ET4340 Electronics for Quantum Computing
Homework 4

Mick van Gelderen
4091566

November 2013

Problem 1: Quantum Fourier Transform

1. Write the 8×8 matrix (in the computational basis) corresponding to the quantum Fourier transform on a 3-qubit register.

The matrix $U_{QFT,8}$ is defined as:

$$U_{QFT,8} = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} e^{\frac{i2\pi lk}{N}} |l\rangle \langle k|$$

where $N = 8$.

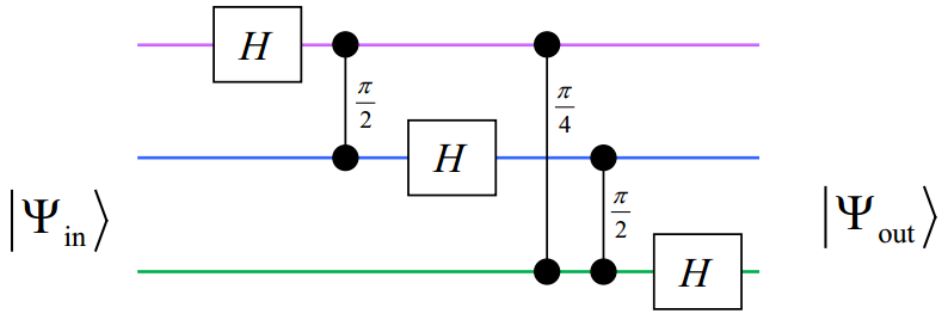
$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \sqrt{2}+\sqrt{2}i & i & -\sqrt{2}+\sqrt{2}i & -1 & -\sqrt{2}-\sqrt{2}i & -i & \sqrt{2}-\sqrt{2}i \\
1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & -\sqrt{2}+\sqrt{2}i & -i & \sqrt{2}+\sqrt{2}i & -1 & \sqrt{2}-\sqrt{2}i & i & -\sqrt{2}-\sqrt{2}i \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\sqrt{2}-\sqrt{2}i & i & \sqrt{2}-\sqrt{2}i & -1 & \sqrt{2}+\sqrt{2}i & -i & -\sqrt{2}+\sqrt{2}i \\
1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & \sqrt{2}-\sqrt{2}i & -i & -\sqrt{2}-\sqrt{2}i & -1 & -\sqrt{2}+\sqrt{2}i & i & \sqrt{2}+\sqrt{2}i
\end{pmatrix}$$

2. Show that this matrix is unitary

Use a computer to calculate $U_{QFT}U_{QFT} = I$. You can easily see that the matrix is symmetric so this is the only thing we have to show. Matlab indeed gives I with some near zero imaginary parts.

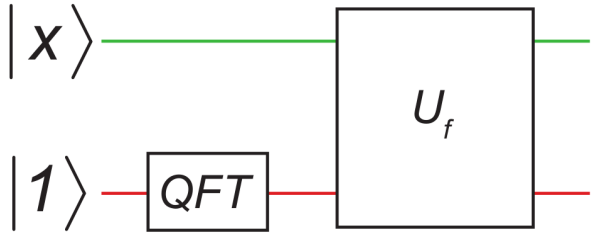
3. Draw the quantum circuit that implements this QFT

From the lecture slides where it was drawn for 4 qubits:



Problem 2: Generalized quantum kick-back

In the lectures, we have seen how the Deutsch and Bernstein-Vazirani quantum games exploit quantum kick-back to efficiently extract properties of n-to-1 bit boolean functions. In this problem, we generalize quantum kick-back to n-to-m bit boolean functions encoded in unitary functions as usual: $U_f |x\rangle |y\rangle = |x\rangle |(y + f(x)) \bmod M\rangle$ for computational states $|x\rangle$ and $|y\rangle$ in the top and bottom registers, respectively. Consider the circuit below.



1. What is the state of the bottom register after application of the m-bit QFT on the initial state $|1\rangle = |00000...1\rangle$?

$U_{QFT} |1\rangle$ equals the last column of U_{QFT}. So $U_{QFT} |1\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{i2\pi(N-1)k}{N}} |k\rangle$

2. Now apply U_f, with the top register starting in a computational state $|x\rangle$. What is the combined state of the top and bottom registers immediately after U_f? Show that this this state can be rewritten as:

$$e^{\frac{-i2\pi f(x)}{M}} |x\rangle \otimes U_{QFT} |1\rangle ,$$

where $M = 2^m$. Evidently, the top and bottom registeres are not entangled, and $f(x)$ is encoded in the quantum phase of the probability amplitude!

