ET4340 Electronics for Quantum Computing
Homework 5

Mick van Gelderen
4091566

December 2013

Problem 1: Warmup exercises

1. Consider an operator M and one of its eigenstates $|\psi\rangle$ (with eigenvalue λ). Consider another operator A that anticommutes with M (i.e., $\{M, A\} \equiv MA + AM = 0$). Show that the state $A|\psi\rangle$ is an eigenstate of M with eigenvalue $-\lambda$.

We can write $M|\psi\rangle = \lambda|\psi\rangle$.

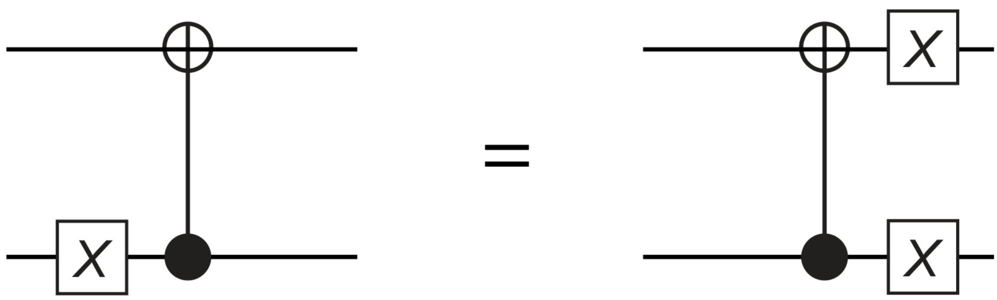
$$\begin{aligned} MA + AM &= 0 \\ MA|\psi\rangle + AM|\psi\rangle &= 0 \\ MA|\psi\rangle + A\lambda|\psi\rangle &= 0 \\ M(A|\psi\rangle) &= -\lambda(A|\psi\rangle) \end{aligned}$$

2. Now consider an operator B that commutes with M (i.e., $[M, B] \equiv MB - BM = 0$). Show that the state $B|\psi\rangle$ is an eigenstate of M with eigenvalue λ.

Assuming M again has an eigenstate $|\psi\rangle$ with eigenvalue λ.

$$\begin{aligned} MB - BM &= 0 \\ MB|\psi\rangle - BM|\psi\rangle &= 0 \\ M(B|\psi\rangle) &= \lambda(B|\psi\rangle) \end{aligned}$$

3. Prove the identity:



Feel free to do this either by multiplying matrices or by manipulating circuit diagrams. From this we see that a single-qubit bit-flip error prior to `CNOT` proliferates into a double bit-flip error.

It is an intuitive identity in my opinion. Flipping the control bit flips the output. So to 'simulate' the flip before a `CNOT` you can flip both the output and the control after the `CNOT`.

By matrice multiplication (note that the order of operations is reversed with respect of the diagram because we compute the combined matrix $M = M_n \dots M_2 M_1$ in $|out\rangle = M|in\rangle$ where M_n is operation n):

$$M_a = \texttt{CNOT}_{01}(I \otimes X) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_b = (X \otimes X)\texttt{CNOT}_{01} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_a = M_b$$

Problem 2: Three-qubit bit flip code

Consider the 3-qubit bit-flip code as covered in the lectures. In this code, a one-qubit state $|\psi\rangle = \alpha|0_2\rangle + \beta|1_2\rangle$ is encoded as $|\Psi\rangle = \alpha|0_3 0_2 0_1\rangle + \beta|1_3 1_2 1_1\rangle$.

1. Suppose the encoded state is distorted by a rotation of 60° about the $+\hat{x}$ axis of qubit 3. What are the possible error syndromes you could measure (i.e., the measurement results m_a and m_b)? Show that the state $|\Psi\rangle$ is recovered after error correction, every time.

Since the distortion is not a full flip you can measure qubit 3 as either -1 or $+1$. The other two bits are not affected and will retain their (equal) states.

The Z (parity) measurement m_a of qubits q_1 and q_2 will always give $m = +1$ because $q_1 = q_2$. The same does not hold for qubits q_2 and q_1, you will measure $m = \pm 1$ by chance.

The error syndromes thus are $m_a = +1, m_b = +1$ and $m_a = +1, m_b = -1$. In the first case you won't have to do anything. In the second case you will have to flip qubit q_3 back.

Thus the final state is always 'correct' or valid so to speak.

