Introduction of Radio Astronomy

1. Introduction

The radio window

The radio frequency band ranges from about 10 MHz to 1 THz. Below the lower boundary charged particles in the atmosphere reflect radio waves back into space. And above the upper bound vibrational transitions of molecules absorb the radiation.

The low-frequency cut-off

The ionosphere consists of a plasma of charged particles with an effective refractive index:

$$n^2 = 1 - \left(\frac{\omega_p}{\omega}\right)^2$$

where $\omega_p = 2\pi\nu_p$ is the plasma frequency ($\nu_p \propto \sqrt{N_e}$), defined as

$$\nu_p = \frac{\omega_p}{2\pi} = \sqrt{\frac{N_e e^2}{4\pi^2 \epsilon_0 m}}$$

One can see that in the case that $\omega_p > \omega$, $n^2 < 1$, which implies total reflection. If $\omega_p < \omega$ there's refraction, and if $\omega_p \ll \omega$ $n^2 \approx 1$. Note that this means that the observation conditions are dependent on the electron density, which is dictated by the UV photons from the solar radiation.

The high-frequency cut-off

Absorption

The high-frequency cut-off is mainly due to absorption of radiation by molecules, which can also emit radiation via thermal emission.

We can define the mass absorption coefficient k for some species i,

$$k_i = \frac{\sigma n_i}{r_i \rho_0}$$

where σ is the collision cross-section, n_i the number density of the species of particles, $r_i = \rho_i/\rho_0$ the mixing ratio, and ρ_0 the mass density of air. We also define the optical depth τ, which is a measure of the absorption and scattering of the radiation by a medium.

$$\tau_i(\lambda, z_0) = \int_{z_0}^{\infty} n_i(z)\sigma \, dz = \int_{z_0}^{\infty} \kappa(\lambda, z) \, dz$$

where the rightmost integral is in terms of the *linear absorption coefficient* $\kappa(\lambda, z) = k_i(\lambda)\rho_i(z)$.

The attenuation of an incident ray of intensity I_0, received at altitude z_0, summed over all absorbing species is given by

$$I(z_0) = I_0 \exp\Big[-\sum_i \tau(\lambda, z_0)\Big] = I_0 \exp\big[-\tau(z)\big]$$

where $\tau(z)$ is the optical depth as a function of zenith angle. Where all absorbing species are considered together. This form of the optical depth is dependent on the path length and thus the airmass $X(z)$: $\tau(z) = \tau_0 X(z)$, where $X(z) = \sec(z)$ and τ_0 is the optical depth at zenith.

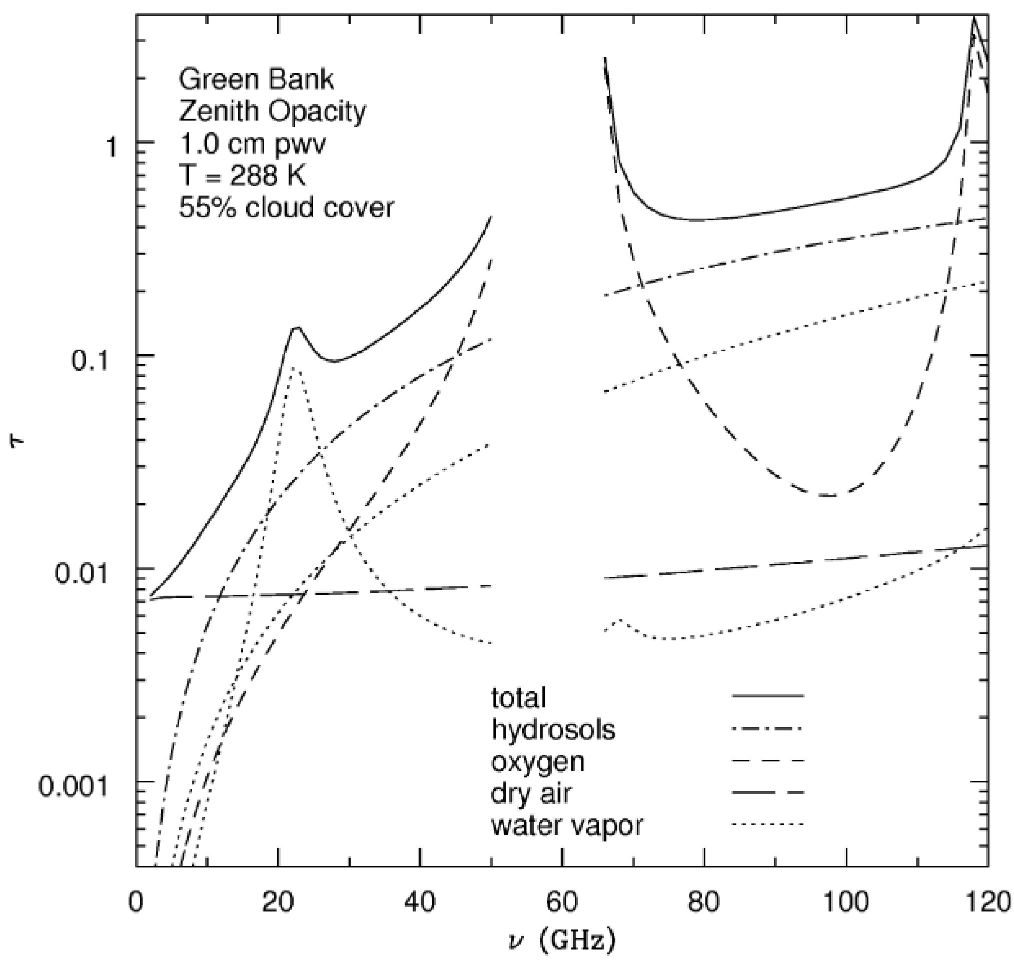

Figure 1: optical depth

The zenith opacity varies with frequency and is a function of path length, so higher altitude and drier locations are favored. To minimize the effect of water vapor.

Emission

The absorbing atmosphere also emits radiation in the radio part of the EM spectrum. This noise we characterize by the system noise (which is defined as an equivalent noise temperature T_{sys}), via $P = kT\Delta\nu$. Or per unit bandwidth or frequency element $P_\nu = kT_{\text{sys}}$. The system noise depends on the CMB, atmospheric emissions, ground emissions, receiver losses, injected noise, and receiver noise. Generally the receiver noise T_{rx} dominates.

The contribution from the sky opacity to the atmospheric noise T_{sky} is

$$T_{\text{sky}} = T_{\text{atm}}\big[1 - \exp(-\tau_\nu)\big]$$

where T_{atm} is the atmospheric kinetic temperature. To minimize this noise, we want to observe in cold and dry places.

Early radio astronomy

The spectral brightness B_ν at frequency ν of a blackbody object is given by

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1}$$

which is Planck's law. In the low frequency regime ($h\nu/kT \ll 1$), using the Taylor expansion of the exponential, we obtain the Rayleigh-Jeans approximation:

$$B_\nu \approx \frac{2kT\nu^2}{c^2}$$

We can then define the flux-density S_ν as the power received per unit area per unit frequency. If the celestial source subtends a small angle $\Omega \ll 1$,

$$S_\nu \approx B_\nu \Omega$$

where the solid angle subtended by a source, with area A (e.g. $A = \pi \times \text{radius}^2$ for a spherical source) at a distance d, is defined as

$$\Omega = \frac{A}{d^2}$$

Note that these radio sources can be thermal and non-thermal (e.g. accelerated electron) in origin.

Radio Telescopes and interferometers

The surface accuracy of a radio telescope is proportional to $\lambda/16$, meaning that if the surface of the dish can have bumps/inaccuracies in the surface of that order, without is affecting the quality of the data.

Note that large single-element radio telescopes can be constructed relatively cheaply, but these will have limited spacial resolution. Given that $\theta \approx \lambda/D$.

Interferometric techniques have been developed to combine several single-element telescopes into a multi-element array. The resolution is now limited by the distance d between the dishes, $\theta \approx \lambda/d$. Using this method, the highest possible angular resolution observations (in astronomy) are made.

2. Radiation Fundamentals

Brightness & Flux

Radiation in empty space can be considered as a stream of particles moving at the speed of light. The energy dE flowing through $d\sigma$ in time dt in the frequency range $\nu + d\nu$ within solid angle $d\Omega$ from an incoming ray at angle θ is,

$$dE = I_\nu \cos\theta d\sigma \, d\Omega \, dt \, d\nu$$

Brightness

This has an associated power, $dP = I_\nu \cos\theta \, d\sigma \, d\Omega \, d\nu$. Such that we define the specific intensity as,

$$I_\nu = \frac{dP}{\cos\theta \, d\sigma \, d\Omega \, d\nu}$$

The specific intensity is conserved along any ray in empty space, meaning: the brightness is independent of distance and is the same at the source and at the detector. Furthermore,

$$I = \int_0^\infty I_\nu \, d\nu$$

is also conserved.

