BASKIN SCHOOL OF ENGINEERING
Department of Applied Mathematics and Statistics

Student number: ____________________

(A random identifying number should be assigned to you and recorded on a separate sheet)

First Year Exam: June 11th 2010

INSTRUCTIONS

You must answer both questions in Part A pertaining to courses AMS 205 and AMS 211.

You must also answer all 4 out of the 8 questions in Part B.

There should be plenty of room to write answers in this booklet. Please use both sides of the paper. If you need to use more paper, please attach it at the appropriate place, put your name on it and make sure it is clear which question it pertains too.

PART A

Problem 1 (AMS 205) :

For each one of the following statements, decide if it is true or false. You must briefly justify your answer (short proof, counterexample and/or argument).
(Each part is worth $\frac{100}{6}\%$)

1. Let $\Phi(x)$ be the cumulative distribution function of the standard normal distribution and let $Z \sim \mathsf{N}(0,1)$. Then $\mathsf{E}(\Phi(Z)) = 1/2$.

2. If $\tilde{\theta}$ is is an unbiased estimator of θ, then $\tilde{\phi} = 1/\tilde{\theta}$ is an unbiased estimator of $\phi = 1/\theta$.

3. Consider a statistical model where $Y_{ijk} \sim \mathsf{N}(\alpha_i\beta_j, \sigma^2)$ for $i = 1, \ldots, 5$, $j = 1, \ldots, 8$, $k = 1, \ldots, 10$ and where $\{\alpha_i\}_{i=1}^5$, $\{\beta_j\}_{j=1}^8$ and σ^2 are all unknown. The resulting model is not identifiable.

4. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with density $f(x|\theta)$ that depends on a unidimensional parameter θ. If $n > 2$, every minimal sufficient statistic for the problem has dimension $k < n$.

5. Let $X_1, \ldots, X_n$ be a random sample where $X_i \sim p(\cdot|\theta)$. The uniformly most powerful test of level α to contrast $H_0 : \theta = \theta_0$ vs. $H_a : \theta = \theta_1$ that is based on the likelihood ratio test is obtained by rejecting H_0 if

$$\Lambda = \frac{\prod_{i=1}^n p(x_i|\theta_1)}{\prod_{i=1}^n p(x_i|\theta_0)} > k$$

for some constant k such that $\mathsf{Pr}(\Lambda > k|H_0) = \alpha$.

6. The power and the level of a statistical test must sum up to one.

Solution:

Problem 2 (AMS 211) :

1. (50%) Classify the following ordinary differential equation and solve it:

$$x\frac{dy}{dx} + 2(1 - x^2)y = 1$$

2. (50%) A function $f(x)$ equals e^{-x} over $0 < x < 1$. Expand $f(x)$ as a Fourier sine series and calculate the Fourier coefficients. Sketch the periodic Fourier solution. Why are the end points of the function of the interval $[0, 1]$ excluded?

Solution :

PART B

Problem 3 (AMS 212A) :

(a) (20%) The following equation

$$x^2 f'' + 2x f' + (x^2 - n(n+1))f = 0 \tag{1}$$

for integer values of n is called the Spherical Bessel Equation.

Show that the Spherical Bessel Equation is a Sturm-Liouville problem, by casting it in the form $(p(x)f')' + q(x)f = -\lambda r(x)f$. You will need to specify what the functions p, q and r are.

(b) (80%) A conducting sphere of radius R has been left at ambient temperature ($t = 20^\circ$) for a long time, and now has a steady, uniform temperature throughough with this value. At time $t = 0$, it is immersed in water at 0°, and slowly cools down. The aim of this problem is to calculate the temperature profile T in the sphere as a function of position within the sphere and of time. You may assume that the temperature of the water remains 0° at all times, and that the conductivity of the sphere is $\kappa = 1$.

(i) Give a complete mathematical description of the problem (governing equation, boundary conditions, initial conditions).

(ii) Is this a hyperbolic, parabolic or elliptic problem? (Pass/Fail question)?

