Harmonic Regression: One component model

$$y_i = \rho \cos(\omega t_i + \eta) + \epsilon_i, \quad i = t_1 : t_T$$

- ω is the angular frequency (cycles per unit of time) in $(0, \pi]$.
- $\lambda = 2\pi/\omega$ is the period. Highest possible frequency is $\omega = \pi$ and lowest possible period is $\lambda = 2$ (obs. per cycle). This is the *Nyquist*.
- The phase η lies between zero and 2π.
- We can write the model as

$$y_i = a \cos(\omega t_i) + b \sin(\omega t_i) + \epsilon_i,$$

with $a = \rho \cos(\eta)$, $b = -\rho \sin(\eta)$, and so $a^2 + b^2 = \rho^2$, and $\eta = \tan^{-1}(-b/a)$.

Harmonic Regression

One-component model

Reference Analysis

Conditional on ω we have

$$p(a, b, v|\omega, y_{t_1:t_T}) \propto p(a, b, v|\omega) \prod_{i=1}^{T} N(y_{t_i}|\mathbf{f}_i'\beta, v),$$

with $\mathbf{f}_i' = (\cos(\omega t_i), \sin(\omega t_i))$ and $\beta = (a, b)'$.

Taking $p(a, b, v|\omega) \propto 1/v$ we have

- $p(\beta|v, \omega, \mathbf{y}) = N(\beta|\hat{\beta}, v(\mathbf{FF}')^{-1})$ and
$p(\beta|\omega, \mathbf{y}) = T_{T-2}(\beta|\hat{\beta}, s^2(\mathbf{FF}')^{-1})$, with $s^2 = R/(T-2)$.
For T large, $p(\beta|\omega, \mathbf{y}) \approx N(\beta|\hat{\beta}, s^2(\mathbf{FF}')^{-1})$.

Harmonic Regression

One-component model

- $p(y_{i_1:i_T}|\omega)$:

$$
\begin{aligned}
p(y_{t_1:t_T}|\omega) &\propto |\mathbf{FF}'|^{-1/2} R^{-(T-2)/2} \\
&\propto |\mathbf{FF}'|^{-1/2} \{1 - \hat{\beta}' \mathbf{FF}' \hat{\beta} / (\mathbf{y}'\mathbf{y})\}^{(2-T)/2},
\end{aligned}
$$

and so,

$$
\begin{aligned}
p(\omega|y_{t_1:t_T}) &\propto p(\omega) p(y_{t_1:t_T}|\omega) \\
&\propto p(\omega) |\mathbf{FF}'|^{-1/2} \{1 - \hat{\beta}' \mathbf{FF}' \hat{\beta} / (\mathbf{y}'\mathbf{y})\},^{(2-T)/2}
\end{aligned}
$$

What happens when $t_i = i$ (equally-spaced data) and we evaluate the functions at the Fourier frequencies $\omega_k = 2\pi k/T$ for $1 \leq k < T/2$?

Harmonic Regression

One-component model

- $p(y_{i_1:i_T}|\omega)$:

$$\begin{aligned} p(y_{t_1:t_T}|\omega) &\propto |\mathbf{FF}'|^{-1/2} R^{-(T-2)/2} \\ &\propto |\mathbf{FF}'|^{-1/2}\{1 - \hat{\beta}'\mathbf{FF}'\hat{\beta}/(\mathbf{y}'\mathbf{y})\}^{(2-T)/2}, \end{aligned}$$

and so,

$$\begin{aligned} p(\omega|y_{t_1:t_T}) &\propto p(\omega)p(y_{t_1:t_T}|\omega) \\ &\propto p(\omega)|\mathbf{FF}'|^{-1/2}\{1 - \hat{\beta}'\mathbf{FF}'\hat{\beta}/(\mathbf{y}'\mathbf{y})\},^{(2-T)/2} \end{aligned}$$

What happens when $t_i = i$ (equally-spaced data) and we evaluate the functions at the Fourier frequencies $\omega_k = 2\pi k/T$ for $1 \leq k < T/2$?

