Homework Assignment #1

Due at start of class on Nov. 22, 2013

Problem 1: *Pauli matrices.*
Throughout this course, we will make extensive use of the Pauli sigma operators. In the computational basis $\{|0\rangle, |1\rangle\}$, they are represented by the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They are often denoted as σ_x, σ_y, and σ_z, respectively. In this problem we ask you to prove several simple properties of these matrices.

1. Show that these matrices are hermitian. (Note: a matrix A is hermitian iff $A = A^\dagger$.)
2. Show that these matrices are unitary. (Note: a matrix A is unitary iff $A^\dagger A = AA^\dagger = I$).
3. Show that $[Y, Z] = 2iX$, and $\{Y, Z\} = 0$, where $[A, B] \equiv AB - BA$ and $\{A, B\} \equiv AB + BA$ denote commutation and anticommutation, respectively.
4. Show that $X^n = X$ for odd integers n, and $X^n = I$ for even integers n. (Note: similar expressions hold for powers of Y and Z.)

Problem 2: *A second reason why the Bloch vector representation is useful.*
In class, we saw how operations on single qubits can be visualized as rotations on the Bloch sphere. Here, you derive a simple connection between the cartesian coordinates of the Bloch vector and the expected value of Pauli measurements.

Consider a qubit in the state

$$|\Psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle.$$

1. Write the cartesian coordinates of the associated Bloch vector ν.
2. Derive an expression for the expected value of a measurement of X, given by $\langle\Psi|X|\Psi\rangle$. How does this quantity relate to the cartesian coordinates of ν?
3. Similarly, derive an expression for the expected value of a measurement of Y, and relate it to a cartesian coordinate of ν.
4. Similarly, derive an expression for the expected value of a measurement of Z, and relate it to a cartesian coordinate of ν.
5. Verify that $\langle X\rangle^2 + \langle Y\rangle^2 + \langle Z\rangle^2 = 1$.

Problem 3: *Quantum state tomography.*
In the lab, we are often faced with the task of inferring the quantum state $|\Psi\rangle$ of a qubit produced at the output of some quantum circuit. This task, known as quantum state tomography, goes as follows: Run the circuit and perform a measurement of X. Repeat, repeat, repeat,...
Run the circuit and perform a measurement of Y. Repeat, repeat, repeat,...
Run the circuit and perform a measurement of Z. Repeat, repeat, repeat,...
Estimate the Bloch vector corresponding to $|\Psi\rangle$ by calculating the average of your X measurements, your Y measurements and your Z measurements.

1. You walk in the lab and Christian reports to you these measurement results:
   Measurements of X: +1, +1, -1, +1, -1, +1, -1, -1, +1, -1,...
   Measurements of Y: always -1.
   Measurements of Z: -1, -1, +1, +1, -1, +1, -1, +1, +1, -1,...
   What is your estimate of $|\Psi\rangle$?

2. You walk in the lab and Christian reports to you the various measurement results:
   Measurements of X: +1, +1, -1, +1, -1, +1, -1, -1, +1, -1,...
   Measurements of Y: always -1.
   Measurements of Z: always +1.
   What do you say to Christian in this case?

Problem 4: *Quantum compiling.*
In programming a quantum computer, it pays to simplify a quantum circuit to the minimal number of gates required. You also have to figure out how to realize a quantum gate using the set of gates that you can actually implement with your hardware. As mentioned in class, the gates that we can easily implement in the lab are the rotation gates $R_{\hat{n}}(\theta) = \cos(\theta/2)\, I - i \sin(\theta/2)\, \hat{n}\cdot\vec{\sigma}$.

1. Show that $H^2 = I$.

2. Show that $HXHHXH = I$.

3. Show that $HXHHYH = iX = R_{\hat{x}}(-\pi)$.

4. Show that $H = iR_{\hat{n}}(\pi)$ for $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{z})$.

5. Show that $H = iR_{\hat{x}}(\pi)R_{\hat{y}}(\pi/2)$.

Problem 5: (Extra Credit) *Exponentiation of Pauli matrices.*

1. Show that
$$e^{i\alpha X/\hbar} = \cos(\alpha)\, I + i\sin(\alpha)\, X.$$
(Hint: matrix exponentiation is defined by a power series: $e^A \equiv \sum_{n=0}^{\infty} A^n/n!$.)

2. Show that
$$e^{i\alpha\hat{n}\cdot\vec{\sigma}} = \cos(\alpha)\, I + i\sin(\alpha)\, \hat{n}\cdot\vec{\sigma},$$
where $\hat{n} = (n_x, n_y, n_z)$ is a unit vector and $\vec{\sigma} = (X, Y, Z)$.