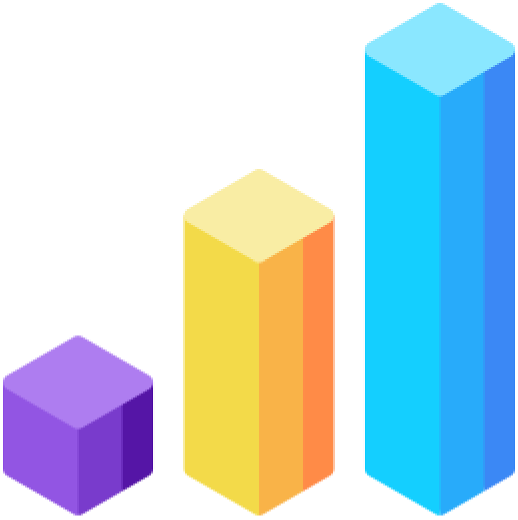

SIT787 Mathematics for AI

Assessment 2

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Question 1

Given the matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We observe that the matrix is in the Reduced Row Echelon Form.

A matrix is in echelon form if the following two conditions hold:

1. All zero rows, if any, are at the bottom of the matrix
2. Each learning nonzero entry in a row is to the right of the leading nonzero entry in the preceding row (Lipschutz et al 2013, p70)

A matrix is in row-reduced echelon form if its is an echelon matrix and if it satisfies the following additional two properties:

1. Each pivot (leading non-zero entry) is equal to 1)
2. Each pivot is the only nonzero entry in its column (Lipschutz et al 2013, p71)

As the matrix A conforms to both constraints for echelon form and reduced-row echelon form, the matrix is in reduced-row echelon form.

Rank

The rank of a matrix A, written rank(A), is equal to the number of pivots in an echelon form of A. (Lipschutz et al 2013, p72). Therefore, as there are two pivots in A:

$$rank(A) = 2$$

Basis For Column Space of A

The column space of A, denoted C(A), is:

$$C(A) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Theorem 4.14 ii) Any linearly independent set $s = \{u_1, u_2 ... u_n\}$ with n elements is a basis of V (Lipschutz et al 2013, p124)

Due to this rule and as A is already in reduced row echelon form and the pivot columns in A are linearly independent, Basis of $C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Basis for Row Space of A

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C(A) = span \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

The next step is to perform Gaussian elimination and then take the pivot columns.
A^T is already in reduced row echelon form therefore the pivot columns (columns 1 and 2) are linearly independent.
Therefore
Basis of $C(A^T) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Basis for Null Space of A

The null space N(A) is the set of vectors that satisfies $Ax = 0$

Given that A is a 3x2 matrix the vector x must be in R^2. To solve $Ax = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

After matrix multiplication we get
$1.x_1 + 0.x_2 = 0 \Rightarrow x_1 = 0$
$0.x_1 + 1.x_2 = 0 \Rightarrow x_2 = 0$
$0.x_1 + 0.x_2 = 0$

Therefore the only solution to $Ax = 0$ is $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Which means that there are no non-zero vectors hence the null space of A is:
$N(A) = \emptyset$

Basis for the Left Null Space

The left null space of A $N(A^T)$ is the set of vectors that satisfies $A^Tx = 0$
The vector(s) are in R^3

$$A^Tx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

After matrix multiplication we get
$1.x_1 + 0.x_2 + 0.x_3 = 0 \Rightarrow x_1 = 0$
$0.x_1 + 1.x_2 + 0.x_3 = 0 \Rightarrow x_2 = 0$
x_3 is free

Therefore the solution to $A^Tx = 0$ is
$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
1 being chosen for the free variable x_3

So the basis for the left null space
$$N(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Question 2

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B = A^T A = \begin{bmatrix} 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

B is a 2x2 identity matrix

Eigenvalue

To find the eigenvalues we solve $det(B - \lambda I) = 0$

$$B - \lambda I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{bmatrix}$$

$$\det \begin{bmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{bmatrix} = (1 - \lambda)(1 - \lambda) - 0.0 = (1 - \lambda)^2$$