(iii) Using separation of variables, show that the temperature as a function of position and time (for $t > 0$ and interior to the sphere) is

$$T(r, \theta, \phi, t) = \sum_{n=1}^{\infty} a_n j_0(\sqrt{\lambda_n} r) e^{-\lambda_n t} \tag{2}$$

where r is radius, θ is co-latitude and ϕ is azimuth. You must find expressions for a_n and λ_n in terms of known quantities in the problem.

Useful mathematical formula for this problem:

1. The spherical geometry Laplacian is:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{3}$$

in the coordinate system (r, θ, ϕ).

2. Solutions to the Spherical Bessel Equation are the Spherical Bessel Functions:

$$\begin{aligned} &j_n(x) \text{ are regular at } x = 0 \\ &y_n(x) \text{ are singular at } x = 0 \end{aligned} \tag{4}$$

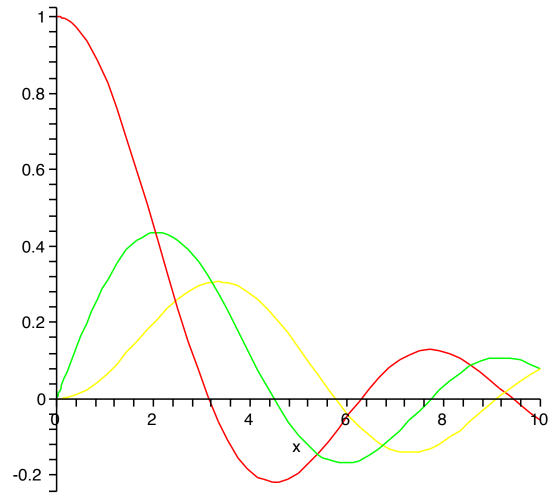

Figure 1: The functions $j_0(x)$, $j_1(x)$ and $j_2(x)$

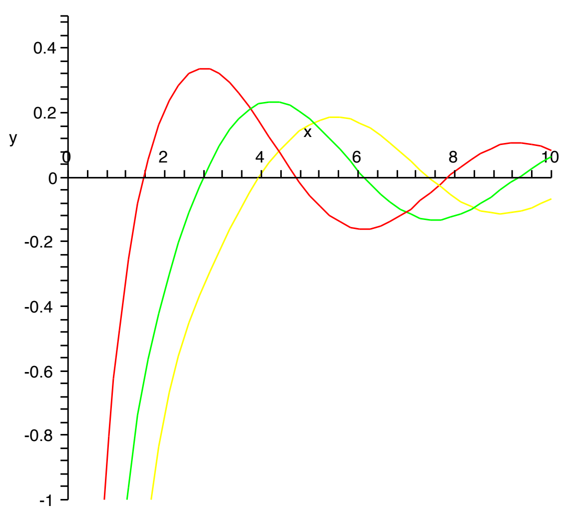

Figure 2: The functions $y_0(x)$, $y_1(x)$ and $y_2(x)$

Spherical Bessel Functions oscillate about 0 (see Figure). The $m-$th zero of the $j_n(x)$ function is denoted as $z_{n,m}$, and the m-th of the $y_n(x)$ function is denoted as $\zeta_{n,m}$.

Solution :

<u>Problem 4 (AMS 212B) :</u>

Use the method of multiple scale expansion to solve the initial value problem

$$\begin{cases} y'' + \varepsilon\,(1+y^2)\,y' + y = 0 \\ y(0) = 1\ , \quad y'(0) = 0 \end{cases} \qquad , \qquad \varepsilon \to 0_+$$

Find the leading term in the expansion.

<u>Solution :</u>

Problem 5 (AMS 213) :

Problem 5.1 [20%] The initial value problem

$$\frac{dy}{dt} = y^{1/3}, \; y(0) = 0$$

has a solution

$$y(t) = (2t/3)^{3/2} \tag{5}$$

Explain why Euler method $y_{k+1} = y_k + h(y_k)^{1/3}$, $y_0 = 0$, does not converge to solution (5).

Problem 5.2 [30%] Consider the following steepest descent method for solving $Ax = b$.

$$\begin{aligned} p_k &= b - Ax_k \\ \alpha_k &= \frac{p_k^T p_k}{p_k^T A p_k} \\ x_{k+1} &= x_k + \alpha_k p_k \end{aligned}$$

Show that the error vector, $e_{k+1} = A^{-1}b - x_{k+1}$, always conjugate to the search direction p_k, i.e., $p_k^T A e_{k+1} = 0$.