Flux

We define the *flux-density* as the spectral power received by a detector per unit projected area, i.e.

$$dS_\nu = I_\nu \cos\theta \, d\Omega$$

So for on-axis sources,

$$S_\nu = \int_{\text{source}} I_\nu(\theta, \phi) \, d\Omega$$

where $d\Omega = \cos\theta \, d\theta \, d\phi$. Note that S_ν is dependent on the distance to the source from the observer.

Luminosity

Using S_ν, we define the spectral luminosity,

$L_\nu = 4\pi r^2 S_\nu$

Which is independent of the distance to the source. If you integrate L_ν over the whole frequency range, you obtain the *bolometric luminosity* L_{bol}.

Radiative Transfer

Absorption only

If we consider a source at $s = 0$ and a detector at $s = s_0$, we can define the linear absorption coefficient as

$$\kappa_\nu = \frac{dp_\nu}{ds}$$

where dp_ν is the probability of a photon being absorbed in a slab of thickness ds. Such that

$$\frac{dI_\nu}{I_\nu} = -dp_\nu$$

If we integrate over, say, an area of space which is not empty containing some absorbing *material*, from s_{in} to s_{out}, we obtain

$$I_\nu(s_{\mathrm{out}}) = I_\nu(s_{\mathrm{in}}) \exp(-\tau_\nu)$$

where τ_ν is the optical depth, defined as,

$$\tau_\nu = -\int_{s_{\mathrm{out}}}^{s_{\mathrm{in}}} \kappa_\nu(s')\, ds'$$

If $\tau_\nu \gg 1$ the absorber is optically thick (opaque) and if $\tau_\nu \ll 1$ the absorber is optically thin (transparent).

Including emission

There's also a chance that a photon may be emitted within some volume $ds\, d\sigma$, which is described by ϵ_ν, the emission coefficient. Such that the radiative transfer equation (RTE) becomes,

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + \epsilon_\nu$$

These coefficients are not independent, when the emitting and absorbing medium are in thermodynamic equilibrium (TE), then $I_\nu = B_\nu(T)$. In that case the LHS of the RTE becomes zero and,

$$\frac{\epsilon_\nu(T)}{\kappa_\nu(T)} = B_\nu(T$$

Which is *Kirchhoff's law* for a TE system. We also have *local thermodynamic equilibrium* (LTE), where the emitting/absorbing material is in thermal equilibrium, even if it's not in equilibrium with the radiation field.

The brightness temperature

From the RJ-approximation we can see that T and I_ν are equivalent, such that we define a quantity called the *brightness temperature* $T_b(\nu)$,

$$T_b(\nu) \equiv \frac{I_\nu c^2}{2k\nu^2}$$

T_b is a practical way to specify the brightness, as radio telescopes are often calibrated by loads of known temperature. The brightness temperature of the atmosphere is given by

$$T_b = T_{\mathrm{atm}}\big\{1 - \exp(-\tau_z \sec z)\big\}$$

In general, the brightness temperature of radiation at position s, of a medium with temperature T and optical depth $\tau_\nu(s)$, is given by:

$$T_b(s) = T_b(0)\exp[-\tau_\nu(s)] + T(1 - \exp[-\tau_\nu(s)])$$

where the first term on the RHS is the attenuation of the original brightness by the medium and the second term is the contribution of the emission from the medium. Note that at centimeter wavelengths, the CMB and atmosphere dominate the signal received from radio telescopes.

Emissivity and Reflectivity

We define the *reflectivity* as the probability that an incident photon of a certain frequency is reflected, such that for an opaque body,

$$a_\nu + r_\nu = 1$$

where a_ν is the absorption coefficient ($a_\nu = 1$ for a back body), r_ν is the reflection coefficient. The brightness temperature of an opaque body in LTE at temperature T is,

$$T_b(\nu) = a_\nu T = (1 - r_\nu)T$$

Note that $a_\nu < 1$ (no perfect absorbers exist), therefore $T_b < T$. And, a perfect reflector has a zero brightness temperature, i.e. it does not emit.

Black Body Radiation

The integrated brightness (radiance) of a black body radiator at temperature T is,

$$B(T) = \frac{\sigma T^4}{\pi}$$

where σ is the Stefan-Boltzmann constant. Note that the Planck function attains a maximum at

$$\nu_{\text{max}}\,[\text{GHz}] = 55.789T$$

We can also define a spectral energy density,

$$u_\nu = \frac{1}{c}\int I_\nu \, d\Omega$$

Which, when $I_\nu = B_\nu$, is given by $u_\nu = 4\pi B_\nu/c$. And the total radiation energy density, given by integrating over all frequencies, is,

$$u = \frac{4\sigma T^4}{c} = \alpha T^4$$

where $\alpha = 7.6 \times 10^{-16}$ J/m^3/K^4.

Radio Waves in a Conducting Medium

A plasma consists of an ionised gas of ions and free electrons, which has no net charge. We generally consider a cold plasma, where the thermal motions of electrons are negligible.

The medium is defined by the parameters μ, ϵ, and σ. Recall from E&M that,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and

$$\nabla \times \mathbf{B} = -\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

where $\mathbf{J} = \sigma \mathbf{E}$. Rewriting the above equations, fully, in terms of $\mathbf{E}$ and $\mathbf{B}$, we obtain the wave equation for $\mathbf{E}$,

$$\nabla^2 \mathbf{E} - \mu_0 \sigma \dot{\mathbf{E}} - \mu_0 \epsilon_0 \ddot{\mathbf{E}} = 0$$

This expression reduces to,

$$k^2 = \frac{\omega^2}{c^2} + i\frac{\sigma\omega}{\epsilon_0 c^2}$$

Note that $c^2 = 1/\mu_0\epsilon_0$. This is an expression for the dispersion, i.e. how the waves change as a function of frequency. We can define the phase velocity of the waves, i.e. the rate that any one frequency component travels through a medium, as,

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon\mu}}$$

Note, that means that we can write the wave vector as $k = \alpha + i\beta$,

$$\mathbf{E}(r,t) = \mathbf{E}_0 \exp(-\beta r)\exp(-i(\omega t - \alpha r))$$

The $\exp(-\beta r)$ term will dampen the waves, in case of a conducting medium (similar to the absorbing effect discussed in optics). The conductivity in the plasma is,

$$\sigma = i\frac{n_e e^2}{m_e^2 \omega}$$

which is purely imaginary. Plugging this expression into the expression for the wave vector k, we obtain,

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{n_e e^2}{\omega^2 \epsilon_0 m_e}\right)$$

where we define the plasma frequency,

$$\omega_p^2 = \frac{n_e e^2}{\epsilon_0 m_e}$$

which defined the natural resonant frequency of the oscillation of the plasma. We can also define the group velocity as the rate that the wave envelop travels through a medium as,

$$v_g = \frac{d\omega}{dk}$$

Note that the group velocity is related to the phase velocity by $v_g v_p = c^2$ and if $\omega_g = \omega$, there's no propagation of radiation through the plasma, and when $\omega > \omega_p$ the group velocity is purely imaginary and we have reflection.

Using the phase velocity we can define the refractive index,

$$n = \frac{c}{v} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

We can investigate the effect of the dispersive effect on the group velocity, considering a set of pulses moving with the group velocity. The time that these pulses will be delayed (due to dispersion) is given by

$$\tau_D = \int_0^L \frac{dl}{v_g} = \frac{L}{c} + \frac{e^2}{8\pi^2 c m_e \epsilon_0} \frac{1}{\nu^2} \int_0^L n_e \, dl$$

Using this expression, we can define the difference in arrival time between two different frequencies as,

$$\Delta\tau_D = \frac{e^2}{8\pi^2 c m_e \epsilon_0} \left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2} \right) \int_0^L n_e \, dl$$

The integral we call the *Dispersion Measure* (DM), i.e.,

$$\mathrm{DM} = \int_0^L n_e \, dl$$

with n_e in cm^{-3} and l in pc. From DM, we see that the larger the electron density or the path length, the larger the dispersive effect is.

Nyquist Theorem & Noise Temperature

A resistor is any electrical device that absorbs all of the electrical power applied to it. The motions of charged particles in a nonzero temperature resistor generates electrical noise (a current). This noise is almost constant in frequency, but would increase with temperature.

The power in such a resistor is dependent on the temperature and the collisional time, $P \propto T/\tau_c$. There's a high frequency cut-off, above this cut-off we have time-scales were the collision become too small for fluctuations, meaning no noise/power. Below this cut-off we have a flat response in spectral space; i.e. random white noise.