?

3. Finally, consider the case that the top register is initialized in the maximal superposition state $\frac{1}{\sqrt{N}}(|0\rangle + \cdots + |N-1\rangle)$. As usual, $N = 2^n$. What will be the final state after application of U_f?

?

Problem 3: Breaking RSA In this exercise, we will break RSA by period finding. N will be small enough that we will find periods by brute force.

1. List the integers $a < N$ that are co-prime with N. Let us pick one of these integers: let us agree to all 'randomly' pick $a = 8$.

I guess $N = 21$.

n	divisible by	co-primes
2	2	1
3	3	1 2
4	2 4	1 3
5	5	1 2 3 4
6	2 3 6	1 5
7	7	1 2 3 4 5 6
8	2 4 8	1 3 5 7
9	3 9	1 2 4 5 7 8
10	2 5 10	1 3 7 9
11	11	1 2 3 4 5 6 7 8 9 10
12	2 3 4 6 12	1 5 7 11
13	13	1 2 3 4 5 6 7 8 9 10 11 12
14	2 7 14	1 3 5 6 9 11 13
15	3 5 15	1 2 4 7 8 11 13 14
16	2 4 8 16	1 3 5 7 9 11 13 15
17	17	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
18	2 3 6 9 18	1 5 7 11 13 17
19	19	1 3 5 7 8 9 10 11 12 13 14 15 16 17 18 19
20	2 4 5 10 20	1 3 7 9 11 13 17 19
21	3 7 21	1 2 4 5 8 10 11 13 16 17 19

2. Compute $8^0 \bmod 21$, $8^1 \bmod 21$, ... until you discover the period r of $f(x) = 8^x \bmod 21$.

n	$8^n \bmod 21$
0	1
1	8
2	1

The period seems to be 2.

3. Find the greatest common denominator (gcd) between, $8^{r/2}+1$ and 21. Check whether the result is a prime factor of 21.

Since $r = 2$, $8^{r/2}+1 = 9$. The gcd of 9 and 21 is 3 which is indeed a prime factor of 21.

4. Similarly, find the gcd between $8^{r/2}-1$ and 21.

The gcd of 7 and 21 is 7 which is a prime factor of 21.

5. Repeat the process for $a = 10$.

Finding the period:

n	10^n mod 21
0	1
1	10
2	6
3	13
4	4
5	19
6	1

The period seems to be 6. The gcd's of $10^3 \pm 1$ and 21 are 3 and 7, magic!

Problem 4: Breaking RSA with Shor's algorithm

Now we will go through the steps of Shor's algorithm in order to find the period r of 8^x mod 21. For simplicity, lest us use just four qubits for the top register which is enough for our choice of $a = 8$. *Note: Please use decimal bra-ket notation instead of binary.*

1. Initialize each register to $|0\rangle$.

Cool it down!

2. Apply the hadamard gate to each qubit in the top register.

The top register ends up in a state that is a combination of all possible 4 qubit computational states: $\frac{1}{\sqrt{16}}(|0\rangle + |1\rangle + \cdots + |15\rangle)$.

3. Add 8^x mod 21 to the bottom register where x is the state of the top register.

We calculate $8^x \bmod 21$ for $x \in \{0, 1, \ldots 15\}$.

x	$8^x \bmod 21$	x	$8^x \bmod 21$
0	1	8	1
1	8	9	8
2	1	10	1
3	8	11	8
4	1	12	1
5	8	13	8
6	1	14	1
7	8	15	8

So, y must be $\frac{1}{\sqrt{16}}(|1\rangle + |8\rangle + |1\rangle + |8\rangle \ldots |1\rangle + |8\rangle)$

4. Rewrite this state so you group all terms with identical $f(x)$ — observe the periodicity in the amplitudes which emerges, and observe also that you cannot efficiently reveal the period by any measurement.

This means that the y register is in the state $\frac{1}{\sqrt{2}}(|1\rangle + |8\rangle)$.

For this simple example you might be able to measure 100 times. If you measure $|1\rangle$ 31 times and $|8\rangle$ 69 times it is probable that the period is 2 because you only measured 2 different outcomes. In general, there is a chance that if you measure k times you will have not measured one or more possible outcomes which causes you to use an r that is too low. For bigger N, the amount of measurements k becomes very, very large to have confidence in your estimation of r.

Using the QFT is far more efficient.

5. Apply the Quantum Fourier Transform to the top register.

We apply the Quantum Fourier Transform to the top register after measurement of the bottom register.

In the case that we measure $y =$