ET4340: Electronics for Quantum Computing December 20, 2013

Homework Assignment #6
Last one!!!
Due at start of class on Jan. 10, 2014

Important note: Throughout this assignment, we use the same labeling conventions (data qubits and syndrome measurements) as in the latest version of the slides for weeks 50 and 51 (in BlackBoard). To facilitate grading, please also use the same labeling conventions!

Problem 1: *Protection against single-qubit relaxation*
We have a single-qubit superposition state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ ($\alpha, \beta \neq 0$ and $|\alpha|^2 + |\beta|^2 = 1$) encoded using Shor's 9-qubit code as $|\Psi_L\rangle = \alpha|0_{\text{Shor}}\rangle + \beta|1_{\text{Shor}}\rangle$. Suppose that data qubit 9 undergoes a relaxation process by interacting with its environment. That is:

$$\begin{aligned} |0_9\rangle|e\rangle &\rightarrow |0_9\rangle|e\rangle \\ |1_9\rangle|e\rangle &\rightarrow \sqrt{1-p}|1_9\rangle|e\rangle + \sqrt{p}|0_9\rangle|e'\rangle, \end{aligned}$$

were p is a relaxation probability. Assume that all other data qubits remain undisturbed.

1. What are the possible combinations of syndrome measurement outcomes?

2. Specify the probability for each possible combination of measurement outcomes.

3. Show that for all possible combinations of measurement outcomes, the correction step will successfully return the logical qubit to its original state $|\Psi_L\rangle$.

Problem 2: *Two-qubit errors in Steane's 7-qubit code*
In this problem, we explore whether Steane's 7-qubit code can, in addition to correcting errors in single qubits, also correct errors on two qubits. Suppose we have a single-qubit superposition state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ ($\alpha, \beta \neq 0$ and $|\alpha|^2 + |\beta|^2 = 1$) encoded as $|\Psi_L\rangle = \alpha|0_{\text{Steane}}\rangle + \beta|1_{\text{Steane}}\rangle$.

1. *Warmup:* Suppose data qubit 6 is corrupted by a phase-flip (Z). What will be the syndrome measurement results (m_a through m_f)?

2. To preserve the logical qubit state, what correction must be applied when such syndrome measurements are obtained?

3. Suppose that data qubits 1 and 2 both undergo a phase-flip each. Explain why the syndrome measurement results will be the same in this case.

4. Naturally, your feedback scheme applies the same correction as in (2) above (it cannot tell that actually a two-qubit error happened). Will this correction fix the problem, or will it add an error at the logical qubit level? If the latter, please specify the logical error incurred.

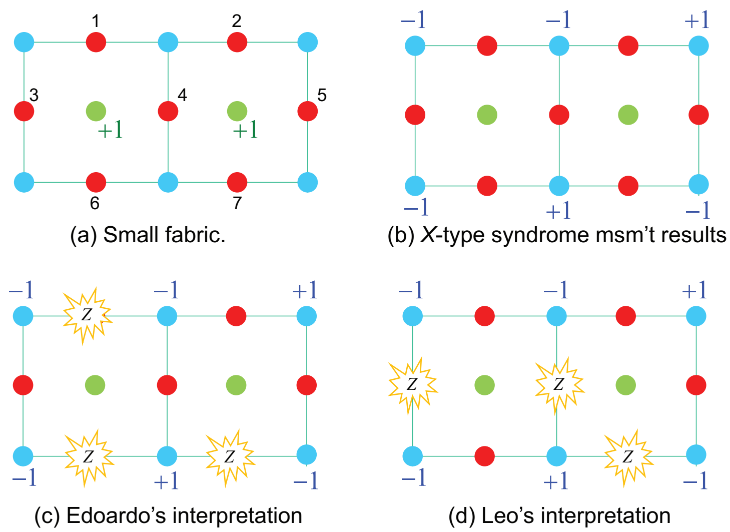

Figure 1: prototype fabric

Problem 3: *Initializing the surface-code fabric.*

Edoardo and Leo are in the lab playing with a prototype surface-code fabric, consisting of 7 (red) data qubits and 8 ancilla qubits (2 green, 6 blue) used for syndrome measurements [see Fig. 1(a)]. They want to initialize the red qubits to the state $|0L\rangle$, defined as the single state that is a common eigenstate of all 8 stabilizers with eigenvalue +1 in all cases. The starting state of the fabric is $|\psi_{\text{init}}\rangle = |0000000\rangle$, i.e., all data qubits in the ground state.
(a) They perform the two green (Z-type) measurements. Specify the two measurement operators. Explain why both measurements will give result +1. What is the state of the fabric after these measurements are done?
(b) They proceed to the blue (X-type) measurements. Specify the 6 measurement operators. They obtain the measurement results shown in Fig. 1(b). Edoardo interprets that the measurements have projected the fabric onto the state $Z_1 Z_6 Z_7 |0_L\rangle$ [Fig. 1(c)]. Would this state be consistent with the error syndromes measured? [Hint: don't try to write down the 7-qubit state!]. Edoardo suggests that applying the correction $Z_1 Z_6 Z_7$ will put the red qubits in $|0_L\rangle$. Show that if Edoardo's interpretation of the post-measurement state is right, then his suggestion is good and the fabric will be initialized to $|0_L\rangle$.
(c) Leo looks puzzled. In his interpretation, the measurements have projected the fabric onto the state $Z_3 Z_4 Z_7 |0_L\rangle$ [Fig. 1(d)]. Thus, he suggests that instead we apply $Z_3 Z_4 Z_7$ to arrive at $|0_L\rangle$. Show that if Leo's interpretation of the post-measurement state is right, then his suggestion is a good one.
(d) You walk into the room. Edoardo and Leo show you the measurement results and ask you

for your interpretation. You provide yet another possibility for how the measurements have projected the fabric. Please specify it. Suggest a different correction.
(e) Now that we are all confused, we visit Chris. Chris smiles and says, *You fools....Even if Edoardo's interpretation is right, Leo's suggestion will work. Even if Leo's interpretation is right, Edoardo's suggestion will work.* Show that this is true.
(f) Show that if Edoardo's interpretation of the post-measurement state is right, your suggestion will work. Similarly, show that if Leo's interpretation is right, your suggestion will work.
(g) Prove $Z_1 Z_6 Z_7 |0_L\rangle = Z_3 Z_4 Z_7 |0_L\rangle$. This means that everyone's conjecture of the post-measurement state is actually the same state!
Conclusion: To initialize the fabric into $|0_L\rangle$ starting from $|0000000\rangle$, perform the X-type measurements, provide a consistent interpretation of the post-measurement state based on the error syndromes and correct according to that interpretation.

Problem 4: *Logical operations in the Ninja Star*
In lecture, we have seen how $X_7 X_5 X_3$ and $Z_9 Z_5 Z_1$ implement the logical-qubit operations X_L and Z_L, respectively.

1. Show that $X_9 X_6 X_3$ also implements X_L. To see this, you can follow several paths. One is to work out how this operation transforms the logical basis states $|0_L\rangle$ and $|1_L\rangle$. The other is to show that $X_9 X_6 X_3$ is equal to a product of $X_7 X_5 X_3$ and some of the 8 stabilizers.
2. Similarly, show that $X_8 X_5 X_2$ also implements X_L.
3. Show that $Z_6 Z_5 Z_4$ implements Z_L.
4. Suggest another product of three data-qubit Z operations that also implements Z_L.