Classical Parameter Estimation

The traditional modeling approach to spatial random fields is to focus on the first two moments of the distribution. In the Gaussian case, these characterize the full stochastic behavior of the process. So the estimation problem consists of determining the mean and covariance functions.

Classical Parameter Estimation

The traditional modeling approach to spatial random fields is to focus on the first two moments of the distribution. In the Gaussian case, these characterize the full stochastic behavior of the process. So the estimation problem consists of determining the mean and covariance functions.

For the mean we usually assume a linear parametric form

$$\mu(s) = \beta_0 + \sum_{j=1}^{p} \beta_j d_j(s)$$

where $\beta_j, j = 0, \ldots, p$ are unknown parameters and $d_j(s)$ are known spatial explanatory variables.

Estimating the Trend

An initial estimation of the trend is usually done using the LSE for β, the vector of linear coefficients. Letting D be the matrix of covariates, we have that the LSE is the solution of the normal equations

$$D'D\hat{\beta} = D'X$$

where X is the vector of observed realizations of the random field. The residuals are then defined as

$$R = X - D\hat{\beta}$$

Estimating the Trend

An initial estimation of the trend is usually done using the LSE for β, the vector of linear coefficients. Letting D be the matrix of covariates, we have that the LSE is the solution of the normal equations

$$D'D\hat{\beta} = D'X$$

where X is the vector of observed realizations of the random field. The residuals are then defined as

$$R = X - D\hat{\beta}$$

When the covariance structure of X is known to be a matrix that is not a multiple of the identity, say V, then the LSE is the solution to

$$D'V^{-1}D\tilde{\beta} = D'V^{-1}X$$

Estimating the Covariance

Once the trends has been removed we can use the residuals to estimate the correlation of the process. We use a method of moments estimator of the semi-variogram. For a stationary process the semi-variogram is given by

$$\gamma(u) = \frac{1}{2}E(X(s) - X(s-u))^2$$

so we can obtain an unbiased estimator of $\gamma(||s_i - s_j||)$ as

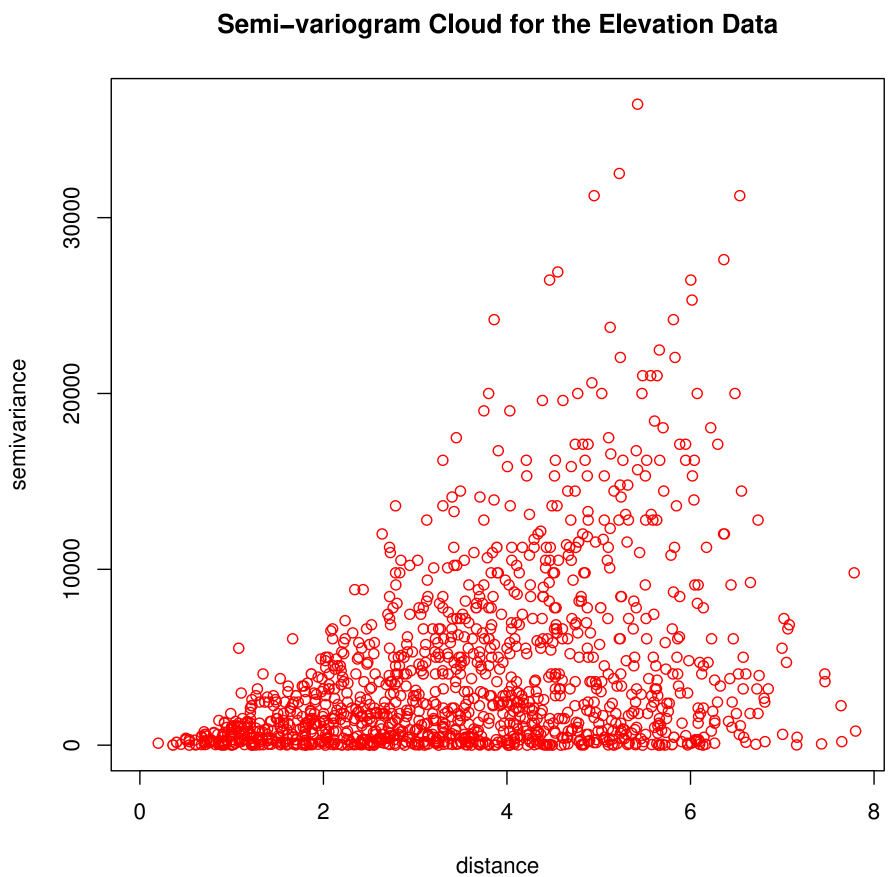
$$\hat{\gamma}(u_{ij}) = (X(s_i) - X(s_j))^2/2$$

