AMS 276 – Project 1

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Frailty model

For observation i and measurement j, we assume the hazard at time y_{ij} is given by

$$h(y_{ij}|\mathbf{x}_{ij}, w_i) = h_0(y_{ij}) w_i \exp(\mathbf{x}_{ij}^\top \boldsymbol{\beta}), \quad i = 1, \ldots, n, \quad j = 1, \ldots, m_i$$

The covariates $\mathbf{x}_{ij}$ is a vector of length 2. The first element is the sex (1 for female, 0 for male) and the second is age. We assume the baseline hazard h_0 is a Weibull, i.e. $h_0(t) = \alpha\gamma t^{\alpha-1}$. This yields the following likelihood

$$\begin{aligned}
L(\mathbf{y}|\mathbf{x}, \boldsymbol{\nu}, \mathbf{w}, \alpha, \gamma, \boldsymbol{\beta}) &= \prod_{i=1}^{n}\prod_{j=1}^{m_i} \left[h(y_{ij}|\mathbf{x}_{ij}, w_i)\right]^{\nu_{ij}} \exp\left(-H(y_{ij}|\mathbf{x}_{ij}, w_i)\right) \\
&= \prod_{i=1}^{n}\prod_{j=1}^{m_i} \left[h_0(y_{ij}) w_i \exp(\mathbf{x}_{ij}^\top \boldsymbol{\beta})\right]^{\nu_{ij}} \exp\left(-H_0(y_{ij}) w_i e^{\mathbf{x}_{ij}^\top \boldsymbol{\beta}}\right) \\
&= \prod_{i=1}^{n}\prod_{j=1}^{m_i} \left[\alpha\gamma y_{ij}^{\alpha-1} w_i \exp(\mathbf{x}_{ij}^\top \boldsymbol{\beta})\right]^{\nu_{ij}} \exp\left(-\gamma y_{ij}^{\alpha} w_i e^{\mathbf{x}_{ij}^\top \boldsymbol{\beta}}\right)
\end{aligned}$$

Priors

We assume the following priors

$$\begin{aligned}
w_i|\kappa &\overset{iid}{\sim} Gamma(\kappa^{-1}, \kappa^{-1}) \\
\eta = \kappa^{-1} &\sim Gamma(\phi_1, \phi_2) \\
\boldsymbol{\beta} &\sim Normal(\bar{\boldsymbol{\beta}}, \boldsymbol{\Sigma}) \\
\gamma &\sim Gamma(\rho_1, \rho_2) \\
\alpha &\sim Gamma(a_1, a_2)
\end{aligned}$$

Equivalent, we could let $\kappa \sim InvGamma(\phi_1, \phi_2)$. We take this approach since there were some issues in sampling η because of a very long right tail. We let $\phi_1 = \phi_2 = 0.001$, $\bar{\boldsymbol{\beta}} = \mathbf{0}$, $\boldsymbol{\Sigma} = 10^3 \mathbf{I}_{2\times 2}$, $\rho_1 = \rho_2 = 0.001$, and $a_1 = a_2 = 0.001$.

Full conditionals

The full posterior is

$$\begin{aligned}
\pi(\mathbf{w}, \gamma, \kappa, \boldsymbol{\beta}, \alpha|\cdot) \propto &\left\{\prod_{i=1}^{n}\prod_{j=1}^{m_i} \left[\alpha\gamma y_{ij}^{\alpha-1} w_i \exp(\mathbf{x}_{ij}^\top \boldsymbol{\beta})\right]^{\nu_{ij}} \exp\left(-\gamma y_{ij}^{\alpha} w_i e^{\mathbf{x}_{ij}^\top \boldsymbol{\beta}}\right)\right\} \times \\
&\left\{\prod_{i=1}^{n} \frac{(\kappa^{-1})^{\kappa^{-1}}}{\Gamma(\kappa^{-1})} w_i^{\kappa^{-1}-1} \exp(-w_i \kappa^{-1})\right\} \times \kappa^{-(\phi_1+1)} \exp\left(-\frac{\phi_2}{\kappa}\right) \times \\
&\exp\left(-\frac{1}{2}(\boldsymbol{\beta} - \bar{\boldsymbol{\beta}})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{\beta} - \bar{\boldsymbol{\beta}})\right) \times \gamma^{\rho_1 - 1} \exp\left(-\gamma\rho_2\right) \times \\
&\alpha^{a_1-1} \exp\left(-\alpha a_2\right)
\end{aligned}$$

