A TVAR(p) process is given by

$$y_t = \phi_{1,t} y_{t-1} + \ldots + \phi_{p,t} y_{t-p} + \epsilon_t, \quad \epsilon_t \sim N(0, v_t).$$

- *Evolution of AR coefficients.*

  $$\boldsymbol{\phi}_t = \boldsymbol{\phi}_{t-1} + \mathbf{w}_t, \;\; \mathbf{w}_t \sim N(\mathbf{0}, \mathbf{W}_t),$$

  with $\boldsymbol{\phi}_t = (\phi_{1,t}, \ldots, \phi_{p,t})'$.
- *DLM representation:* $\{\mathbf{F}_t, \mathbf{I}_p, v_t, \mathbf{W}_t\}$ with $\mathbf{F}_t' = (y_{t-1}, \ldots, y_{t-p})$.

$$y_t = x_t + \nu_t, \;\; x_t = \mathbf{F}'\theta_t, \;\; \theta_t = \mathbf{G}_t\theta_{t-1} + \mathbf{w}_t,$$

- Let $\mathbf{G}_t$ be a $p \times p$ matrix with p different eigenvalues $\alpha_{t,1}, \ldots, \alpha_{t,p}$. Assume C pairs of complex eigenvalues $r_{t,j}\exp(\pm i\omega_{t,j})$, $j = 1 : C$, and $R = p - 2C$ real eigenvalues $r_{t,j}$, $j = (C+1) : (C+R)$.
- Then,

$$\mathbf{G}_t = \mathbf{B}_t\mathbf{A}_t\mathbf{B}_t^{-1}$$

  where $\mathbf{A}_t$ is the diagonal matrix of eigenvalues (in arbitrary but fixed order) and $\mathbf{B}_t$ is a corresponding matrix of eigenvectors.

Define $\mathbf{H}_t = \mathrm{diag}(\mathbf{B}_t'\mathbf{F})\mathbf{B}_t^{-1}$. Transform $\boldsymbol{\theta}_t$ and $\mathbf{w}_t$ via $\boldsymbol{\gamma}_t = \mathbf{H}_t\boldsymbol{\theta}_t$ and $\boldsymbol{\delta}_t = \mathbf{H}_t\mathbf{w}_t$. Then,

$$y_t = x_t + \nu_t, \quad x_t = \mathbf{1}'\boldsymbol{\gamma}_t, \quad \boldsymbol{\gamma}_t = \mathbf{A}_t\mathbf{K}_t\boldsymbol{\gamma}_{t-1} + \boldsymbol{\delta}_t,$$

where $\mathbf{1}' = (1, \ldots, 1)$ and $\mathbf{K}_t = \mathbf{H}_t\mathbf{H}_{t-1}^{-1}$.

General decomposition: $y_t = x_t + \nu_t$, with

$$x_t = \sum_{j=1}^{C} x_{t,j}^{(1)} + \sum_{j=1}^{R} x_{t,j}^{(2)}.$$

- $x_{t,j}^{(1)}$ real process related to $r_{t,j}\exp(\pm i\omega_{t,j})$, $j = 1 : C$.
- $x_{t,j}^{(2)}$ real process related to $r_{t,C+j}$, $j = 1 : R$.

AR(p) decomposition

Using the DLM representation $\{\mathbf{E}_p, \mathbf{G}(\phi), 0, \mathbf{W}\}$ we have:

- The eigenvalues of $\mathbf{G}(\phi)$ are the reciprocal roots of the characteristic polynomial, $\alpha_1, \ldots, \alpha_p$, with C pairs of complex reciprocal roots $\alpha_{2j-1} = r_j \exp(-i\omega_j)$, $\alpha_{2j} = r_j \exp(i\omega_j)$, for $j = 1:C$, and $\alpha_j = r_j$ for $j = (2C+1):p$.
- $\mathbf{A}_t = \mathbf{A} = \text{diag}(\alpha_1, \ldots, \alpha_p)$, $\mathbf{B}_t = \mathbf{B}$, and $\mathbf{K}_t = \mathbf{I}$.
- Decomposition: $y_t = \sum_{j=1}^{C} x_{t,j}^{(1)} + \sum_{j=1}^{R} x_{t,j}^{(2)}$, with
  - $x_{t,j}^{(1)} \sim \text{ARMA}(2,1)$ with constant modulus r_j and frequency ω_j, $j = 1:C$;
  - $x_{t,j}^{(2)} \sim \text{AR}(1)$ with constant modulus r_{C+j}, $j = 1:R$.