Harmonic Regression

One-component model

At $\omega = \omega_k$, $\mathbf{FF}' = (T/2)\mathbf{I}_2$ in each case, and the MLEs $\hat{\beta}_k = (\hat{a}_k, \hat{b}_k)'$ given by

$$\begin{aligned} \hat{a}_k &\equiv \hat{a}(\omega_k) = (2/T)\sum_{i=1}^{T} y_i \cos(\omega_k i), \\ \hat{b}_k &\equiv \hat{b}(\omega_k) = (2/T)\sum_{i=1}^{T} y_i \sin(\omega_k i). \end{aligned}$$

In addition,

$$\hat{\beta}_k' \mathbf{FF}' \hat{\beta}_k = I(\omega_k) \equiv \frac{T[\hat{a}^2(\omega_k) + \hat{b}^2(\omega_k)]}{2},$$

and

$$p(\omega|y_{1:T}) \propto p(\omega)\{1 - I(\omega)/\mathbf{y}'\mathbf{y}\}^{(2-T)/2}$$

Harmonic Regression

One-component model

- When T is large and ω not too small, $p(\omega|y_{1:T})$ can be closely approximated by $p(\omega)\{1 - I(\omega)/\mathbf{y}'\mathbf{y}\}^{(2-T)/2}$ not just at the Fourier frequencies.
- The function

$$I(\omega) = \frac{T}{2}(\hat{a}(\omega)^2 + \hat{b}(\omega)^2),$$

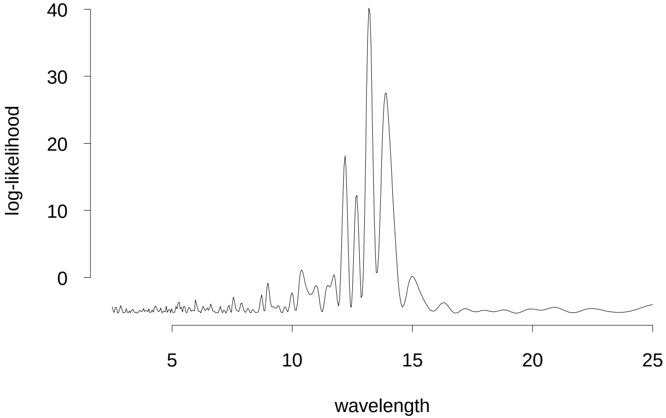
is known as the *periodogram*. If $I(\omega)$ is large for a given ω, then such frequency is "important". The periodogram can be efficiently computed via the fast Fourier transform.

Some useful R functions

- **spec.pgram**: Computes the periodogram using the fast Fourier transform. It can smooth the result via Daniell smoothers. Note that the periodogram is not a consistent estimator of the spectral density, but adjacent values are asymptotically independent and so a consistent estimator can be derived by smoothing the raw periodogram.
- **spec**: Estimates the spectral density using the periodogram or the AR representation of the process.
- **spec.ar**: Fits an AR to the data and estimates the spectral density using such representation.

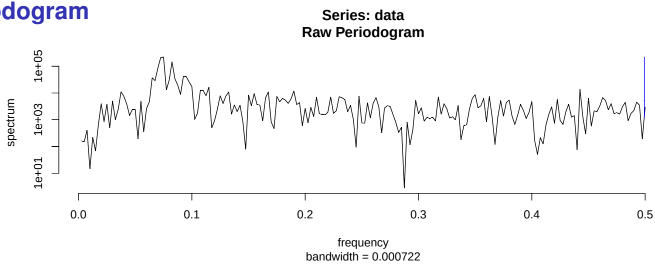
Harmonic Regression: One-component model

