Homework Assignment #5

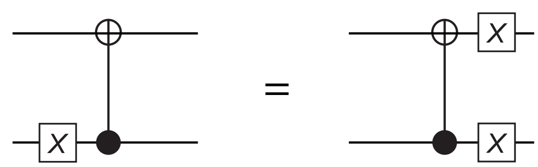
Due at start of class on Dec. 20, 2013

Problem 1: *Warmup exercises*

1. Consider an operator M and one of its eigenstates $|\psi\rangle$ (with eigenvalue λ). Consider another operator A that anticommutes with M (i.e., $\{M, A\} \equiv MA + AM = 0$). Show that the state $A|\psi\rangle$ is an eigenstate of M with eigenvalue $-\lambda$.

2. Now consider an operator B that commutes with M (i.e., $[M, B] \equiv MB - BM = 0$). Show that the state $B|\psi\rangle$ is an eigenstate of M with eigenvalue λ.

3. Prove the identity:



   Feel free to do this either by multiplying matrices or by manipulating circuit diagrams. From this we see that a single-qubit bit-flip error prior to C-NOT proliferates into a double bit-flip error.

Problem 2: *Three-qubit bit flip code*
Consider the 3-qubit bit-flip code as covered in lecture. In this code, a one-qubit state $|\psi\rangle = \alpha|0_2\rangle + \beta|1_2\rangle$ is encoded as $|\Psi\rangle = \alpha|0_3 0_2 0_1\rangle + \beta|1_3 1_2 1_1\rangle$.

1. Suppose the encoded state is distorted by a rotation of 60° about the $+\hat{x}$ axis of qubit 3. What are the possible error syndromes you could measure (i.e., the measurement results m_a and m_b)? Show that the state $|\Psi\rangle$ is recovered after error correction, every time.

2. Suppose now that instead the encoded state is distorted by a rotation of 45° about $+\hat{y}$ axis of qubit 2, but you don't know it and stick to using the bit-flip code without modifications. What are the possible error syndromes you would measure? Can you recover the state $|\Psi\rangle$ every time? When do you succeed and when do you not? When you don't recover it, what is the erroneous final state of the logical qubit?

Problem 3: *Shor's 9-qubit code*
Consider Shor's 9-qubit code as covered in the lecture.

1. Suppose a phase flip occurred on qubit 4. What error syndromes $(m_a,\ldots,m_h)$ will you measure? Suppose a phase flip occurred on qubit 5. What error syndromes will you measure in this case? Finally, suppose a phase flip occurred on qubit 6. What error syndromes will you measure in this case? You should find something surprising, perhaps puzzling... Explain why this is or is not a problem.

2. Suppose you measure the syndromes $m_a = m_b = m_c = m_d = m_g = m_h = -1$, $m_e = m_f = 1$. What error do these syndromes detect? Hint: it is not a single-qubit error. Interestingly, this shows that Shor's code can correct at least some two-qubit errors!

Problem 4: *Operations on logical qubits*
One aspect that makes certain error-correction codes more practical than others is the ability to perform logical operations without having to decode, operate, and re-encode. By the way, this is one of the reasons why the most "economical" 5-qubit code has never been very popular. Consider again the Shor 9-qubit code.

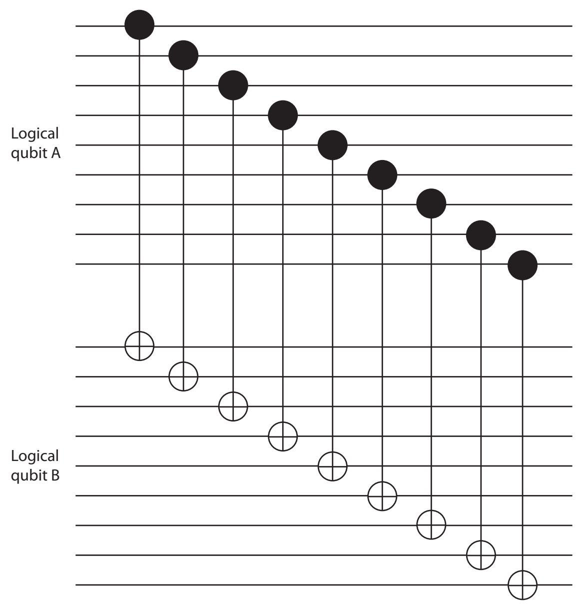
1. Show that the operation $X_9X_8X_7X_6X_5X_4X_3X_2X_1$ realizes a logical Z operation (i.e., $|0_{\text{Shor}}\rangle \to |0_{\text{Shor}}\rangle$, $|1_{\text{Shor}}\rangle \to -|1_{\text{Shor}}\rangle$.

2. Show that $Z_9Z_8Z_7Z_6Z_5Z_4Z_3Z_2Z_1$ realizes the logical X operation (i.e., $|0_{\text{Shor}}\rangle \to |1_{\text{Shor}}\rangle$, $|1_{\text{Shor}}\rangle \to |0_{\text{Shor}}\rangle$.

3. What logical operation does $Y_9Y_8Y_7Y_6Y_5Y_4Y_3Y_2Y_1$ do?

4. Can you think of a simpler way to realize this logical X operation?

5. Finally, consider two qubits, A and B, each encoded using Shor's code, and the transversal quantum circuit below:



What operation does this circuit perform on the two logical qubits? (Hint: it's not quite what you think of at first glance!)