We now derive the full conditionals for each parameter.

$$
\begin{aligned}
\pi(w_i|\cdot) \quad &\propto \quad \prod_{j=1}^{m_i} w_i^{\nu_{ij}} \exp\left(-\gamma y_{ij}^{\alpha} w_i e^{\mathbf{x}_{ij}^{\top}\boldsymbol{\beta}}\right) \times w_i^{\kappa^{-1}-1} \exp(-w_i \kappa^{-1}) \\
&= \quad w_i^{\sum_{j=1}^{m_i} \nu_{ij}} \exp\left(-w_i \sum_{j=1}^{m_i} \gamma y_{ij}^{\alpha} e^{\mathbf{x}_{ij}^{\top}\boldsymbol{\beta}}\right) \times w_i^{\kappa^{-1}-1} \exp(-w_i \kappa^{-1}) \\
&= \quad w_i^{(\kappa^{-1}+\sum_{j=1}^{m_i} \nu_{ij})-1} \exp\left(-w_i(\kappa^{-1} + \sum_{j=1}^{m_i} \gamma y_{ij}^{\alpha} e^{\mathbf{x}_{ij}^{\top}\boldsymbol{\beta}})\right) \\
\implies \quad & w_i|\cdot \sim Gamma\left(\kappa^{-1} + \sum_{j=1}^{m_i}, \kappa^{-1} + \sum_{j=1}^{m_i} \gamma y_{ij}^{\alpha} e^{\mathbf{x}_{ij}^{\top}\boldsymbol{\beta}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\pi(\gamma|\cdot) \quad &\propto \quad \left\{\prod_{i=1}^{n}\prod_{j=1}^{m_i} \gamma^{\nu_{ij}} \exp\left(-\gamma y_{ij}^{\alpha} w_i e^{\mathbf{x}_{ij}^{\top}\boldsymbol{\beta}}\right)\right\} \times \gamma^{\rho_1-1} \exp(-\gamma\rho_2) \\
&= \quad \gamma^{\sum_{i=1}^{n}\sum_{j=1}^{m_i} \nu_{ij}} \exp\left(-\gamma \sum_{i=1}^{n}\sum_{j=1}^{m_i} y_{ij}^{\alpha} w_i e^{\mathbf{x}_{ij}^{\top}\boldsymbol{\beta}}\right) \times \gamma^{\rho_1-1} \exp(-\gamma\rho_2) \\
&= \quad \gamma^{(\rho_1+\sum_{i=1}^{n}\sum_{j=1}^{m_i} \nu_{ij})-1} \exp\left(-\gamma(\rho_2 + \sum_{i=1}^{n}\sum_{j=1}^{m_i} y_{ij}^{\alpha} w_i e^{\mathbf{x}_{ij}^{\top}\boldsymbol{\beta}})\right) \\
\implies \quad & \gamma|\cdot \sim Gamma\left(\rho_1 + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \nu_{ij}, \rho_2 + \sum_{i=1}^{n}\sum_{j=1}^{m_i} y_{ij}^{\alpha} w_i e^{\mathbf{x}_{ij}^{\top}\boldsymbol{\beta}}\right)
\end{aligned}
$$

$$
\pi(\kappa|\cdot) \quad \propto \quad \left\{\prod_{i=1}^{n} \frac{(\kappa^{-1})^{\kappa^{-1}}}{\Gamma(\kappa^{-1})} w_i^{\kappa^{-1}-1} \exp(-w_i\kappa^{-1})\right\} \times \kappa^{-(\phi_1+1)} \exp\left(-\frac{\phi_2}{\kappa}\right)
$$