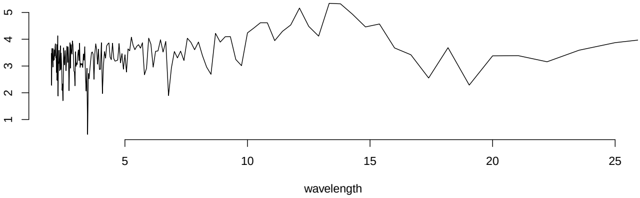
Example: EEG data



EEG data

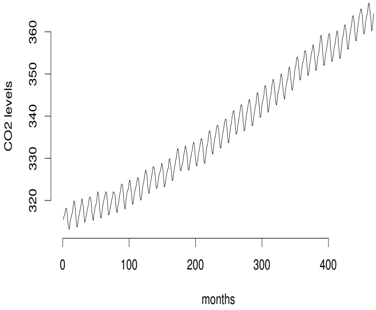
Raw periodogram

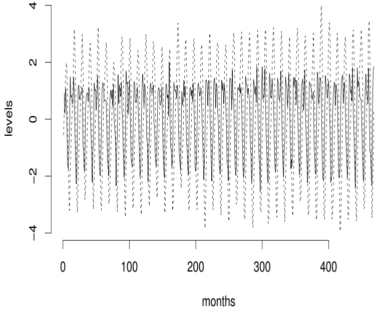




Harmonic Regression: One-component model

Mauna Loa CO_2 data: monthly measures of ground level carbon dioxide concentrations from Jan 1959 to Dec 1975.

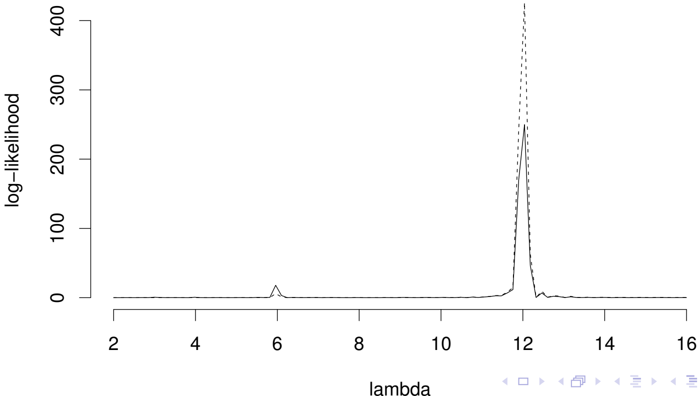




Harmonic Regression

One-component model

Mauna Loa Data



Harmonic Regression

Several frequency components

Two-component model: $y_i = \mathbf{f}_i'\beta + \epsilon_i$ with $\beta = (a_1, b_1, a_2, b_2)'$, and

$$\mathbf{f}_i' = (\cos(\omega_1 t_i), \sin(\omega_1 t_i), \cos(\omega_2 t_i), \sin(\omega_2 t_i)).$$

Under the reference prior $p(\beta, v|\omega_1, \omega_2) = p(\beta, v) \propto v^{-1}$, we have

$$p(y_{t_1:t_T}|\omega_1, \omega_2) \quad \propto \quad |\mathbf{FF}'|^{-1/2}\{1 - \hat{\beta}'\mathbf{FF}'\hat{\beta}'/(\mathbf{y}'\mathbf{y})\}^{(p-T)/2}.$$

What happens if $t_i = i$ at the Fourier frequencies?

Harmonic Regression

Several frequency components

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What happens if $t_i = i$ at the Fourier frequencies?

Harmonic Regression

Two frequency components

Let $\omega_{1,k} = 2\pi k/T$ and $\omega_{2,l} = 2\pi l/T$. Then, when T is large and ω_1, ω_2 are not too small, $p(y_{1:T}|\omega_1, \omega_2)$ can be approximated by

$$p(y_{t_1:t_T}|\omega_1, \omega_2) \propto \{1 - [I(\omega_1) + I(\omega_2)]/\mathbf{y}'\mathbf{y}\}^{(4-T)/2},$$

where

$$I(\omega_{1,k}) + I(\omega_{2,l}) = \frac{T}{2}(a^2_{1,k} + b^2_{1,k} + a^2_{2,l} + b^2_{2,l}),$$

and

$$a(\omega_{1,k}) = (2/T)\sum_{t=1}^{T} y_t \cos(\omega_{1,k} t), \qquad b(\omega_{1,k}) = (2/T)\sum_{t=1}^{T} y_t \sin(\omega_{1,k} t),$$

$$a(\omega_{2,l}) = (2/T)\sum_{t=1}^{T} y_t \cos(\omega_{2,l} t), \qquad b(\omega_{2,l}) = (2/T)\sum_{t=1}^{T} y_t \sin(\omega_{2,l} t).$$

