BASKIN SCHOOL OF ENGINEERING
Department of Applied Mathematics and Statistics

Student number: ____________________

First Year Exam: June 15th 2009

INSTRUCTIONS

You must answer both questions in Part A pertaining to courses AMS 205 and AMS 211.

You must also answer 4 out of the 8 questions in Part B. Please indicate clearly which problems you are selecting to be graded.

PART A

Problem 1 (AMS 205) :

For each one of the following statements, decide if it is true or false. You must justify your answer with a short proof, counterexample, or reference to a standard theorem. (**Note:** All parts of the problem have equal weight.)

1. If $\tilde{\theta}(X)$ is an unbiased estimator for θ, it is also consistent.

   Answer:

2. If the maximum likelihood estimator exists and is unique, then it is a function of a minimal sufficient statistic for the problem.

   Answer:

3. Let $X_1, \ldots, X_n$ be a random sample where $X_i \sim \text{Uniform}(0, \theta)$. Suppose you want to test $H_0 : \theta = \theta_0$ vs. $H_a : \theta \neq \theta_0$ using the likelihood ratio test. Then, as $n \to \infty$, $-2 \log \Lambda \xrightarrow{D} \chi^2_1$, where Λ is the likelihood ratio.

   Answer:

4. In hypothesis testing, the p-value is the probability that H_0 is true.

   Answer:

5. Let $X_1, \ldots, X_m$ be a random sample where $X_i \sim \text{Binomial}(n, \theta)$. The maximum likelihood estimator for $\phi = \log\left(\frac{\theta}{1-\theta}\right)$ is $\hat{\phi} = \log\left(\frac{\bar{x}}{n-\bar{x}}\right)$.

   Answer:

6. Let $X_1, \ldots, X_n$ be a random sample where $X_i \sim p(\cdot|\theta)$, and assume that $p(\cdot|\theta)$ satisfies the usual regularity conditions. If an unbiased estimator $\tilde{\theta}$ for θ has variance equal to $1/\mathsf{E}\left(\left[\frac{\partial}{\partial\theta} \log \prod_{i=1}^n p(x_i|\theta)\right]^2\right)$, then it is the minimum variance unbiased estimator.

   Answer:

Problem 2 (AMS 211) :

(a) [33%] Find and classify the stationary points of the function

$$f(x,y) = 3xy - x^3 - y^3$$

(b) [34%] A discrete dynamical system (or Markov chain) is defined by a transition matrix $\mathbf{A}$ between one state $\mathbf{x}_k$ and the next $\mathbf{x}_{k+1}$ such that $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$. Show that any steady state of the system is described by a particular eigenvalue/eigenvector problem. If

$$A = \begin{pmatrix} 2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix},$$

find the general form of the steady-state solutions $\mathbf{x_s}$.

(c) [33%] Solve

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$$

subject to

$$y(0) = 0,\ y(\pi/3) = 1$$

Solution :

PART B

Problem 3 (AMS 206) :

Suppose $X_1, \ldots, X_n \mid \mu, \sigma^2 \stackrel{i.i.d.}{\sim} N(\mu, \sigma^2)$ with both μ and σ^2 unknown, and suppose we use noninformative priors, $f(\mu) \propto 1$ and $f(\sigma^2) \propto \frac{1}{\sigma^2}$. Show that the marginal posterior distribution for μ is a location-scale t with $\nu = n - 1$ degrees of freedom. How does this compare to the frequentist result for the sampling distribution of the MLE? (You don't need to re-derive the frequentist result.)

Information on distributions that you may find helpful:

Distribution	Density Function
Inverse Gamma	$f(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\beta/x}$
Normal	$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$
Location-Scale t	$f(x \mid \nu, \mu, \sigma^2) = \frac{\Gamma\left[\frac{1}{2}(\nu+1)\right]}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi\sigma^2}} \left[1 + \frac{1}{\nu\sigma^2}(x-\mu)^2\right]^{-(\nu+1)/2}$

Solution :

Problem 4 (AMS 207) :

1. (40%) Consider the following model

$$\begin{aligned} y_{i,j} \mid \theta_i &\overset{ind.}{\sim} N(\theta_i, v), \quad i = 1:I, j = 1:n_i, && (1) \\ \theta_i &\overset{i.i.d.}{\sim} N(\mu, \sigma^2), \quad i = 1:I, && (2) \end{aligned}$$

   where v, μ and σ^2 are assumed known.
   Write the model as a linear regression model and find $p(\theta_1, \ldots, \theta_I|\mathbf{y})$, where $\mathbf{y} = \{y_{i,j}, i = 1:I; j = 1:n_i\}$. What is the mode of this distribution?

