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In order to specify a Gaussian random field we need a mean,

$$m(s) = E(X(s)), \quad \forall s \in S$$

and a covariance function,

$$C(s, s') = \text{cov}(X(s), X(s'), \quad \forall s, s' \in S.$$

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Kolmogorov Existence Theorem: If a system of finite-dimensional distributions F satisfies

$$F_{s_1,\ldots,s_k}(x_1,\ldots,x_k) = F_{\pi s_1,\ldots,\pi s_k}(x_{\pi 1},\ldots,x_{\pi k})$$

for any permutation π, and

$$F_{s_1,\ldots,s_{k-1}}(x_1,\ldots,x_{k-1}) = F_{s_1,\ldots,s_k}(x_1,\ldots,x_{k-1},\infty)$$

then there exists, on some probability space $(\Omega, \mathcal{F}, P)$, a random field $X(s), s \in S$, having $F_{s_1,\ldots,s_k}$ as its finite dimensional distributions.

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For a multivariate normal distribution it is easy to check that the above conditions are satisfied.

Definition: $C(s, s')$ is **positive definite** if for any positive integer n, $s_j \in S$ and $c_j \in \mathbb{R}$ for $j = 1, \ldots, n$

$$\sum_{i,j} c_i c_j C(s_i, s_j) \geq 0$$

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Definition: The **variance function** is defined as

$$\sigma^2(s) = C(s, s)$$

Definition: A random field is **strictly stationary** if for any finite collection of sites $s_1, \ldots, s_n$ and any $u \in S$, the joint distributions of $(X(s_1), \ldots, X(s_n))$ and $(X(s_1 + u), \ldots, X(s_n + u))$ are the same.

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Definition: A random field is **weakly stationary** if $m(s) = m, \forall s \in S$ and $C(s, s + u) = C(u)$. Notice that any stationary process must have constant variance, so $C(u) = \sigma^2 \rho(u)$.

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Definition: A random field is **weakly stationary** if $m(s) = m, \forall s \in S$ and $C(s, s + u) = C(u)$. Notice that any stationary process must have constant variance, so $C(u) = \sigma^2 \rho(u)$.

Remark: Strict stationarity implies weak stationarity. For a Gaussian process the opposite is also true.

Definition: Assuming that $E(X(s+u) - X(s)) = 0$, the **variogram** is defined as $E(X(s+u) - X(s))^2 = \mathrm{var}(X(s+u) - X(s))$.

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Definition: A stationary random field is **isotropic** if the covariance function depends on distance alone, i.e. $C(s, s') = C(\tau)$ where $\tau = ||s - s'||$. This is a very strong condition on the radial symmetry of the covariance.

Model	$C(\tau)$
Spherical	$C(\tau) = \begin{cases} \sigma^2(1 - \frac{3}{2}\tau/\phi + \frac{1}{2}(\tau/\phi)^3) & \text{if } 0 < \tau \leq \phi \\ 0 & \text{if } \tau \geq \phi \end{cases}$
Powered Exponential	$C(\tau) = \sigma^2 \exp(-\lvert\tau/\phi\rvert^\nu) \;\; \tau > 0 \;\; 0 < \nu \leq 2$
Rational Quadratic	$C(\tau) = \sigma^2 \left(1 - \frac{\tau^2}{(\phi^2+\tau^2)}\right) \;\; \tau > 0$
Wave	$C(\tau) = \sigma^2 \frac{\sin(\tau/\phi)}{\tau/\phi} \;\; \tau > 0$
Matérn	$C(\tau) = \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)}(\tau/\phi)^\nu K_\nu(\tau/\phi) \;\; \tau > 0 \;\; \nu > 0$

Model	$\gamma(\tau)$
Spherical	$\gamma(\tau) = \begin{cases} \sigma^2(\frac{3}{2}\tau/\phi - \frac{1}{2}(\tau/\phi)^3) & \text{if } 0 < \tau \leq \phi \\ \sigma^2 & \text{if } \tau \geq 1/\phi \end{cases}$
Powered exponential	$\gamma(\tau) = \sigma^2(1 - \exp(-\lvert\tau/\phi\rvert^\nu)) \;\; \tau > 0 \;\; 0 < \nu \leq 2$
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Matérn	$\gamma(\tau) = \sigma^2\left(1 - \frac{(\tau/\phi)^\nu}{2^{\nu-1}\Gamma(\nu)}K_\nu(\tau/\phi)\right) \;\; \tau > 0 \;\; \nu > 0$

The isotropic covariances considered are defined by at least two parameters. The **scale** σ and the **range** ϕ. σ^2 corresponds to the variance of the process.

The range determines the decay of the covariance function. So, for example, consider an exponential covariance. If the distance for which the correlation is equal to 5% is τ_0, then the range is equal to $\tau_0/2.3$. Note that in the parameterizations used in the tables, the range is measured in the same units used to obtain distances.

Definition: We can obtain **geometric anisotropy** by considering the norm $||s||_K = \sqrt{s' K s}$ for a positive definite matrix K. If ρ is a valid correlation function for an isotropic random field, we can define $\rho_K(s, s') = \rho(||s - s'||_K)$.

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For $s \in \mathbb{R}^2$, we can obtain a parametric representation of K by letting $K = P\Lambda P^T$, where

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\theta \in (-\pi, \pi)$ and $\lambda_1, \lambda_2 > 0$. In three dimensions we need three rotation matrices and one diagonal matrix. The factorization can be generalized to higher dimensions.

Definition: **Separability** is achieved in multidimensional settings by taking products of stationary covariances in each dimension. Let $s = (s_1, \ldots, s_k),\ k \leq n,\ s_i \in \mathbb{R}^{n_i}\ n = \sum_i n_i,$, then

$$C(s, s') = C_1(s_1, s_1') \cdots C_k(s_k, s_k')$$

Notice that stationarity of the individual components implies stationarity of the field. Isotropy of the individual components does not imply isotropy of the resulting random field. In fact, separability is a simple way of obtaining anisotropy.

To see that this produces a valid covariance function, consider the two dimensional case. Consider the mean zero process $X(s)$ with covariance function $C_X(s, s') = C_1(s_1, s_1')$ and the mean zero process $Y(s_2)$, independent of X, with $C_Y(s, s') = C_2(s_2, s_2')$. Then define the process $Z(s) = X(s)Y(s)$. We have that

$$\text{cov}(Z(s), Z(s')) = \text{cov}(X(s)Y(s), X(s')Y(s')) =$$

$$\text{cov}(X(s)X(s'))\text{cov}(Y(s)Y(s')) = C_1(s_1, s_1')C_2(s_2, s_2')$$

Thus, the product of the covariances corresponds to the covariance of a valid random field.

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Thus, the product of the covariances corresponds to the covariance of a valid random field.

Separability implies that the correlation between $Z(s_1,s_2)$ and $Z(s'_1,s_2)$ does not depend on s_2, which can be unrealistic in some cases.

More General Covariance Functions

The class of covariance functions is closed under positive sums, limiting and integration.

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One way to obtain general anisotropic correlations is to consider a mapping, say g of S, so that

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BAD IDEA!: Make the correlation dependent on covariates, as it is hard to make sure that the result is positive definite.