Harmonic model with known period p

Let $y_t = \mu(t) + \epsilon_t$, with $\mu(t)$ a periodic function with period p. Then,

$$y_t = \sum_{k=1}^{m} \{\alpha_{1,k} \cos(2\pi kt/p) + \alpha_{2,k} \sin(2\pi kt/p)\} + \epsilon_t.$$

The MLEs are

$$\begin{aligned} \hat{\alpha}_{1,k} &= (2/T) \sum_{t=1}^{T} y_t \cos(2\pi kt/p), \\ \hat{\alpha}_{2,k} &= (2/T) \sum_{t=1}^{T} y_t \sin(2\pi kt/p), \end{aligned}$$

for $k < m = \lfloor p/2 \rfloor$, $\hat{\alpha}_{1,p/2} = (1/T) \sum_{t=1}^{T} (-1)^{t-1} y_t$, and $\hat{\alpha}_{2,p/2} = 0$ in the case of even p.

Harmonic model with known period p

The sum of squares partions as

$$\frac{T}{2}\sum_{k=1}^{m-1}(\hat{\alpha}_{1,k}^2+\hat{\alpha}_{2,k}^2)+T\hat{\alpha}_{1,m}^2=\sum_{k=1}^{m}I(\omega_k),$$

where $I(\cdot)$ is the periodogram evaluated at $\omega_k = 2\pi k/p$ for $k < m$. Under the reference prior:

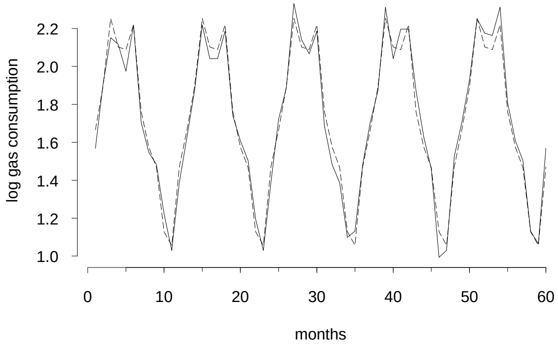
- The joint posterior for all coefficients is a multivariate Student-t $\Rightarrow$ for any harmonic k but the Nyquist, $(\alpha_{1,k}, \alpha_{2,k})'$ has a bivariate Student-t distribution with $\nu = T - p + 1$ (there are $p - 1$ parameters in the regression) d.f.;
- The mode of the bivariate Student-t is $\hat{\beta}_k = (\hat{\alpha}_{1,k}, \hat{\alpha}_{2,k})'$, and the scale matrix $(2s/T) \times \mathbf{I}_2$ where $s^2 = \mathbf{e}'\mathbf{e}/\nu$.

Harmonic model with known period p

- The posterior density contour running through zero has probability content $p_k = Pr(F_{2,\nu} \leq z_k)$ with $z_k = (\hat{\alpha}^2_{1,k} + \hat{\alpha}^2_{2,k})T/4s = I(\omega_k)/2s$.
- If p is even: the univariate posterior Student-t distribution for $\alpha_{1,p/2}$ leads to $p_{p/2} = Pr(F_{1,\nu} \leq z_{p/2})$ with $z_{p/2} = \hat{\alpha}^2_{1,p/2} T/s = I(\pi)/s$.
- The harmonic k is significant if $1 - p_k$ is small.

Harmonic model with known period p

UK gas consumption series. Logged values of monthly estimates of UK inland natural gas consumption data over the period Oct 1979 to Sep 1984 in log millions of tons of coal equivalent. We consider a model with $p = 12$, $T = 60$, and $\nu = 49$.