TVAR(p) decomposition

Using the DLM representation $\{\mathbf{E}_p, \mathbf{G}_t(\phi), 0, \mathbf{W}\}$ with

$$\mathbf{G}_t(\phi) = \begin{pmatrix} \phi_{1,t} & \phi_{2,t} & \phi_{3,t} & \cdots & \phi_{p-1,t} & \phi_{p,t} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

we have $\mathbf{A}_t = \text{diag}(\alpha_{1,t}, \dots, \alpha_{p,t})'$, $\mathbf{B}_t$ the corresponding matrix of eigenvectors, and $\mathbf{K}_t = \mathbf{H}_t \mathbf{H}_{t-1}^{-1}$.
If ϕ_t does not vary a lot over time $\mathbf{K}_t \approx \mathbf{I}_p$. Assuming that C and R are constant over time we have the following decomposition...

TVAR(p) decomposition

$$y_t = \sum_{j=1}^{C} x_{t,j}^{(1)} + \sum_{j=1}^{R} x_{t,j}^{(2)},$$

- $x_{t,j}^{(1)} \approx$ TVARMA$(2, 1)$ with time-varying modulus $r_{t,j}$ and frequency $\omega_{t,j}$, $j = 1 : C$;
- $x_{t,j}^{(2)} \approx$ TVAR(1) with time-varying modulus $r_{t,C+j}$, $j = 1 : R$.

1. The structure of the underlying processes is *approximate,* but calculation of component processes is *exact.* The approximation depends on how close $\mathbf{K}_t$ is to $\mathbf{I}_p$.
2. R and C do not need to be constant over time.

Interpreting latent TVAR structure

How close are the latent processes in the decomposition to TVAR(1) and TVARMA(2, 1) obtained when $\mathbf{K}_t = \mathbf{I}_p$?

$$\mathcal{M}_1: \begin{array}{rcl} y_t &=& x_t + \nu_t \\ x_t &=& \mathbf{1}'\gamma_t \\ \gamma_t &=& \mathbf{A}_t\mathbf{K}_t\gamma_{t-1} + \delta_t \end{array} \qquad \mathcal{M}_2: \begin{array}{rcl} y_t &=& x_t + \nu_t \\ x_t &=& \mathbf{1}'\gamma_t \\ \gamma_t &=& \mathbf{A}_t\gamma_{t-1} + \delta_t. \end{array}$$

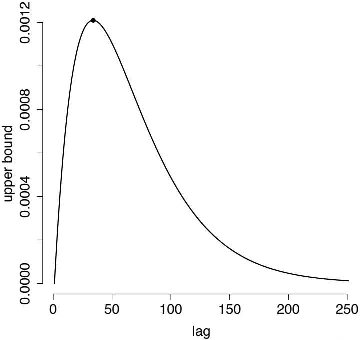
$$\begin{array}{rcl} f_t^{(1)}(h) &=& E(x_{t+h} \mid \gamma_t, \mathcal{M}_1) = \mathbf{1}'\mathbf{A}_{t+h}\mathbf{K}_{t+h}\ldots\mathbf{A}_{t+1}\mathbf{K}_{t+1}\gamma_t, \\ f_t^{(2)}(h) &=& E(x_{t+h} \mid \gamma_t, \mathcal{M}_2) = \mathbf{1}'\mathbf{A}_{t+h}\mathbf{A}_{t+h-1}\ldots\mathbf{A}_{t+1}\gamma_t. \end{array}$$

$$\frac{|f_t^{(1)}(h) - f_t^{(2)}(h)|}{\|\gamma_t\|_\infty} \leq (\lambda^*)^h \times [(1+\epsilon^*)^h - 1],$$

with $\lambda^* = \max_{0\leq j\leq h-1}(\max_{1\leq i\leq p}|\lambda_{((t+h-j),i)}|)$;
$\epsilon^* = \max_{1\leq j\leq h}\|\mathcal{E}_{t+j}\|_\infty$, $\mathbf{K}_{t+j} = \mathbf{I}_p + \mathcal{E}_{t+j}$.

