ET4340 Electronics for Quantum Computing
Homework 2

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Problem 1: The Bell basis

In class, we introduced the *Bell states*

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$
$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

Like the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, this set of states forms a basis for the fourdimensional Hilbert space of two qubits. Show that this basis is ortho-normal. That is:

1. Show that the inner product of every Bell state with itself is unity. In other words, show that $\langle\Upsilon|\Upsilon\rangle = 1|\forall\Upsilon \in \{\Psi_+, \Psi_-, \Phi_+, \Phi_-\}$.

The bell states can be written as simple vectors as follows:

$$
\begin{aligned}
2\,|\Psi_+\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 1 & 0\end{pmatrix}^\top \\
|\Psi_-\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix}^\top \\
|\Phi_+\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}^\top \\
|\Phi_-\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix}^\top
\end{aligned}
$$

It is very easy to see that for each vector, taking the inner product with itself results in 1. For example:

$$\langle\Psi_+|\Psi_+\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \cdot (0^2 + 1^2 + 1^2 + 0^2) = 1$$

2. Show that the inner product of every Bell state with every other bell state equals zero.

Since the inner product is commutative we only have to do a small set of computations. Also, it is easy to see that the inner product between a Ψ and a Φ Bell state equals 0 since there is a zero in every row. So the remaining cases are:

$$
\begin{aligned}
\langle\Psi_+|\Psi_-\rangle &= \left(\frac{1}{\sqrt{2}}\right)^2 \cdot (0^2+1^2+1\cdot(-1)+0^2) = 0 \\
\langle\Phi_+|\Phi_-\rangle &= \left(\frac{1}{\sqrt{2}}\right)^2 \cdot (1^2+0^2+0^2+1\cdot(-1)) = 0
\end{aligned}
$$

3. Write a matrix that transforms the coordinates of a state in the computational basis to the coordinates of the same state in the Bell basis. To be clear, find a matrix M that translates $|00\rangle$ to $|\Psi_+\rangle$, $|01\rangle$ to $|\Psi_-\rangle$, and so forth.

Lets say we want to express the state $|00\rangle = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}$ in the computational basis in the Bell basis. We must find a lineair combination of the bell states that adds up to $|00\rangle$.

We could do this by reducing a matrix but that should not be necessary. You can see that we need to add $|\Phi_+\rangle$ and $|\Phi_-\rangle$ and normalize it.

$$
\begin{aligned}
|00\rangle &= x \cdot (|\Phi_+\rangle + |\Phi_-\rangle) \\
&= x \cdot 2 \cdot \frac{1}{\sqrt{2}} |00\rangle \qquad \text{where } x \cdot 2 \cdot \frac{1}{\sqrt{2}} = 1 \text{ so } x = \frac{1}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} \cdot (|\Phi_+\rangle + |\Phi_-\rangle) \\
&= \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}^\top
\end{aligned}
$$

If we repeat these steps for all the computational states we get the following matrix:

$$
M = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}
$$

Problem 2: Generalized `CPHASE` gates

1. Write the four-by-four matrix transformation which flips the phase of the green qubit and does nothing to the red qubit, i.e., implements $Z \otimes I$.

Lets notate the two-qubit state as $|gr\rangle$. It can be proven that $(A \cdot |g\rangle) \otimes (B \cdot |r\rangle) = (A \otimes B) \cdot (|g\rangle \otimes |r\rangle)$. This property can be used to conjure multi-cubit transformations from single-cubit transformations.

To find the transformation that flips the green and retains the red we compute $Z \otimes I$

$$
\begin{aligned}
Z \otimes I &= \left(\begin{array}{c|c} Z_{1,1} I & Z_{1,2} I \\ \hline Z_{2,1} I & Z_{2,2} I \end{array} \right) \\
&= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
\end{aligned}
$$

2. Now do the reverse, flip the red qubit and do nothing to the green qubit.

$$I \otimes Z = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

3. Give the matrix which flips the phase of the green qubit if and only if the red qubit is in $|0\rangle$.

This gate is a lot like the $Z \otimes I$ transformation except that we don't want to flip when the red qubit is in $|1\rangle$. So we flip the sign of the fourth column which corresponds to $|11\rangle$:

$$\texttt{CPHASE}_s{}^{1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

Similarly, we can imagine two more generalized `CPHASE` gates. Show by example how using Z gates we can change any one of these transformations into any of the other three.

You can calculate $(Z \otimes I)\texttt{CPHASE}_s$ and $(Z \otimes Z)\texttt{CPHASE}_s$ to get two new gates with a -1 in the respectively the fourth and second columns (since $Z \otimes Z$ has $\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}$ as the diagonal).

Problem 3: From `CNOT` to `CPHASE` using experimentalist's one-qubit gates

In class we discussed that $\texttt{CPHASE} = H_g \texttt{CNOT}_{rg} H_g$.

