Lecture Notes Mathematical Physics

Contents

1 Part 1: Sequences & Series

1.1 Sequences basics

A sequence is thought of as a list of numbers in definite order, denoted as $\{a_n\}$ meaning $\{a_1, a_2, ..., a_n\}$. This sequence has a limit namely

$$\lim_{n\to\infty} a_n = L \tag{1}$$

If $\forall\epsilon > 0\ \exists N \ \Rightarrow\ n > N$ then $|a_n - L| < \epsilon$. And if this limit exists, the sequence converges otherwise it is divergent.

Squeeze theorem

Like limits of functions we can also define the squeeze theorem for sequences, namely if $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \quad \Rightarrow \quad \lim_{n\to\infty} b_n = L \tag{2}$$

Theorem

If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Definition increasing / decreasing

A sequence is increasing if $a_n < a_{n+1}$ for $n > 0$. And decreasing if $a_n > a_{n+1}$ for $n > 0$. And a sequence is called monotonic if it is either one.

Definition bounded from above / below

A sequence is bounded above if there is a number M such that $a_n \leq M$ for $n > 0$. And it is bounded from below if there is a number m such that $a_n \geq m$ for $n > 0$. It is called bounded if it is bounded in both *directions*.

Monotonic sequence theorem

Every bounded, monotonic sequence is convergent!

1.2 Series basics

A series is the infinite sum of a sequence, denoted by $\sum_{n=1}^{\infty} a_n$. If a series is convergent and the limit of the sequence is s then we can write the sum as $\sum_{n=1}^{\infty} a_n = s$.

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. (The converse is not true in general!)

Absolutely convergent

A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Conditionally convergent

A series $\sum a_n$ is absolutely convergent, then it is convergent.

Theorem

A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

Series rearrangement

We can rearrange a series by changing the order of the terms. If $\sum a_n$ is an absolutely convergent series with sum s, then any rearrangement of $\sum a_n$ has the same sum s. On the other hand if the series is conditionally convergent and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has a sum equal to r.

1.3 Geometric series

The geometric series is defined as

$$a + ar + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \tag{3}$$

For which we can obtain (for $|r| < 1$)

$$s_n = a\left[\frac{1-r^n}{1-r}\right] \quad \Rightarrow \quad \lim_{n\to\infty} s_n = \frac{a}{1-r} \tag{4}$$

Therefore

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1 \tag{5}$$

1.4 P-Series

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

1.5 Power series

A power series is a series of the form

$$\sum_{n=0}^{\infty} x^n = c_0 + c_1 x + c_2 x^2 + \ldots \tag{6}$$

More generally,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots \tag{7}$$

For the last series, (i) it converges when $x = a$, (ii) it converges for all x, and (iii) there is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. Where R is called the **radius of convergence** of the power series. $R = 0$ for (i) and $R = \infty$ for (ii). And the **interval of convergence** of a power series is the interval of all x values for which the series converges.

1.6 Tests for divergence & convergence

Divergence test

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent.

Integral test

Let f be a continuous, positive, decreasing function and let $a_n = f(n)$. Then the series $\sum a_n$ is convergent iff the improper integral from 1 to ∞ is convergent.

1. If $\int_1^\infty f(x)\,dx$ is convergent then $\sum_1^\infty a_n$ is convergent.
2. If $\int_1^\infty f(x)\,dx$ is divergent then $\sum_1^\infty a_n$ is divergent.

We need to estimate the remainder $R_n = s - s_n$. Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^\infty f(x)\,dx \leq R_n \leq \int_n^\infty f(x)\,dx \tag{8}$$

Comparison test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
2. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Limit comparison test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \tag{9}$$

where c is a finite number and $c > 0$, then either both series converge or both diverge. [Note: if $c = 0$ then the series either converges of diverges.]

Alternating series test

An alternating series is a series whose terms are alternately positive and negative. If the alternating series

$$\sum_{n=1}^\infty (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \ldots \quad b_n > 0 \tag{10}$$

satisfies $b_{n+1} \leq b_n$ for all n and $\lim_{n\to\infty} b_n = 0$. Then the series is convergent. Then the remainder is estimated as $|R_n| = |s - s_n| \leq b_{n+1}$.

