Homework Assignment #2

Due at start of class on Nov. 29, 2013

Problem 1: *The Bell basis.*
In class, we introduced the *Bell states*

$$\begin{aligned}
|\Psi_+\rangle &= \tfrac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \\
|\Psi_-\rangle &= \tfrac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \\
|\Phi_+\rangle &= \tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
|\Phi_-\rangle &= \tfrac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right).
\end{aligned}$$

Like the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, this set of states forms a basis for the four-dimensional Hilbert space of two qubits. Show that this basis is ortho-normal. That is:

1. Show that the inner product (i.e., $\langle\Psi_-|\Psi_-\rangle$ and so on) of every Bell state with itself is unity.
2. Show that the inner product between any two different Bell states (i.e., $\langle\Psi_-|\Psi_+\rangle$ and so on) is zero.
3. Write the matrix that transforms the coordinates of a state in the computational basis to the coordinates of the same state in the Bell basis.

Problem 2: *Generalized* CPHASE *gates.*

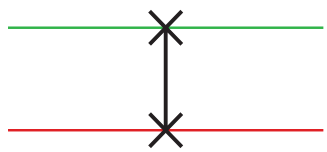
1. Write the four-by-four matrix transformation which flips the phase of the green qubit and does nothing to the red qubit, i.e., implements $Z \otimes I$.
2. Write the four-by-four matrix transformation which flips the phase of the red qubit and does nothing to the green qubit, i.e., implements $I \otimes Z$.
3. Write the four-by-four matrix which flips the phase of the green qubit if and only if the red qubit is in $|0\rangle$. Like the standard CPHASE introduced in class, this transformation differs from the identity only in that one of the diagonal elements is -1. But, which one? Similarly, we can imagine two more generalized CPHASE gates. Show by example how using Z gates we can change any one of these transformations into any of the other three.

Problem 3: *From* CNOT *to* CPHASE *using experimentalist's one-qubit gates.*
In class, we discussed that CPHASE $= H_g \mathrm{CNOT}_{rg} H_g$.

1. Show that CPHASE $= R_{y,g}(-\pi/2)\mathrm{CNOT}_{rg}R_{y,g}(\pi/2)$. This gives a recipe for implementing CPHASE using gates that are more easily implemented in the lab.

2. Draw the quantum circuit implementing these three operations.

Problem 4: *The SWAP operation.*
The two-qubit unitary SWAP, as the name implies, exchanges the quantum states of the red and green qubits, i.e., it implements the transformation $|\psi\rangle \otimes |\phi\rangle \to |\phi\rangle \otimes |\psi\rangle$. In quantum circuit language, SWAP is represented by



1. Write the four-by-four matrix representing this unitary transformation.

2. Show that SWAP $= \mathrm{CNOT}_{rg}\mathrm{CNOT}_{gr}\mathrm{CNOT}_{rg}$ (Note: recall that the left-most subscript index denotes the control, and the right-most index denotes the target.) The easiest way to show this is to multiply the CNOT matrices out.

3. Similarly, show that SWAP $= \mathrm{CNOT}_{gr}\mathrm{CNOT}_{rg}\mathrm{CNOT}_{gr}$.

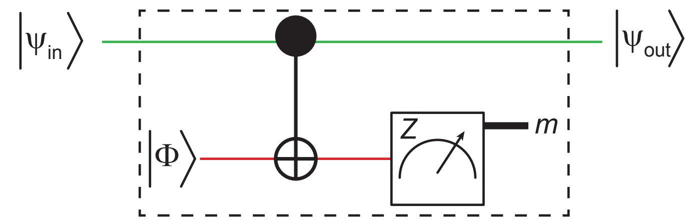
Problem 5: *Measurements on Bell states.*
Edoardo and Chris each have one qubit from a pair prepared in the Bell state $|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$.

1. Edoardo and Chris agree to each perform a measurement of Z on their qubit. Edoardo measures -1. Can he predict the result of Chris' measurement?

2. Suppose that instead Edoardo and Chris agree to each perform a measurement of X on their qubit. Edoardo measures $+1$. Can he predict the result of Chris' measurement?

3. Lastly, suppose that they agree that Edoardo measures Y on his qubit and that Chris measures Z on his. Edoardo measures -1. Can he predict the result of Chris' measurement?

Problem 6: *Gate by measurement.*
Consider the quantum circuit below. The initial state of the system is a product state with green qubit in $|\Psi_{\text{in}}\rangle = \alpha|0\rangle + \beta|1\rangle$ and red qubit in $|\phi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + e^{i\phi}|1\rangle\big)$.



1. Use the generalized Born rule to write the output state $|\Psi_{\rm out}\rangle$ of the green qubit when the measurement result is $m = +1$. Show that for this case, the transformation $|\Psi_{\rm in}\rangle \to |\Psi_{\rm out}\rangle$ is equivalent to a z rotation of the green qubit by ϕ.

2. Similarly, write the output state of the green qubit when the measurement result is $m = -1$. For this case, what is the equivalent transformation $|\Psi_{\rm in}\rangle \to |\Psi_{\rm out}\rangle$?