Such that for a single resistor the spectral power from rms noise fluctuations is,

$$P_\nu = kT$$

Which is independent on the resistance; it depends only on the temperature. The expression for P_ν is *Nyquists formula.*

Radiation From An Accelerated Charge

Accelerated charged emit (non-thermal) radiation. If we consider the radial and tangential electric fields, they are related via,

$$E_\theta = E_r \frac{t \sin\theta}{c} \frac{dv}{dt}$$

where we can use that $E_r = q/r^2$ from Coulomb's law. In case there's no acceleration, then there will be a zero tangential field and hence no radiation, such that,

$$E_\theta = \frac{qa\sin\theta}{rc^2}$$

The power per unit area is defined as,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$$

And for plane waves (in cgs units) $|\mathbf{E}| = |\mathbf{H}|$, such that

$$S = \frac{1}{4\pi}\frac{q^2a^2}{c^3}\frac{\sin^2\theta}{r^2}$$

Due to the sine squared term the radiation emitted is highly anisotropic. Such that the total power emitted in all directions is given by,

$$P = \int S\,dA = \frac{2}{3}\frac{q^2a^2}{c^3}$$

This expression for the power is called *the Larmor formula*, it describes the total power radiated by any accelerated charged particle, such as electrons and protons.

The total energy emitted is given by,

$$W = \int_{-\infty}^{\infty} P(t)\,dt = \frac{2}{3}\frac{e^2}{c^3}\int_{-\infty}^{\infty} a(t)\,dt$$

3. Dipole Antennas

Fundamentals

An *antenna* is a device for converting EM radiation in space into electrical currents.

The E and H-fields

If we consider an antenna of length $\Delta\ell$, we find the E-field and induction H to be,

$$H_\psi = -i\frac{I\Delta\ell}{2\lambda}\frac{\sin\theta}{r}\left(1 - \frac{1}{ikr}\right)\exp\big(-i\big[\omega t - kr\big]\big)$$

$$E_\theta = -i\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{I\Delta\ell}{2\lambda}\frac{\sin\theta}{r}\left(1 - \frac{1}{ikr} + \frac{1}{(ikr)^2}\right)\exp\big(-i\big[\omega t - kr\big]\big)$$

Note that the radiation field dominates in the far-field limit, the induction field goes as $1/r^2$, and the static part of the E-field goes as $1/r^3$.

We can determine the time-averaged Poynting vector, for a constant current distibution, is,

$$\langle \mathbf{S} \rangle = \left| \mathrm{Re}\left\{ \mathbf{E} \times \mathbf{H} \right\} \right| = \frac{1}{2}\sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{I\Delta\ell}{2\lambda} \right)^2 \frac{\sin^2\theta}{r^2}$$

Such that the time-averaged power is given by integrating over the Poynting flux, to this extent we find,

$$\langle P \rangle = \frac{\pi}{3c\epsilon_0} \left(\frac{I\Delta\ell}{\lambda} \right)^2$$

Now if we consider a varying current, $\langle P \rangle$ will be smaller by a factor 4.

The length of a dipole antenna is typically half a wavelength (i.e. $\ell = \lambda/2$), because the standing wave is highest at the centre when each half of the dipole is $\lambda/4$ long.

Radiation Resistance

For a resistor with resistance R,

$$\langle P_{\mathrm{heat}} \rangle = \frac{1}{2} I_0^2 R$$

For some time-varying current $I = I_0 \cos\omega t$. We define that radiation resistance of an antenna as,

$$R_{\mathrm{rad}} = \frac{2\langle P_{\mathrm{rad}} \rangle}{I_0^2}$$

We want the radiation resistance to be high, as it is defined as the resistance at the antenna feed point that is due to EM radiation. For the short dipole ($\Delta\ell \ll \lambda$) with $I = I(t)$,

$$\langle R_{\mathrm{rad}} \rangle = \frac{\pi}{6c\epsilon_0} \left(\frac{\Delta\ell}{\lambda} \right)^2$$

Note: a larger dipole is a better receiver. The radiation resistance is related to the geometry of the antenna, the ohmic resistance related to the material of the antenna, and the impedance of free space ($Z_0 = 1/c\epsilon_0 \approx 120\pi\ \Omega$). We can define the time-averaged input power as a combination of the radiation resistance (R_{rad}) and the resistance of the antenna (R_{load}).

$$P_{\mathrm{in}} = \langle VI \rangle = \frac{V_0^2}{2(R_{\mathrm{rad}} + R_{\mathrm{load}})}$$

Therefore, the power transferred to the load is,

$$P_{\mathrm{load}} = \langle I^2 R_{\mathrm{load}} \rangle$$

Such that the power re-radiated by the antenna is,

$$P_{\text{rad}} = \langle I^2 R_{\text{rad}} \rangle$$

The maximum power transferred to the load (our antenna) is when $R_{\text{rad}} = R_{\text{load}}$, found via setting $dP_{\text{load}}/dR_{\text{load}} = 0$. In this case half of the power into the system is re-radiated.

Antenna Power Patterns

The normalized power pattern of a simple short dipole is,

$$P_n(\theta, \phi) = \frac{1}{P_{\max}} P(\theta, \phi)$$

We define the *power gain* $G(\theta, \phi)$, the power transmitted per unit solid angle in the direction (θ, ϕ) divided by the power transmitted per unit solid angle from an isotropic antenna with the same total power. We often express the gain in decibells, $G_{\text{dB}} = 10 \log_{10} G$.

For a lossless isotropic antenna, by conservation of energy, the average gain should be unity. For such an antenna the gain depends only on the angular distribution of radiation from that antenna.

The higher the gain, the narrower the radiation pattern (the directivity),

$$\Delta\Omega = \frac{4\pi}{G_{\max}}$$

Where $\Delta\Omega$ is the solid angle over which the antenna is most sensitive.

Effective Collecting Area

The receiving counterpart of the transmitting gain is the effective collecting area, defined as the product of the geometric area and the incident spectral power per unit area.

$$A_e \equiv \frac{P_\nu}{S_{(\text{matched})}} = \eta_a A_g$$

where η_a is the aperture efficiency and A_g is the geometric area. Any antenna with a single output measures only one polarization. E-fields perpendicular to the antenna wires do not produce currents in the antenna. To *collect* both polarizations, a pair of crossed dipoles are needed.

Therefore,

$$S_{(\text{matched})} = \frac{S_\nu}{2}$$

Recall that,

$$P_\nu = kT$$

Such that, using the R-J equation, we obtain,

$$\int_{4\pi} A_e(\theta,\phi)\, d\Omega = \lambda^2$$

Meaning the average collecting area will equal,

$$\langle A_e \rangle = \frac{\lambda^2}{4\pi}$$

Meaning the effective collecting area is independent of the antenna environment, so this is valid for any type of radiation. In case of an isotropic antenna,

$$A_e(\theta,\phi) = \langle A_e \rangle$$

Therefore, dipoles are used at long wavelengths. To get a decent collecting area of short wavelengths, we need many dipoles.

Reciprocity Theorems

- Strong: If a voltage is applied to the terminals of an antenna A and the current is measured at the terminals of another antenna B, then an equal current (in both amplitude and phase) will appear at the terminals of A if the same voltage is applied to B. Note that, for any antenna pair, $I_A V_A = I_B V_B$.
- Weak: The power pattern of an antenna is the same for transmitting and receiving.

  $$G(\theta,\phi) \propto A_e(\theta,\phi)$$

  For an isotropic antenna we know that $G(\theta,\phi) = 1$ and $A_e(\theta,\phi) = \langle A_e \rangle$, such that

  $$A_e(\theta,\phi) = \frac{\lambda^2}{4\pi} G(\theta,\phi)$$

Antenna Temperature

The power output from a receiving antenna,

$$T_A \equiv \frac{P_\nu}{k} = \frac{A_e S_\nu}{2k}$$

for an unpolarised source. Using this measure we can calibrate the antenna and compare it to the source signal, receiver noise, or atmospheric noise. More generally,

$$T_A = \frac{A_e}{2k} \int I_\nu(\theta,\phi)\, d\Omega$$

We can also define the *beam area* Ω_A, the area through which all the power radiated by the antenna would stream if $P(\theta, \phi)$ maintained its maximum value over the solid angle and is zero everywhere else. Such that the *beam solid angle* is,

$$\Omega_A \equiv \int_{4\pi} P_n(\theta, \phi)\, d\Omega$$

The power received is also a function of the power pattern, such that the true antenna temperature is,

$$T_A = \frac{A_e}{2k} \iint I_\nu(\theta, \phi) P_n(\theta, \phi)\, d\Omega$$

Where,

$$P_n(\theta, \phi) = \frac{G(\theta, \phi)}{G_{\max}}$$

We define the following two elements of the beam:

- Main beam solid angle: the area containing the principle response out to the first zero.

  $$\Omega_{\mathrm{MB}} = \int_{\mathrm{MB}} P_n(\theta, \phi)\, d\Omega$$

  with efficiency (the so-called *main beam efficiency*),

  $$\eta_B = \frac{\Omega_{\mathrm{MB}}}{\Omega_A}$$

- Side-lobes: areas outside the principle response that are non-zero.