Problem 5.3 [50%] Consider the following Cauchy problem

$$\begin{aligned} u_t + a u_x &= 0, \; -\infty < x < \infty \\ u(x,0) &= g(x), \\ \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx &< \infty, \; \forall t \geq 0 \end{aligned}$$

where $a > 0$ is a constant; and the following finite difference scheme

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} + a \frac{u_k^n - u_{k-1}^n}{\Delta x} = 0$$

where u_k^n is the discrete solution at $x = k\Delta x$ and $t = n\Delta t$.

1. Show that the scheme is consistent; [25%]
2. Use Fourier transform to show that for $a\frac{\Delta t}{\Delta x} \leq 1$, the scheme is stable. [25%]

Solution :

(Q5 cont)

Problem 6 (AMS 214) :

Question 6.1: [50%] Consider the system

$$\dot{x} = x - y - x^3$$
$$\dot{y} = x + y - y^3$$

Using a change of variable from polar to Cartesian to show that

$$r - r^3 \leq \dot{r} \leq r - \frac{1}{2} r^3 \tag{6}$$

where $r^2 = x^2 + y^2$. Hint: you may need to use the trigonometric relation

$$\cos^4 \theta + \sin^4 \theta = \frac{3}{4} + \frac{1}{4} \cos 4\theta \tag{7}$$

By constructing an appropriate trapping region, show the existence of a limit cycle in this system.

Question 6.2: [50%]
(a) Consider the non-dimensional equation:

$$\ddot{\theta} + b\dot{\theta} + \sin \theta = 0$$

where $b > 0$. What physical system could this represent? Cast this equation into a system of two first-order ODEs.

(b) Find the fixed points, and study their stability. Show that there is a critical value of the parameter, b_c, where the qualitative behavior of the solutions changes.

(c) Draw an accurate phase portrait for two values of b, one larger than b_c and one smaller than b_c.

(d) Explain in words how these solutions relate to the original physical system studied.

(e) Can this system exhibit chaos? If yes, discuss the kind of solutions expected in the chaotic regime. If no, explain why.

Solution :

(Q6 cont)

Problem 7 (AMS 206) :

1. (50%) Let $X_1, \ldots, X_n$ be a random sample from a binomial with unknown probability of success θ, i.e., $f(x|n,\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, and put a prior on θ such that $f(\theta) = \frac{4}{3} - \theta^2$ for $\theta \in [0,1]$. Give an MCMC algorithm to sample from the posterior (please write out all of the complete conditionals and use as many Gibbs steps as practical).

2. (50%) Suppose you are doing MCMC and you need to generate your own samples from an arbitrary binomial distribution with $n = 2$. Write out the steps for using the inverse CDF method to sample from a binomial distribution with $n = 2$ and $p = \theta$.

Solution :

Problem 8 (AMS 207) :

Consider the following random-effects hierarchical model, which is useful in meta-analysis and other applications:

$$\begin{aligned} (\mu, \sigma^2) &\sim p(\mu, \sigma^2) \\ (\gamma_i | \mu, \sigma^2) &\stackrel{\text{IID}}{\sim} N(\mu, \sigma^2) \\ (y_i | \gamma_i) &\stackrel{\text{indep}}{\sim} N(\gamma_i, \tau_i^2), \end{aligned} \tag{8}$$

for $i = 1, \ldots, I$, in which the τ_i^2 are assumed known.

1. (40%) With $\theta = (\mu, \sigma^2)$ and an appropriate choice for latent data z, specify the two distributions $p(\theta | z, y)$ and $p(z | \theta, y)$ needed to carry out an EM algorithm to find the posterior mode of θ given $y = (y_1, \ldots, y_I)$, making an appropriate conditionally conjugate choice for the prior distribution on θ, and use this to specify the E and M steps of your EM algorithm.

2. (60%) Specify a Gibbs sampler to make random draws from the augmented posterior distribution $p(\mu, \sigma^2, z | y)$, by providing details on the full-conditional distributions $p(\mu | \sigma^2, z, y)$, $p(\sigma^2 | \mu, z, y)$ and $p(z | \mu, \sigma^2, y)$.

