ET4340 Electronics for Quantum Computing
Homework 6

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Januari 2014

Problem 1: Protection against single-qubit relaxation

We have a single-qubit superposition state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ ($\alpha, \beta \neq 0$ and $|\alpha|^2 + |\beta|^2 = 1$) encoded using Shor's 9-qubit code as $|\Psi_L\rangle = \alpha|0_{Shor}\rangle + \beta|1_{Shor}\rangle$. Suppose that data qubit 9 undergoes a relaxation process by interacting with its environment. That is:

$$
\begin{aligned}
|0_9\rangle|e\rangle &\to |0_9\rangle|e\rangle \\
|1_9\rangle|e\rangle &\to \sqrt{1-p}\,|1_9\rangle|e\rangle + \sqrt{p}\,|0_9\rangle|e'\rangle
\end{aligned}
$$

where p is a relaxation probability. Assume that all other data qubits remain undisturbed.

1. What are the possible combinations of syndrome measurement outcomes?

$$|0_9\rangle = \alpha_9|0_9\rangle + \beta_9|1_9\rangle$$

Applying the relaxation gives:

$$\alpha_9|0_9\rangle + \beta_9|1_9\rangle \to \alpha_9|0_9\rangle|e\rangle + \beta_9\sqrt{p}\,|0_9\rangle|e'\rangle + \beta_9\sqrt{1-p}\,|1_9\rangle|e\rangle$$

Simplifying and discarding the environment state:

$$\to (\alpha_9 + \beta_9\sqrt{p})|0_9\rangle + \beta_9\sqrt{1-p}\,|1_9\rangle$$

We can write $M|\psi\rangle = \lambda|\psi\rangle$.

$$
\begin{aligned}
MA + AM &= 0 \\
MA|\psi\rangle + AM|\psi\rangle &= 0 \\
MA|\psi\rangle + A\lambda|\psi\rangle &= 0 \\
M(A|\psi\rangle) &= -\lambda(A|\psi\rangle)
\end{aligned}
$$

2. Now consider an operator B that commutes with M (i.e., $[M, B] \equiv MB - BM = 0$). Show that the state $B|\psi\rangle$ is an eigenstate of M with eigenvalue λ.

Assuming M again has an eigenstate $|\psi\rangle$ with eigenvalue λ.

$$\begin{aligned} MB - BM &= 0 \\ MB|\psi\rangle - BM|\psi\rangle &= 0 \\ M(B|\psi\rangle) &= \lambda(B|\psi\rangle) \end{aligned}$$

3. Prove the identity:

   Feel free to do this either by multiplying matrices or by manipulating circuit diagrams. From this we see that a single-qubit bit-flip error prior to `CNOT` proliferates into a double bit-flip error.

It is an intuitive identity in my opinion. Flipping the control bit flips the output. So to 'simulate' the flip before a `CNOT` you can flip both the output and the control after the `CNOT`.

By matrice multiplication (note that the order of operations is reversed with respect of the diagram because we compute the combined matrix $M = M_n \dots M_2 M_1$ in $|out\rangle = M|in\rangle$ where M_n is operation n):

$$M_a = \texttt{CNOT}_{01}(I \otimes X) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_b = (X \otimes X)\texttt{CNOT}_{01} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_a = M_b$$