2. (60%) Consider again the model from part 1, but now assume that μ is unknown, with

$$\begin{aligned} \mu \mid \phi &\sim N(m, \phi), \\ \phi &\sim Inv - \chi^2(\nu), \end{aligned}$$

   where m and ν are fixed (and v and σ^2 are still assumed known).
   Write down the joint posterior $p(\theta_1, \ldots, \theta_I, \mu, \phi|\mathbf{y})$ (up to a proportionality constant), and show that samples from this posterior distribution can be obtained via Gibbs sampling. Provide the details of this algorithm (i.e., find all the full conditional distributions).

Some useful distributions.

- inverse-χ^2 : $x \sim Inv - \chi^2(n)$,

$$p(x) \propto x^{-(n/2+1)} e^{-1/2x};$$

- scaled-inverse-χ^2 : $x \sim Inv - \chi^2(n, s^2)$,

$$p(x) \propto x^{-(n/2+1)} e^{-ns^2/(2x)}.$$

Solution :

Problem 5 (AMS 212A) :

The problem contains 2 questions: (A) and (B). Wrong answers to a pass/fail (P/F) question will give you 0 on the whole AMS212A problem.

(A) (20%) Consider the linear PDE

$$au_{xx} + bu_{xy} + cu_{yy} = 0 \tag{3}$$

- Describe how to use coefficients a, b and c to determine the type of the PDE. What condition needs to be satisfied for this PDE to be (a) elliptic (b) parabolic and (c) hyperbolic.
- (P/F question) For each of the 3 types of equations named above, give one well-known example of real physical equation which is of the same type .

(B) (80%) Solve the following problem using whichever method you prefer, for $\omega \neq 2c$.

$$\begin{aligned}
&u_{tt} = c^2 u_{xx} + \sin(\omega t)\sin(2x) \\
&u(0,t) = u(\pi,t) = 0 \\
&u(x,0) = 0 \\
&u_t(x,0) = 0
\end{aligned}$$

(c, ω are known constants). What happens for $\omega = 2c$? You may answer the question by solving the problem mathematically or by explaining what happens physically.

Solution :

Problem 6 (AMS 212B) :

This problem contains 2 questions: (A) and (B).

(A) (50%) Find **the leading term** expansion of

$$I = \int_0^2 t \cdot \exp\left[\lambda(3t - t^3)\right] dt \qquad \text{as } \lambda \to +\infty$$

Hint: You may need the expansion formulas for the Laplace method.. If you do not remember these expansion formulas, you can derive them from $\int_{-\infty}^{+\infty} \exp(-x^2) dx = \sqrt{\pi}$.

(B) (50%) Find **the leading term** expansion of

$$I = \int_0^1 \sqrt{1+t} \cdot \sin\left[\lambda(e^t - t)\right] dt \qquad \text{as } \lambda \to +\infty$$

Hint: You may need the expansions of $\int_0^a \cos(\lambda s^2)\, ds$ and $\int_0^a \sin(\lambda s^2)\, ds$. If you do not remember these expansions, you can derive them from $\int_0^{+\infty} \cos(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ and $\int_0^{+\infty} \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Solution :

Problem 7 (AMS 213) :

Problem 5.1 [20%] Consider the following Adams Bashforth method

$$x_{k+1} = x_k + \frac{h}{2}\left(3f(x_k, t_k) - f(x_{k-1}, t_{k-1})\right), \qquad k = 1, 2, \cdots$$

for the initial value problem

$$\frac{dx}{dt} = f(x,t), \qquad x(0) = x_0$$

Is this scheme explicit or implicit, single step or multistep? What is the order of the local truncation error? (only a brief explanation is needed.)

Problem 5.2 [40%] Let A be a nonsingular matrix. Consider the following iterative process for solving $Ax = b$.

$$Px^{k+1} = b - Qx^k, \quad k = 0, 1, 2, \cdots$$

where P and Q are $n \times n$ matrices satisfying

$$\begin{aligned} P + Q &= A \\ \|P^{-1}\| \cdot \|Q\| &= \delta < 1 \end{aligned}$$

for some induced norm $\|\cdot\|$. Show that, from any x^0, x^k converges to the solution of $Ax = b$.

Problem 5.3 [40%] Let $u(x,t)$ be a sufficiently smooth function of x and t. Denote $u_i^n = u(i\Delta x, n\Delta t)$, where Δx and Δt are the mesh sizes of a rectangular grid. Show that the finite difference scheme

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - \frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{\Delta x^2} = 0$$

1. is a consistent approximation of $u_t - u_{xx} = 0$, if $\Delta t = \Delta x^2$;
2. is a consistent approximation of $u_t + u_{tt} - u_{xx} = 0$, if $\Delta t = \Delta x$.