4. Dish Telescopes & Horn Antennas

Basics

We use parabolic reflectors at short wavelengths, for which it must have a collecting area $> \lambda^2/4\pi$ of an isotropic antenna. It provides a much larger angular resolution, and much more directive.

Most radio telescopes use large reflectors to collect and focus power onto simple feed antennas connected to receivers.

The Focal Length

Plane wave fronts from a distant source arrive perpendicular to the z-axis. From a wavefront at height h above the vertex, the ray path lengths at all radial offsets r down to the reflector and up tot the focal point at $z = f$ must be the same to remain in phase. Which gives us the requirement that,

$$z = \frac{r^2}{4f}$$

The focal length is related to the diameter of the reflector, by the ratio f/D. If this is too high, the support structure is too large. If its too small the field of view becomes limited. Typically, $f/D \approx 0.4$.

Parabolic reflectors are useful because,

1. The effective area can approach the geometric area.
2. Simpler than an array of dipoles.
3. Feed antenna can be *swapped out* to observe over a wide range of frequencies. Note that as the frequency goes up the feed horn size goes down.

The Far-Field Distance

We can consider spherical waves emitted from a certain distance as plane waves.

$$R_{\text{ff}} \approx \frac{2D^2}{\lambda}$$

Unless the distance is larger than R, path-length errors will introduce significant phase errors in the waves coming from the off-axis component of our reflector, reducing the effective area and the antenna power pattern.

The Aperture Illumination Pattern

An aperture is the opening which all rays pass. For a paraboloidal reflector of diameter D, the aperture is a plane circle with diameter D.

In determining the power gain as a function of position for a circular aperture, we assume,

1. A 1D aperture of width D and $R \gg R_{\text{ff}}$.
2. Consider a current density $J(x,t) = J(x)\exp(-i\omega t)$ for $-D/2 \leq x \leq D/2$ and zero otherwise. Note that, the radiation between two points separated by x on the aperture, will travel an extra distance $\Delta r = x \sin\theta$.

For a receiving antenna, the electric field pattern is $f(\ell)$ and the electric field illuminating the aperture is $g(u)$,

$$f(\ell) = \int_{\text{aperture}} g(u)\exp(-i2\pi\ell u)\,du$$

In the far-field limit, the electric field pattern is the Fourier transform of the electric field illuminating the aperture.

Uniform illumination

The normalised electric field pattern of a uniformly illuminated 1D aperture of width D at wavelength λ. Note $g(u)$ = constant and $-D/2\lambda < u < D/2\lambda$. So we consider a unit rectangle of size $D = \lambda$, so $\Pi(u) = 1$ and $-1/2 < u < 1/2$ and its zero outside those bounds. We find the pattern,

$$f(\ell) = \frac{\sin(\pi\ell)}{\pi\ell} = \text{sinc}(\ell)$$

where $\ell = \sin\theta$. Using the similarity theorem for Fourier transforms: $f(1/a)/|a|$ is the FT of $g(au)$ for $a \neq 0$. Then,

$$f(\ell) \propto \text{sinc}(\ell D/\lambda)$$

And for large apertures (D/λ) $\ell \approx \theta$, i.e. $f(\ell) \propto \text{sinc}(\theta D/\lambda)$. And the power pattern scales as,

$$P(\ell) \propto \text{sinc}^2\left(\frac{\ell D}{\lambda}\right)^2$$

Normalised power spectrum

Such that the normalised power pattern, $P_n(\ell)$, for a 1D uniformly illuminated aperture is,

$$P_n(\ell) = \text{sinc}^2\left(\frac{\theta D}{\lambda}\right)^2$$

The central peak of the power pattern between the first minima is called the main beam. The smaller secondary peaks are called side-lobes. These minima are for,

$$\frac{\pi\theta D}{\lambda} = n\pi$$

The first minimum is for $n = 1$. Such that we can define the zero-power bream width,

$$\theta_{\text{ZPBW}} = \frac{2\lambda}{D}$$

Which is the angular size of the main beam. However, we generally use the half-power beam width as the definition for the main beam,

$$\theta_{\text{HPBW}} \approx \frac{0.89\lambda}{D}$$

The angular size for which the half of the power is received, assuming full illumination. This is the width of the point spread function. And is defined as the resolution of a telescope.

More generally, the HPBW is given by,

$$\theta_{\text{HPBW}} \approx \frac{k\lambda}{D}$$

where k is dependent on the illumination pattern.

2D apertures

We now consider a 2D rectangular aperture. The field pattern in 2D is,

$$f(\ell, m) \propto \iint g(u, v) \exp(-i2\pi(\ell u + mv))\, du\, dv$$

where $v = y/lambda$, $m = \sin\theta_y$, and similarly for u and ℓ. The electric field pattern will therefore be,

$$f(\ell, m) \propto \text{sinc}\left(\frac{\ell D_x}{\lambda}\right) \text{sinc}\left(\frac{m D_y}{\lambda}\right)$$

With a normalised power pattern like,

$$P_n(\ell, m) = \text{sinc}^2\left(\frac{\ell D_x}{\lambda}\right) \text{sinc}^2\left(\frac{m D_y}{\lambda}\right)$$

We can determine the peak power gain G_0 in any direction,

$$G_0 = \frac{4\pi D_x D_y}{\lambda^2} \propto A_{\text{geom}} = D_x D_y$$

Such that the power gain is,

$$G = G_0 \text{sinc}^2\left(\frac{\theta_x D_x}{\lambda}\right) \text{sinc}^2\left(\frac{\theta_y D_y}{\lambda}\right)$$

From the definition of the maximum effective collecting area we find that the largest effective area is the geometric area of the telescope. And A_e is independent of the wavelength.

An ideal illuminated aperture has an (aperture) efficiency $\eta_A = 1$, given the definition,

$$\eta_A \equiv \frac{\max(A_e)}{A_{\text{geom}}}$$

If we want to achieve the maximum angular resolution be look for a narrow main beam width in an illumination pattern. To limit the side-lobe structure we want for instance a Gaussian current grading, which has minimal/none side-lobe structure.

The Aperture Efficiency

Not all of the aperture is used for observation. Such that the aperture efficiency is smaller than unity. This also affects the angular resolution and side-lobe structure. Note we define $\eta_A = \eta_{\text{sp}}\eta_t\eta_b\eta_s$,

- η_{sp}: feed spillover; radiation from the feed falls outside the antenna.
- η_t: feed illumination taper; outer parts of the antenna are illuminated less than the centre. For instance, in case of a Gaussian illumination pattern the beam width is about 35% larger. Which reduces the aperture efficiency by about 30%. It does also reduce the spillover and reduces the first side-lobe level by an order of magnitude.
- η_b: aperture blockage, for instance by the receiver system. This term is defined as,

$$\eta_b = \left(1 - \frac{\text{effective blocked area}}{\text{total area}}\right)^2$$

- η_s: surface errors; random errors on the parabolic shape introduce phase changes. Caused by manufacturing errors, thermal changes across the antenna, deformation due to gravity, or strong winds. The surface efficiency is given by the Ruze equation, in terms of the rms of the surface error fluctuations,

$$\eta_s = \exp\left[-\left(\frac{4\pi\sigma}{\lambda}\right)^2\right]$$

Antenna Mount & Feed Position

- Equatorial mount: the mount is aligned with the polar axis. Rotation of polar axis set the RA. DEC axis moves to set the DEC.
  - Pros: beam does not rotate over time, better tracking accuracy.
  - Cons: higher cost, poorer gravity performance, non-intersecting axis.
- Alt-Az mount: two moving axes that are fixed relative to the Earth.
  - Pros: lower cost, better gravity performance.
  - Cons: beam rotates over time.

5. Radiometers

Basics

Recall that the noise temperature is defined as,

$$T_N = \frac{P_\nu}{k}$$

And the system temperature is equivalent to the total noise power from all sources referenced to the input of an ideal receiver connected to the output of a radio telescope.

$$P_{\text{sys}} = \sum_i P_i$$

and

$$T_{\mathrm{sys}} = \sum_i T_i$$

These components T_i include noise from the CMB, the target source, the atmosphere, losses at the receiver, the injected calibration system, and the receiver as a whole. Often the source is the smallest and the receiver noise dominates.

These receivers generate noise equivalent to a circuit consisting of an ideal noiseless receiver whose input is a resistor of temperature T_{rec}.

Band Limited Noise

Band limited noise is due to the voltage at the output of a radio telescope, which is the sum of noise voltages from many independent random contributions. The central limit theorem states that the amplitude distribution tends to a Gaussian.

