SIT787 - Mathematics for AI

Trimester 1, 2025

Due: no later than the end of Week 4, Sunday 30 March 2025, 8:00 pm AEST

Important notes:

- Your submission can be handwritten but it must be legible. If your submission is not legible, it will not be marked and will result in a zero mark. A proper way of presenting your solutions is part of the assessment.
- Please follow the order of questions in your submission.
- All steps (workings) to arrive at the answer must be clearly shown. No marks will be awarded for answers without workings.
- Generally, you need to keep your answers in the form of quotients and surds (e.g. $\frac{2}{3}$ and $\sqrt{3}$). Rarely, you may convert your solutions into decimals for plotting or comparing purposes. However, you need to show the final answer in terms of quotients and surds before converting them into decimals.
- Only (scanned) electronic submission would be accepted via the unit site (CloudDeakin).
- Your submission must be in ONE pdf file. Multiple files and/or in different file format, e.g. .jpg, will NOT be accepted. It is your responsibility to ensure your file is not corrupted and can be read by a standard pdf viewer. Failure to comply will result in a zero mark.

Piecewise defined functions: Sometimes a function is not differentiable at all points. For example, consider the absolute value function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

This function is not differentiable at $x = 0$. As you can see, the graph of the function changes direction (from decreasing to increasing) abruptly when $x = 0$.

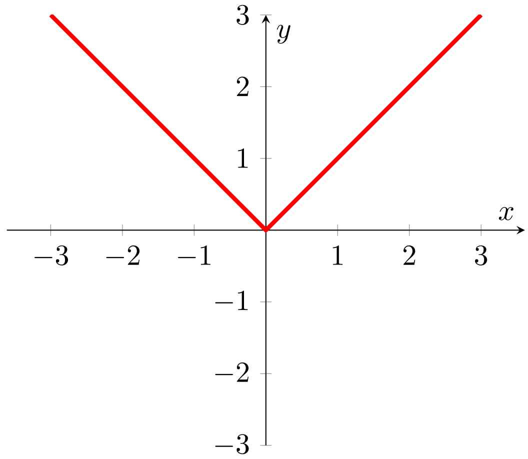

Figure 1: Absolute value function

In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of the function does not have a unique tangent at this point, and f is not differentiable there. In addition, if a function is not continuous at a point $x = a$, then f is not differentiable at a. So, at any discontinuity, f fails to be differentiable. A third possibility for a function to be non-differentiable is that the curve has a vertical tangent line at $x = a$. Let's see some examples with their plots.

(a) $f(x) = x^{\frac{2}{3}}$

(b) $g(x) = x^{\frac{1}{3}}$

(c) $h(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ \sin(x) & \text{if } x < 0. \end{cases}$

Let's see why these functions are not differentiable.

(a) $f(x) = x^{\frac{2}{3}}$. $f'(x) = \frac{2}{3\sqrt[3]{x}}$. The derivative function is not defined at $x = 0$. In Figure 2 (a), you see that $f(x)$ has a corner at $x = 0$, and the tangent is vertical. This function is differentiable everywhere except $x = 0$.

(b) $g(x) = x^{\frac{1}{3}}$. $g'(x) = \frac{1}{3\sqrt[3]{x^2}}$. The derivative function is not defined at $x = 0$. In Figure 2 (b), you see that the tangent is vertical at $x = 0$. This function is differentiable everywhere except $x = 0$.

(c) $h(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ \sin(x) & \text{if } x < 0. \end{cases}$ This function is not continuous at $x = 0$ as you can see in Figure 2 (c). How can you check whether the function is not continuous at $x = 0$? well, if