Solution :

Problem 8 (AMS 214) :

The problem contains 3 questions: (A), (B) and (C). They are all short questions, you should not be spending more than 10 minutes on each. Wrong answers to a pass/fail (P/F) question will give you 0 on the whole AMS214 problem.

(A) (20%) Consider the 1D system $\dot{x} = 5 - re^{-x^2}$.

- Find the expression for the fixed points as a function of r, and determine their stability (P/F question).
- Find the bifurcation point.
- Draw a complete annotated bifurcation diagram.
- Name the bifurcation.

(B) (40%) Consider the system

$$\dot{R} = aR - bRF$$
$$\dot{F} = -cF + dRF$$

where a,b,c and d are positive. You can also assume that $R \geq 0$ and $F \geq 0$.

- Find the fixed points of the model and study their stability.
- Show formally that the model predicts many periodic orbits. (Hint: you may want to find a conserved quantity).
- Draw a complete annotated phase portrait for the system.
- For some initial condition (R_0, F_0) that is not one of the fixed points, sketch on the same diagram $R(t)$ and $F(t)$.

(C) (40%) Consider the Tent map

$$x_{n+1} = rx_n \text{ if } x_n \in [0, 1/2]$$
$$x_{n+1} = r - rx_n \text{ if } x_n \in (1/2, 1]$$

where $r > 0$.

- Find the fixed points and study their stability (P/F question).
- At what value of r (call it r_c) do the fixed points bifurcate? Draw two cobweb diagrams: one for r slightly below r_c and one for r slightly above r_c.
- Prove that for $r > r_c$ nearly all trajectories are chaotic.

Solution :

Problem 9 (AMS 256) :

Consider the model defined as follows. For $i = 1 : N$, let

$$\begin{aligned} y_i &= \epsilon_i, \quad i \neq k, i \neq k+1, i \neq k+2 \\ y_k &= \lambda_1 + \epsilon_k, \\ y_{k+1} &= -\lambda_1 + \lambda_2 + \epsilon_{k+1}, \\ y_{k+2} &= -\lambda_2 + \epsilon_{k+2}, \end{aligned}$$

where k is a fixed integer, $1 \leq k \leq N-2$ and the ϵ_is are iid $N(0, \sigma^2)$ variables. Let $\boldsymbol{\beta} = (\lambda_1, \lambda_2)'$ and suppose σ^2 is known.

1. (40%) Derive the LSE of $\boldsymbol{\beta}$, denoted as $\hat{\boldsymbol{\beta}}$.
2. (30%) Find the distribution of $\hat{\boldsymbol{\beta}}$. Are $\hat{\lambda}_1$ and $\hat{\lambda}_2$ independent? (Justify your answer).
3. (30%) Find a 95% confidence interval for $\lambda_1 + \lambda_2$.

Solution :

Problem 10 (AMS 274) :

1. Consider the Poisson distribution with mean μ (denoted by Poisson$(\cdot;\mu)$).
   • Write the probability mass function of the distribution in the exponential dispersion family form, and provide the expressions for the natural parameter and the variance function.

   (20%)

2. Let y_i be realizations of independent random variables Y_i with Poisson$(\cdot;\mu_i)$ distributions, for $i = 1,\ldots,n$ (where $\mathrm{E}(Y_i) = \mu_i$). Consider the Poisson generalized linear model (glm) based on link function $\log(\mu_i) = \boldsymbol{x}_i^T \boldsymbol{\beta}$, where $\boldsymbol{x}_i$ is the covariate vector for response y_i and $\boldsymbol{\beta}$ is the vector of regression coefficients.
   • Obtain the deviance for the comparison of the Poisson glm above with the *full* model that includes a different μ_i for each y_i.

   (40%)

3. Consider a Bayesian formulation for the special case of the Poisson glm from part 2 based on a single covariate with values x_i. That is,

$$y_i \mid \beta_1, \beta_2 \overset{ind.}{\sim} \text{Poisson}(y_i; \mu_i = \exp(\beta_1 + \beta_2 x_i)), \quad i = 1, \ldots, n$$

   with $\mathrm{N}(a_j, b_j^2)$ priors for β_j, $j = 1, 2$.
   • Develop a Metropolis-Hastings algorithm to sample from $p(\beta_1, \beta_2 \mid \text{data})$, the posterior distribution of (β_1, β_2), where data $= \{(y_i, x_i) : i = 1, \ldots, n\}$.

   (40%)

Solution :