Ratio test

(i) If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1 \tag{11}$$

then the series $\sum a_n$ is absolutely convergent. (ii) If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1 \quad \text{or} \quad \infty \tag{12}$$

then the series $\sum a_n$ is divergent. (iii) If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1 \tag{13}$$

then the test is inconclusive.

Root test

(i) If

$$\lim_{n\to\infty} |a_n|^{1/n} = L < 1 \tag{14}$$

then the series $\sum a_n$ is absolutely convergent. (ii) If

$$\lim_{n\to\infty} |a_n|^{1/n} = L > 1 or \infty \tag{15}$$

then the series $\sum a_n$ is divergent. (iii) If

$$\lim_{n\to\infty} |a_n|^{1/n} = L = 1 \tag{16}$$

then the test is inconclusive.

1.7 Function representation by power series

If the power series $\sum c_n(x-a)^n$ has $R > 0$, then for $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ on the interval $(a-R, a+R)$ we can differentiate $f(x)$ as

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x-a)^n\right] \tag{17}$$

and integrated as

$$\int f(x)\ \mathrm{d}x = \sum_{n=0}^{\infty} \int c_n(x-a)^n\ \mathrm{d}x \tag{18}$$

Note that although R stays the same after integration and / or differentiation, the interval of convergence might be changed.

1.8 Taylor series

Definition

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a| < R$ then

$$c_n = \frac{f^{(n)}(a)}{n!} \tag{19}$$

then we obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{20}$$

Which we call a **Taylor series**, if $a = 0$ we call it a **Maclaurin series**.

Theorem

If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n \to \infty} R_n(x) = 0 \tag{21}$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

1.9 Binomial series

If k is a real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!} + \dots \tag{22}$$

where

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} \tag{23}$$

2 Part 2: Second Order Differential Equations

Second order differential equations are of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x) \tag{24}$$

If $G(x) = 0$ then the equation is called a homogeneous linear equation.

2.1 Linear homogeneous equations

Theorem

If $y_1(x)$ and $y_2(x)$ are both solutions for the homogeneous equation then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{25}$$

is also a solution.

Theorem

If $y_1(x)$ and $y_2(x)$ are linearly independant solutions for the homogeneous equation, and $P(x) \neq 0$, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{26}$$

2.2 Solving a linear homogeneous equation

Assume a differential equation

$$ay''(t) + by'(t) + cy(t) = 0 \tag{27}$$

Then we can use a substitution with $y = e^{rt}$ so that we get the auxiliary equation / characteristic equation

$$ar^2 + br + c = 0 \tag{28}$$

With solutions

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{29}$$

Then the general solutions to the differential equation are given as follows:

1. If r_1 and r_2 are real and distinct then $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
2. If $r_1 = r_2 = r$ then $y = c_1 e^{rt} + c_2 t e^{rt}$.
3. If r_1 and r_2 ($r_{1,2} = \alpha \pm i\beta$) are complex then $y = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$.

2.3 Nonhomogeneous linear equations

Again they are of the form

$$ay'' + by' + cy = G(t) \tag{30}$$

Theorem

The solution of the nonhomogeneous equation is given as

$$y(t) = y_p(t) + y_c(t) \tag{31}$$

where y_c is the complementary solution (the solution to the homogeneous case) and y_p is a particular solution to the nonhomogeneous equation.

The method of undetermined coefficients

If we have $ay'' + by' + cy = G(t)$ then $y_p(t)$, the particular solution can be guessed by this method as

1. If $G(t) = P(t)e^{kt}$, where $P(t)$ is an nth-degree polynomial, then try

$$y_p(t) = Q(t)e^{kt}$$

2. If $G(t) = P(t)e^{kt}\cos(mt)$ or $G(t) = P(t)e^{kt}\sin(mt)$, then try

$$y_p(t) = Q(t)e^{kt}\cos(mt) + R(t)e^{kt}\sin(mt)$$

And if terms of $y_p(t)$ are a solution to the complementary equation then multiply it by some power of t.