1. Show that $\texttt{CPHASE} = R_{y,g}(-\frac{\pi}{2})\texttt{CNOT}_{rg}R_{y,g}(\frac{\pi}{2})$.

[1] $\texttt{CPHASE}_s$ denotes the gate discussed in class

Lets start with computing $R_y(\pm\frac{\pi}{2})$:

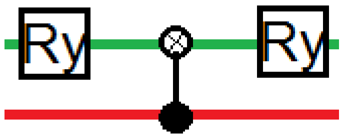
$$
\begin{aligned}
R_y(-\frac{\pi}{2}) &= \cos(-\frac{\pi}{4})I - i\sin(-\frac{\pi}{4})Y \\
&= \frac{1}{\sqrt{2}}I + i\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \\
R_y(\frac{\pi}{2}) &= \cos(\frac{\pi}{4})I - i\sin(\frac{\pi}{4})Y \\
&= \frac{1}{\sqrt{2}}I - i\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}
\end{aligned}
$$

By using the lessons learnt in the previous assignment we can compute the transformation matrix for the operation $\left(R_y(-\frac{\pi}{2})\otimes I\right)\texttt{CNOT}_{rg}\left(R_y(\frac{\pi}{2})\otimes I\right)|gr\rangle$

$$
\begin{aligned}
M &= \frac{1}{2}\begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\\ -1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}2 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}
\end{aligned}
$$

So M is actually a `CPHASE` gate.

2. Draw the quantum circuit implementing these three operations.



Problem 4: The `SWAP` operation

The two-qubit unitary `SWAP`, as the name implies, exchanges the quantum states of the green and red qubits, i.e., it implements the transformation $|\psi\rangle\otimes|\phi\rangle \to |\phi\rangle\otimes|\psi\rangle$. In quantum circuit language, the `SWAP` gate is notated as a line between the qubits with crosses at the intersections.

1. Write the four-by-four matrix representing this unitary transformation.

$|00\rangle$ and $|11\rangle$ are unaffected. $|01\rangle$ and $|10\rangle$ are interchanged thus the matrix will be:

$$\texttt{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. Show that $\texttt{SWAP} = \texttt{CNOT}_{rg}\texttt{CNOT}_{gr}\texttt{CNOT}_{rg}$

This is a simple matrix multiplication and it turns out to be correct.

3. Show that $\texttt{SWAP} = \texttt{CNOT}_{gr}\texttt{CNOT}_{rg}\texttt{CNOT}_{gr}$

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Problem 5: Measurements on Bell states

Edoardo and Chris each have one qubit from a pair prepared in the Bell state $|\Phi_+\rangle$.

1. Edoardo and Chris agree to each perform a measurement of Z on their qubit. Edoardo measures -1. Can he predict the result of Chris' measurement?

No, the qubits in the Bell states are entangled. Performing a measurement with respect to any angle on one of the qubits will change both of the qubits.

2. Suppose that instead Edoardo and Chris agree to each perform a measurement of X on their qubit. Edoardo measures $+1$. Can he predict the result of Chris' measurement?

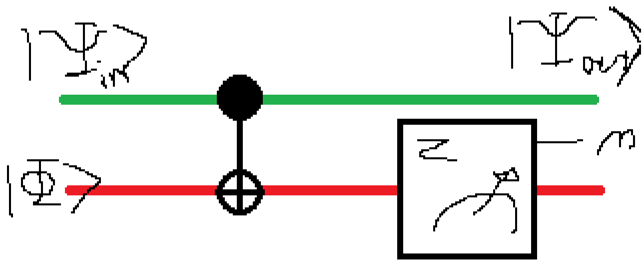
Again, no.

3. Lastly, suppose that they agree that Edoardo measure Y on this qubit and that Chris measures Z on his. Edoardo measures -1. Can he predict the result of Chris' measurement?

Nope.

Problem 6: Gate by measurement

Consider the quantum circuit below. The initial state of the system is a product state with green qubit in $|\Psi_{in}\rangle = \alpha|0\rangle + \beta|1\rangle$ and the red qubit in $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$.



1. Use the generalized Born rule to write the output state $|\Psi_{out}\rangle$ of the green qubit when the measurement result is $m = +1$. Show that for this case, the transformation $|\Psi in\rangle \to |\Psi_{out}\rangle$ is equivalent to a z rotation of the green qubit by ϕ.

Lets compute the circuit:

$$\texttt{CNOT}_{gr}\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{i\phi}\end{pmatrix}\right) = \begin{pmatrix}\alpha\\ \alpha e^{i\phi}\\ \beta e^{i\phi}\\ \beta\end{pmatrix}$$

Since $m = +1$ when measuring the red qubit with respect to Z we need to look at the states $|00\rangle$ and $|10\rangle$. So we get:

$$\begin{aligned}|\Psi_{out}\Phi\rangle &= \frac{1}{\sqrt{2}}\left(\alpha|00\rangle + \beta e^{i\phi}|10\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(\alpha|0\rangle + \beta e^{i\phi}|1\rangle\right) \otimes |0\rangle\end{aligned}$$

We can see that only the green bit is affected and it was rotated by ϕ.

2. Similarly, write the output state of the green qubit when the measurement result is $m = -1$. For this case, what is the equivalent transformation $|\Psi_{in}\rangle \to |\Psi_{out}\rangle$?

Through similar means we get $|\Psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(\alpha e^{i\phi}|0\rangle + \beta|1\rangle\right)$. The transformation is an y rotation of the green qubit by ϕ.