where $u_{ij} = ||s_i - s_j||$.

A plot of all the pairs $\hat{\gamma}(u_{ij})$ versus u_{ij} is known as a **Variogram Cloud**.

Empirical Semi-Variogram

An example of such a plot for the elevation data is presented in the panel. The plot was created with the geoR command `plot(variog(elevation, max.dist=8, op="cloud"),col=2, main='Semi-variogram Cloud for the Elevation Data')`



The variogram cloud has limited inferential utility as there is too much scatter in the plot. There are two problems with $\hat{\gamma}(u_{ij})$:

- Under Gaussian assumption for the observations $\hat{\gamma}(u_{ij})$ follows a χ^2_1, which is a very skewed distribution.
- There are very strong correlations between the points, as only n observations are used to obtain $n(n-1)/2$ estimates.

Empirical Semi-Variogram

The variogram cloud has limited inferential utility as there is too much scatter in the plot. There are two problems with $\hat{\gamma}(u_{ij})$:

- Under Gaussian assumption for the observations $\hat{\gamma}(u_{ij})$ follows a χ^2_1, which is a very skewed distribution.
- There are very strong correlations between the points, as only n observations are used to obtain $n(n-1)/2$ estimates.

In order to solve the scattering problem in the variogram cloud we smooth the empirical variogram

Smoothing the Semi-Variogram

For points on a regular grid a very obvious way of smoothing the empirical semi-variogram is to take averages of $\hat{\gamma}(u_{ij})$ for each distinct u_{ij}. For irregular designs we have to introduce binning, which may cause the estimated semi-variogram to be biased. We proceed as follows:

1. We consider a bin width h
2. We define
$$\gamma_k = \text{ave}_{u_{ij} \in ((k-1)h, kh]} \left(\hat{\gamma}(u_{ij})\right)$$
3. Define $u_k = (k - 0.5)h$, the midpoint of the k-th interval.
4. Plot γ_k versus u_k

Smoothing the Semi-Variogram

For points on a regular grid a very obvious way of smoothing the empirical semi-variogram is to take averages of $\hat{\gamma}(u_{ij})$ for each distinct u_{ij}. For irregular designs we have to introduce binning, which may cause the estimated semi-variogram to be biased. We proceed as follows:

1. We consider a bin width h

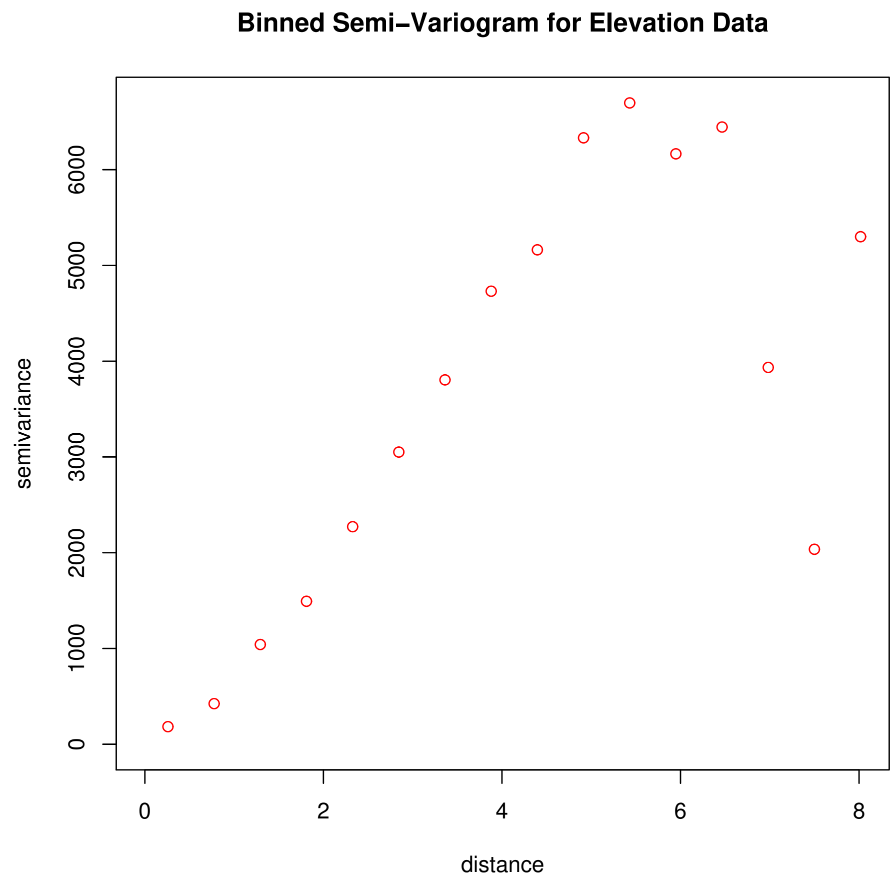
2. We define
$$\gamma_k = \text{ave}_{u_{ij} \in ((k-1)h, kh]} \left(\hat{\gamma}(u_{ij})\right)$$

3. Define $u_k = (k - 0.5)h$, the midpoint of the k-th interval.

4. Plot γ_k versus u_k

As γ_k is an approximately unbiased estimate of $\gamma(u_k)$, this plot will give a smoothed estimate of the semi-variogram curve.

Smoothed semi-variogram for the elevation data using `plot(variog(elevation, uvec=16),col=2, main='Binned Semi-Variogram for Elevation Data')`



For large distances we see the effects of (a) strong correlations and (b) small number of points

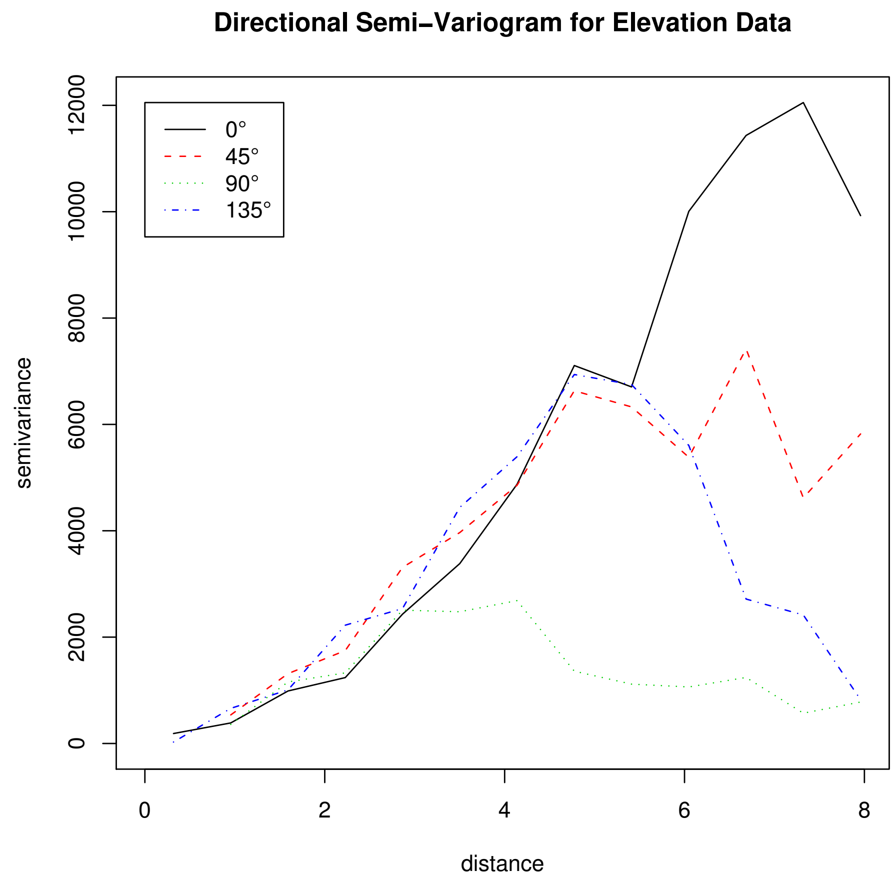
More elaborate forms of smoothing can be considered. In general this is a problem of non-parametric regression for non-independent observations. For EDA it is, in most cases, sufficient to consider a subjectively chosen bin size and a fast a simple form of smoothing like averaging.

