ET4340 Electronics for Quantum Computing
Homework 5

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Problem 1: Warmup exercises

1. Consider an operator M and one of its eigenstates $|\psi\rangle$ (with eigenvalue λ). Consider another operator A that anticommutes with M (i.e., $\{M, A\} \equiv MA + AM = 0$). Show that the state $A|\psi\rangle$ is an eigenstate of M with eigenvalue $-\lambda$.

We can write $M|\psi\rangle = \lambda|\psi\rangle$.

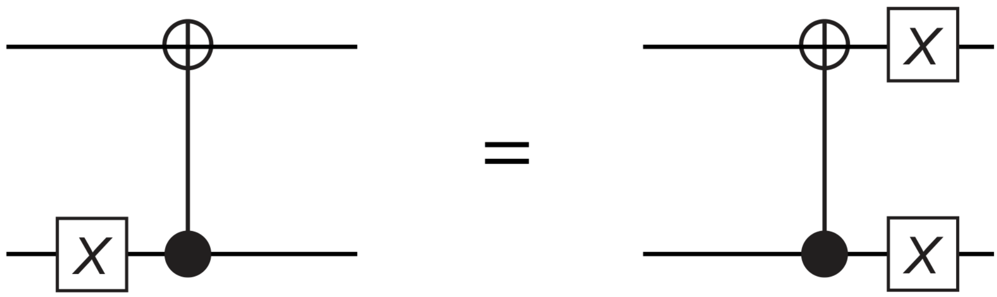
$$
\begin{aligned}
MA + AM &= 0 \\
MA|\psi\rangle + AM|\psi\rangle &= 0 \\
MA|\psi\rangle + A\lambda|\psi\rangle &= 0 \\
M(A|\psi\rangle) &= -\lambda(A|\psi\rangle)
\end{aligned}
$$

2. Now consider an operator B that commutes with M (i.e., $[M, B] \equiv MB - BM = 0$). Show that the state $B|\psi\rangle$ is an eigenstate of M with eigenvalue λ.

Assuming M again has an eigenstate $|\psi\rangle$ with eigenvalue λ.

$$
\begin{aligned}
MB - BM &= 0 \\
MB|\psi\rangle - BM|\psi\rangle &= 0 \\
M(B|\psi\rangle) &= \lambda(B|\psi\rangle)
\end{aligned}
$$

3. Prove the identity:



Feel free to do this either by multiplying matrices or by manipulating circuit diagrams. From this we see that a single-qubit bit-flip error prior to `CNOT` proliferates into a double bit-flip error.

It is an intuitive identity in my opinion. Flipping the control bit flips the output. So to 'simulate' the flip before a `CNOT` you can flip both the output and the control after the `CNOT`.

By matrice multiplication (note that the order of operations is reversed with respect of the diagram because we compute the combined matrix $M = M_n \dots M_2 M_1$ in $|out\rangle = M\,|in\rangle$ where M_n is operation n):

$$
\begin{aligned}
M_a = \mathtt{CNOT}_{01}(I \otimes X) \quad &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
M_b = (X \otimes X)\mathtt{CNOT}_{01} \quad &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
M_a = M_b &
\end{aligned}
$$

Problem 2: Three-qubit bit flip code

Consider the 3-qubit bit-flip code as covered in the lectures. In this code, a one-qubit state $|\psi\rangle = \alpha\,|0_2\rangle + \beta\,|1_2\rangle$ is encoded as $|\Psi\rangle = \alpha\,|0_3 0_2 0_1\rangle + \beta\,|1_3 1_2 1_1\rangle$.

1. Suppose the encoded state is distorted by a rotation of 60 deg about the $+\hat{x}$ axis of qubit 3. What are the possible error syndromes you could measure (i.e., the measurement results m_a and m_b)? Show that the state $|\Psi\rangle$ is recovered after error correction, every time.

Since the distortion is not a full flip you can measure qubit 3 as either -1 or $+1$. The other two bits are not affected and will retain their (equal) states.

The Z (parity) measurement m_a of qubits q_1 and q_2 will always give $m = -1$ because $q_1 = q_2$. Since the same does not hold for qubit q_3 you will measure $m = \pm 1$.

The error syndromes thus are $m_a = -1, m_b = -1$ and $m_a = -1, m_b = +1$. In the first case you won't have to do anything. In the second case you will have to flip qubit q_3 back.

2. Suppose now that instead the encoded stat is distorted by a rotation of 45 deg about the $+\hat{y}$ axis of qubit q_2, but you don't know it and stick to using the bit flip code without modifications. What are the possible error syndromes you would measure? Can you recover the state $|\Psi\rangle$ every time? When do you succeed and when do you not? When you don't recover it, what is the rroneous final state of the logical qubit?