GLMM Parameter Estimation via Approximation Methods

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This is a review of “Approximate Inference in Generalized Linear Mixed Models” by Breslow and Clayton in 1993.

Hierarchical model

The ith of n observations has univariate response y_i with vectors $\mathbf{x}_i$ and $\mathbf{z}_i$ is covariates.

We have a q-vector $\mathbf{b}$ of random effects, where $\mathbf{b} \sim N(\mathbf{0}, \mathbf{D}(\boldsymbol{\theta}))$.

In vector notation, the conditional mean of $\mathbf{y} = (y_1, \ldots, y_n)^\top$ given $\mathbf{b}$ is assumed to satisfy

$$\mathrm{E}(\mathbf{y}|\mathbf{b}) = \boldsymbol{\mu} = h(\mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}\mathbf{b})$$

and $\mathrm{Var}(y_i|\mathbf{b}) = \phi a_i v(\mu_i)$ for $i = 1, \ldots, n$, for known constant a_i and variance function $v(\cdot)$. Note, we do not have any distributional assumption on y_i.

Quasi-likelihood function

For independent observations $y_i, \ldots, y_n$, the quasi-likelihood function is defined as

$$\begin{aligned} \frac{\partial K(y_i, \mu_i)}{\partial \mu_i} &= \frac{y_i - \mu_i}{a_i v(\mu_i)} \\ \Longrightarrow K(y_i, \mu_i) &= \int_{y_i}^{\mu_i} \frac{y_i - u_i}{a_i v(u_i)} du_i \end{aligned}$$

This has properties similar to the log-likelihood of a distribution from the exponential family.

The integrated quasi-likelihood function is defined by

$$\begin{aligned} e^{ql(\boldsymbol{\alpha},\boldsymbol{\theta})} &\propto |\mathbf{D}|^{-1/2}\int \exp\left[\frac{1}{\phi}\sum_{i=1}^{n} K(y_i,\mu_i) - \frac{1}{2}\mathbf{b}^\top\mathbf{D}^{-1}\mathbf{b}\right] d\mathbf{b} \\ &= c|\mathbf{D}|^{-1/2}\int \exp\left[\kappa(\mathbf{b})\right] d\mathbf{b} \end{aligned}$$

where c is some multiplicative constant. Using Laplace's method for integral approximation (and ignoring the constants), we have

$$ql(\boldsymbol{\alpha},\boldsymbol{\theta}) \approx -\frac{1}{2}\log|\mathbf{D}| - \frac{1}{2}\log|\kappa''(\tilde{\mathbf{b}})| - \kappa(\tilde{\mathbf{b}})$$

where $\tilde{\mathbf{b}}$ is the solution to $\kappa'(\mathbf{b}) = 0$.

Penalized quasi-likelihood (PQL)

Marginal quasi-likelihood (MQL)

Estimating parameters

Roughbutts and Whitesides