$\therefore (1 - \lambda)^2 = 0$

The roots are therefore $\lambda_1 = 1, \lambda_2 = 1$

Eigenvectors

Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

To find the eigenvectors we solve

$$(B - 1.\lambda)v = \begin{bmatrix} 1 - 1 & 0 \\ 0 & 1 - 1 \end{bmatrix} v = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} v = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \Rightarrow 0x + 0y = 0$$

Any value of x and y solves the equation, so choose x = 1 and y = 1. The eigenspace is R^2 therefore the eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Sigma

The positive eigenvalues are $\lambda_1 = 1, \lambda_2 = 1$

Therefore

$$\sigma_1 = \sqrt{1} = 1$$
$$\sigma_2 = \sqrt{1} = 1$$
$$\therefore D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Question 3

From Question 2 we have the eigenvectors being:

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} v_1 v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Orthogonality Test

Two conditions are required for a set of vectors to be orthonormal:

1. Each vector must be a unit vector (norm equal to 1)
2. Vectors must be orthogonal (dot product equal to 0)

Norm Checks

$$\|v_1\| = \sqrt{1^2 + 0^2} = \sqrt{1} = 1$$
$$\|v_2\| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$$

Therefore both vectors are unit vectors as their norm's are 1.

Orthogonality Checks

$$v_1.v_2 = 1.0 + 0.1 = 0 + 0 = 0$$

Therefore the vectors v_1, v_2 are orthogonal as their dot product is 0.

Since v_1, v_2 both have norm of 1 and they are orthogonal, they are already orthonormal.

Question 4

$$G = AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 0 + 1 \cdot 0 \\ 0 \cdot 1 + 0 \cdot 0 & 0 \cdot 0 + 0 \cdot 1 & 0 \cdot 0 + 0 \cdot 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues

To find the eigenvalues need to solve:

$$det(G - \lambda I) = 0$$

$$G - \lambda I = \begin{bmatrix} 1 & -\lambda & 0 & 0 \\ & 0 & 1 & -\lambda & 0 \\ & 0 & 0 & -\lambda \end{bmatrix}$$

The determine equation for a 3x3 matrix is:

$$det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Hence

$$det(G - \lambda I) = (1 - \lambda)(1 - \lambda)(-\lambda)$$

Set determinant to zero:

$$(1 - \lambda)^2(-\lambda) = 0$$

Therefore the eigenvalues are:

$\lambda_1 = 1\lambda_2 = 1\lambda_3 = 0$

And $\lambda_1 \geq \lambda_2 \geq \lambda_3$

Eigenvectors

We can find the eigenvectors u, by solving $(G - \lambda I) = 0$

For $\lambda = 1$ we get

$\lambda_1 = 1, \lambda_2 = 1$

$$(G-1I)u = \begin{bmatrix} 1-1 & 0 & 0 \\ 0 & 1-1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-u_3 = 0 \therefore u_3 = 0$.
u_1, u_2 are free variables so set to 1.
Eigenspace for lambda equal to one is two dimensional and the third component is zero. Se we can pick two independent eigenvectors

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 0$
Need to solve
$Gu = 0$

Therefore

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus
$u_1 = u_2 = 0$ with u3 free. So

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Orthonormality</u>

Norm Checks
$\|u_1\| = \sqrt{1^2 + 0^2 + 0^2} = \sqrt{1} = 1$
$\|u_2\| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1$
$\|u_3\| = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{1} = 1$
Therefore all norms are unit vectors

Orthogonality Checks
$u_1 \cdot u_2 = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0$
$u_1 \cdot u_3 = 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 = 0$
$u_2 \cdot u_3 = 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$
Therefore all vectors are orthogonal as their dot products equal 0

Final U Matrix

$$U = [u_1 u_2 u_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

U is orthogonal if $U^T \cdot U = I$ and $U \cdot U^T = I$.

$$U^T \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$U \cdot U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since both products equal the identity matrix, U is an orthogonal matrix.