The method of variation of parameters

Suppose we found the complementary solution $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$ to $ay'' + by' + cy = G(t)$. To find a particular solution we will try to find a pair of functions such that $c_1 = u_1(t)$ and $c_2 = u_2(t)$. So that we obtain a particular solution like

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{32}$$

To find the pair of functions we make the following assumption

$$u_1' y_1 + u_2' y_2 = 0 \tag{33}$$

Next we make the assumption that $a = 1$, so that we obtain

$$u_1' y_1' + u_2' y_2' = G(t) \tag{34}$$

From the last two equations we can, by substituting the former into the latter, solve for either one of the functions. Therefore

$$u_1(t) = -\int \frac{y_2 G(t)}{y_1 y_2' - y_1' y_2}\,dt \quad \text{and} \quad u_2' = \int \frac{y_1 G(t)}{y_1 y_2' - y_1' y_2}\,dt \tag{35}$$

Given that $y_1 y_2' - y_1' y_2 \neq 0$. In the end we obtain

$$y_p(t) = -y_1 \int \frac{y_2 G(t)}{y_1 y_2' - y_1' y_2}\,dt + y_2 \int \frac{y_1 G(t)}{y_1 y_2' - y_1' y_2}\,dt \tag{36}$$

2.4 Applications of 2nd order differential equations

Vibrating springs

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{37}$$

where k is the spring constant, a positive constant. We can solve it by noting it has the auxiliary equation $mr^2 + k = 0$, which has the solution $r = \pm\sqrt{k/m} = \pm\omega$. So it has a general solution

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t) \quad \Rightarrow \quad x(t) = A\cos(\omega t + \delta) \tag{38}$$

In the latter case $A = \sqrt{c_1^2 + c_2^2}$, $\cos\delta = c_1/A$, and $\sin\delta = -c_2/A$.

Damped vibrations

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{39}$$

where c is the damping constant, again a positive constant. To solve it we first find the auxiliary equation $mr^2 + cr + k = 0$, with roots

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \tag{40}$$

Thus we have three possible cases for a solution:

1. Overdamping ($c^2 - 4mk > 0$): We have a solution of the form

   $$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

   with $\sqrt{c^2 - 4mk} < c$, so r_1 and r_2 should both be negative. Therefore $x(t) \to 0$ as $t \to 0$.

2. Critical damping ($c^2 - 4mk = 0$): In this case $r_1 = r_2 = -c/2m$, and we have a solution of the form

   $$x(t) = (c_1 + c_2 t)e^{-(c/2m)t}$$

   here the damping is just sufficient to suppress any vibrations.

3. Underdamping ($c^2 - 4mk < 0$): The roots are now complex such that

   $$r_{1,2} = -\frac{c}{2m} \pm \omega i$$

   where

   $$\omega = \frac{\sqrt{4mk - c^2}}{2m}$$

   such that the solution has the form

   $$x(t) = e^{-(c/2m)t}(c_1 \cos \omega t + c_2 \sin \omega t)$$

   thus the oscillations are damped by the factor $e^{-(c/2m)t}$. Again we have that $x(t) \to 0$ as $t \to 0$.

Forced vibrations

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \tag{41}$$

$F(t) = F_0 \cos \omega_0 t$ would be a commonly occurring type of external force, note that $\omega_0 \neq \omega$. To solve this equation one could use the method of undetermined coefficients. Note also that if $\omega_0 = \omega$, then the applied frequency will reinforce the natural frequency and resonance will occur.

Electric circuits

Another application would be electric circuits, such that one would have an equation like

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t) \tag{42}$$

2.5 Series solutions

Often differential equations cannot be solved in terms of finite combinations of simple familiar functions, in this case we try a solution in the form of a power series

$$y = \sum_{n=0}^{\infty} c_n t^n$$

such that

$$y' = \sum_{n=1}^{\infty} c_n n t^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) t^{n-2}$$

3 Fourier Series

3.1 Definition

A **Fourier Series** is a series of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

defined for $x \in [-\pi, \pi]$, so that f is a piecewise continuous function. To find equations for a_n and b_n (so called **Fourier coefficients**) we integrate $f(x)$ over the boundaries on x. We find

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad \text{for} \quad n = 1, 2, 3, \ldots$$

and for a_0 we find

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

3.2 Fourier Convergence Theorem

If f is a periodic function with period 2π and $f \& f'$ are piecewise continuous on $[-\pi, \pi]$, then the Fourier series is convergent.