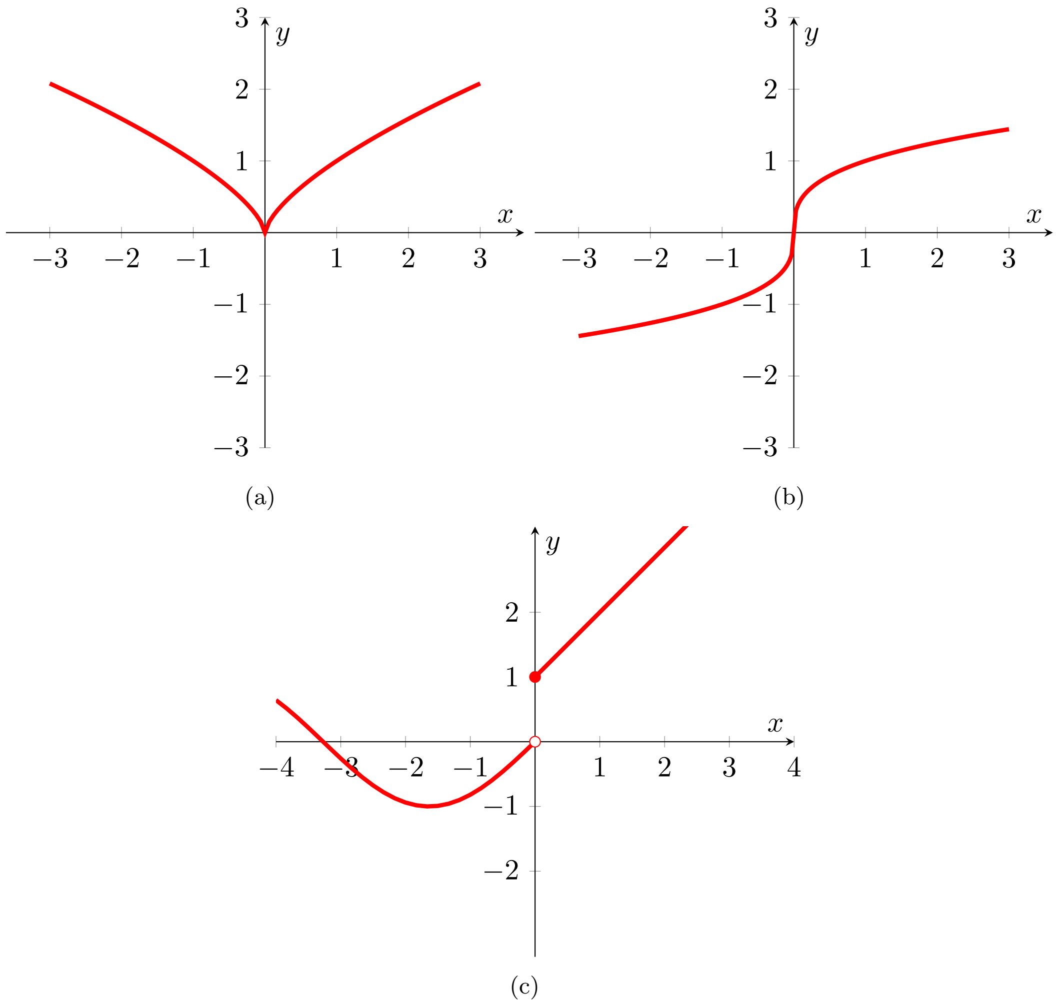

Figure 2: (a) $f(x)$ (b) $g(x)$ (c) $h(x)$

you plug $x = 0$ in the upper and lower rules, you get different values. $(0) + 1 \neq \sin(0)$, or $1 \neq 0$. If a function is not continuous at a point, it is not differentiable there. Furthermore, if we exclude 0, when $x > 0$, $h(x) = x + 1$. Its derivative for $x > 0$ is $h'(x) = 1$. When $x < 0$, $h(x) = \sin(x)$. On this interval, $h'(x) = \cos(x)$. Therefore, the derivative of this functions is

$$h'(x) = \begin{cases} 1 & \text{if } x > 0 \\ \cos(x) & \text{if } x < 0. \end{cases}$$

This function is differentiable everywhere except $x = 0$.

The derivative of the absolute value function $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$ is computed as follows;

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

You see that the derivative of the function just to the left of $x = 0$ is -1 and just to the right of $x = 0$ is $+1$. The absolute value function is differentiable everywhere except for $x = 0$. You always need to be careful with the break points. We will learn how to find the derivative of this kind of function soon.