The *Nyqvist-Shannon sampling theorem* states that any signal (even if we just consider noise) of limited bandwidth $\Delta\nu$ and duration τ can be represented by a minimum of $2N$ independent samples.

A Simple Total-Power Radiometer

This measures the time-averaged power of the input noise in some well-defined RF (radio frequency) range,

$$\left[\nu_{\mathrm{RN}} - \frac{\Delta\nu_{\mathrm{RN}}}{2}, \quad \nu_{\mathrm{RN}} + \frac{\Delta\nu_{\mathrm{RN}}}{2}\right]$$

where Δ_{RF} is the receiver bandwidth.

The simplest radiometer consists of four stages in series:

1. An ideal lossless bandpass filter that passed input noise only in the desired frequency range. Now the voltages are no longer random. It now resembles a sine wave of frequency $\sim \nu_{\mathrm{RF}}$, whose amplitude fluctuates on time scales $\Delta\tau \approx \Delta\nu_{\mathrm{RF}}^{-1}$. Note that $\Delta\nu_{\mathrm{RF}} < \nu_{\mathrm{RF}}$.
2. An ideal *square-law detector* whose output voltage is proportional to the square of its input voltage. Suppose $V_{\mathrm{in}} \approx \cos(2\pi\nu_{\mathrm{RF}}t)$, the output is $V_{\mathrm{out}} \propto \cos^2(2\pi\nu_{\mathrm{RF}}t)$, those mean is the average power of the input signal.
3. An integrator, which smoothes out the rapidly fluctuating detector output. It is a rapidly varying component at frequencies $2\nu_{\mathrm{RF}}$. Any unwanted rapid variations can be suppressed by taking the arithmetic mean of the detected envelope over some timescale, integrating or averaging the detector output.
4. An recording device, such as a voltmeter to measure and record the smoothed voltage.

The Radiometer Equation

Our ability to measure a signal is dependent on the noise properties of our complete system. The variations are estimated by,

1. In a time interval τ there are a minimum $N = 2\Delta\nu\tau$ independent samples of the total noise power T_{sys}.
2. The uncertainty in the noise power is about $\sqrt{2}T_{\text{sys}}$.
3. The rms error in the average of $N \gg 1$ independent samples is reduced by the factor $\sqrt{N}$.

Such that the rms uncertainty in kelvin is given by,

$$\sigma_T \approx \frac{T_{\text{sys}}}{\sqrt{\Delta\nu_{\text{RF}}\tau}}$$

Typically the system temperature is significantly larger than the source temperature. Therefore we need the rms uncertainty in the system temperature to be as low as possible. We could, for instance, increase the observing bandwidth, observe longer, or decrease the receiver temperature.

The SNR of our target source is,

$$\frac{S}{N} = \frac{T_{\text{source}}}{\sigma_T}$$

The rms uncertainty can be expressed in terms of the system equivalent flux density (SEFD),

$$\text{SEFD} = \frac{2kT_{\text{sys}}}{A_e}$$

to obtain the rms uncertainty in Jansky. It is a good way to compar ethe sensitivity of telescopes because it takes both the receiver system and the effective area into account.

In terms of flux density the ideal radiometer equation is,

$$\sigma_{S_\nu} = \frac{2kT_{\text{sys}}}{A_e\sqrt{\tau\Delta\nu}} = \frac{\text{SEFD}}{\sqrt{\tau\Delta\nu}}$$

Coherent Total Power Radiometers

Typical coherent receivers preserve phase information. Most practical radiometers systems are heterodyne receivers. In which a mixer (after an amplifier) multiplies the RF signal by a sine wave of a frequency ν_{LO} of the Local Oscillator.

1. Amplifiers: increase the signal before further processing. They need to have low internal noise, large fractional bandwidths ($\Delta\nu/\nu$), linearity, gain stability. The amplifier gain is the ratio between the input and output power,

   $$G = \frac{P_{\text{out}}}{P_{\text{in}}}$$

   Typically we need amplification of an order $\sim 100\,\text{dB}$, most often multiple amplifiers are used. Such that the total gain is given by,

   $$G = \prod_i G_i$$

Note that each of these amplifiers add (temperature) noise to the system,

$$T_{\mathrm{eq}} = T_1 + \frac{T_2}{G_1} + \frac{T_3}{G_1 G_2} + \dots$$

So, given that the gains are generally large, the noise is dominated by the first amplifier. Note that lossy devices (mixers and filters) have $G < 1$. Therefore, they are often proceeded with another amplifier.

2. Mixers: change (typically lower) the frequency of the received emission. This is useful to (1) avoid feedback of amplified signals back into the ront end, (2) choose a frequency which is easier to deal with, considering the amplifiers/digitizers. The process mixes the RF signal with the sine wave from the Local Oscillator, producing a (known) phase shift. This adds a small amount of noise, but the signal is at a new intermediate frequency (IF), $\nu_{\mathrm{IF}} = \nu_{\mathrm{RF}}\nu_{\mathrm{LO}}$. The upper IF and higher harmonics can be removed using a low pass filter, leaving only the lower IF. Such that we can split the spectral window into multiple IFs.
3. Filters: limit the frequencies passing through the system, or change the phase of the input. Some examples are
   - A band pass filter: allow frequencies between two frequencies.
   - A low pass filter: only allow frequencies below some frequency.
   - A high pass filter: only allow frequencies above some frequency.
   - A band stop filter: eliminate frequencies between two frequencies.
   - An all pass filter: allows all frequencies, but shifts the phase.
4. Digitalization: an *analog-to-digital (A/D) converter* converts the analog input into a digital output. I.e. convert voltages to bits. Note that this conversion results in some loss of the signal. However, a 2-bit system already allows for 90% of the signal to be reconstructed. The number of bits used to describe the analog input needs to be as low as possible, without loss of information. E.g. LOFAR uses 16-bits, JVLA uses 3 and 8-bits. The sampling must satisfy the Nyqvist-Shannon theorem.

Receiver Calibration

Measuring the receiver temperature is needed to determine the noise power scale at the receiver input. For coherent receiver systems, this requires the noise temperature increment ΔT at the receiver input to be related to the receiver output power (Δz).

This is done by connecting two or more known power sources, typically matched resistive loads of known temperature to the system. The receiver input is related to the receiver output (z) via the receiver gain G,

$$z_L = (T_L + T_R)G$$

and

$$Z_H = (T_H + T_R)G$$

giving

$$T_R = \frac{T_H - T_L y}{y - 1}$$

where T_R is the receiver temperature and,

$$y = \frac{z_H}{z_L}$$

This calibration method assumes that the receiver is a linear power measuring device. And we assume that the gain is close to constant in time.

Limitations to the Radiometer Equation

1. Gain instability: the ratio of the input to the output power (i.e. the gain) might fluctuate over time due to changes in the system. If the gain changes by some amount ΔG,
$$\frac{\Delta T}{T_a + T_{\rm rec}} = \frac{\Delta G}{G}$$
The variations due to noise and gain changes are random processes, therefore,
$$\sigma^2_{\rm total} = \sigma^2_{\rm noise} + \sigma^2_G$$
Such that we obtain a practical total-power radiometer equation,
$$\sigma_T \approx T_{\rm sys}\left[\frac{1}{\Delta_{\rm RF}\tau} + \left(\frac{\Delta G}{G}\right)^2\right]^{1/2}$$
Astronomical signals are typically 10^{-4} of the total system temperature. Therefore, we need to make sure that the gain variations are similarly small. To that extent we need to calibrate very often.
2. Atmospheric fluctuations: fluctuations in atmospheric emission also add to the noise in the output of a simple receiver. This effect (i.e. the fluctuations in receiver gain and atm. emission) can be minimized by observing two parts of the sky by two different feeds. There's a switch which switches between which of the two feeds is used. This is called Dicke Switching. The disadvantage of this method is that, to reach the same noise level, observations take twice as long. Because half of the time is spent off-source.
3. Source confusion: single-dish telescopes have large collecting areas, but relatively poor angular resolution at long wavelengths. Sky-brightness fluctuations caused by faint sources that enter the beam are called *confusion*. We consider the rms confusion σ_c for specifying the width of the confusion distribution. The confusion noise can be minimized by decreasing the angular resolution.
4. Baseline ripples: for spectral line observations, the signal from the receiver and bright sources will get reflected between the receiver and reflector, which could cause ripples (a standing wave) in the frequency response of the radio telescope. We can minimize the effects of the ripple by shifting the focus by $\pm\lambda/8$, observing a calibrator source and dividing out the ripple, and/or use an offset gregorian reflector (as there is essentially no ability for standing waves to form between the receiver and reflector).

Multi-beam Systems

We can include multiple single pixel receivers. This is useful to:

1. Increase the effective field-of-view of the telescope. Each feed samples a different portion of the sky, with the same solid angle. useful for surveys with large telescopes.
2. Can help with calibration, since multiple feeds see a different part of the sky.

