Fourier Representation

A deterministic periodic function function $f(s)$ with period $2p$, that is absolutly integrable over $[-p, p]$ can be written as

$$f(s) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(\pi ns/p) + b_n \sin(\pi ns/p))$$

where

$$a_n = \frac{1}{p}\int_{-p}^{p} f(s)\cos(\pi ns/p)ds \quad b_n = \frac{1}{p}\int_{-p}^{p} f(s)\sin(\pi ns/p)ds$$

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The energy is defined as $2p\sum c_i^2$ and the power is the energy per unit time, $\sum c_i^2$. Here $c_0 = a_0/2$ and $c_i^2 = (a_i^2 + b_i^2)/2$. The power spectrum is the graph of c_i^2 vs $i/(2p)$.

Continuous Fourier Representation

Suppose that g is a real or complex valued deterministic function that is integrable over $\mathbb{R}^d$. Then the Fourier transform of g is

$$G(\omega) = \int_{\mathbb{R}^d} g(s) \exp(i\omega' s) ds.$$

If G is integrable on $\mathbb{R}^d$ then

$$g(s) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} G(\omega) \exp(-i\omega' s) d\omega,$$

and

$$\int_{\mathbb{R}^d} |g(s)|^2 ds = \int_{\mathbb{R}^d} |G(\omega)|^2 d\omega.$$

Theorem: A real function $\rho(\tau)$ on $\mathbb{R}^n$ is a correlation function if and only if it can be represented in the form

$$\rho(\tau) = \int_{\mathbb{R}^n} e^{i\tau' k} dF(k) = \int_{\mathbb{R}^n} \cos(\tau' k) dF(k)$$

where the function $F(k)$ on $\mathbb{R}^n$ is an n-dimensional distribution function. The second equality is justified by the fact that ρ is a real function.

F is denoted as the **Spectral Distribution Function**.

Theorem: The correlation function of a stationary random field is the characteristic function of some n-dimensional random variable X. Conversely, the characteristic function of any random variable is a correlation function for a stationary random field in $\mathbb{R}^n$.

In other words, given a correlation function ρ we can write

$$\rho(\tau) = Ee^{i\tau' X}$$

for some random variable X.

When F is continuous, a spectral density f exists and

$$f(k) = \frac{\partial^n F(k)}{\partial k_1 \dots \partial k_n}$$

and so the spectral representation becomes

$$\rho(\tau) = \int_{\mathbb{R}^n} e^{i\tau' k} f(k) dk = \int_{\mathbb{R}^n} \cos(\tau' k) f(k) dk$$

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Note that, for small values of τ, the behavior of ρ is controlled by large values of k. In other words, the smoothness of the random field is related to its high frequency properties.

Theorem: A function $f(k)$ on $\mathbb{R}^n$ is the spectral density function of a stationary correlation function on $\mathbb{R}^n$ if and only if $f(k) \geq 0$ and $\int_{\mathbb{R}^n} f(k)dk = 1$.

Then, using the formula for the inversion of the Fourier transform

$$f(k) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\tau' k} \rho(\tau) d\tau \quad .$$

Obtaining Correlation Functions

Since a non-negative function is the spectral density of a valid correlation, a general strategy for determining if a given function is a valid correlation is to evaluate its spectral density and check if it is non-negative for any $k \in \mathbb{R}^n$.

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On the other hand, a general strategy for creating valid correlation functions is to consider a non-negative function as a spectral density and find its Fourier transform.

The spectral density for the Gaussin correlation is

$$f(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\phi\tau^2 - ik\tau} d\tau = \frac{1}{\sqrt{\pi}} e^{-k^2/(4\phi)}.$$

So, for very large ϕ compared to k, $f(k)$ is almost constant.

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So, for very large ϕ compared to k, $f(k)$ is almost constant.

We define the **white noise** as a Gaussian process with constant spectrum. This corresponds to a correlation whose mass is all concentrated at zero. Or a Gaussin correlation with range equals zero. This gives a discontinuous process.