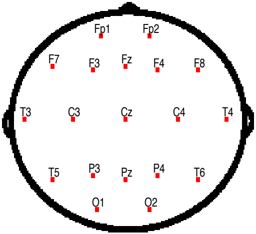
TVAR(p) such that, for all $j > t$, $\|\mathcal{E}_{t+j}\|_\infty \leq 10^{-4}$ and that all the characteristic reciprocal roots have moduli less than 0.97.

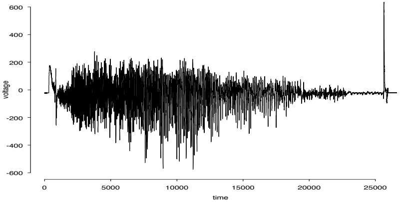
$$\frac{|f_t^{(1)}(h) - f_t^{(2)}(h)|}{\|\gamma_t\|_\infty} \leq 0.97^h \times [(1.0001)^h - 1].$$

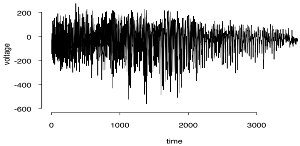


Example: Inferring latent structure in EEGs

- **Data:** 3,600 EEG observations recorded on a patient who received electroconvulsive therapy (ECT). Subsampled central portion of channel *Cz*. 18 additional channels.







Example: Inferring latent structure in EEGs

- **Model:**

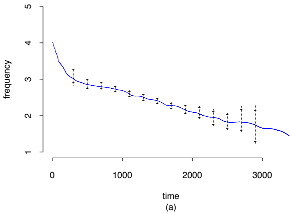
$$
\begin{aligned}
y_t &= \sum_{j=1}^{p} \phi_{j,t} y_{t-j} + \nu_t, \quad \nu_t \sim N(0, v_t), \\
\phi_t &= \phi_{t-1} + \boldsymbol{\xi}_t, \quad \boldsymbol{\xi}_t \sim N(\boldsymbol{\xi}_t \mid \mathbf{0}, \mathbf{U}_t(\delta_\phi)), \\
v_t &= \delta_v v_{t-1}/\eta_t, \quad \eta_t \sim Be(\eta_t \mid a_t, b_t),
\end{aligned}
$$

with δ_ϕ and δ_v discount factors in $(0, 1]$. Optimal values of δ_ϕ, δ_v, and p obtained by maximizing

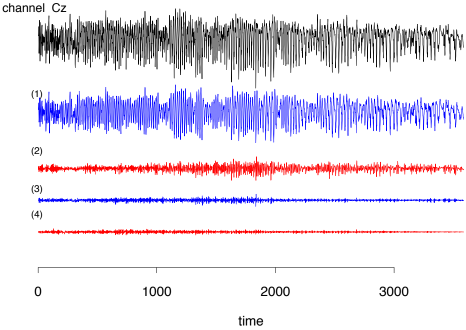
$$
l(\delta_\phi, \delta_v, p) \equiv \log[y_{(p^*+1):T} \mid \mathcal{D}_{p^*})] = \sum_{t=p^*+1}^{T} \log[p(y_t \mid \mathcal{D}_{t-1})],
$$

over $[0.9, 1] \times [0.9, 1] \times [4, 20]$, with $p^* = 20$. $\hat{\delta}_\phi = 0.994$, $\hat{\delta}_v = 0.95$, and $\hat{p} = 12$.