2. Suppose now that instead the encoded state is distorted by a rotation of 45° about the $+\hat{y}$ axis of qubit q_2, but you don't know it and stick to using the bit flip code without modifications. What are the possible error syndromes you would measure? Can you recover the state $|\Psi\rangle$ every time? When do you succeed and when do you not? When you don't recover it, what is the erroneous final state of the logical qubit?

This time we apply a maybe-Y gate to qubit q_2. Since $Y = iXZ$ and the measurements we perform are two Z measurement, we can make some conclusions about the different cases.

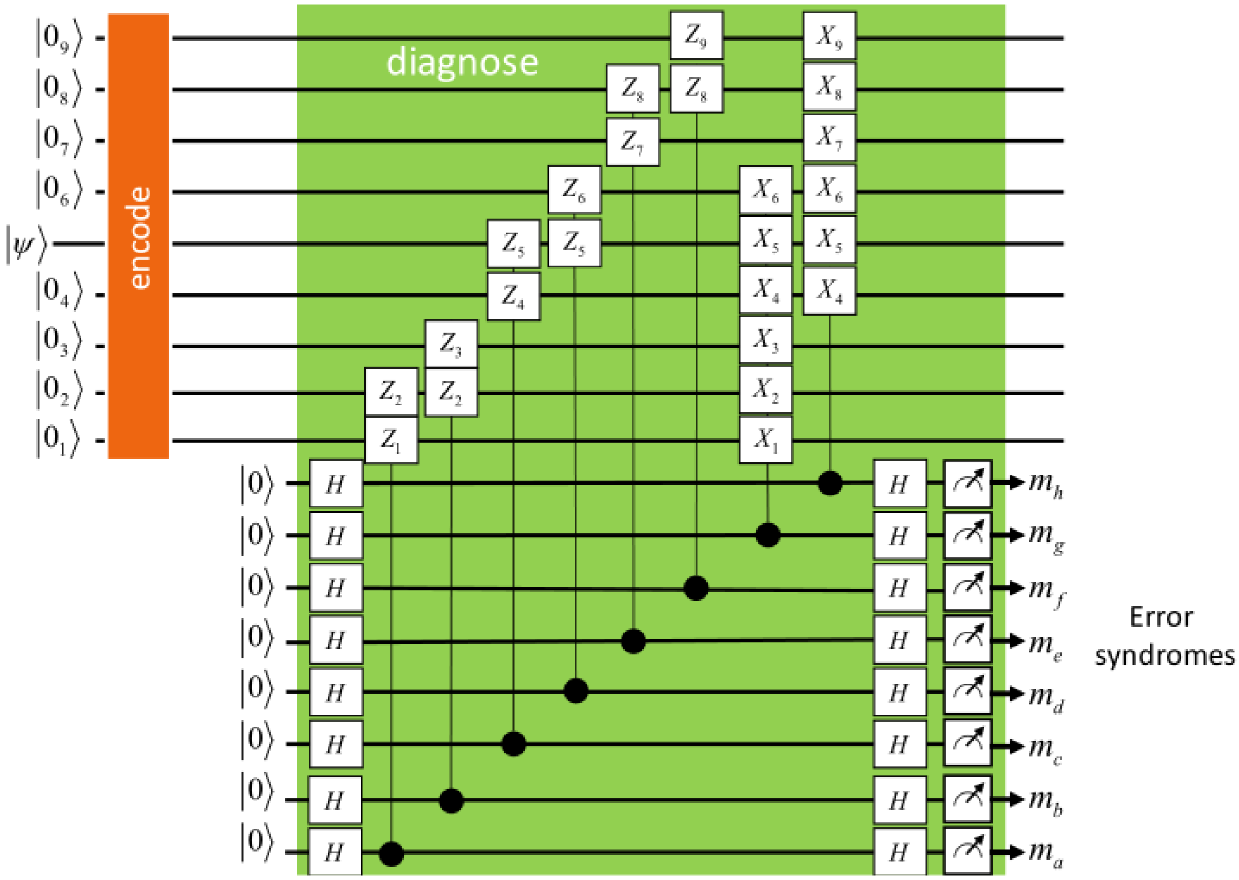
We can recover the error when the q_2 becomes Xq_2 because the syndrome is $m_a = -1, m_b = -1$. This syndrome induces a flip on q_2 which makes $XXq_2 = q_2$.

When q_2 becomes Zq_2, the syndrome is $m_a = +1, m_b = +1$. The final state in this case is $|\Psi\rangle = \alpha\,|0_3 +_2 1_1\rangle + \beta\,|0_3 -_2 1_1\rangle$.