The sum of the Fourier series is equal to $f(x)$ at all numbers x where f is continuous. At the number x where f is discontinuous, the sum of the Fourier series is the average of the right and left limits, that is

$$\frac{1}{2} \Big[f(x^+) + f(x^-) \Big]$$

3.3 Functions with period $2L$

We consider all functions where $f(x+2L) = f(x)$ for all x. So if f is a piecewise function on $[-L, L]$, if we let $t = \pi x/L$ its Fourier series is given by

$$g(t) = f(x) = f(Lt/\pi) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

The coefficients can be found by using the substitution $x = Lt/\pi$, $t = \pi x/L$, such that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

a_n is given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos nt\, dt = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

and b_n is given by

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin nt\, dt = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

We can look at the Fourier series for even functions, in this case $b_n = 0$, and for odd functions $a_n = 0$.

3.4 Handy trigonometric integrals

Here are a few identities that are useful for calculating Fourier series:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\, dx = \pi\delta_{mn}$$
$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = \pi\delta_{mn}$$
$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\, dx = 0$$

where δ_{mn} is the kronecker delta which is equal to 1 for $m = n$, and 0 for $m \neq n$.

3.5 Complex Fourier notation

For $f(x)$ periodic in $[-\pi, \pi]$

Now the series is given as

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx}$$

where

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

For $f(x)$ periodic in $[-L, L]$

And now as

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i\left(\frac{n\pi x}{L}\right)}$$

where

$$A_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\left(\frac{n\pi x}{L}\right)}\, dx$$

3.6 Parseval's theorem

If we let

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad , \quad 0 < x < L$$

Then, by applying the Kronecker delta function to the integral $\int_0^L \left[f(x)\right]^2 dx$, we find that

$$\frac{2}{L} \int_0^L \left[f(x)\right]^2 dx = \sum_{n=1}^{\infty} b_n^2$$

Now for a full Fourier Series on $[-L, L]$ Parseval's theorem says that for

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

we get that

$$\frac{1}{L} \int_{-L}^{L} \left[f(x)\right]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

4 Fourier Transforms

Let us consider the Fourier Series periodic in $[-L/2, L/2]$,

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i\left(\frac{2n\pi x}{L}\right)} \quad \text{where} \quad A_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-i\left(\frac{2n\pi x}{L}\right)}\, dx$$

Actually the Fourier Transform is a generalization of the complex Fourier series in the limit as $L \to \infty$, so that $A_n \to F(k)\, dk$ and $n/L \to k$. We then obtain

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi i k x}\, dk$$

where

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x}\, dx$$

Here we call $F(k) = \mathcal{F}_x\big[f(x)\big](k)$ the forward $(-i)$ Fourier transform. And $f(x) = \mathcal{F}_k^{-1}\big[F(k)\big](x)$ is called the inverse $(+i)$ Fourier transform.

Note: Some authors write the transforms in terms of the angular frequency $\omega = 2\pi k$ instead of the oscillation frequency k, such that

$$H(k) = \mathcal{F}\big[h(t)\big] = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, dt$$

$$h(t) = \mathcal{F}^{-1}\big[H(\omega)\big] = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) e^{i\omega t}\, d\omega$$

To fix this asymmetry often the following convention is used:

$$g(y) = \mathcal{F}\big[f(t)\big] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-iyt}\, dt$$

$$f(t) = \mathcal{F}^{-1}\big[g(y)\big] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y) e^{iyt}\, dy$$

