SIT787 -Mathematics for AI

Assignment 2

Trimester 1, 2025

Due: no later than the end of Week 8, Sunday 4 May 2025, 8:00 pm AEST

Important notes:

- Your submission can be handwritten but it must be legible. If your submission is not legible, it will not be marked, resulting in a zero mark. A proper way of presenting your solutions is part of the assessment.
- Please follow the order of questions in your submission.
- All steps (workings) to arrive at the answer must be clearly shown. No marks will be awarded for answers without workings.
- Generally, you need to keep your answers in the form of quotients and surds (e.g. $\frac{2}{3}$ and $\sqrt{3}$). Rarely, you may convert your solutions into decimals for plotting or comparing purposes. However, you need to show the final answer in terms of quotients and surds before converting them into decimals.
- Only (scanned) electronic submission would be accepted via the unit site (CloudDeakin).
- Your submission must be in ONE pdf file. Multiple files and/or in different file formats, e.g. .jpg, will NOT be accepted. It is your responsibility to ensure your file is not corrupted and can be read by a standard PDF viewer. Failure to comply will result in a zero mark.

Question) Consider the following matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(1) Find rank(A), a basis for the column space of A, $C(A)$, a basis for the row space of A, $C(A^T)$, a basis for the null space of A, $N(A)$, and a basis for the left null space of A, $N(A^T)$.

Hint: For a $m \times n$ matrix A, the column space of A is defined as $C(A) = \{A\mathbf{x} | \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$, and the row space of A is defined as $C(A^T) = \{A^T\mathbf{y} | \mathbf{y} \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$. Also, the nullspace $N(A)$ is defined as $\{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$. For the same matrix, the left nullspace $N(A^T)$ is defined as $\{\mathbf{y} \in \mathbb{R}^m | A^T\mathbf{y} = \mathbf{0}\} \subseteq \mathbb{R}^m$. Also, we have

$$\dim(C(A)) + \dim(N(A^T)) = m$$

$$\dim(C(A^T)) + \dim(N(A)) = n.$$

[Note: You need to use Gaussian elimination to answer this question. For **all** the linear systems in this assignment, you need to use Gaussian elimination to solve them.]

(2) Construct $B = A^T A$, and find eigenvalues and eigenvectors of B. For positive eigenvalues of B, define $\sigma_i = \sqrt{\lambda_i}$, where $\lambda_1 \geq \lambda_2$. Construct matrix D as follows.

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

(3) If the eigenvectors of B are not orthonormal, orthonormalise them. Make a matrix V using the orthonormal vectors you obtained. The ordering of the columns of V should be the same as the ordering of the eigenvalues, that is $V = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} \end{bmatrix}$. In other words, $\boldsymbol{v_1}$ is the eigenvector corresponding λ_1 and $\boldsymbol{v_2}$ is the eigenvector corresponding to λ_2.

(4) Find eigenvalues and eigenvectors of the matrix $G = A^T A$, and orthonormalise them. Call them $\boldsymbol{u_1}, \boldsymbol{u_2},$ and $\boldsymbol{u_3}$ according to the order of $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of eigenvalues of G. Construct the matrix $U = \begin{bmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} \end{bmatrix}$, and check whether it is an orthogonal matrix.

(5) Find three vectors $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ so that

$$\boldsymbol{w_i} = \frac{1}{\sigma_i} A\boldsymbol{u_i}, i = 1, 2$$

$$\boldsymbol{w_3} \perp \boldsymbol{w_1}, \boldsymbol{w_3} \perp \boldsymbol{w_2}, \text{ and } ||\boldsymbol{w_3}|| = 1.$$

Compare $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ with $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ and comment similar characteristic of these sets.

(6) Compute the product UDV^T.

(7) Do the questions (1), (2), (3), (4) for the slightly different version of A:

$$A_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix}$$

Construct $\sigma_{p_i}, \boldsymbol{v_{p_i}}, \boldsymbol{u_{p_i}}, V_p, D_p,$ and U_p. Justify the changes you observe. (In this question you may use decimal numbers.)

(8) Construct a low-rank approximation of A and A_p following this definition using their corresponding eigenvectors:

$$A \approx \sigma_1 \boldsymbol{v_1} \boldsymbol{u_1}^T$$

$$A_p \approx \sigma_{p_1} \boldsymbol{v_{p_1}} \boldsymbol{u_{p_1}}^T$$

Here $\sigma_{p_1}, \boldsymbol{v_{p_1}},$ and $\boldsymbol{u_{p_1}}$ are corresponding to the eigenvalues and eigenvectors related to A_p. Compare these two matrices and justify the changes if you observe any.

(9) **(For D and HD levels)** Consider the slightly altered version of A:

$$A_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \epsilon & 0 \end{bmatrix}$$

(9.1) Find eigenvalues and eigenvectors of $A_\epsilon A_\epsilon^T$ and $A_\epsilon^T A_\epsilon$ in terms of ϵ. Compute $\sigma_i = \sqrt{\lambda_i}$ in terms of ϵ for positive eigenvalues.

(9.2) Construct $V_\epsilon, D_\epsilon,$ and U_ϵ.

(9.3) Critically analyse the impact of ϵ on the σ_i, λ_i, eigenvectors, $V_\epsilon, D_\epsilon,$ and U_ϵ.

[**1**:10+**2**:10+**3**:5+**4**:10+**5**:10+**6**:5+**7**:15+**8**:10+(**9.1**:10+**9.2**:5+**9.3**:10)=100 marks]