I think my answer is in the right direction but I'm having trouble writing it down and proving it mathmatically.

Shor's 9-qubit code

Consider Shor's 9-qubit code as covered in the lecture.



1. Suppose a phase flip occurred on qubit q_4. What error syndromes $(m_a, \ldots, m_h)$ will you measure?

A phase flip means a Z transition so the error syndrome for Z_4 is m_g, m_h. I will write down only the abnormal measurements ($m = -1$).

2. Now suppose a phase flip occured on qubit q_5.

The syndrome for Z_5 is m_g, m_h as well.

3. Now suppose a phase flip occured on qubit q_6.

The syndrome for Z_6 is m_g, m_h as well.

4. Explain the results from the previous three assignments.

Apparently we get the same syndrome when a phase flip occurs on qubits q_4, q_5 and q_6. Judging from Shor's previous work there has to be something clever going on behind the scenes here.

The fact is that we can fix a phase flip on a qubit q_z that is in any of the sets of three qubits $q_1 \cdots q_3$, $q_4 \cdots q_6$ and $q_7 \cdots q_9$ by performing a phase flip in any one (or all three) of the qubits in the set of qubits that q_z is in. So Z_4 can be fixed by doing a Z on qubit q_4, q_5 or q_6 (or all three).

This property follows from the fact that if you do an even number of Z transformations, the state of the logical cubit will stay the same.

Lets see what Z_2 does:

$$(I \otimes Z \otimes I)(\alpha |000\rangle + \beta |111\rangle) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\beta \end{bmatrix}$$

This just demonstrates that a Z transformation indeed does not affect $\alpha |000\rangle$ and flips the phase of $\beta |111\rangle$.

5. Suppose you measure the syndromes $m_a = m_b = m_c = m_d = m_g = m_h = -1, m_e = m_f = +1$. What error does this syndrome detect? Hint: it is not a single-qubit error. Interestingly, this shows that Shor's code can correct at least some two-qubit errors!

Table 1 table describes what cubits parity measurements are affected by which transformations (X, Y or Z errors) on each qubit.

We are looking for the syndrome in the first row:

?	•	•	•	•		•	•	5
Y_5			•	•		•	•	5
X_2	•	•				◦		2

As you can see from the table, this syndrome can be produced by an Y on q_5 and an X on q_2. This is the only 2-qubit error that will generate this syndrome.

	Y								
	X						Z		
	a	b	c	d	e	f	g	h	
9						•		•	9
8					•	•		•	8
7					•			•	7
6				•			•	•	6
5			•	•			•	•	5
4			•				•	•	4
3		•					•		3
2	•	•					•		2
1	•						•		1
	a	b	c	d	e	f	g	h	

Table 1: Pauli operations affecting qubits in Shor's 9-qubit code

Problem 4: Operations on logical qubits One aspect that makes certain error-correction codes more practical than others is the ability to perform logical operations directly on the encoded qubits. This is the reason why the most 'economical' 5-qubit code has never been very popular. Consider again the Shor 9-qubit code.

1. Show that the operation $X_9 \otimes X_8 \dots X_1$ performs the logical Z operation (i.e., $|0_{Shor}\rangle \to |0_{Shor}\rangle$, $|1_{Shor}\rangle \to -|1_{Shor}\rangle$).

$$|0_{Shor}\rangle = \frac{1}{\sqrt{8}}\left(|000\rangle + |111\rangle\right)^{\otimes^3}$$
$$|1_{Shor}\rangle = \frac{1}{\sqrt{8}}\left(|000\rangle - |111\rangle\right)^{\otimes^3}$$

By flipping all the qubits in $|0_{Shor}\rangle$ we get:

$$(X_9 \otimes X_8 \dots X_1)\,|0_{Shor}\rangle = \frac{1}{\sqrt{8}}\left(|111\rangle + |000\rangle\right)^{\otimes^3} = |0_{Shor}\rangle$$

By doing the same for $|1_{Shor}\rangle$ we obtain:

$$\begin{aligned}(X_9 \otimes X_8 \dots X_1)\,|1_{Shor}\rangle &= \frac{1}{\sqrt{8}}\left(|111\rangle - |000\rangle\right)^{\otimes^3} \\ &= (-1)^3\frac{1}{\sqrt{8}}\left(|000\rangle - |111\rangle\right)^{\otimes^3} = -\,|1_{Shor}\rangle\end{aligned}$$

2. Show that the operation $Z_9 \otimes Z_8 \dots Z_1$ performs the logical X operation (i.e., $|0_{Shor}\rangle \to |1_{Shor}\rangle$, $|1_{Shor}\rangle \to |0_{Shor}\rangle$).