$$
\pi(\boldsymbol{\beta}|\cdot) \quad \propto \quad \left\{\prod_{i=1}^{n}\prod_{j=1}^{m_i} \left[\exp(\mathbf{x}_{ij}^{\top}\boldsymbol{\beta})\right]^{\nu_{ij}} \exp\left(-\gamma y_{ij}^{\alpha} w_i e^{\mathbf{x}_{ij}^{\top}\boldsymbol{\beta}}\right)\right\} \times \exp\left(-\frac{1}{2}(\boldsymbol{\beta}-\bar{\boldsymbol{\beta}})^{\top}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}-\bar{\boldsymbol{\beta}})\right)
$$

$$
\pi(\alpha|\cdot) \quad \propto \quad \left\{\prod_{i=1}^{n}\prod_{j=1}^{m_i} \left[\alpha y_{ij}^{\alpha-1}\right]^{\nu_{ij}} \exp\left(-\gamma y_{ij}^{\alpha} w_i e^{\mathbf{x}_{ij}^{\top}\boldsymbol{\beta}}\right)\right\} \times \alpha^{a_1-1} \exp(-\alpha a_2)
$$

We will use Metropolis-Hastings updates for $\boldsymbol{\beta}$, κ, and α, while we can sample directly for each w_i and γ. After a significant burn-in, we retain $200,000$ posterior samples. Trace plots on the posteriors of $\boldsymbol{\beta}$, κ, and α showed no concern of convergence. Acceptance rates for the M-H parameters were around $0.20 - 0.26$, a desirable range. Chains that started at various starting locations all ended up in the same location.

Results and interpretation of parameters

Posterior distributions are given in Figures 1 and 2. A summary for κ, $\boldsymbol{\beta}$, γ, and α are shown in Table 1. My posteriors are very comparable to those given in your analysis.

	mean	sd	0.025%	0.975%
κ	0.600	0.307	0.125	1.324
β_1	-1.937	0.576	-3.147	-0.865
β_2	0.007	0.013	-0.017	0.034
γ	0.017	0.015	0.002	0.057
α	1.234	0.162	0.931	1.578

Table 1: Posterior summaries for some parameters.

κ describes the homogeneity of the cluster frailties. For a new cluster $w^\star$, the predictive mean is 1, and the predictive variance is $\kappa^\star$, where $\kappa^\star$ is the posterior for κ. So for small κ, the clusters are more similar. For large κ, we would see more w_i's that are much different than the rest, but this is not quite the case from our analysis.

The parameters (γ, α) are from the Weibull distribution. Here, they may best be interpreted relative to their role in the hazard function. There is good evidence that $\alpha > 1$, meaning as time goes on, the hazard increases. γ is a scale parameter so no one cares about it.

The parameters (β_1, β_2) can be used to compare survival times for the male and female groups as well as for the age of the subjects. Using the posterior samples of β_1, we compute $E(e^{\beta_1}) = 0.168$. This means that the hazard function for females is 0.168 times that as for the males. There is weak evidence that $\beta_2 > 0$, which is to say that as age increases, the hazard function increases.

We also fit a frequentist model in R:

```
coxph(Surv(time, nu) ~ as.factor(Sex) + Age + frailty(cluster, theta = 1),
data = data.frame(dat))
```

We fix $\kappa = \theta = 1$, which is comparable to our prior that w_i has prior mean 1.

	m.l.e.	s.e.
β_1: Female	-1.93574	0.58022
β_2: Age	0.00832	0.01577

Table 2: Frequentist estimates for β_1 and β_2.

These estimates are very similar to ours. But it isn't Bayesian so it's obviously not as good. Plus, the algorithm broke when I didn't fix κ.