Harmonic model with known period p

UK gas consumption series

k	$1 - p_k$,	
1	0	(*)
2	0	(*)
3	0.07	
4	0	(*)
5	0.08	
6	0.37	

Representation of the ACF

Let $\gamma(h)$ be the autocovariance of a stationary process $\{y_t\}$. Then,

$$\gamma(h) = \int_{-\pi}^{\pi} e^{i\omega h} dF(\omega),$$

where $F(\omega)$ is the spectral distribution. Note that $V(y_t) = \gamma(0) = \int_{-\pi}^{\pi} dF(\omega)$ and so, $F(\omega)/\gamma(0)$ is a probability distribution on $(-\pi, \pi)$. By symmetry of the ACF we have

$$\gamma(h) = 2\int_{0}^{\pi} \cos(\omega h) dF(\omega),$$

for $h \geq 0$. If $F(\omega)$ is continuous and differentiable, with $f(\omega) = dF(\omega)/d\omega$,

$$\gamma(h) = 2\int_{0}^{\pi} \cos(\omega h) f(\omega) d\omega.$$

Representation of ACF

Under certain conditions,

$$
\begin{aligned}
f(\omega) &= \frac{1}{2\pi}\{\gamma(0) + 2\sum_{h=1}^{\infty}\gamma(h)\cos(\omega h)\} \\
&= \frac{\gamma(0)}{2\pi}\{1 + 2\sum_{h=1}^{\infty}\rho(h)\cos(\omega h)\}.
\end{aligned}
$$

Example

White Noise. Assume $\gamma(h) = 0$ for $h > 0 \Rightarrow f(\omega) = \gamma(0)/2\pi$.

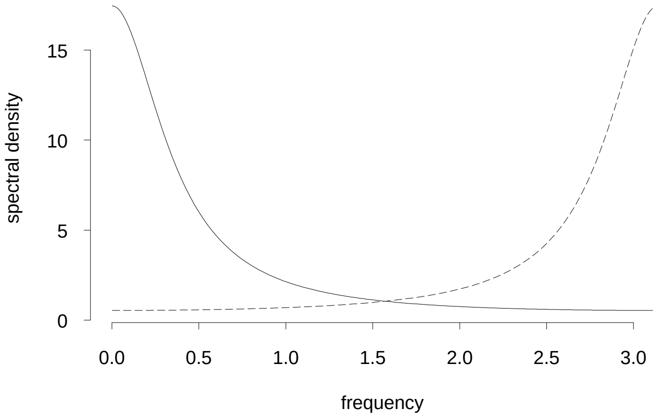
Example

AR(1) **Process.** $y_t = \phi y_{t-1} + \epsilon_t$ with $E(\epsilon_t) = 0$ and $V(\epsilon_t) = v$ and $|\phi| < 1$. Then, $\gamma(0) = v/(1-\phi^2)$, $\gamma(h) = \gamma(0)\phi^h$ and

$$f(\omega) = \frac{v}{2\pi}[1 + \phi^2 - 2\phi\cos(\omega)]^{-1}.$$

Spectral Densities

Spectra AR(1) with $\phi = 0.7$ and $\phi = -0.7$



Spectral Densities

Example

MA(1) Process. $y_t = \epsilon_t - \theta\epsilon_{t-1}$, $\theta < 1$ and $E(\epsilon_t) = 0$, $V(\epsilon_t) = v$. Then, $\rho(h) = 0$ for all $h > 1$ and $\rho(1) = -\theta/(1+\theta^2)$ and

$$f(\omega) = \frac{v}{2\pi}[1 + \theta^2 - 2\theta\cos(\omega)].$$

General Linear Processes

$y_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$ with $E(\epsilon_t) = 0$, $V(\epsilon_t) = 0$ and $\psi_0 = 1$. Then,

$$\begin{aligned} \gamma(h) &= v\sum_{j=0}^{\infty} \psi_j \psi_{j+h}, \quad \text{and} \\ f(\omega) &= \frac{v}{2\pi}\psi(e^{-i\omega})\psi(e^{i\omega}) = \frac{v}{2\pi}|\psi(e^{-i\omega})|^2, \end{aligned}$$