Solution :

Problem 9 (AMS 356) :

Consider the model defined by the following equation:

$$y_{i,j} = \mu + \alpha_i + \epsilon_{i,j}, \quad \epsilon_{i,j} \sim N(0, \sigma^2) \tag{9}$$

for $i = 1:2$, $j = 1:3$ and with all $\epsilon_{i,j}$s independent.

1. Part I.
   (a) (10%) Write the model in matrix form $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$. What is the rank of $\boldsymbol{X}$? What is the rank of $\boldsymbol{X}'\boldsymbol{X}$?
   (b) (10%) Is $H_0: \alpha_1 = \alpha_2$ testable? Justify your answer.
   (c) (10%) Is $H_0: \mu = 0$ testable? Justify your answer.
   (d) (5%) Are the LSEs of μ, α_1, and α_2 unique? Justify your answer.
   (e) (5%) Is the LSE of $\alpha_1 - \alpha_2$ unique? Justify your answer.
2. Part II: Add the restriction $\alpha_1 = 0$ to the model in Equation (9).
   (a) (20%) Find the LSEs of μ and α_2 and denote them as $\hat{\mu}$ and $\hat{\alpha}_2$. Are these unique?
   (b) (15%) What is the distribution of $\hat{\mu}$ and $\hat{\alpha}_2$? Are $\hat{\mu}$ and $\hat{\alpha}_2$ independent?
   (c) (10%) Give the form of a 95% C.I. for $\mu + \alpha_2$.
   (d) (10%) Find the B.L.U.E. of $\mu + \alpha_2$ and its associated variance.
   (e) (5%) Is model (9) with the restriction $\alpha_2 = 0$ equivalent to model (9) with the restriction $\alpha_1 + \alpha_2 = 0$? Justify your answer.

Solution :

Problem 10 (Stats combo question) :

Consider a Bernoulli regression model for a vector of covariates $\boldsymbol{x}$ and a binary response y, with observed values $\boldsymbol{x}_i$ and y_i, $i = 1, ..., n$, respectively. Specifically,

$$y_i \mid \boldsymbol{\beta} \overset{ind.}{\sim} \{\Phi(\boldsymbol{x}_i^T \boldsymbol{\beta})\}^{y_i} \{1 - \Phi(\boldsymbol{x}_i^T \boldsymbol{\beta})\}^{1-y_i}, \quad i = 1, ..., n \tag{M.1}$$

where $\Phi(\cdot)$ denotes the standard normal c.d.f., and $\boldsymbol{\beta}$ is the vector of regression coefficients, which is assigned a flat prior. Assume that the design matrix of covariates $\boldsymbol{X}$ is of full rank.

Moreover, assume that the binary response y arises by discretizing an underlying (real-valued) continuous response z. Specifically, we observe $y_i = 1$ if $z_i > 0$ or $y_i = 0$ if $z_i \leq 0$, where z_i is the latent (unobserved) continuous response corresponding to observed response y_i, for $i = 1, ..., n$. Consider the following augmented regression model

$$\begin{aligned} y_i \mid z_i &\overset{ind.}{\sim} \{1(z_i > 0)1(y_i = 1) + 1(z_i \leq 0)1(y_i = 0)\}, \quad i = 1, ..., n \\ z_i \mid \boldsymbol{\beta} &\overset{ind.}{\sim} \mathrm{N}(\boldsymbol{x}_i^T \boldsymbol{\beta}, 1), \quad i = 1, ..., n \end{aligned} \tag{M.2}$$

where $1(v \in A)$ denotes the indicator function with value 1 when $v \in A$. Hence, note that the first stage of model (M.2) denotes the (degenerate) distribution for y_i conditional on the sign of z_i as described above.

1. (30%) Show how model (M.1) can be obtained from model (M.2).
2. (70%) Use the model (M.2) representation to develop a Gibbs sampler for the posterior distribution of $\boldsymbol{\beta}$. Provide details of the posterior full conditionals for $\boldsymbol{\beta}$ and for the z_i, $i = 1, ..., n$.

Solution :