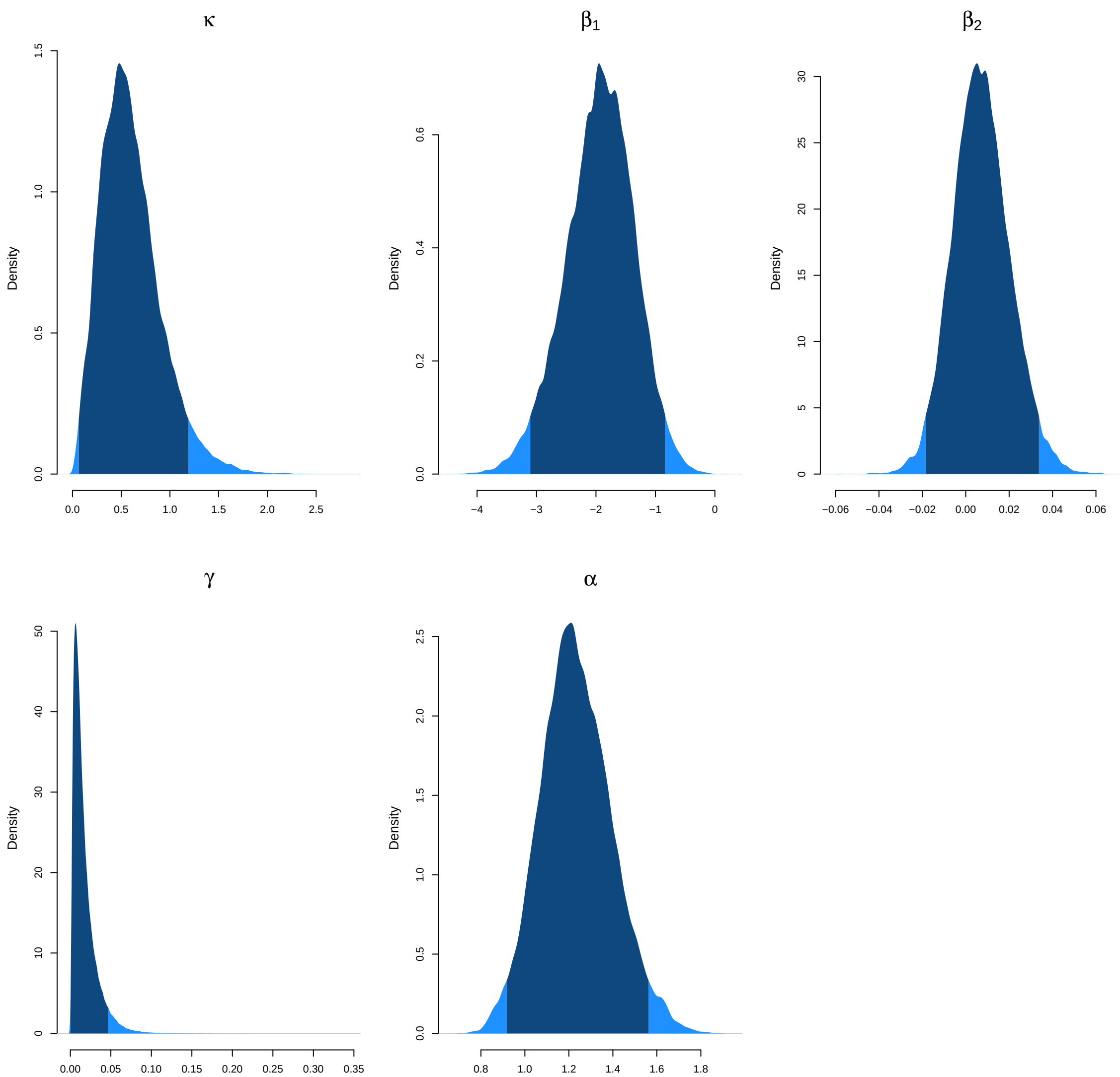

Figure 1: Posterior distributions and summary statistics for κ, $\boldsymbol{\beta}$, γ, and α.