<u>Question 5</u>

Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\sigma_1 = 1, \sigma_2 = 1$ and $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

And the equation:

$$w_i = \frac{1}{\sigma_i} A v_i, \quad i = 1, 2$$

<u>Calculate w_1</u>

$$w_1 = \frac{1}{1} A v_1 = 1 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \times \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 1 + 0 \cdot 0 \end{bmatrix} = 1 \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

<u>Calculate w_2</u>

$$w_2 = \frac{1}{1} A v_1 = 1 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \times \begin{bmatrix} 1 \cdot 0 + 0 \cdot 1 \\ 0 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 0 + 0 \cdot 1 \end{bmatrix} = 1 \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So $w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

<u>Calculate w_3</u>

To find the vector w_3 that have norm of 1 and is orthogonal to w_1, w_2 we must show:

1. $w_3 \perp w_1$ that is $w_3 \cdot w_1 = 0$
2. $w_3 \perp w_2$ that is $w_3 \cdot w_2 = 0$
3. $\|w_3\| = 1$

Let $w_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Orthogonal condition 1

$w_3 \cdot w_1 = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x \cdot 1 + y \cdot 0 + z \cdot 0 = 0$$

$x = 0$ to satisfy equation

Orthogonal condition 2
$w_2 \perp w_1 = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = x \cdot 0 + y \cdot 1 + z \cdot 0 = 0$$

$y = 0$ to satisfy equation

Norm condition 2

Currently $w_3 = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$

We must solve $\|w_3\| = 1$
$\|w_3\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + 0^2 + z^2} = \sqrt{z^2} = |z|$

z can be 1 or -1. Choosing one we get:

$$w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Vectors are:

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The set $\{w_1, w_2, w_3\} = \{u_1, u_2, u_3\}$ which means that the matrix made from w_1, w_2, w_3 is an identity matrix and also orthonormal.

Question 6

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \; V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U \cdot D = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore UDV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 0 & 0 \cdot 0 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Question 7

$$A_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix}$$

$$\text{RREF of } \mathrm{A_p} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Rank: 2

Basis for Column Space of A_p

Columns of A_p are linearly independent, therefore the basis of the column space of A_p is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0.1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Basis for Row Space of A_p

$row_1 = [1, 0]$
$row_2 = [0, 1]$
$row_3 = [0.1, 0]$

Row 3 is linearly dependent on row 1 (0.1 * 1 = 0.1) therefore doesnt add any new information to the row space

Therefore the basis for row space of A_p is $\{(1, 0), (0, 1)\}$

Basis for Null Space of A_p

Need to solve $A_p x = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

System of equations

$1 \cdot x_1 + 0 \cdot x_2 = 0 \implies x_1 = 0$
$0 \cdot x_1 + 1 \cdot x_2 = 0 \implies x_2 = 0$
$0.1 \cdot x_1 + 0 \cdot x_2 = 0 \implies 0.1x_1 = 0 \implies x_1 = 0$

Therefore $Null(A_p) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

Basis for the Left Null Space of A_p

$$A_p^T = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \end{bmatrix}$$

Need to solve $A_p^T y = 0$

$$\begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

System of equations

$1 \cdot y_1 + 0 \cdot y_2 + 0.1 \cdot y_3 = 0 \Rightarrow y_1 = -0.1y3$

$0 \cdot y_1 + 1 \cdot y_2 + 0 \cdot y_3 = 0 \Rightarrow y_2 = 0$

y_3 is free so

$$N(A_p^t) = \left\{ \begin{bmatrix} -0.1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Eigenvalues and Eigenvectors of $B = A_p^t A_p$ **(Question 3)**

$$B = A_p^T A_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1.1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalues of a diagonal matrix are the elements along the diagonal

Therefore $\lambda_1 = 1.1, \lambda_2 = 1$

For each eigenvalue solve $(B - \lambda I)v = 0$

For λ_1:

$$\begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

X is free and y must equal 0 therefore $v_{p1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For λ_2:

$$\begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Y is free and x must equal 0 therefore $v_{p2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\sigma_{p1} = \sqrt{1.1} = 1.05, \sigma_{p2} = \sqrt{1} = 1$$

$$D_p = \begin{bmatrix} 1.05 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$V_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Perform orthonormality checks