Semi-variogram based parameter estimation consists of using the empirically estimated semi-variogram and using one of the many families considered in the literature and fit their parameters using least squares. This produce is likely to be affected by (a) lack of independence; (b) uneven number of observations per bin and (c) outlying observations.

To explore the possible geometric anisotropies one can bin the data according to specific directions.

To explore the possible geometric anisotropies one can bin the data according to specific directions.

A directional semi-variogram can be obtained using the command `plot(variog4(elevation, max.dist=9))`



Trend or covariance?

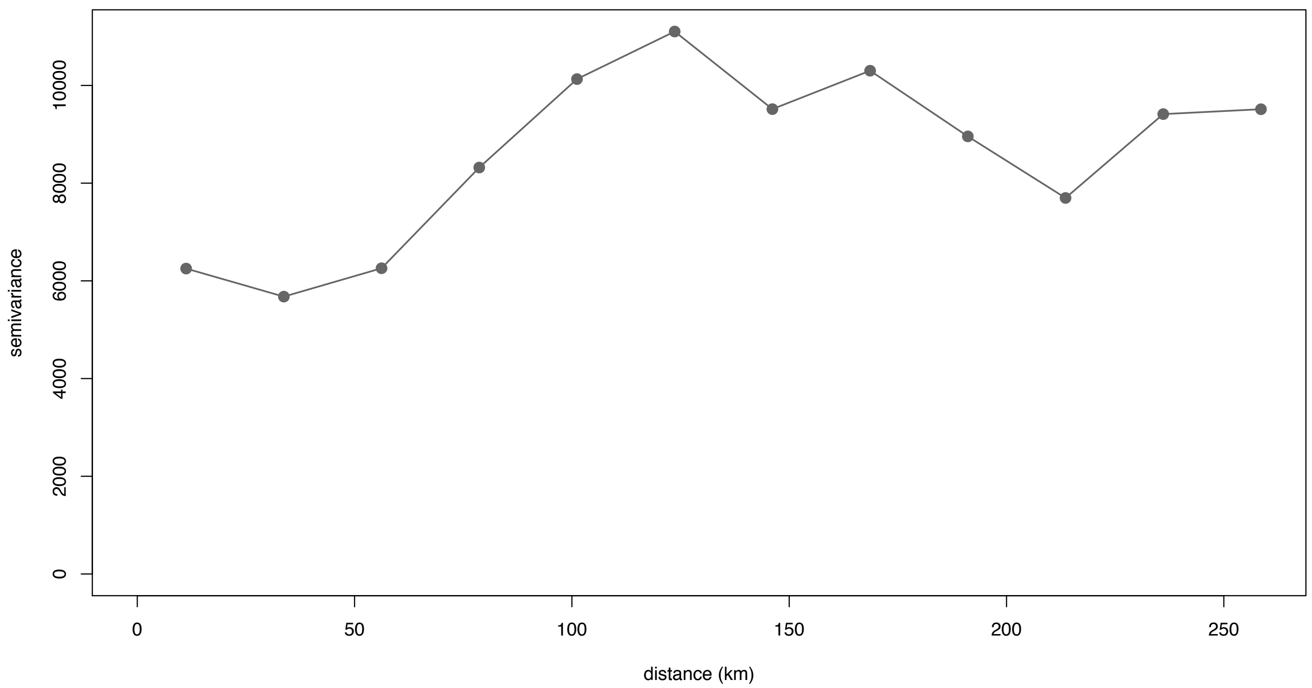
Consider a stochastic process with mean function $\mu(s) = D(s)'\beta$ and covariance function $C(s, s')$. Then, if we assume that the components of β are random variables with covariance matrix equal to $\gamma^2 I$, we have that

