BASKIN SCHOOL OF ENGINEERING

Department of Applied Mathematics and Statistics

First Year Exam: June 10, 2013

INSTRUCTIONS

If you are on the Applied Mathematics track, you are required to complete problems 1(AMS 203), 2(AMS 211), 3(AMS 212A), 4(AMS 212B), 5(AMS 213), and 6(AMS 214).

If you are on the Statistics track, you are required to complete problems 1(AMS 203), 2(AMS 211), 7(AMS 205B), 8(AMS 206B), 9(AMS 207), and 10(AMS256).

Please complete all required problems on the **supplied exam papers**. Write your exam ID number and problem number on each page. Use only the **front side** of each page.

Problem 1 (AMS 203):

An experiment consists of measuring the reaction time X (in seconds) to a certain stimulus. The stimulus is such that the reaction time cannot be less than 1 second and more than 3 seconds. The probability density function for X is given by

$$f(x) = \begin{cases} \frac{3}{2}x^{-2} & \text{for } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

1. (15 points) Compute the expected reaction time.
2. (15 points) Obtain the median reaction time.
3. (40 points) If an individual takes more than 1.5 seconds to react, a light comes on and stays on either until one further second has elapsed or until the person reacts (whichever happens first). Let T denote the amount of time the light remains lit. Find the expectation of random variable T.
4. (30 points) Suppose that the reaction time to the stimulus is measured for each of 10 individuals. Assume that reaction times for the individuals are independent, and that

the probability density function of the reaction time is the same for all 10 individuals, given by $f(x)$ above. Provide the distribution, including the values of its parameters, for random variable Y defined as the number of individuals with reaction time less than 2.5 seconds.

Problem 2 (AMS 211):

2.1 [40%] Using a Lagrange multiplier, find the maximum and minimum value of the function $f(x,y) = 2x^2 + y^2$ on the circle $x^2 + y^2 = 9$

2.2 [40%]

(a) Write down (or derive, if necessary) the formula for the line integral of the scalar function $f(x(t), y(t))$, where t is a parameter, over a curve $C(x(t), y(t))$ with respect to the arc length down the curve between starting point $C(t = t_0)$ and ending point $C(t = t_1)$.

(b) Find the total mass of an (infinitely thin) circle of wire whose location is described by $x^2 + y^2 = 1$ and whose density function is given by $f(x,y) = 3 + x + y$

2.3 [20%] Find the solution of the following homogeneous ODE:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0, \quad y(0) = 0, \quad y(\pi/4) = e^{3\pi/4}$$

Problem 3 (AMS 212A):

Consider a circular drum with a central hole. Its outer radius is a and inner radius is b. The drum skin is tightly pinned at the outer and inner boundary. It supports oscillations that propagate at wave-speed c.

Find the spatial eigenmodes of this "holy" drum and show that their corresponding oscillations frequencies ω_{nm}

satisfy the following equation (which you do not need to solve):

$$J_n\left(\frac{\omega_{nm}b}{c}\right)Y_n\left(\frac{\omega_{nm}a}{c}\right)=J_n\left(\frac{\omega_{nm}a}{c}\right)Y_n\left(\frac{\omega_{nm}b}{c}\right) \tag{1}$$

where J_n and Y_n are Bessel functions (see below), and m and n can be any positive integers. Clearly explain *every* step of your calculation, starting with a mathematical description of the model (equation and boundary conditions). Stuck

To answer this question, you will need the following information.

- The Laplacian operator in polar coordinates is:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \tag{2}$$

- The Bessel Equation is

$$x^2\frac{d^2 f}{dx^2}+x\frac{df}{dx}+(x^2-n^2)f=0 \tag{3}$$

  and has two types of solutions. The regular Bessel function $J_n(x)$ and the singular Bessel function $Y_n(x)$.

Problem 4 (AMS 212B):

1. (85%) Use the matched asymptotic expansion to solve the boundary value problem

$$\begin{cases} \varepsilon y'' + y' + y^2 = 0 \\ y(0)=0\,, \quad y(1)=\frac{1}{2} \end{cases}, \qquad \varepsilon \to 0_+$$

   Find the leading term of the composite expansion.