4.1 Properties of Fourier transforms

Linearity

If $f(x)$ and $g(x)$ have the Fourier transforms transforms $F(k)$ and $G(k)$, then

$$\begin{aligned}\mathcal{F}_x\big[af(x) + bg(x)\big](k) &= \mathcal{F}_x\big[af(x)\big](k) + \mathcal{F}_x\big[bg(x)\big](k)\\ &= a\mathcal{F}_x\big[f(x)\big](k) + b\mathcal{F}_x\big[g(x)\big](k)\end{aligned}$$

Derivatives

the Fourier transform of the nth derivative $f^{(n)(x)}$, of $f(x)$, is given by

$$\mathcal{F}_x\big[f^{(n)}(x)\big](k) = (i2\pi k)^n \mathcal{F}_x\big[f(x)\big](k)$$

4.2 The Dirac delta function

The Dirac delta function $\delta(x)$ is a distribution for which

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a)$$

since

$$\delta(x) = \begin{cases} \infty\,, & x = 0 \\ 0\,, & x \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Also note that

$$\delta(ax) = \frac{1}{|a|}\delta(x) \quad \text{and} \quad \delta(-x) = \delta(-x)$$

The delta function can also be given as follows, in the limit for $a \to 0$,

$$\delta_a(x) = \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$$

The delta function as Fourier transform

As a Fourier transform it is given as

$$\delta(k) = \mathcal{F}_x[1](k) = \int_{-\infty}^{\infty} e^{-i2\pi kx}\, dx$$

and similarly

$$\delta(x) = \mathcal{F}_k[1](x) = \int_{-\infty}^{\infty} e^{-i2\pi kx}\, dk$$

And therefore we can derive the following expression

$$\mathcal{F}_x[\delta(x-x_0)](k) = \int_{-\infty}^{\infty} \delta(x-x_0) e^{-i2\pi kx}\, dx = e^{-i2\pi kx_0}$$

Limits and the delta function

Some solutions to limits also contain a term with a delta function, such as

$$\lim_{\alpha\to 0} \frac{1}{\alpha + i\omega} = \lim_{\alpha\to 0} \frac{\alpha}{\alpha^2+\omega^2} - \frac{i\omega}{\alpha^2+\omega^2} = \frac{1}{\pi}\delta(\omega) - \frac{i}{\omega}$$

This limit can be solved by noting another definition of the delta function (also known as the Poisson kernel), namely

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon\to 0} \frac{\epsilon}{x^2+\epsilon^2}$$

5 Partial differential equations

We shall solve PDEs using separation of variables, so we must be working with linear homogeneous PDEs with linear homogeneous boundary conditions.

5.1 The heat equation

The heat equation is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \quad u = u(x,t)$$

Where u denotes the temperature at position $0 < x < L$ at time t. We consider two problems:

1. Dirichlet problems: $u(0,t) = 0$ and $u(L,t) = 0$.
2. Neumann problems: $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial t}(L,t) = 0$.

Dirichlet problems

We assume a solution of the form

$$u(x,t) = F(x)G(t)$$

Such that

$$\frac{\partial u}{\partial t} = FG' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G \quad \Rightarrow \quad FG' = c^2 F''G$$

And by separation of the variables we find

$$\frac{G'}{c^2 G} = \frac{F''}{F} = \text{cst} = -p^2$$

Now we have the following two differential equations

$$G' + c^2 p^2 G = 0 \quad \text{and} \quad F'' + p^2 F = 0$$

Starting with F we find that $F(0) = F(L) = 0$, such that the general solution is given by

$$F = A\cos(px) + B\sin(px)$$

We find that $A = 0$, and that $\sin(pL) = 0$ $(B \neq 0)$, so

$$pL = n\pi \quad \Rightarrow \quad p = p_n = \frac{n\pi}{L} \quad (n = 1, 2, 3, ...)$$

Therefore F is given as

$$F = F_n = \sin(p_n x) = \sin\left(\frac{n\pi}{L}x\right) \quad (n = 1, 2, 3, ...)$$

If we let the equation for G become

$$G' + \lambda_n^2 = 0, \quad \lambda_n = \frac{cn\pi}{L}$$

Then the general solution for G is given as

$$G = G_n = B_n e^{-\lambda_n^2 t} \quad (n = 1, 2, 3, ...)$$

From this we find that the equation for u is given as

$$u_n(x,t) = F_n(x)G_n(t) = B_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

Such that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

where

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right)$$

Satisfying the initial condition $u(x,0) = f(x)$.