Spectral densities

Example

AR(2) **process.** $y_t = \phi_1 y_{t-1} + \phi_t y_{t-2} + \epsilon_t$, and $\psi(u) = 1/(1 - \phi_1 u - \phi_2 u^2)$, $\Rightarrow$

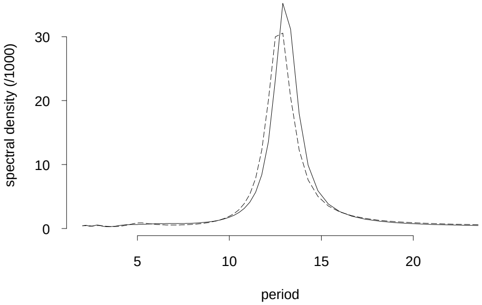
$$
\begin{aligned}
f(\omega) &= \frac{v}{2\pi} |(1 - \phi_1 e^{-i\omega} - \phi_2 e^{-2i\omega})|^{-2} \\
f(\omega) &= \frac{v}{2\pi[1 + \phi_1^2 + 2\phi_2 + \phi_2^2 + 2(\phi_1\phi_2 - \phi_1)\cos(\omega) - 4\phi_2 \cos^2(\omega)]}.
\end{aligned}
$$

If the process is stationary there are constraints on ϕ_1, ϕ_2. If the roots are real, $f(\omega)$ has a mode at either 0 or π; otherwise, the roots are complex conjugates and $f(\omega)$ is unimodal at $\omega = \arccos[-\phi_1(1 - \phi_2)/4\phi_2] \in (0, \pi)$.

Spectral densities

AR(p) process

$$f(\omega) = \frac{v}{2\pi|\Phi(e^{-i\omega})|^2} = \frac{v}{2\pi|(1 - \phi_1 e^{-i\omega} - \ldots - \phi_p e^{-ip\omega})|^2}.$$



Spectral densities

ARMA Processes

y_t a causal ARMA(p, q) process, $\Phi(B)y_t = \Theta(B)\epsilon_t$, with $\epsilon_t \sim N(0, v)$, then

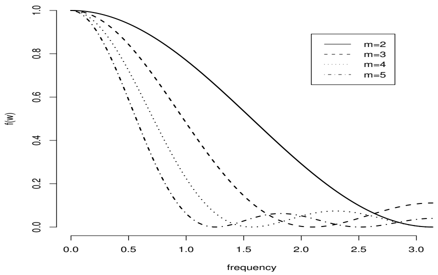
$$\begin{aligned} f(\omega) &= \frac{v}{2\pi}\frac{|\Theta(e^{-i\omega})|^2}{|\Phi(e^{-i\omega})|^2} \\ &= \frac{v}{2\pi}\frac{1+\sum_{j=1}^{q}\theta_j^2 + 2\sum_{k=1}^{q}(\sum_{j=k}^{q}\theta_j\theta_{t-j})\cos(k\omega)}{1+\sum_{j=1}^{p}\phi_j^2 + 2\sum_{k=1}^{p}(\sum_{j=k}^{p}\phi_j^*\phi_{j-k}^*)\cos(k\omega)}, \end{aligned}$$

with $\theta_0 = \phi_0^* = 1$ and $\phi_j^* = -\phi_j$ for $1 \leq j \leq p$.

Spectral densities

Example

MA process. Let $y_t = \sum_{j=0}^{m-1} c_j x_{t-j}$, with $c_j = 1/m$. Then, $f_y(\omega) = f_x(\omega)|c(e^{i\omega})|$, with $|c(e^{i\omega})| = (1-\cos(m\omega))/[m^2(1-\cos(\omega))]$. These processes preserve low frequencies (e.g., trends) and dampen high frequencies (e.g., noise) $\Rightarrow$ *low-pass filters.*



Spectral densities

Example

Differencing. $y_t = x_t - x_{t-1}$ and so, $f(\omega) = 2(1 - \cos(\omega))$. This enhances high frequencies and damps low frequencies $\Rightarrow$ *high-pass filter.*