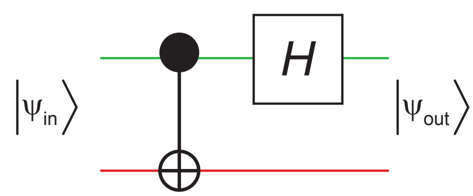
Homework Assignment #3

Due at start of class on Dec. 6, 2013

Problem 1: *Bell state analyzer*
In the previous homework, we introduced the *Bell states*

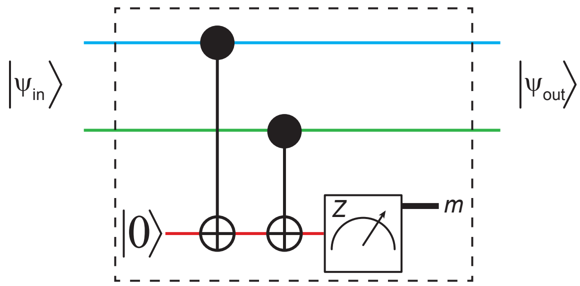
$$\begin{aligned} |\Psi_+\rangle &= \tfrac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \\ |\Psi_-\rangle &= \tfrac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \\ |\Phi_+\rangle &= \tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\ |\Phi_-\rangle &= \tfrac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right). \end{aligned}$$

Consider the quantum circuit below:



- What is the 4×4 unitary matrix describing the transformation on the two input qubits? (Please write it using the computational basis)
- What will be the output state when the input state is $|\Psi_+\rangle$?
- What will be the output state when the input state is $|\Psi_-\rangle$?
- What will be the output state when the input state is $|\Phi_+\rangle$?
- What will be the output state when the input state is $|\Phi_-\rangle$?
- For each of the four input states above, what will be the outcome of measurements performed on the two qubits at the end?

Problem 2: *Entanglement by measurement*
Consider the circuit below:



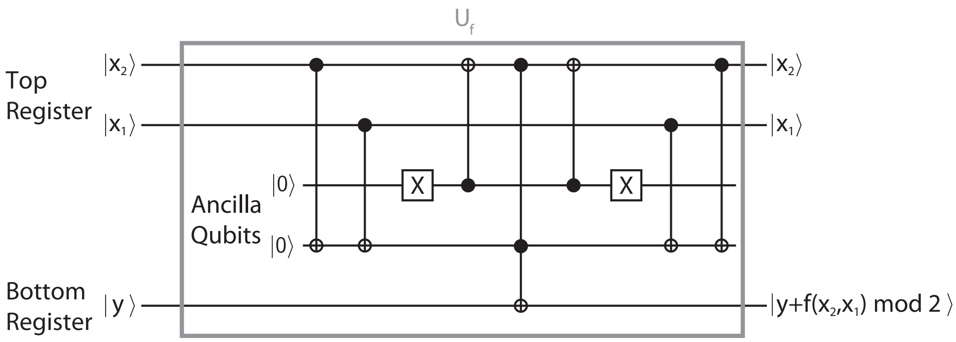
The two input qubits are initially prepared in the maximal superposition state

$$|\Psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right),$$

and the ancilla qubit in $|0\rangle$.

- What is the state of the three-qubit system after the two CNOT gates and before the measurement?
- Using the generalized Born rule, find the probability of the ancilla measurement giving result $m = +1$.
- What is the final state of the two qubits when the ancilla measurement gives $m = +1$? What is the concurrence of this state?
- What is the final state of the two qubits when the ancilla measurement gives $m = -1$? What is the concurrence of this state?
- Suppose that when $m = -1$, and only then, you apply X to one of the qubits (This implements a form of feedback control). What will be the final state in this case?
- Can you think of a strategy to deterministically produce $|\Psi_-\rangle$ using measurement?

Problem 3: *Encoding boolean functions in unitaries*



Above, you see a more complicated quantum circuit than in Quiz #3. As in Quiz #3, the top register has two qubits, and the lower register only 1. However, the quantum unitary uses two internal ancilla qubits.

1. What is the state of the ancilla qubits immediately after the unitary? Does the answer depend on the input state of the top and bottom registers?

2. Assume you enter a maximal superposition state

$$\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

   in the top register, and $|0\rangle$ in the bottom register. After the unitary, you measure the lower-register qubit in the z basis and get $m = +1$. Is the final state of the two qubits in the top register an entangled or a product state?

3. Give the truth table with input and output of both registers. Why are the ancillas not included here?

4. Deduce the encoded boolean function $f(x)$.

5. Can you device another quantum circuit that encodes this boolean and does not make use of any ancilla qubits?

Problem 4: *Grover's algorithm*
In this problem, we will look further into Grover's algorithm. You can approach this problem in whichever way is more comfortable for you, analytically or by writing a simple program in your favorite language (Matlab, C, anything you like). If you do it with a program, please print your code!

1. In class, we showed that the $N \times N$ matrix corresponding to Grover's analysis step, implementing *inversion about the mean*, has elements $M_{ij} = 2/N$ for $i \neq j$, and $M_{ii} = -1 + 2/N$. Verify that this matrix is unitary. That is, show $M^\dagger M = MM^\dagger = I$.

2. Consider the situation where the search function has two solutions, instead of just one, and suppose $N = 2^{10}$. Will Grover's algorithm work? If so, is it equally likely to find either of the two solutions, or will it always find one of them?

3. How many Grover iterations will it require to find one of the answers?

Problem 5: (EXTRA CREDIT) *Fidelity to a Bell state as an entanglement witness*
The fidelity of a two-qubit state $|\psi\rangle$ to a reference state $|\psi_{\mathrm{ref}}\rangle$ is defined as

$$\mathcal{F} \equiv |\langle\psi|\psi_{\mathrm{ref}}\rangle|^2.$$

Consider the set of all untentangled two-qubit states. That is, all states of the form

$$|\psi(\theta_2, \phi_2, \theta_1, \phi_1)\rangle = \Big(\cos(\theta_2/2)|0\rangle + e^{i\phi_2}\sin(\theta_2/2)|1\rangle\Big) \otimes \Big(\cos(\theta_1/2)|0\rangle + e^{i\phi_1}\sin(\theta_1/2)|1\rangle\Big)$$

- Prove that the fidelity of $|\psi(\theta_2, \phi_2, \theta_1, \phi_1)\rangle$ to the Bell state $|\Phi_+\rangle$ cannot exceed 50% for any choice of θ_2, ϕ_2, θ_1, ϕ_1.

- Note that the converse is not true: less than 50% fidelity of a state to $|\Phi_+\rangle$ does not guarantee that the state is not entangled. Provide a simple example.