Incoherent Receivers

Incoherent receivers don't maintain the phase information. An example is a bolometer, mainly used at mm and sub-mm wavelengths due to heterodyne receivers being comparatively expensive.

Bolometer radiometers

These use the varying resistance R of a material as a function of temperature T, to measure the intensity of the incoming radiation. They are broadband (~ 100 GHz) systems, due to the thermal effect being nearly independent on the frequency.

Essentially, the radiation from our source P_0 will change the temperature ΔT of a material coupled to a heat sink of temperature T_0. From energy balance, we can relate the thermal capacitance C and thermal conductance G via,

$$C\frac{d\Delta T}{dt} + G\Delta T = P$$

For a steady flow,

$$\Delta T = \frac{P}{G}$$

If the power source is switches off, then the temperature after some time is,

$$\Delta T = \frac{P}{G}\exp\left(-\frac{t}{\tau}\right)$$

where $\tau = C/G$ is the thermal time constant. Generally,

$$P = P_0 \exp(i2\pi\nu t)$$

Such that,

$$\Delta T = \frac{P_0 \exp(i2\pi\nu t)}{G(1 + i2\pi\nu\tau)}$$

The noise is dominated by the photon noise from te incoming radiation. We define the noise equivalent power (NEP): the power required to fall on the detector in order to raise the output by an amount equal to the rms of the detector. Given that we know that $\Omega A = \lambda^2$, including the emissivity ϵ of the background radiation:

$$\text{NEP}_{\text{ph}} = 2\epsilon k T_{\text{bg}} \sqrt{\Delta\nu}$$

6. Inferferometers

Basics

Each of the telescopes part of the interferometer has a resolution of $\theta \approx \lambda/D$. The combined resolution of the entire interferometer is,

$$\theta \approx \frac{\lambda}{B}$$

which is obtained by correlating the signals from different telescopes to effectively increase the diameter of a single telescope to an arbitrarily large distance B called the baseline length.

Pros: - High angular resolution, down to (less than) 1 mas. - Better sensitivity, due to the collecting area being $N \times \pi D^2/4$, where N is the number of telescopes. - Large field-of-field in the case of phased array feeds.

Cons: - Field-of-field is limited by the *fov* of the individual telescopes.

Young's Double Slit Experiment

If we consider a plane wave passing through a slit a fringe pattern is created due to constructive and destructive interference.

Constructive interference fringes occur when the path difference is an integer number of wavelength (i.e. the waves add in phase),

$$d \sin\theta = n\lambda$$

Similarly, for destructive interference,

$$d \sin\theta = (n + 1/2)\lambda$$

When the distance y between the slits is much smaller than the distance to where the pattern is formed L the maxima are located at,

$$y_c = \frac{n\lambda L}{d}$$

and

$$y_d = \frac{(n + 1/2)\lambda L}{d}$$

The spacing between successive fringes is,

$$\Delta y = \frac{\lambda L}{d}$$

or rather $\theta \approx \lambda/D$ in terms of the angular distance.

A Simple Two-Element Interferometer

A two element interferometer is similar to Young's double slit experiment. The radiation to antenna 1 travels an extra distance $\mathbf{b} \cdot \hat{\mathbf{s}} = b\cos\theta$, where $\mathbf{b}$ is the baseline length.

Expressed in a geometric delay we have,

$$\tau_g = \frac{\mathbf{b} \cdot \hat{\mathbf{s}}}{c}$$

Which is the time delay in the signal at antenna 1. The combined output signal is the time-averaged multiplied voltage of the two telescopes. Note that we average over a time much larger than the inverse of the frequency of the voltage, $\Delta t \gg (2\omega)^{-1}$, such that we obtain the following response (output voltage),

$$R = \langle V_1 V_2 \rangle = \frac{V^2}{2}\cos(\omega\tau_g)$$

where ω is the frequency of the voltage, which is related to the observing frequency. The sinusoidal variations we call fringes, such that we can define the fringe phase $\phi = \omega\tau_g = b\cos\theta/c$. Differentiating the phase w.r.t. to the angle to the source is,

$$\frac{d\phi}{d\theta} = \frac{2\pi b\sin\theta}{\lambda}$$

The fringe period $\Delta\phi = 2\pi$, corresponds to an angular change of $\Delta\theta = \lambda/b\sin\theta$. Thus, for large b, interferometers can measure very accurately the positions of sources.

For sources much larger than the adjacent positive and negative fringes will cancel out. E.g. the CMB is everywhere and therefore the signal will completely cancel (approximately). Small baselines are needed for extended sources and conversely large baselines are necessary for compact sources.

Extended Sources

A spatailly incoherent extended source with sky brightness $I_\nu(\hat{\mathbf{s}})$ near frequency $\nu = \omega/2\pi$ can be considered as the sum of independent point sources. The response is then,

$$R_C = \int I_\nu(\hat{\mathbf{s}})\cos(2\pi\mathbf{b}\cdot\hat{\mathbf{s}}/\lambda)\,d\Omega$$

The output from the correlator also contains a complex part,

$$R_s = \int I_\nu(\hat{\mathbf{s}}) \sin(2\pi \mathbf{b} \cdot \hat{\mathbf{s}}/\lambda) \, d\Omega$$

This allows us to define the (complex) visibility:

$$V = r_c - iR_s = Ae^{-i\phi}, \quad A = \sqrt{R_c^2 + R_2^2}, \quad \phi = \arctan(R_s/R_c)$$

The amplitude contains information about the brightness of the object. The phase contains information about the position of the source in the sky.

For a two element interferometer the response in terms of the visibility is given by,

$$V_\nu = \int I_\nu(\hat{\mathbf{s}}) \exp(2\pi \mathbf{b} \cdot \hat{\mathbf{s}}/\lambda) \, d\Omega$$

The General Response of an Interferometer

We define a set of coordinates, the l, m, n-coordinates. l is east-west, m is north-south, n is up-down, and note that the source is located in the $\hat{\mathbf{s}}$ direction.

Or we define the (u, v, w)-plane. Where u is east-west, w is up-down, and v is north-south. These are defined as,

$$\frac{\mathbf{b}}{\lambda} = (u, v, w)$$

and

$$\mathbf{s} = (l, m, \sqrt{1 - l^2 - m^2})$$

Such that we can describe the visibility as,

$$V_\nu(u, v, w) = \int \frac{I_\nu(l, m)}{\sqrt{1 - l^2 - m^2}} \exp\big[-i2\pi(ul + vm + wn)\big] \, dl \, dm$$

In case $w = 0$,

$$V_\nu(u, v, w) = \int \frac{I_\nu(l, m)}{\sqrt{1 - l^2 - m^2}} \exp\big[-i2\pi(ul + vm)\big] \, dl \, dm$$

So the response of an interferometer is the inverse Fourier transform of the (apparent) sky brightness distribution.

The Visibility Function

Each visibility samples a discrete point in the Fourier plane, giving information about the amount of power on some projected angular size on the sky.

To make images of the sky that are closest to the true surface brightness distribution we need a completely (well) samples (u, v)-plane.

To obtain a better sampling of the Fourier plane,

1. Add more antennas: increase the number of baselines by $N(N-1)$
2. Add more frequencies: the (u, v)-coordinates are a function of the wavelength, using more frequency channels, the radial sampling will be increased.
3. Add more time: each visibility coordinate is dependent on the projected baseline length b. As the Earth rotates, b changes and a different part of the (u, v)-plane is sampled.

Aperture synthesis uses the Earth's rotation to change the projected baseline length, to increase the sampling.

Note that the amplitude and phase of the visibilities as a function of baseline length and are dependent on the structure of the source.

- A point source has a constant visibility amplitude and if the source is offset from the center it will be constantly varying over time.
- A Gaussian source has a decreasing amplitude and constantly varying phase.

The Delay Beam

Consider the effect of increasing the integration time and the bandwidth of our observation (to increase the SNR and the sampling of the (u, v)-plane). For a constant source brightness, we find that the brightness function over a bandwidth $\Delta\nu$ is,

$$V = \int I_\nu(\hat{\mathbf{s}})\mathrm{sinc}(\Delta\nu\tau_g)\exp(-i2\pi\nu_c\tau_g)\,d\Omega$$

We can consider 3 beams:

1. Primary beam: due to the power pattern of the individual antennas of the baselines.
2. Synthesised beam: due to the sinusoidal response of the two antennas of the baseline.
3. Delay beam: due to the attenuation produced by the finite bandwidth of the observation.

The effect of the delay beam can be corrected for by defined a delay/phase center. It can be defined by adding a phase shift in the correlation, or by introducing an extra path length in the cable connecting the antennas, this is the compensating delay $\tau_0 \approx \tau_g$.