2. (15%) Consider the function defined below

$$F(x) \equiv \exp(-x^3)\int_0^x \exp(t^3)dt$$

   We want to find a few terms in the expansion of $F(x)$ as $x \to +\infty$.

   What is the key step in expanding $F(x)$?

   You will get extra credit if you carry out the expansion and find the first 2 terms.

Problem 5 (AMS 213):

~~Problem 5.1~~ [50%] Consider the following conjugate gradient method for solving $Ax = b$

$$\begin{cases} \alpha_k &= \frac{p_k^T r_k}{p_k^T A p_k} \\ x_{k+1} &= x_k + \alpha_k p_k \\ r_{k+1} &= b - A x_{k+1} \\ p_{k+1} &= r_{k+1} - \frac{p_k^T A r_{k+1}}{p_k^T A p_k} p_k \end{cases} \tag{4}$$

with given initial condition x_0, $r_0 = b - Ax_0$ and $p_0 = r_0$. Let CG iteration (4) be applied to $Ax = b$ with A be symmetric and positive definite. At step $k \geq 1$, show that for all $\Delta x \in span\{r_0, r_1, \cdots, r_{k-1}\}$, $f(x_k + \Delta x) \geq f(x_k)$, where $f(x) = \frac{1}{2}x^T Ax - x^T b$.

Problem 5.2 [50%] Consider the Cauchy problem

$$\begin{aligned} u_t + au_x &= 0, \ -\infty < x < \infty \\ u(x,0) &= g(x), \\ \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx &< \infty, \ \forall t \geq 0 \end{aligned}$$

where a is a constant; and the following finite difference scheme

$$\frac{u_k^{n+1} - \frac{1}{2}(u_{k+1}^n + u_{k-1}^n)}{\Delta t} + a\frac{u_{k+1}^n - u_{k-1}^n}{2\Delta x} = 0$$

where u_k^n is the discrete solution at $x = k\Delta x$ and $t = n\Delta t$. Use Fourier transform to show that the scheme is stable if $\left|a\frac{\Delta t}{\Delta x}\right| \leq 1$.

~~Problem 6 (AMS 214):~~

Consider $\dot{x} = x - y - x(x^2 + 5y^2)$, $\dot{y} = x + y - y(x^2 + y^2)$.

a) (30%) Classify the fixed point at the origin.

b) (20%) Rewrite the system in polar coordinates using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.

c) (15%) Determine the circle of maximum radius r_1 with the center in the origin such that all trajectories on the circle have a radially outward component.

d) (15%) Determine the circle of minimum radius r_2 with the center in the origin such that all trajectories on the circle have a radially inward component.

d) (20%) Assuming that no fixed point exists in the band limited by the circles from (c)-(d), what can be concluded based on the Poincaré-Bendixson theorem?

$\exists$ limit cycle

Problem 7 (AMS 205B):

Consider the following model for how long You'll have to wait for an event E, of interest to You, to happen:

$$p(t_i|\theta) = \frac{\theta}{t_i^2} I(t_i \geq \theta) \tag{5}$$

for some $\theta > 0$, where $I(A)$ denotes the indicator function for set A. You obtain a (conditionally IID, given θ) random sample $t = (t_1, \dots, t_n)$ of these waiting times, and Your goal is to estimate θ and attach a well-calibrated measure of uncertainty to your estimate.

(1) (15 points) Find the method-of-moments estimate of θ based on t.

(2) (30 points) Identify a sufficient statistic for θ, and show that the maximum-likelihood estimate of θ based on t is given by $\hat{\theta}_n = \min_i t_i$.

(3) (20 points) Using Theorem 1 below (You don't need to prove this Theorem), show that the repeated-sampling distribution of $\hat{\theta}_n$ is

$$p(\hat{\theta}_n|\theta) = \frac{n\,\theta^n}{\left(\hat{\theta}_n\right)^{n+1}} I(\hat{\theta}_n \geq \theta)\,. \tag{6}$$

Theorem 1. Let $Y_{(1)}, \dots, Y_{(n)}$ denote the order statistics of an IID random sample $(Y_1, \dots, Y_n)$ from a continuous population with CDF $F_Y(y)$ and PDF $f_Y(y)$. Then the PDF of $Y_{(i)}$ is

$$f_{Y_{(i)}}(y) = \frac{n!}{(i-1)!\,(n-i)!} f_Y(y)\,[F_Y(y)]^{i-1}\,[1 - F_Y(y)]^{n-i}\,. \tag{7}$$

(4) (35 points) Use equation (6) to construct an equal-tailed $100(1-\alpha)\%$ confidence interval for θ that's valid for all $n \geq 1$. (*Hint:* Is there a pivotal quantity suggested by the form of (6)?)