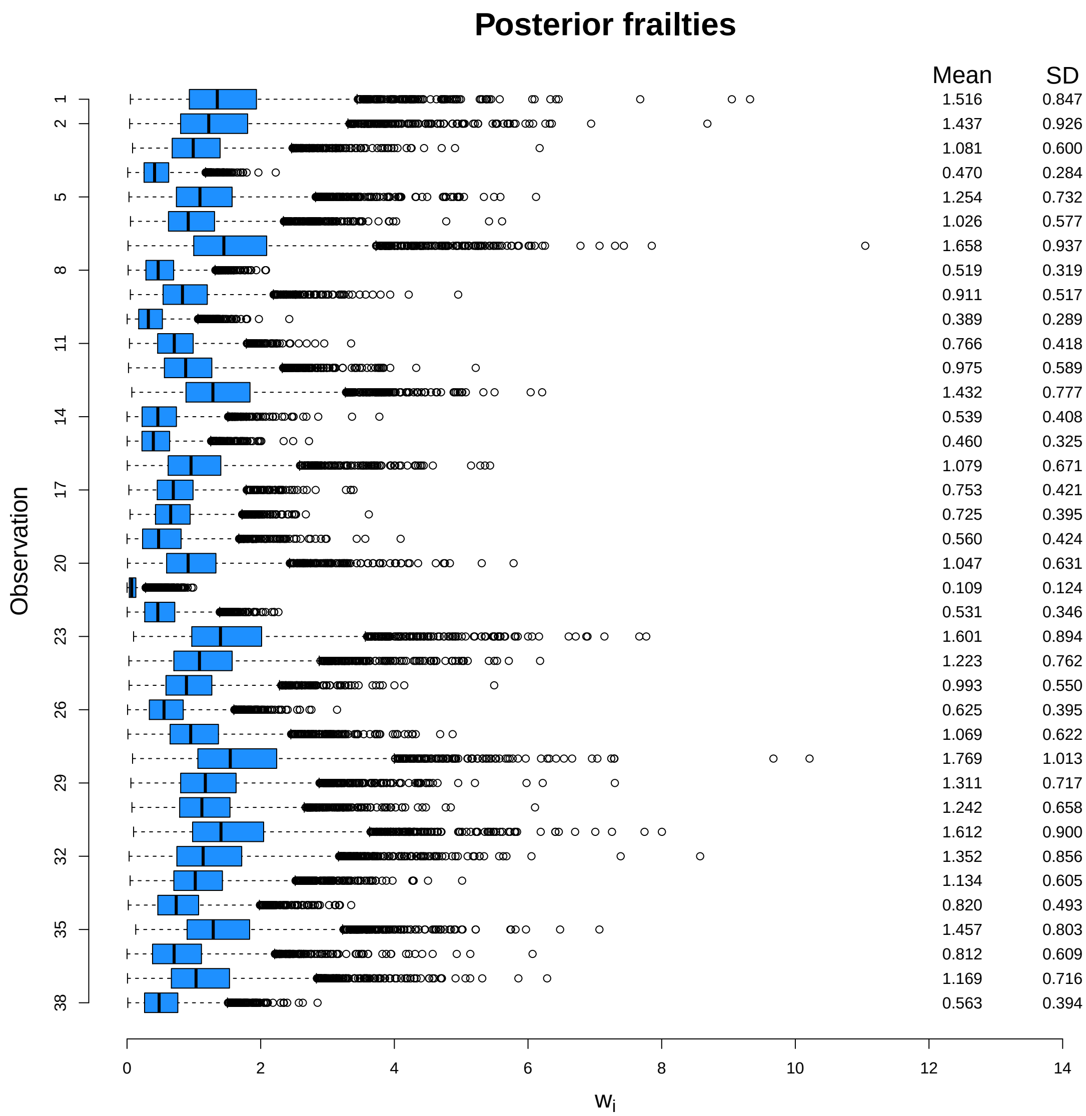

Figure 2: Boxplots for the $n = 38$ frailty parameters w_i with mean and standard deviations.