5.2 1D Wave equation

We want to find a solution to the wave equation for a fixed string

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

for $0 \leq x \leq L$. We can fix the string at endpoints with $u(0,t) = u(L,t) = 0$ (boundary conditions). And the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x),$$

$$f(0) = f(L) = g(0) = g(L) = 0$$

We want a solution $u(x,t) = X(x)T(t)$, then

$$\frac{\partial^2 u}{\partial t^2} = XT'' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T \quad \Rightarrow \quad XT'' = a^2 XT''$$

from which we obtain

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda^2 \quad \Rightarrow \quad X'' + \lambda^2 X = 0 \quad \text{and} \quad T'' + a^2\lambda^2 T = 0$$

Solving the first equation yields

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$$

we find that $c_1 = 0$ and that $c_2 \neq 0$ for $\lambda_n = \pi n/L$, such that

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Now for the second equation use $\lambda = \lambda_n$ yielding

$$T(t) = A\cos(a\lambda_n t) + B\sin(a\lambda_n t) = A\cos\left(\frac{an\pi}{L}t\right) + B\sin\left(\frac{an\pi}{L}t\right)$$

We then have that

$$u_n(x,t) = \left[\sin\left(\frac{n\pi x}{L}\right)\right]\left[A_n \cos\left(\frac{an\pi}{L}t\right) + B_n \sin\left(\frac{an\pi}{L}t\right)\right]$$

so

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

Then to satisfy the initial condition and find A_n and B_n we differentiate the entire expression with respect to t and find the coefficients as we would in a regular Fourier series.

5.3 Laplace's equation

Suppose we want to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $0 < x < L$, $0 < y < H$, where we have the boundary conditions

$$u(0,y) = g(y)$$

$$u(L,y) = 0$$

$$u(x,0) = 0$$

$$u(x,H) = 0$$

and then let $u(x,y) = X(x)Y(y)$, such that

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

so

$$X'' = \lambda X$$

$$Y'' = \lambda Y$$

Note $Y(0) = Y(H) = 0$ so

$$Y = c\sin\left(\frac{n\pi y}{H}\right) \quad \text{for} \quad \lambda = \left(\frac{n\pi}{H}\right)^2$$

and that

$$X(x) = Ae^{n\pi x/H} + Be^{-n\pi x/H} \quad \text{or} \quad X(x) = a_1 \cosh\left[\frac{n\pi}{H}(x-L)\right] + a_2 \sinh\left[\frac{n\pi}{H}(x-L)\right]$$

Since $u(L,y)=0$, we impose that $X(L)=0$, so $a_1=0$ and

$$X(x) = a_2 \sinh\left[\frac{n\pi}{H}(x-L)\right]$$

Such that (note $ca_2 = a_n$),

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi}{H}(x-L)\right] \sin\left(\frac{n\pi y}{H}\right)$$

We know that $u(0,y)=g(y)$, such that

$$u(0,y) = \sum_{n=1}^{\infty} a_n \sinh\left[-\frac{n\pi}{H}L\right] \sin\left(\frac{n\pi y}{H}\right) = g(y)$$

and $g(0)=g(h)=0$, so $g(y)$ can be expressed as a Fourier sine series,

$$g(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right)$$

where $0 \le y \le H$, and

$$A_n = \frac{2}{H}\int_0^H g(y)\sin\left(\frac{n\pi y}{H}\right)dy$$

Such that

$$a_n = \frac{A_n}{\sinh\big(-n\pi L/H\big)}$$

Such that we find the solution to be

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{H\sinh\big(-n\pi L/H\big)}\left[\int_0^H g(y)\sin\left(\frac{n\pi y}{H}\right)dy\right]\sinh\left[\frac{n\pi}{H}(x-L)\right]\sin\left(\frac{n\pi y}{H}\right)$$