Z flips the phase so $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ becomes $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ and vice versa. Evidently, $\frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)^{\otimes 3}$ becomes $\frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)^{\otimes 3}$ and vice versa because of the 9 phase flips. So the operation $Z_9 \otimes Z_8 \ldots Z_1$ performs the logical X operation because $|0_{Shor}\rangle \to |1_{Shor}\rangle$ and $|1_{Shor}\rangle \to |0_{Shor}\rangle$.

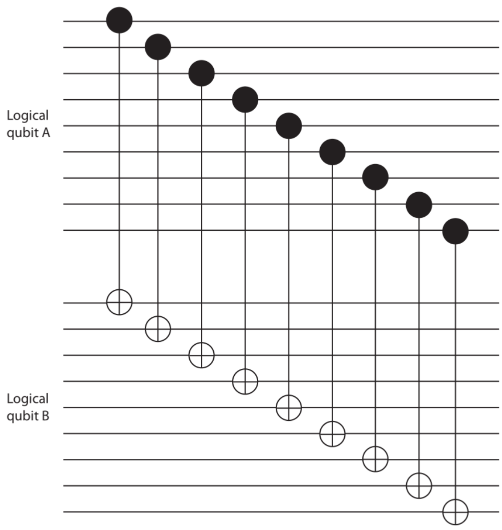
3. What logical operation does $Y_9 \otimes Y_8 \ldots Y_1$ do?

Since $Y = iXZ$ and taking the previous results into account, the operation will implement the logical Y.

4. Can you think of a simpler way to realize this logical X operation?

We saw in problem 2.4 we can get away with a single Z transformation in each set of qubits $q_9 \cdots q_7$, $q_6 \cdots q_4$ and $q_3 \cdots q_1$.

5. Finally, consider two logical qubits, A and B, each encoded using Shor's 9-qubit code, and the transversal quantum circuit below:



What operation does this circuit perform on the two logical qubits? (Hint: It's not quite what you think of at first glance!).

There are four cases to be considered because both A and B can be in either $|0_{Shor}\rangle$ or $|1_{Shor}\rangle$.

The Shor states consist of three parts as seen in previous exercises and we will look at them one at a time.

The set is in the state $\frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$. Since we do a `CNOT` on single qubits from A and B we can look at the problem from the perspective of qubits $q_{A,n}$ and $q_{B,n}$. But, I want to write them shorter so these qubits are now a and b respectively.

Using the following definitions:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

We can calculate the result of applying $\texttt{CNOT}_{ab}$ to the 4 possible combinations of the states of a and b:

$$\begin{array}{ccc} a \otimes b & & \texttt{CNOT}_{ab} \\ |+\rangle \otimes |+\rangle & \rightarrow & |+\rangle \otimes |+\rangle \\ |+\rangle \otimes |-\rangle & \rightarrow & |-\rangle \otimes |-\rangle \\ |-\rangle \otimes |+\rangle & \rightarrow & |-\rangle \otimes |+\rangle \\ |-\rangle \otimes |-\rangle & \rightarrow & |+\rangle \otimes |-\rangle \end{array}$$

The strange thing is that a actually gets phase flipped if $b = |-\rangle$. Now we can extend this to the level of logical qubits and we get that:

$$\begin{aligned} |0_{Shor}\rangle \otimes |0_{Shor}\rangle &\rightarrow |0_{Shor}\rangle \otimes |0_{Shor}\rangle \\ |0_{Shor}\rangle \otimes |1_{Shor}\rangle &\rightarrow |1_{Shor}\rangle \otimes |1_{Shor}\rangle \\ |1_{Shor}\rangle \otimes |0_{Shor}\rangle &\rightarrow |1_{Shor}\rangle \otimes |0_{Shor}\rangle \\ |1_{Shor}\rangle \otimes |1_{Shor}\rangle &\rightarrow |0_{Shor}\rangle \otimes |1_{Shor}\rangle \end{aligned}$$

So, the $\texttt{CNOT}_{ab}$'s actually implement a logical $\texttt{CNOT}_{BA}$.

These were mostly fun and valuable exercises :), thank you for your effors so far I really appreciate it!

Happy holidays!