Dot product is equal to zero check

$v_{p1} \cdot v_2 = (1)(0) + (0)(1) = 0$

Unit length checks

$\|v_{p1}\| = \sqrt{1^2 + 0^2} = \sqrt{1} = 1$

$\|v_{p2}\| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$

Both dot product check and unit length checks reveal that the eigenvectors of B are orthonormal

Construction of G Matrix (Question 4)

$$G_p = A_p A_P^T = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 0.001 \end{bmatrix}$$

Eigenvalues are:

$\lambda_1 = -0.0089$

$\lambda_2 = 1$

$\lambda_3 = 1.00991$

Eigenvectors are:

$$v_1 = \begin{bmatrix} -0.099 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 10.089 \\ 0 \\ 1 \end{bmatrix}$$

Orthonormality checks:

Check dot product of eigenvectors

$v_1 \cdot v_3 = -0.001189 \neq 0$ Therefore G_p is not orthonormal

Orthonormalise:

To normalise a vector, divide it by its norm.

$$\|v_1\| = \sqrt{-0.0089^2 + 1^2 + 1.00991^2} = 1.421210399$$
$$\|v_2\| = \sqrt{0^2 + 1^2 + 0^2} = 1$$
$$\|v_3\| = \sqrt{10.089^2 + 0^2 + 1^2} = 10.138437799$$

$$u_{p1} = \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{-0.0089}{1.4212} \\ 0 \\ \frac{1}{1.4212} \end{bmatrix} = \begin{bmatrix} -0.006262268 \\ 0 \\ 0.703625586 \end{bmatrix}$$

$$u_{p2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\|v_2\|$ is 1 so the normalised vector is equal to the original vector

$$u_{p3} = \frac{v_3}{\|v_3\|} = \begin{bmatrix} \frac{10.089}{10.1384} \\ 0 \\ \frac{1}{10.1384} \end{bmatrix} = \begin{bmatrix} 0.9951 \\ 0 \\ 0.0986 \end{bmatrix}$$

$$U_p = \begin{bmatrix} -0.0062 & 0 & 0.9951 \\ 0 & 1 & 0 \\ 0.7036 & 0 & 0.0986 \end{bmatrix}$$

The addition of 0.1 in the first column of A_p resulted in stretching in one dimension. This resulted in identical V matrices, but U and D had differences in their matrices between the original and p version. The first and third column of U_p were rotated substantially. D_p had its first row, first column change from 1 to 1.05 slightly increasing the magnitude of this component

<u>Question 8</u>

The two equations in the assignment specification sheet were:
$A \approx \sigma_1 v_1 u_1^T$
$A_p \approx \sigma_{p_1} v_{p_1} u_{p_1}^T$
However this results in a 2x3 matrix, whereas the final approximated matrix should be 3x2

I believe the correct equations might be:

$$A \approx \sigma_1 u_1 v_1^T = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_p \approx \sigma_{p_1} u_{p_1} v_{p_1}^T = 1.05 \cdot \begin{bmatrix} -0.0062 \\ 0 \\ 0.7036 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -0.00651 & 0 \\ 0 & 0 \\ 0.73878 & 0 \end{bmatrix}$$

The low rank approximation of A and Ap are different in that their principal direction and scale differs.