Problem 8 (AMS 206B):

Consider $x \mid \theta \sim \text{Binomial}(n, \theta)$ with n known.

1. (45 points) Find Jeffreys prior, $\pi^J(\theta)$, and the corresponding posterior distribution $\pi^J(\theta \mid x)$ for this model.

2. (20 points) Consider the quadratic loss function $L(\theta, \theta^*) = (\theta - \theta^*)^2$. Under the assumptions in part (1), find the estimator $\theta^J(x)$ that minimizes the Bayesian posterior loss.

3. (35 points) Let $x^* \mid \theta \sim \text{Binomial}(m, \theta)$ with m known. Under the assumptions in part (1), find the posterior predictive distribution, i.e., find $p^J(x^* \mid x)$.

Hint: The probability density function of the Beta distribution with shape parameters $\alpha > 0$ and $\beta > 0$ is given by

$$f(y \mid \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}, \quad y \in (0,1)$$

where $\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) dt$, for $a > 0$, denotes the gamma function.

Problem 9 (AMS 207):

Consider the linear model

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim N_n(\boldsymbol{0}, (1/\phi)\boldsymbol{I})$$

where $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{X}$ is a full rank matrix of dimensions $n \times p$, $\boldsymbol{\beta} \in \mathbb{R}^p$, $\boldsymbol{I}$ is a $n \times n$ identity matrix and $\phi > 0$. Consider the prior $p(\boldsymbol{\beta}|\gamma, \phi) = \prod_i p(\beta_i|\gamma_i, \phi)$ where

$$p(\beta_i|\gamma_i, \phi) = (1 - \gamma_i)N(\beta_i|0, 1/\phi) + \gamma_i N(\beta_i|0, c_i/\phi).$$

Here $c_i > 0$ is a known constant, and γ_i is a binary variable taking value 1 with probability ω_i and 0 with probability $1 - \omega_i$. Thus

$$p(\gamma) = \prod_{i=1}^{p} \omega_i^{\gamma_i}(1 - \omega_i)^{1-\gamma_i}, \quad \gamma_i = 0, 1, \quad \omega_i \in (0, 1).$$

To complete the model we assume that $p(\phi) = Ga(\phi|a_\phi, b_\phi)$, and

$$p(\omega) = \prod_{i=1}^{p} Beta(\omega_i|a_\omega, b_\omega).$$

Write the full conditional distributions needed to explore the posterior distribution of the parameters in the model, using a Gibbs sampler.

Hint:

$$(x-a)'A(x-a) + (x-b)'B(x-b) = (x-c)'(A+B)(x-c) + (a-b)'A(A+B)^{-1}B(a-b)$$

where

$$c = (A+B)^{-1}(Aa + Bb)$$

Problem 10 (AMS 256):

For $i = 1, \ldots, N$, let

$$\begin{aligned} Y_i &= \epsilon_i, \quad i \neq k, k+1, k+2 \\ Y_k &= \lambda_1 + \epsilon_k \\ Y_{k+1} &= -c\lambda_1 + \lambda_2 + \epsilon_{k+1} \\ Y_{k+2} &= -c\lambda_2 + \epsilon_{k+2}, \end{aligned}$$

where k is a fixed integer, $1 \leq k \leq N-2$, $|c| < 1$ is a known constant, and the ϵ_i are i.i.d. $N(0, \sigma^2)$ variables, for $i = 1, ..., N$. Let $\beta = (\lambda_1, \lambda_2)'$, and suppose σ^2 is known.

(a) (30 points) Derive the least squares estimate of β, and the variance of the estimate.

(b) (35 points) Derive the least squares estimate of β subject to the restriction $\lambda_1 + \lambda_2 = 0$. What is the variance of this estimate?

(c) (35 points) Derive a statistic for testing $H : \lambda_1 + \lambda_2 = 0$ versus the alternative hypothesis that λ_1 and λ_2 are unrestricted.