The time that the incoming waves can be considered coherent for (i.e. have known phase) is the coherence time, $\tau = 1/\Delta\nu$.

We find that the delay beam can only be compensated for in a single direction, so that theta angular radius of the usable FoV $\Delta\theta$, is determined by the variation in the geometric delay τ_g. We find that,

$$\Delta\theta\Delta\nu \ll \theta_s\nu$$

where $\Delta\theta$ is the angular off-set from the phase centre, $\Delta\nu$ is the visibility bandwidth, θ_s is the synthesised beam width, and ν is the observing frequency. This causes smearing for wide-field images, which we can solve by,

1. splitting the bandwidth into many smaller channels. But, this increases the data volume.
2. Shift the phase centre is post-processing, and average the result after the fact.

The smearing leads to the objects being stretched in the sky surface brightness plane and a lowering of the measured surface brightness. It will be radially stretched in the uv-plane.

Furthermore, the correlator requires an averaging time sufficiently short such that the Earth's rotation will not move the source by more than the synthesised beam width in the interferometer frame. Otherwise this will lead to time smearing. This will tangential broadening of the source in the uv-plane.

$$\Delta\theta\Delta t \ll \theta_s \times 1.37 \times 10^4 \text{ s}$$

Point Source Sensitivity

We can compute the sensitivity using the radiometer equation, including the number of antennas used,

$$\sigma_{I_\nu} = \frac{\text{SEFD}}{\sqrt{N(N-1)\Delta\nu_{\text{RF}}\tau}}$$

in terms of the SEFD. In terms of the system temperature,

$$\sigma_{S_\nu} = \frac{2kT_{\text{sys}}}{A_{\text{eff}}\sqrt{N(N-1)\Delta\nu_{\text{RF}}\tau}}$$

Note that the practical sensitivity of a real interferometer is also affected by the efficiency of the correlator, with an efficiency around ~ 0.89.

Extended Source Sensitivity

An interferometer is sensitive to structure on some angular-scale, which is given in terms of the surface brightness. The surface brightness sensitivity of an interferometer is much worse than a single dish with the same effective area because the beam solid angle is much smaller, by a factor $(D/b)^2$.

The sensitivity (in terms of T_b) is expressed as,

$$\sigma_{T_b} = \left(\frac{\sigma_{S_\nu}}{\Omega_A}\right)\frac{\lambda^2}{2k}$$

The synthesised beams produced by interferometers are typically Gaussian in shape, so the beam solid angle is given by,

$$\Omega_A = \frac{\pi\theta_S^2}{4\ln 2}$$

Aperture Arrays

In the same way that an interferometer works, the receiving elements are added together by taking into account the delay due to the waves arriving at different times, from different directions.

1. Low costs
2. Better effective area at low (radio) frequencies
3. Large field-of-view and flexible electronic beam forming
4. With a very high (angular) resolution

It is simply an array of dipoles, which can function as an interferometer. Having a very narrow beam, allowing for very strong directional sensitivity, contrary to a single dipole.

To change the pointing we add an artificial delay to the output before combining the individual voltages. We can add multiple artificial delays, such that we can look at different parts of the sky at the same time.

We generally place the antennas in an aperture arrays at different distances from one another to have each sample at a different baseline, such that none of them observe the exact same part of the Fourier plane.

Focal Plane Arrays

These are phased array feeds, where the receivers are off-set in the focal plane of the telescope such that they see slightly different parts of the sky.

1. Can provide a much larger field-of-view
2. Still limited by the mechanics of the telescope
3. Combining different beams results in a uniform response to the sky

7. Continuum Emission Mechanisms

Overview

- Thermal emission: the thermal motions (due to $T > 0$ K) of particles results in the emission of thermal radiation. Such as,
  - Black body emission, e.g. the moon, quiet Sun, planets
  - Grey body emission, e.g. heated dust
  - Free-free emission, thermal bremsstrahlung from ionized hydrogen clouds
- Non-thermal emission: charged particles are accelerated in magnetic fields to produce synchrotron radiation (independent of the temperature). E.g. accelerated within radio jets from black holes or shocked acceleration by expanding SN remnants.

Spectral energy distributions allow us to study the broad band emission such that we can determine whether the source is thermal or non-thermal. Furthermore, they allow us to study the physical conditions within the sources (density, temperature, magnetic fields, star-formation, energetics).

Note that for instance synchrotron radiation goes as $S_\nu \propto \nu^{-0.7}$, free-free emission goes as $S_\nu \propto \nu^{-0.1}$, and dust goes up according to a (modified) black body spectrum.

Emission from Heated Dust

Dust grains mainly consist of silicate and graphite (carbon) material, of non-spherical shape. They are intermixed with (molecular) hydrogen clouds and therefore trace the molecular hydrogen used for star formation.

The UV radiation from young stars is absorbed by the dust and re-radiated at FIR wavelengths (at a much lower temperature). As structures have a variety of temperatures, the dust is not perfectly non-reflective and there is a variation of the dust opacity with frequency, therefore, we use a modified black body (grey-body) spectrum.

From radiative transfer we find that the spectrum is defined by,

$$S_\nu = (1 - e^{-\tau_D}) B_\nu(T) \Delta\Omega$$

Note that at mm/sub-mm wavelengths the medium can be considered optically thin ($\tau_D \ll 1$). Furthermore, the optical depth is not constant,

$$\tau_D = \left(\frac{\nu}{\nu_0}\right)^\beta$$

where ν_0 is the frequency when $\tau_D = 1$ and β is the emissivity. Or in terms of the opacity,

$$\kappa_\nu = \kappa_0 \left(\frac{\nu}{\nu_0}\right)^\beta$$

Such that we find that the spectrum is defined by,

$$S_\nu = \frac{2h}{c^2} \frac{\Delta\Omega}{\nu_0^\beta} \frac{\nu^{3+\beta}}{e^{h\nu/kT} - 1}$$

Typical values are $\beta = 1$ to 2 and a dust temperature of $T = 1$ to 2 K.

As the dust is heated by young stars the luminosity of the dust is directly related to the star-formation rate of the galaxy, such that,

$$\mathrm{SFR}\,[\mathrm{M}_\odot/\mathrm{yr}] = 1.71 \times 10^{10}\, L_{\mathrm{IR}}\,[L_\odot]$$

where L_{IR} is the integrated IR luminosity.

Free-Free Emissions (Breaking Radiation)

Free-free radiation is related to star formation. And it is produced by free electrons in an ionised plasma (HII regions) scattering off ions without being captured (they remain free).

The free electrons are attracted by the positive ions, therefore will endure an electro-static acceleration, which produces pulses of radiation. As derived in week 2 the power of this radiation is given by Lamor's formula,

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3}$$

where a is the tangential (perpendicular) acceleration, whose magnitude is determined from Coulomb's law.

The energy of an emitted photon is much smaller than the energy of an electron. Therefore, the deflection of the electron is very small, we can consider it to (almost) remain a straight line.

During the interaction, the electron is accelerated in the radial and tangential directions to its almost straight path, such that respectivelly,

$$F_{||} = \frac{Ze^2}{b^2}\sin\psi\cos^2\psi$$

and,

$$F_{\perp} = \frac{Ze^2}{b^2}\cos^3\psi$$

where b is the impact parameter (the distance of closest approach).

The tangential component of the acceleration is roughly Gaussian, which produces a short pulse that dominates radio wavelengths. The radial component is somewhat sinusoidal, containing both acceleration and deceleration, dominating IR wavelengths.

Only considering the radial component, we find that the power is,

$$P = \frac{2}{3}\frac{Z^2 e^6}{m_e^2 c^3}\left(\frac{\cos^3\psi}{b^2}\right)^2$$

The total energy radiated is therefore given by $W = \int P\,dt$, which gives,

$$W = \frac{\pi}{4}\frac{Z^2 e^6}{m_e^2 c^3 b^4}\frac{1}{v}$$

Which is the total energy radiated by a charge e if it moves in the field of an ion with a charge Ze. The energy is emitted in a single pulse of duration $\tau \approx b/v$, called the interaction time. The frequency of such an emission is,

$$\nu = \frac{1}{2\pi\tau}$$

which is the frequency over which the power spectrum is approximately flat.

The total energy radiated per unit frequency is,

$$W_\nu = \frac{\pi^2}{2}\frac{Z^2e^6}{m_e^2c^3}\frac{1}{b^2v^2}$$

To determine the free-free radiation from an HII cloud, we need to determine the distribution of velocities and its dependence on the electron temperature T_e. For which we assume that the velocities are distributed according to a non-relativistic Maxwellian distribution. And the distribution of impact parameters and its dependence on the electron and ion number densities N_e and N_i.