Theorem: If ρ is a stationary correlation function that is continuous everywhere except possibly at zero, then

$$\rho(\tau) = a\rho_w(\tau) + b\rho_c(\tau), \quad a, b \geq 0$$

where $\rho_w(0) = 1$ and $\rho_w(\tau) = 0$, if $\tau \neq 0$. ρ_c is a stationary correlation function that is continuous everywhere.

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So, a random field X can be decomposed into a completely chaotic part, say X_w, and a continuous part, say X_c. Furthermore, the two components are independent. Such decomposition provides a justification for the use of a nugget effect, as is customary in the geostatistical literature.

Consider the stationary and separable exponential correlation in $\mathbb{R}^n$

$$\rho(\tau) = \exp\{-a_1|\tau_1| - \cdots - a_n|\tau_n|\}$$

The spectral density is given by

$$f(k) = \frac{1}{\pi^n} \frac{a_1 \cdots a_n}{(k_1^2 + a_1^2) \cdots (k_n^2 + a_n^2)} \geq 0 \ \ \forall k$$

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Consider

$$\rho(\tau) = a_1^{|\tau_1|} \cdots a_n^{|\tau_n|}, \ \ |a_i| < 1, \ \forall i$$

Then

$$f(k) = \frac{1}{(2\pi)^n} \frac{(1 - a_1^2) \cdots (1 - a_1^n)}{|e^{ik_1} - a_1|^2 \cdots |e^{ik_n} - a_n|^2} \geq 0 \ \ \forall k$$

For isotropic correlation functions the Wiener-Khintchine's Theorem takes a simpler form. This is because the n-dimensional Fourier integral can be replaced by a one dimensional integral.

Theorem: A real function $\rho(\tau)$, $\tau \in \mathbb{R}$ is a correlation function if and only if

$$\rho(\tau) = 2^{(n-2)/2}\Gamma(n/2)\int_0^\infty \frac{J_{(n-2)/2}(k\tau)}{(k\tau)^{(n-2)/2}} d\Phi(k),$$

where Φ is a distribution function on $\mathbb{R}$ and J_i is a Bessel function of the first kind.

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The representation of an isotropic correlation depends on n. The conditions for an isotropic correlation are more restrictive for higher than for lower dimensions. A correlation that is valid in $\mathbb{R}^n$ must be valid in $\mathbb{R}^{n-1}$, but not the opposite.

Denote by $\mathcal{D}_n$ the class of valid isotropic correlations in $\mathbb{R}^n$ then we have that $\mathcal{D}_1 \supset \mathcal{D}_2 \supset \cdots \supset \mathcal{D}_\infty$.

When a spectral isotropic density exists it is related to Φ by the formula

$$\Phi(k) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^k w^{n-1} f(w) dw \quad .$$

and

$$\rho(\tau) = \tfrac{1}{2} \int_0^\infty \cos(k\tau) f(k) dk \quad \text{for} \rho \in \mathcal{D}_1$$

$$\rho(\tau) = \int_0^\infty J_0(k\tau) f(k) dk \quad \text{for} \rho \in \mathcal{D}_2$$

$$\rho(\tau) = \int_0^\infty \tfrac{\sin(k\tau)}{k\tau} k^2 f(k) dk \quad \text{for} \rho \in \mathcal{D}_3 \quad .$$

The isotropic exponential correlation in $\mathbb{R}^n$

$$\rho(\tau) = \exp\{-a\tau\},\ a, \tau > 0$$

has spectral density given by

$$f(k) \propto \frac{1}{(k^2 + a^2)^{(n+1)/2}} \quad \forall k$$

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Consider

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Then

$$f(k) \propto \exp\{-k^2/(4a)\},\ \ \forall k > 0$$

The Matèrn correlation family in $\mathbb{R}^n$

$$\rho(\tau) \propto (a\tau)^\nu K_\nu(a\tau) \; a, \nu, \tau > 0$$

has spectral density given by

$$f(k) \propto \frac{1}{(k^2 + a^2)^{\nu + n/2}} > 0 \;\; \forall k$$

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$$\rho(\tau) \propto \frac{1}{(a^2 + \tau^2)^{\nu}}, \quad a, \tau, \nu > 0$$

Then

$$f(k) \propto (a\tau)^{\nu - n/2} K_{\nu - n/2}(ak) > 0 \;\; \forall k$$