ET4340 Electronics for Quantum Computing
Homework 6

Mick van Gelderen

4091566

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Problem 1: Protection against single-qubit relaxation

We have a single-qubit superposition state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ ($\alpha, \beta \neq 0$ and $|\alpha|^2 + |\beta|^2 = 1$) encoded using Shor's 9-qubit code as $|\Psi_L\rangle = \alpha |0_{Shor}\rangle + \beta |1_{Shor}\rangle$. Suppose that data qubit 9 undergoes a relaxation process by interacting with its environment. That is:

$$|0_9\rangle |e\rangle \rightarrow |0_9\rangle |e\rangle$$

$$|1_9\rangle |e\rangle \rightarrow \sqrt{1-p} |1_9\rangle |e\rangle + \sqrt{p} |0_9\rangle |e'\rangle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where p is a relaxation probability. Assume that all other data qubits remain undisturbed.

1. What are the possible combinations of syndrome measurement outcomes?

> Qubit 9 only affects measurements f and h. In the case a bit flip occurs on qubit 9, Z_9 will trigger and X_9 will not; the measurement outcomes are $m_f = -1$, $m_h = +1$. In the case nothing happens, both measurements will be $+1$.

2. Specify the probability for each possible combination of measurement outcomes.

> $$|0_9\rangle = \alpha_9 |0_9\rangle + \beta_9 |1_9\rangle$$
>
> Applying the relaxation gives:
>
> $$\alpha_9 |0_9\rangle + \beta_9 |1_9\rangle \rightarrow \alpha_9 |0_9\rangle |e\rangle + \beta_9 \sqrt{p} |0_9\rangle |e'\rangle + \beta_9 \sqrt{1-p} |1_9\rangle |e\rangle$$
>
> Simplifying and discarding the environment state:
>
> $$\rightarrow (\alpha_9 + \beta_9 \sqrt{p}) |0_9\rangle + \beta_9 \sqrt{1-p} |1_9\rangle$$

Feel free to do this either by multiplying matrices or by manipulating circuit diagrams. From this we see that a single-qubit bit-flip error prior to `CNOT` proliferates into a double bit-flip error.

It is an intuitive identity in my opinion. Flipping the control bit flips the output. So to 'simulate' the flip before a `CNOT` you can flip both the output and the control after the `CNOT`.

By matrice multiplication (note that the order of operations is reversed with respect of the diagram because we compute the combined matrix $M = M_n \ldots M_2 M_1$ in $|out\rangle = M |in\rangle$ where M_n is operation n):

$$
M_a = \texttt{CNOT}_{01}(I \otimes X) \quad =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
$$

$$
M_b = (X \otimes X)\texttt{CNOT}_{01} \quad =
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
$$

$$M_a = M_b$$