AMS 223
Problems 3.3, 3.4
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Model

We consider a single-component harmonic regression model

$$y_i = a\cos(\omega t_i) + b\sin(\omega t_i) + \epsilon_i, \quad i = 1, \ldots, T$$

where $\epsilon_i \overset{iid}{\sim} N(0, v)$. Let $\boldsymbol{\beta} = (a, b)^\top$ and $\mathbf{f}_i = (\cos(\omega t_i), \sin(\omega t_i))^\top$, then we have $y_i \sim N(\mathbf{f}_i^\top \boldsymbol{\beta}, v)$. Define $\mathbf{y} = (y_1, \ldots, y_T)^\top$ and $\mathbf{F}$ be the $2 \times T$ matrix whose columns are $\mathbf{f}_i$. Then we have

$$\mathbf{y}|\boldsymbol{\beta}, v, \omega \sim N\left(\mathbf{F}^\top\boldsymbol{\beta}, v\mathbf{I}\right).$$

Using the prior $p(\boldsymbol{\beta}, v, \omega) = p(\boldsymbol{\beta}, v|\omega)p(\omega) \propto v^{-1}p(\omega)$, the full conditionals for $\boldsymbol{\beta}$ and v are

$$\begin{aligned}
\boldsymbol{\beta}|v, \omega, \mathbf{y} &\sim N\left(\hat{\boldsymbol{\beta}}, v(\mathbf{F}\mathbf{F}^\top)^{-1}\right) \\
v|\boldsymbol{\beta}, \omega, \mathbf{y} &\sim IG\left(T/2, (\mathbf{y} - \mathbf{F}^\top\boldsymbol{\beta})^\top(\mathbf{y} - \mathbf{F}^\top\boldsymbol{\beta})/2\right),
\end{aligned}$$

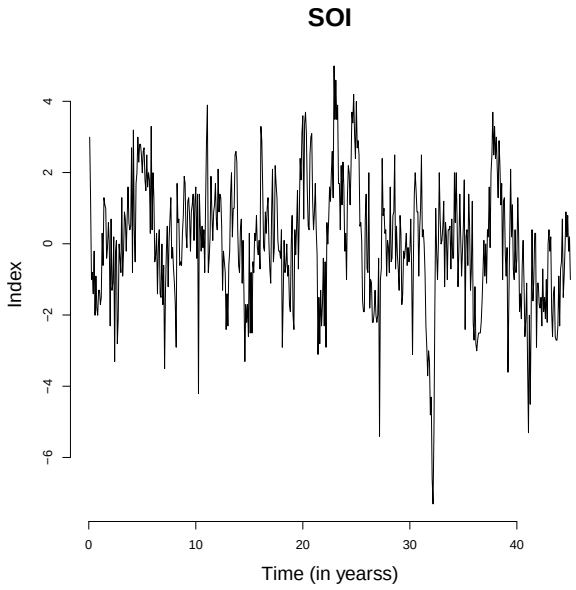
where $\hat{\boldsymbol{\beta}} = (\mathbf{F}\mathbf{F}^\top)^{-1}\mathbf{F}\mathbf{y}$ is the maximum likelihood estimate for a fixed ω. For this analysis, interest is primarily in the angular frequency ω (and the corresponding period $\lambda = 2\pi/\omega$). The marginal posterior for ω has the form

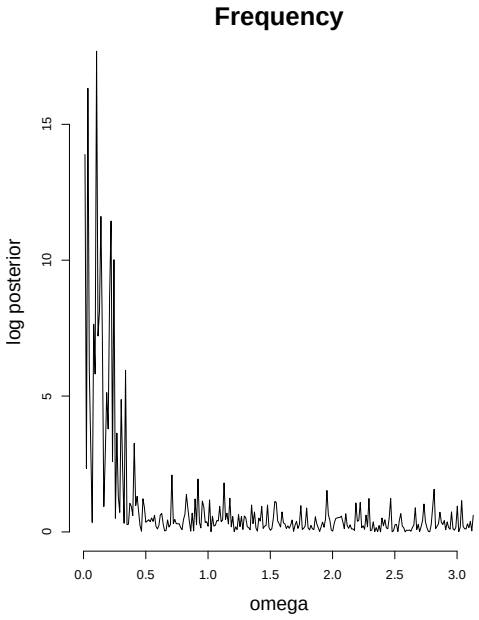
$$p(\omega|\mathbf{y}) \propto p(\omega)\left|\mathbf{F}\mathbf{F}^\top\right|^{-1/2}\left[1 - \hat{\boldsymbol{\beta}}^\top\mathbf{F}\mathbf{F}^\top\hat{\boldsymbol{\beta}}/(\mathbf{y}^\top\mathbf{y})\right]^{(2-T)/2}.$$