The number of electron-ion encounters per unit volume is given by,

$$N(v,b)\,dv\,db = \left[2\pi\,db\right]\left[vf(v)\,dv\right]N_eN_i$$

Therefore the emission coefficient is given by,

$$\epsilon_\nu = \frac{\pi^2Z^2e^6N_eN_i}{4c^3m_e^2}\left(\frac{2m_e}{\pi kT}\right)^{1/2}\ln\left[\frac{b_{\max}}{b_{\min}}\right]$$

Such that,

$$I_\nu = \int \epsilon_\nu\,ds \propto \int N_e^2T^{-1/2}\,ds$$

assuming that the hydrogen plasma is optically thin, which means that $N_e = N_i$. The observed I_ν will also depend on the absorption within the HII region, which is, by Kirchoff's law, given by,

$$\kappa_\nu = \frac{\epsilon_\nu c^2}{2kT\nu^2} = \frac{1}{\nu^2T^{3/2}}\left[\frac{Z^2e^6}{c}N_eN_i\frac{1}{\sqrt{2\pi(m_ek)^3}}\right]\frac{\pi^2}{4}\ln\left[\frac{b_{\max}}{b_{\min}}\right]$$

note $b_{\min}$ has a quantum mechanical limit from the uncertainty principle, and $b_{\max}$ has the limit where a significant amount of power at some relevant radio frequency can be generated. Such that, approximately,

$$b_{\min} = \frac{\hbar}{m_ev}$$

and,

$$b_{\max} = \frac{v}{2\pi\nu}$$

Given that $b_{\min}$ is a function of frequency, the natural log is weakly related on the frequency, leading to,

$$\kappa_\nu \propto \nu^{-2.1}$$

And the optical depth along the line-of-sight is given by,

$$\tau_\nu \propto \int \frac{N_e^2}{\nu^{2.1} T^{3/2}}\, ds$$

Such that in the optically thick regine,

$$S_\nu \propto \nu^2$$

consistent with a black body. However, in the optically thin regime we find,

$$S_\nu \propto \nu^{-0.1}$$

creating a very flat spectrum.

In a more involved treatment, considering the proper Gaunt factor, gives the following optical depth,

$$\tau_\nu \approx 3.28 \times 10^{-7} \left(\frac{T_e}{10^4}\right)^{-1.35} \left(\frac{\nu}{\mathrm{GHz}}\right)^{-2.1} \left(\frac{\mathrm{EM}}{\mathrm{pc\,cm^{-6}}}\right)$$

where EM is the emission measure,

$$\frac{\mathrm{EM}}{\mathrm{pc\,cm^{-6}}} = \int_{\mathrm{los}} \left(\frac{N_e}{\mathrm{cm^{-3}}}\right)^2 d\left(\frac{s}{\mathrm{pc}}\right)$$

Note that measurements of the flux-density as a function of frequency can be used to determine the optical depth (and hence the electron number density), which can be used to infer the level of star formation from the number of ionising photons from young stars.

Synchrotron Emission

This consists of charged particles that can be accelerated by magnetic fields, to produce magneto-bremsstrahlung radiation (defined by the particle's velocity). We can consider three regimes,

1. Gyro radiation: $v \ll c$
2. Cyclotron radiation: $v < c$
3. Synchotron radiation: $v \approx c$

Gyro Radiation

Consider particles in a circular orbit with radius R, perpendicular to the B-field. We can define the gyro frequency,

$$\omega_G = \frac{eB}{m_e c}$$

We generally (not impossible) do not see gyro-radiation at radio wavelengths because its too weak.

Synchrotron emissions are produced by cosmic rays (electrons and protons) with extremely high energies. Such that we need special relativity to describe the behaviour of the radiation.

The inertial mass of ultrarelativistic particles is higher (for an electron $m = \gamma m_e$). Thus their orbital frequency is larger.

$$\omega_B = \frac{\omega_G}{\gamma}$$

where the Lorentz factor is defined as,

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}, \ \beta \equiv \frac{v}{c}$$

The power of synchrotron radiation is given by,

$$P = 2\sigma_T \beta^2 \gamma^2 c U_B \sin^2 \alpha$$

where $\beta = v/c$, γ is the Lorentz factor, $U_B = B^2/8\pi$ the magnetic energy density, and σ_T is the Thompson cross-section.

Note that the power radiated by synchrotron radiation is strongly dependent on the magnetic field B and the velocity v of the accelerated particle. Synchrotron radiation is much more powerful than cyclotron radiation due to the relativistic velocities of the electrons, i.e. $P_{\text{sync}} \propto \gamma^2\beta^2$. The average power emitted by an electron over its lifetime (10^3 to 10^6 years), is given by,

$$\langle P \rangle = \frac{4}{3}\sigma_T \beta^2 \gamma^2 c U_B$$

We can define the synchrotron lifetime of the emitting electron as the ratio of the energy of the electron divided by the power,

$$t_{\text{lifetime}} = \frac{\gamma m_e c^2}{\langle P \rangle} \approx 16.4 \text{ years } \frac{1}{\gamma}\left[\frac{1 \text{ Gauss}}{B^2}\right]$$

So electrons with higher γ and/or stronger B-field will lose their energy faster. Note that an old source will have a lot of low energy electrons and a young source will have a lot of high energy electrons.

The power pattern of the radiation is also changed by relativistic effects, the electrons appear to catch up with their own emission, called relativistic beaming.

The observed synchrotron power spectrum is the Fourier transform of the time series of the pulses. This power spectrum is almost continuous,

$$P(\nu) = \frac{\sqrt{3}e^3 B \sin\alpha}{mc^2} \left(\frac{\nu}{\nu_c}\right) \int_{\nu/\nu_c}^{\infty} K_{5/3}(\eta)\, d\eta$$

where the critical frequency is $\nu_c = 3\gamma^2 \nu_G \sin\alpha/2$. The see that most of the energy is radiated around the critical frequency.

The spectrum of a synchrotron source is the superposition of the spectrum of the individual electrons. The ensemble of electrons is thought to have a power-law electron energy distribution (i.e. a non-thermal process),

$$N(E)\, dE \approx K E^{-\delta}\, dE$$

where $N(E)$ is the number density of electrons. The spectral energy distribution of synchrotron emission is given by,

$$S_\nu \propto \nu^\alpha$$

where,

$$\alpha = \frac{1-\delta}{2}$$

and we used that,

$$\epsilon_\nu \propto B^{(\delta+1)/2} \nu^{(1-\delta)/2}$$

Note that the synchrotron spectrum of a power-law energy distribution is itself a power-law. The spectral index α is given by,

$$\alpha = \frac{\log(S_{\nu_2}/S_{\nu_1})}{\log(\nu_2/\nu_1)}$$

From this index we can ascertain the coefficient δ in the energy distribution of electrons. From a spectrum of the synchrotron radiation we find that we can determine an age for the electron distribution and the energetics.

The synchrotron spectrum will have a $\nu^{2.5}$ dependence at low frequencies (optically thick regime), then at higher frequencies it will go as $\nu^{-0.7}$ (optically thin), and thus is dominated by the power-law energy distribution of electrons. At even larger frequencies we see *synchrotron cooling*, where the spectrum drops off, relatively steep, due to higher energy (i.e. higher frequency) electrons radiating away their energy first.

There is also the *low-energy cut-off* in the electron energy distribution, due to the spectrum of individual electrons ($S_\nu \propto \nu^{1/3}$).

If the energy distribution of the electrons is Maxwellian, then *synchrotron self-absorption* will prevent the brightness temperature of the synchrotron radiation from exceeding the kinetic temperature of the emitting electrons. In an relativistic plasma the effective temperature is $T_e \propto \nu^{1/2} B^{-1/2}$, and thus in the optically thick regime $T_b = T_e$, and the flux density will go as $\nu^{5/2}$. Even if the ensemble has a nonthermal energy distribution. Thus at low frequencies the spectrum of a synchrotron self-absorbed. Which is independent of the slope of the electron-energy spectrum! Such that at the turn-over we can estimate,

$$\left(\frac{B}{\text{Gauss}}\right) \approx 1.4 \times 10^{12} \left(\frac{\nu}{\text{Hz}}\right) \left(\frac{T_b}{\text{K}}\right)^{-2}$$

Continuum from normal galaxies is dominated by a combination of, free-free emission from HII regions and synchrotron radiation from cosmic ray (CR)-electrons, mainly accelerated in SNRs of massive stars. For these sources the galactic B-field is generally of the order of 5-100 micro Gauss.

The synchrotron emission is related to the supernovae remnant rate, which is directly related to the star-formation rate via the *FIR-Radio correlation*,

$$q_{\text{IR}} = \log_1 0 \left(\frac{L_{\text{IR}}}{\left[3.75 \times 10^{12} \text{ Hz}\right] L_{1.4 \text{ GHz}}} \right)$$

This quantity q is typically of the order $q = 2.4 \pm 0.25$ dex.