Question 9

$$A_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \epsilon & 0 \end{bmatrix}$$

Question 9.1

$$A_\epsilon^T = \begin{bmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \end{bmatrix}$$

For $A_\epsilon A_\epsilon^T$

$$A_\epsilon A_\epsilon^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \epsilon & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 1 & 1 \cdot \epsilon + 0 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 & 0 \cdot \epsilon + 1 \cdot 0 \\ \epsilon \cdot 1 + 0 \cdot 0 & \epsilon \cdot 0 + 0 \cdot 1 & \epsilon \cdot \epsilon + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \\ \epsilon & 0 & \epsilon^2 \end{bmatrix}$$

To find eigenvalues solve

$$det(A_\epsilon A_\epsilon^T - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} 1-\lambda & 0 & \epsilon \\ 0 & 1-\lambda & 0 \\ \epsilon & 0 & \epsilon^2 - \lambda \end{bmatrix} = 0$$

Use row 2 for Laplace's expansion

$$\det \begin{bmatrix} 1-\lambda & 0 & \epsilon \\ 0 & 1-\lambda & 0 \\ \epsilon & 0 & \epsilon^2-\lambda \end{bmatrix}$$

$$= 0 \begin{bmatrix} 0 & \epsilon \\ 0 & \epsilon^2-\lambda \end{bmatrix} + (1-\lambda) \begin{bmatrix} 1-\lambda & \epsilon \\ \epsilon & \epsilon^2-\lambda \end{bmatrix} + 0 \begin{bmatrix} 1-\lambda & 0 \\ \epsilon & 0 \end{bmatrix}$$

$$= (1-\lambda)(-\lambda - \epsilon^2\lambda + \lambda^2)$$

$$= -\lambda^3 + \epsilon^2\lambda^2 + 2\lambda^2 - \epsilon^2\lambda - \lambda$$

Solve for $-\lambda + \varepsilon^2\lambda^2 + 2\lambda - \varepsilon \quad -\lambda = 0$

$$\lambda(-\varepsilon^2 - 1) + \lambda^2(\varepsilon^2 + 2) - \lambda^3 = 0$$

$$\lambda(-\lambda + \varepsilon^2 + 1)(\lambda - 1) = 0$$

$$-\lambda + \varepsilon^2 + 1 = 0, \quad \lambda - 1 = 0, \quad \lambda = 0$$

Eigenvalues are $\lambda_1 = \epsilon^2 + 1, \lambda_2 = 1, \lambda_3 = 0$

Eigenvectors are found by solving $(A_\epsilon A_\epsilon^T - I\lambda)v = 0$

$$\begin{bmatrix} 1-\lambda & 0 & \epsilon \\ 0 & 1-\lambda & 0 \\ \epsilon & 0 & \epsilon^2-\lambda \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

For $\lambda = 0$

$$\begin{bmatrix} 1 & 0 & \varepsilon \\ 0 & 1 & 0 \\ \varepsilon & 0 & \varepsilon^2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$$

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & \varepsilon & 0 \\ 0 & 1 & 0 & 0 \\ \varepsilon & 0 & \varepsilon^2 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & \varepsilon & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

system of equations

$$V_1 + \varepsilon V_3 = 0$$

$$V_2 = 0$$

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -\varepsilon V_3 \\ 0 \\ V_3 \end{bmatrix}$$

let $V_3 = 1$

$$V = \begin{bmatrix} -\varepsilon \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 1$

$$\begin{bmatrix} 0 & 0 & \varepsilon \\ 0 & 0 & 0 \\ \varepsilon & 0 & \varepsilon^2-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 0 & \varepsilon^2-1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} \varepsilon & 0 & \varepsilon^2-1 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} \varepsilon & 0 & \varepsilon^2-1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ rref}$$

so $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$v_1 = 0$
$v_3 = 0$
v_2 free

so $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 1 + \varepsilon^2$

$$\begin{bmatrix} -\varepsilon^2 & 0 & \varepsilon \\ 0 & -\varepsilon^2 & 0 \\ \varepsilon & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{ccc|c} -\varepsilon^2 & 0 & \varepsilon & 0 \\ 0 & -\varepsilon^2 & 0 & 0 \\ \varepsilon & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -\varepsilon^2 & 0 & \varepsilon & 0 \\ 0 & -\varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} \varepsilon^2 & 0 & \varepsilon & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{\varepsilon} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ rref}$$

system of equations

$$v_1 - \frac{v_3}{\varepsilon} = 0 \Rightarrow v_1 = \frac{v_3}{\varepsilon}$$

$v_2 = 0$

v_3 is free

choosing $v_3 = 1$

$v_1 = 1/\varepsilon$, $v_2 = 0$, $v_3 = 0$ $\quad v = \begin{bmatrix} 1/\varepsilon \\ 0 \\ 1 \end{bmatrix}$