Generally, functions are represented as $y = f(x)$ which consists of a single rule on their domain. However, sometimes functions are defined by different rules on different parts of their domain. Such functions are called piecewise defined functions. For example,

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1. \end{cases} \qquad g(x) = \begin{cases} \frac{1}{2} - \frac{x}{2} & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1. \end{cases}$$

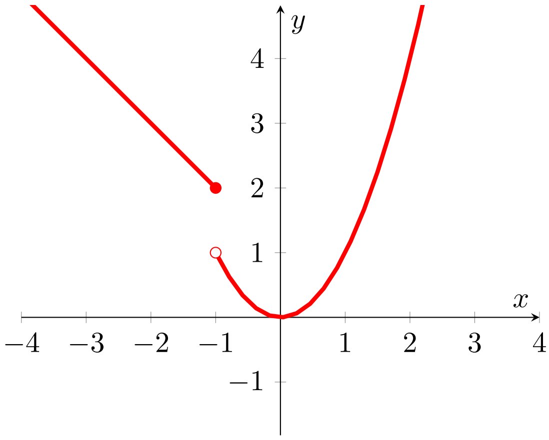

Figure 3: $f(x)$

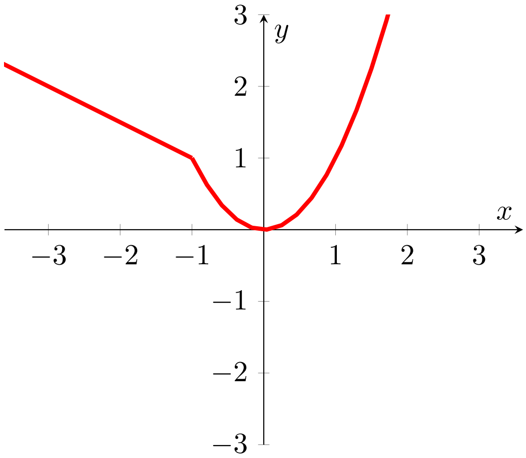

Figure 4: $g(x)$

The most famous function of this type is the absolute value function.

They even can have more than two rules on different parts of their domains.

$$f(x) = \begin{cases} -3 - x & \text{if } x \leq -3 \\ x + 3 & \text{if } -3 \leq x \leq 0 \\ 3 - 2x & \text{if } 0 \leq x \leq 3 \\ \frac{x}{2} - \frac{9}{2} & \text{if } 3 \leq x \end{cases}$$

A plot of this function is shown below.

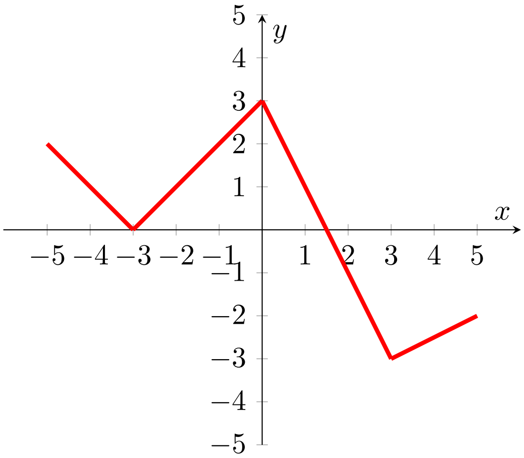

Figure 5: A function with 4 rules

In its general form, a piecewise-defined function is represented as

$$y = f(x) = \begin{cases} f_1(x) & \text{if } a \le x < b \\ f_2(x) & \text{if } b \le x < c \\ f_3(x) & \text{if } c \le x < d \end{cases}$$

A function of this type is continuous if its constituent functions are continuous on the corresponding intervals and there is no discontinuity at each breakpoint. In other words, $f_1(x)$ should be continuous on $[a, b)$, $f_2(x)$ should be continuous on $[b, c)$, and $f_3(x)$ should be continuous on $[c, d)$. In addition, there should not be a discontinuity on the breakpoints $x = b$ and $x = c$.

