(-c) = 1 - (-c) = 1 + c$
- $ac^2 - bc + 2 = 1 + c$

We have a system of equations for when the function is continuous. The system of equations being:
$ac^2 + bc + 2 = c - 1 \Rightarrow ac^2 + bc - c + 3 = 0$
$ac^2 - bc + 2 = 1 + c \Rightarrow ac^2 - bc - c + 1 = 0$

Solving this system of equations:
$(ac^2 + bc - c + 3) - (ac^2 - bc - c + 1) = 0$
$2bc + 2 = 0$
$b = -\frac{1}{c}$

Substituting b into the first equation:
$ac^2 + -\frac{1}{c}.c - c + 3 = 0$
$ac^2 - 1 - c + 3 = 0$
$ac^2 = c + 2$
$a = \frac{c+2}{c^2}$

Therefore when $a = \frac{c+2}{c^2}, b = -\frac{1}{c}, f(x)$ is continuous

Question D2

This question is similar to Question 2, but with c not being defined.

To determine if a function is differentiable, as described in the assignment specification sheet, the following conditions must be met:

1. Its constituent functions are differentiable on the corresponding open intervals
2. At the points where two subintervals touch, the corresponding derivatives of the two neighbouring subintervals should match.

The derivative of the function:

$$f(x) = \begin{cases} x - 1 & 1 \leq x \leq c \\ 1 - x, & -c \leq x < 1 \\ ax^2 + bx + 2, & x < -c \text{ or } x > c \end{cases}$$

is:

$$f'(x) = \begin{cases} 1 & 1 \leq x \leq c \\ -1, & -c \leq x < 1 \\ 2ax + b, & x < -c \text{ or } x > c \end{cases}$$

Each of the constituent functions of f(x) were differentiable on open intervals due to the power rule therefore the first condition of a differentiable function has been met.

The points where the subintervals touch are $x = -c, x = 1, x = c$
We test either side of each point to determine if they match

<u>Boundary Point x = 1</u>

$\lim_{x \to 1^-} f'(x) = -1$

$\lim_{x \to 1^+} f'(x) = 1$

Since $\lim_{x \to 1^-} f'(x) \neq \lim_{x \to 1^+} f'(x)$ the derivatives of these two neighbouring subintervals do not match therefore the function f(x) is not differentiable.

Therefore it is not possible to find a,b and c so the the function is differentiable as the two intervals $1 \leq x \leq c \text{ and } -c \leq x < 1$ are never differentiable, regardless of the values of a, b and c in the whole of $\mathbb{R}$

Question D3

Given:

$$f(x) = \begin{cases} x-1 & 1 \le x \le c \\ 1-x, & -c \le x < 1 \\ ax^2+bx+2, & x < -c \text{ or } x > c \end{cases}$$

and:

$$f'(x) = \begin{cases} 1 & 1 \le x \le c \\ -1, & -c \le x < 1 \\ 2ax+b, & x < -c \text{ or } x > c \end{cases}$$

To determine if a function is differentiable, as described in the assignment specification sheet, the following conditions must be met:

1. Its constituent functions are differentiable on the corresponding open intervals
2. At the points where two subintervals touch, the corresponding derivatives of the two neighbouring subintervals should match.

Also If f is differentiable at a, then f is continuous at a. (Stewart 2013, p88)

We will check the boundary points of $x = c, x = -c$, and obtain a system of equations for each point using the continuity and differentiability requirement.

For boundary point $x = c$

Continuity at $x = c$ requires $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x)$
That is:

$$\lim_{x \to c^-} f(x) = x - 1 = c - 1$$
$$\lim_{x \to c^+} f(x) = ax^2 + bx + 2 = ac^2 + bc + 2$$

Therefore to be continuous at $x = c$:

$$ac^2 + bc + 2 = c - 1 \Rightarrow ac^2 + bc - c + 3 = 0$$

Differentiability at $x = c$ requires $\lim\limits_{x \to c^-} f'(x) = \lim\limits_{x \to c^+} f'(x)$
That is:

$$\lim_{x \to c^-} f'(x) = 1$$
$$\lim_{x \to c^+} f'(x) = 2ax + b = 2ac + b$$

Therefore to be differentiable at $x = c$:

$$2ac + b = 1 \Rightarrow 2ac + b - 1 = 0$$

We now have a system of equations:

$ac^2 + bc - c + 3 = 0$
$2ac + b - 1 = 0$

Solving this:

$b = 1 - 2ac$ (from equation 2)
$ac^2 + (1 - 2ac)c - c + 3 = 0$ (substitute into equation 1)
$-ac^2 + 3 = 0$
$-ac^2 = -3$
$a = \frac{3}{c^2}$
$b = 1 - 2ac \Rightarrow b = 1 - 2(\frac{3}{c^2})c \Rightarrow b = 1 - \frac{6}{c}$

Therefore $f(x)$ is differentiable at $x = c$ when $a = \frac{3}{c^2}, b = 1 - \frac{6}{c}$

<u>For boundary point</u> $x = -c$

Continuity at $x = -c$ requires $\lim_{x \to -c^-} f(x) = \lim_{x \to -c^+} f(x)$
That is:

$\lim_{x \to -c^-} f(x) = ax^2 + bx + 2 = ac^2 + bc + 2$
$\lim_{x \to -c^+} f(x) = 1 - x = 1 - (-c) = 1 + c$

Therefore to be continuous at $x = c$:

$a(-c)^2 + b(-c) + 2 = 1 - c \Rightarrow ac^2 - bc - c + 1 = 0$

Differentiability at $x = -c$ requires $\lim_{x \to -c^-} f'(x) = \lim_{x \to -c^+} f'(x)$
That is:

$\lim_{x \to -c^-} f'(x) = -1$
$\lim_{x \to -c^+} f'(x) = 2ax + b = -2ac + b$

Therefore to be differentiable at $x = -c$:

$-2ac + b = -1 \Rightarrow -2ac + b + 1 = 0$

We now have a system of equations:

$ac^2 - bc - c + 1 = 0$
$-2ac + b + 1 = 0$

Solving this:

$b = 2ac - 1$ (from equation 2)
$ac^2 - (2ac - 1)c - c + 1 = 0$ (substitute into equation 1)
$-ac^2 + 1 = 0$
$ac^2 = -1$

$$a = \frac{1}{c^2}$$

$$b = 2ac - 1 \Rightarrow b = 2(\frac{1}{c^2})c - 1 \Rightarrow b = \frac{2}{c} - 1$$

Therefore $f(x)$ is differentiable at $x = -c$ when $a = \frac{1}{c^2}, b = \frac{2}{c} - 1$

Reference

Stewart J., . (2013). Essential calculus. Belmont, CA: Brooks/Cole.