Eigenvectors are:

$$v_1 = \begin{bmatrix} -\varepsilon \\ 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1/\varepsilon \\ 0 \\ 1 \end{bmatrix}$$

For $A_\epsilon^T A_\epsilon$

$$A_\epsilon^T A_\epsilon = \begin{bmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \epsilon & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 + \epsilon \cdot \epsilon & 1 \cdot 0 + 0 \cdot 1 + \epsilon \cdot 0 \\ 0 \cdot 1 + 1 \cdot 0 + 0 \cdot \epsilon & 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} \epsilon^2 + 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To find the eigenvalues solve:

$det(A_\epsilon^T A_\epsilon - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} \epsilon^2 + 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{bmatrix} = 0$

The determinant is the product of its diagonal elements in a diagonal matrix.
Therefore the determinant is $(\epsilon^2 + 1 - \lambda)(1 - \lambda)$
Solving for $(\epsilon^2 + 1 - \lambda)(1 - \lambda) = 0$
Setting any factor to 0 satisfies the equation therefore eigenvalues are:
$\lambda_1 = \epsilon^2 + 1, \lambda_2 = 1$

Eigenvectors are found by solving $(A_\epsilon^T A_\epsilon - I\lambda)v = 0$

$\begin{bmatrix} \epsilon^2 + 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

For $\lambda_1 = \varepsilon^2 + 1$

$$\begin{bmatrix} \varepsilon^2 + 1 - (\varepsilon^2 + 1) & 0 \\ 0 & 1 - (\varepsilon^2 + 1) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & -\varepsilon^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

v_1 is free, $-\varepsilon^2 v_2 = 0 \quad \therefore v_2 = 0$

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 1$

$$\begin{bmatrix} \varepsilon^2 + 1 - 1 & 0 \\ 0 & 1 - 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

V_2 is free, $\varepsilon^2 V_1 = 0 \therefore V_1 = 0$

$\therefore V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Compute σ for positive values

$\sigma_1 = \sqrt{\varepsilon^2 + 1}$, $\sigma_2 = \sqrt{1} = 1$

for $\varepsilon^2 > 0$

Question 9.2

V_ϵ are the eigenvectors of $A_\epsilon^T A_\epsilon$

$$V_\epsilon = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D_\epsilon = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\epsilon^2+1} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

U_ϵ are the eigenvectors of $A_\epsilon A_\epsilon^T$

$$U_\epsilon = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} -\epsilon & 0 & \frac{1}{\epsilon} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Question 9.3

The following describe the impact of epsilon on the various components.

Impact on σ_i
σ_2 is a constant so has no relationship to epsilon.
$\sigma_1 = \sqrt{\epsilon^2 + 1}$ As the magnitude of epsilon increases sigma 1 also increases.

Impact on λ_i
Epsilon only impacts $\lambda_1 = \epsilon^2 + 1$ as the other two eigenvalues are constant. Lamda 1 has a quadratic dependence on epsilon and therefore a strong directional variance along the principal component.

Impact on V_ϵ

Epsilon has no impact on the matrix V_ϵ as all elements are constants.

Impact on D_ϵ

σ_1 in D_ϵ has a non-linear dependence on epsilon. For values less than 1 epsilon has a minor effect on sigma 1. For values larger than 1, sigma 1 grows approximately linearly with epsilon.

Impact on U_ϵ

The dominant left singular column is highly sensitive to epsilon. It becomes more aligned with the negative x-axis the larger epsilon is.

The second column has no impact from epsilon.

The third column has an inverse relationship to epsilon and therefore its direction changes as epsilon varies.

Reference

Lipschutz, S., Lipson, M. (2013). *Schaum's Outline of Linear Algebra*. New York: McGraw-Hill.