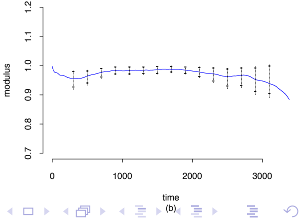
Frequency



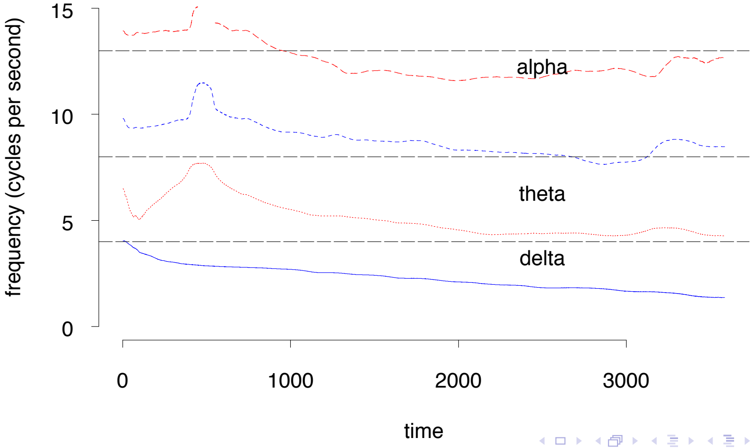
EEG decomposition



Modulus

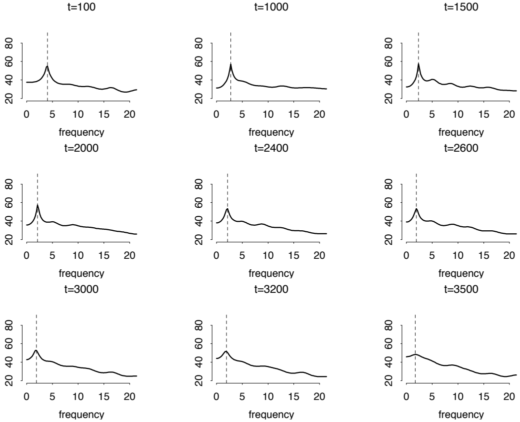


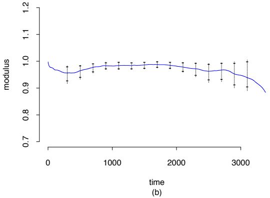
Ordering of the latent components

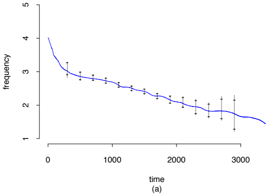


EEG data: Estimated time-varying spectra

$r_{t,j}, \omega_{t,j}$





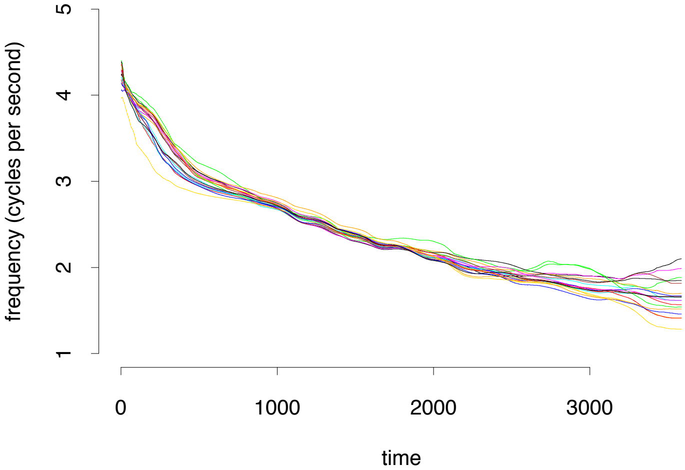


TVAR models for multiple time series

Case study: ECT data

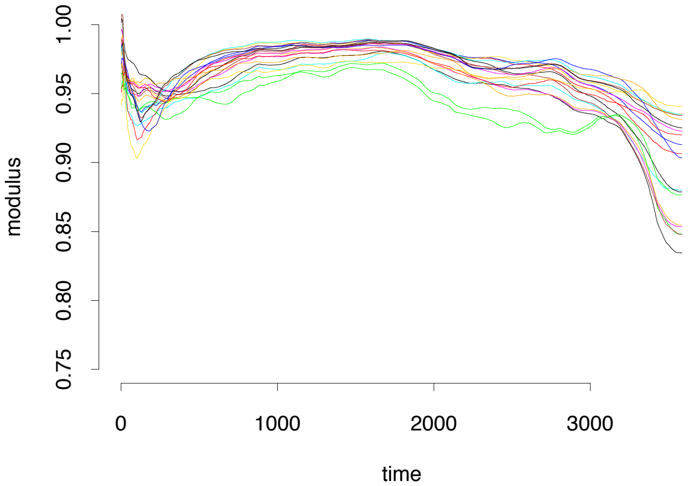
- Fit TVAR(12) models to each of the 19 EEG signals.
- Extract dominant frequencies and moduli.

EEG data: Trajectories of dominant frequencies



(a)

EEG data: Moduli of dominant frequencies



(b)