$$\operatorname{var}(X(s)) = \gamma^2 D'(s)D(s) + C(s, s)$$

so there is a contribution of the linear drift to the overall variance of the process.

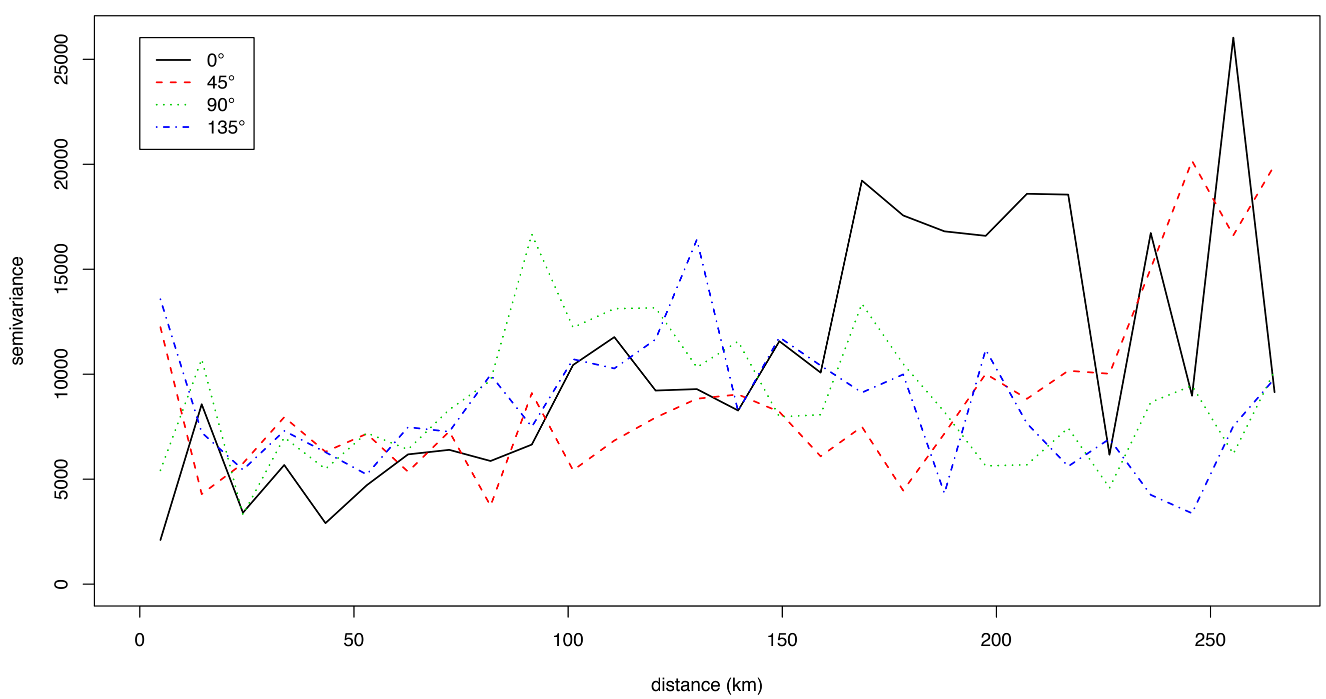
We can thus expect that the estimation of the mean of the process will affect the estimation of its covariance. In general, after subtracting a trend the estimated variogram will be lower than the one obtained from the original data.