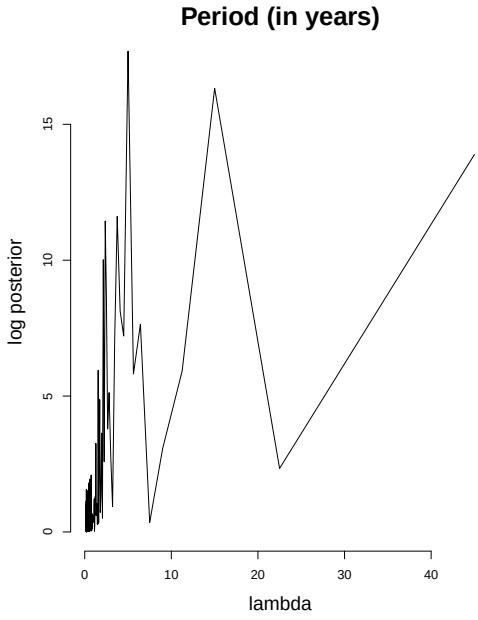
Direct sampling from the marginal posterior of ω is not avaiable. However, since we're only working with one dimension, we can simply discretize the space for ω, compute $p(\omega|\mathbf{y})$ and sample with replacement based on these probabilities.

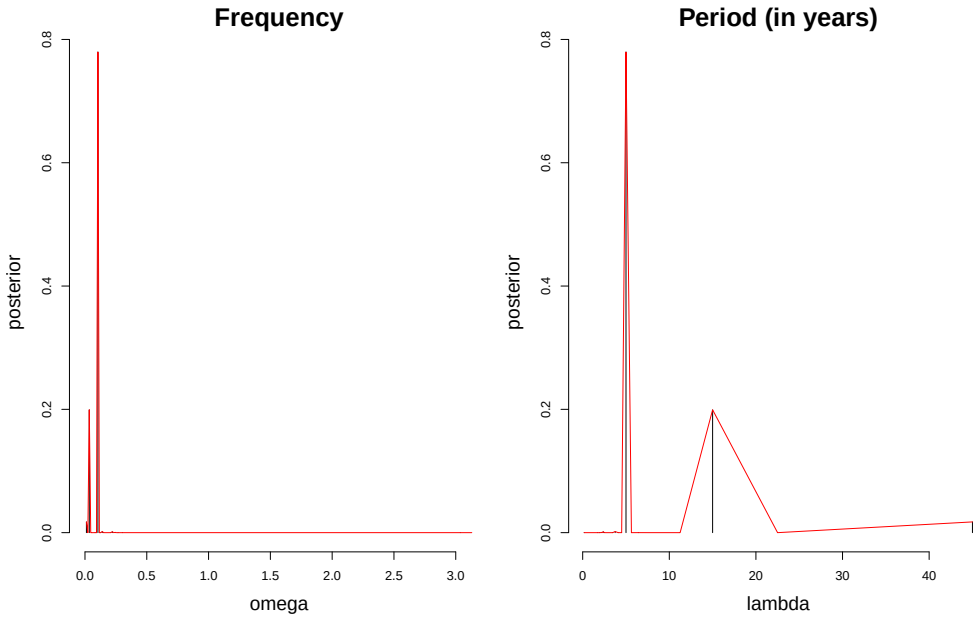
We show plots of the frequency and period calculated from the marginal posterior as well as based off samples.