For example, the function $f(x)$ and $g(x)$ in Figure 3 are continuous, and the function $h(x)$ is discontinuous. Similarly in Figure 4, the function $f(x)$ is discontinuous, and $g(x)$ is continuous.

A piecewise function is differentiable on a given interval in its domain if the following conditions are satisfied in addition to those for continuity mentioned above:

- its constituent functions are differentiable on the corresponding open intervals,
- at the points where two subintervals touch, the corresponding derivatives of the two neighboring subintervals should match.

For example, for $f(x) = \begin{cases} 1 - x & \text{if } x \le -1 \\ x^2 & \text{if } x > -1. \end{cases}$

its derivative can be found as

- On the open interval $x < -1$ or $(-\infty, -1)$, $f(x) = 1 - x$, and $f'(x) = -1$.
- On the open interval $x > 1$ or $(-1, \infty)$, $f(x) = x^2$, and $f'(x) = 2x$.
- We visually see that the function is not continuous at $x = -1$, therefore it is not differentiable there. In other words, if you plug $x = -1$ in the upper and lower rules, you get different values. In addition, at $x = -1$, the derivative of the first rule is -1 and of second rule is $2(-1)$. As $-1 \neq -2$ it is not differentiable at $x = -1$. The derivative of this function is

$$f'(x) = \begin{cases} -1 & \text{if } x < -1 \\ 2x & \text{if } x > -1. \end{cases}$$

  We observe that the derivative function is not defined at $x = -1$.

Another example is $g(x) = \begin{cases} \frac{1}{2} - \frac{x}{2} & \text{if } x \le -1 \\ x^2 & \text{if } x > -1. \end{cases}$

- On the open interval $x < -1$ or $(-\infty, -1)$, $g(x) = \frac{1}{2} - \frac{x}{2}$, and $g'(x) = \frac{-1}{2}$.
- On the open interval $x > -1$ or $(-1, \infty)$, $g(x) = x^2$, and $g'(x) = 2x$.
- At $x = -1$, the function is continuous (as $\frac{1}{2} - \frac{-1}{2} = (-1)^2$), but using the first rule, $g'(-1) = \frac{-1}{2}$ and using the second rule $g'(-1) = 2(-1) = -2$. However, $\frac{-1}{2} \neq -2$. This function is not differentiable at $x = -1$. Then, the derivative of this function is

$$g'(x) = \begin{cases} \frac{-1}{2} & \text{if } x < -1 \\ 2x & \text{if } x > -1. \end{cases}$$

There are some piecewise defined functions that are continuous and differentiable everywhere. For example

$$f(x) = \begin{cases} x - \frac{1}{4} & \text{if } x < \frac{1}{2} \\ x^2 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Optimisation: With optimisation, we want to find the maximum and minimum of a function. We have two types of maximum and minimum points. Absolute maximum and absolute minimum, and local minimum and local maximum. Let's define them properly for a function $y = f(x)$.

- Absolute (or global) maximum: A point $x = c$ is called an absolute maximum of f, if for any value in its domain ($x \in \text{Dom}(f)$), $f(x) \leq f(c)$
- Absolute (or global) minimum: A point $x = c$ is called an absolute minimum of f, if for any value in its domain ($x \in \text{Dom}(f)$), $f(x) \geq f(c)$
- Local maximum: A point $x = c$ is called a local maximum of f, if for every value x near c, $f(x) \leq f(c)$
- Local minimum: A point $x = c$ is called a local minimum of f, if for every value x near c, $f(x) \geq f(c)$.

In the following plot, the function has a local maximum at $x = x_0$, a local minimum at $x = x_1$, and a local maximum at $x = x_2$. The point $x = x_2$ is a global maximum as well. There is no absolute minimum for this function.

However, if we are given a closed interval, we can find the absolute maximum and minimum of the function in that interval. For example, in the following plot, in the closed interval $[a, b]$, the function has an absolute maximum at $x = x_0$ and an absolute minimum at $x = x_1$. To find

all these interesting points of a function, we need to find its critical points. When we find them, then we need to classify each critical point as a global/local maximum/minimum. The critical points for a given function $y = f(x)$ are

- the points $x = c$ where $f'(c) = 0$, or
- the points $x = c$ where $f'(c)$ does not exist.