Guárico Rainfall Variogram Analysis



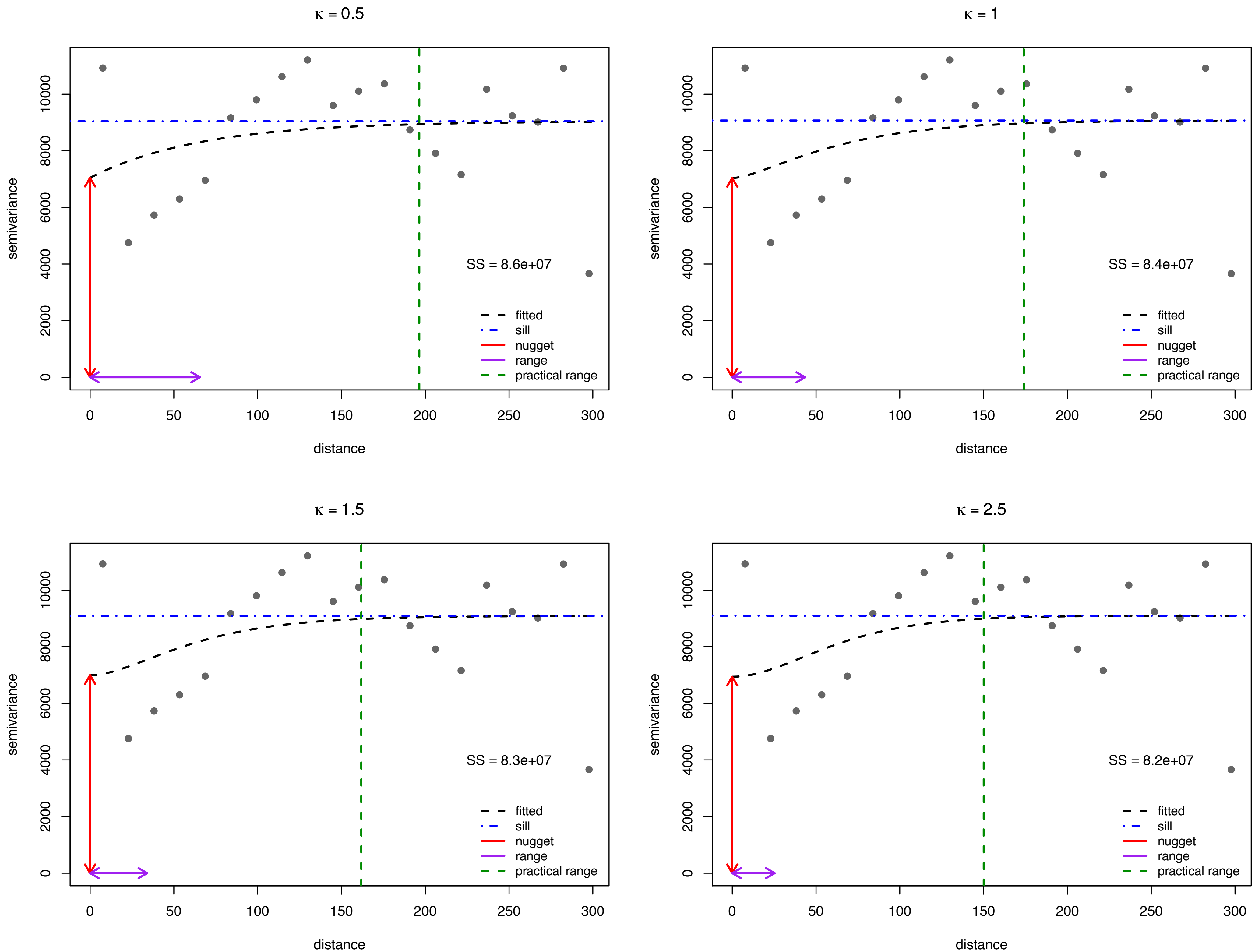
We consider the Guárico rainfall data after removing a quadratic trend in both latitude and height

The variogram provides clear evidence of a nugget effect



The directional variogram does not indicate any obvious geometric anisotropies

Least Squares Fit of Matern Family Correlations



Fits for different values of the smoothness parameter of a Matern correlation function