P&W 3.3 – Southern Oscillation Index

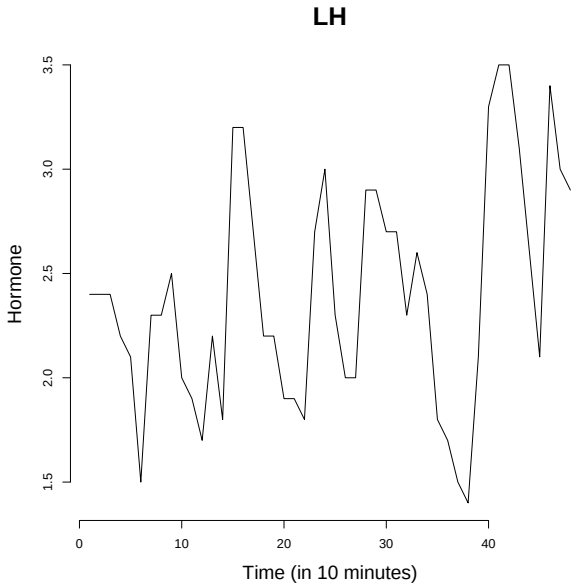


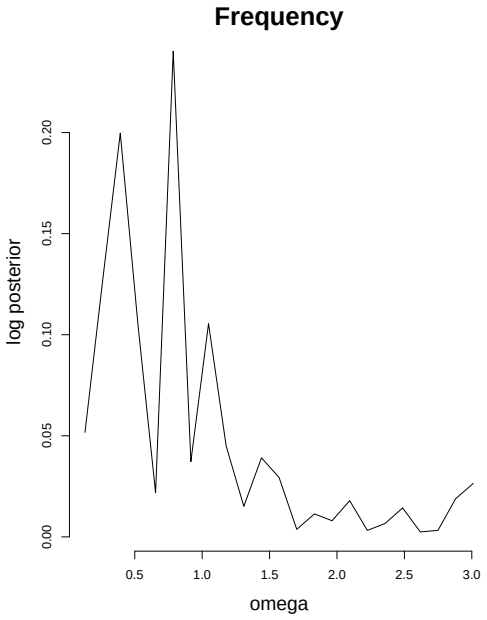


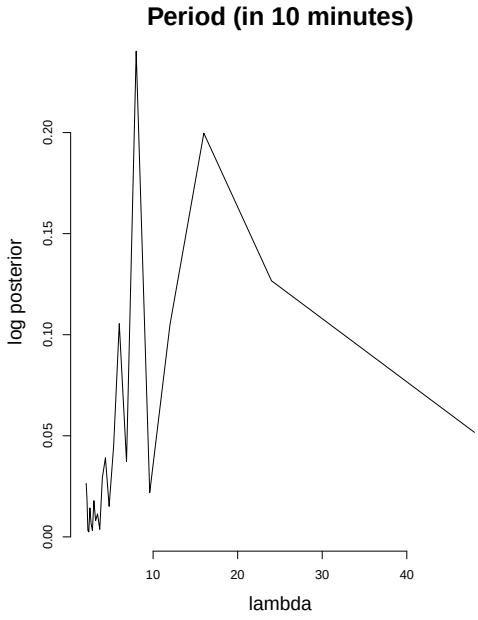


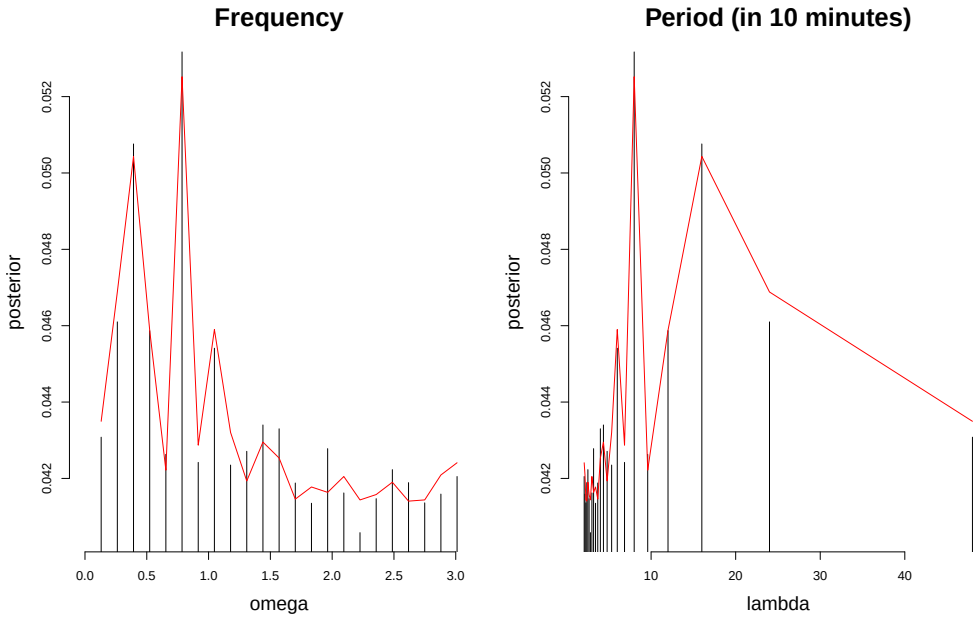

Frequency
posterior
0.0
0.2
0.4
0.6
0.8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
omega
Period (in years)
posterior
0.0
0.2
0.4
0.6
0.8
0
10
20
30
40
lambda

P&W 3.4 – Luteinizing hormone in blood samples








Frequency
posterior
0.042
0.044
0.046
0.048
0.050
0.052
0.0
0.5
1.0
1.5
2.0
2.5
3.0
omega
Period (in 10 minutes)
posterior
0.042
0.044
0.046
0.048
0.050
0.052
10
20
30
40
lambda