Optimisation problems type 1: Finding absolute maximum and minimum of $y = f(x)$ in a given closed interval $[a, b]$:
To solve this problem,

(a) Find critical points of $f(x)$, and evaluate f at these points.
(b) Find $f(a)$ and $f(b)$.

The maximum value of items in (a) and (b) is the absolute maximum of $f(x)$ in $[a, b]$, and the minimum value of items in (a) and (b) is the absolute minimum of $f(x)$ in $[a, b]$.

Optimisation problems type 2 (First derivative test): Find local maximum and minimum of $y = f(x)$ using first derivative:

- Find all critical points of $f(x)$.
- If for a critical point $x = c$, f' changes from positive to negative (f changes from increasing to decreasing), $x = c$ is a local maximum point.
- If for a critical point $x = c$, f' changes from negative to positive (f changes from decreasing to increasing), $x = c$ is a local minimum point.

Optimisation problems type 2 (Second derivative test): Find local maximum and minimum of $y = f(x)$ using second derivative:

- Find all critical points of $f(x)$.
- For a critical point $x = c$, if $f'(c) = 0$ and $f''(c) > 0$, $x = c$ is a local minimum.
- For a critical point $x = c$, if $f'(c) = 0$ and $f''(c) < 0$, $x = c$ is a local maximum.
- if $f''(c) = 0$, the test is inconclusive. It does not give any useful information, and we need to use other techniques to decide the type of stationary point.

Question The function

$$f(x) = \begin{cases} \sqrt{x^2 - 2x + 1} & |x| \le c \\ ax^2 + bx + 2 & |x| > c \end{cases}$$

is defined for some $c \in \mathbb{N}$. Answer the following questions:

- Set c as the most right non-zero digit in your student ID.
  (1) Find the equivalent form of this function as a piecewise defined function by removing absolute value and square root signs.
  (2) Find the derivative of $f(x)$ and determine where the function is not differentiable.
  (3) Find all the intercepts of $y = f(x)$.
  (4) Find all critical points of $y = f(x)$.
  (5) Classify all critical points of $y = f(x)$.
  (6) Find intervals that the function is increasing or decreasing.
  (7) Find the second derivative of $y = f(x)$.
  (8) Find points of inflection of $y = f(x)$.
  (9) Draw the function **by hand** using all the information you obtained so far. Annotate important information on your graph.
- (For D and HD levels) For any $c \in \mathbb{N}$
  (D_1) Find a and b in terms of c so that the function is continuous in whole $\mathbb{R}$.
  (D_2) Is it possible to find a, b and c so that the function is differentiable in whole $\mathbb{R}$?
  (D_3) For what values of c the function is differentiable at $x = c$ or $x = -c$?

Things you need to know:

- $\mathbb{N} = \{1, 2, 3, \ldots\}$
- $\sqrt{x^2} = |x|$
- $|x| \le a \equiv -a \le x \le a$
- $|x| \ge a \equiv x \le -a$ or $x \ge a$
- If a function is not continuous at a point $x = d$, then it is not differentiable at that point.
- For a function

$$f(x) = \begin{cases} f_1(x) & \text{if } a < x \le b \\ f_2(x) & \text{if } b < x < c \end{cases}$$

  to be continuous at $x = b$ we need to have $f_1(b) = f_2(b)$. In addition to be differentiable at $x = b$ we also need to have $f_1'(b) = f_2'(b)$.
- You do not need to know or use the concept of

$$\lim_{x \to a} f(x)$$

  to solve this problem.
- A point of inflection occurs at a point where $f''(x) = 0$, **AND** the second derivative changes sign.

[(1: 10, 2: 10, 3: 10, 4: 10, 5: 5, 6: 5, 7: 5, 8: 5, 9: 10, D_1:10, D_2:10, D_3:10) = 100 marks]