

AMS 223
 Problems 3.3, 3.4
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Model

We consider a single-component harmonic regression model

$$y_i = a \cos(\omega t_i) + b \sin(\omega t_i) + \epsilon_i, \quad i = 1, \dots, T$$

where $\epsilon_i \stackrel{iid}{\sim} N(0, v)$. Let $\boldsymbol{\beta} = (a, b)^\top$ and $\mathbf{f}_i = (\cos(\omega t_i), \sin(\omega t_i))^\top$, then we have $y_i \sim N(\mathbf{f}_i^\top \boldsymbol{\beta}, v)$. Define $\mathbf{y} = (y_1, \dots, y_T)^\top$ and \mathbf{F} be the $2 \times T$ matrix whose columns are \mathbf{f}_i . Then we have

$$\mathbf{y} | \boldsymbol{\beta}, v, \omega \sim N(\mathbf{F}^\top \boldsymbol{\beta}, v \mathbf{I}).$$

Using the prior $p(\boldsymbol{\beta}, v, \omega) = p(\boldsymbol{\beta}, v | \omega) p(\omega) \propto v^{-1} p(\omega)$, the full conditionals for $\boldsymbol{\beta}$ and v are

$$\begin{aligned} \boldsymbol{\beta} | v, \omega, \mathbf{y} &\sim N(\hat{\boldsymbol{\beta}}, v(\mathbf{F}\mathbf{F}^\top)^{-1}) \\ v | \boldsymbol{\beta}, \omega, \mathbf{y} &\sim IG(T/2, (\mathbf{y} - \mathbf{F}^\top \boldsymbol{\beta})^\top (\mathbf{y} - \mathbf{F}^\top \boldsymbol{\beta})/2), \end{aligned}$$

where $\hat{\boldsymbol{\beta}} = (\mathbf{F}\mathbf{F}^\top)^{-1} \mathbf{F}\mathbf{y}$ is the maximum likelihood estimate for a fixed ω . For this analysis, interest is primarily in the angular frequency ω (and the corresponding period $\lambda = 2\pi/\omega$). The marginal posterior for ω has the form

$$p(\omega | \mathbf{y}) \propto p(\omega) |\mathbf{F}\mathbf{F}^\top|^{-1/2} \left[1 - \hat{\boldsymbol{\beta}}^\top \mathbf{F}\mathbf{F}^\top \hat{\boldsymbol{\beta}} / (\mathbf{y}^\top \mathbf{y}) \right]^{(2-T)/2}.$$

Direct sampling from the marginal posterior of ω is not available. However, since we're only working with one dimension, we can simply discretize the space for ω , compute $p(\omega | \mathbf{y})$ and sample with replacement based on these probabilities.

We show plots of the frequency and period calculated from the marginal posterior as well as based off samples.

P&W 3.3 – Southern Oscillation Index

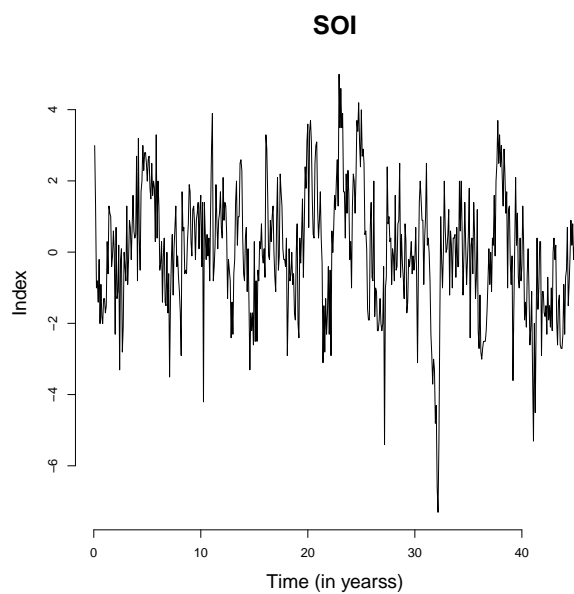


Figure 1: Plot of the SOI data.

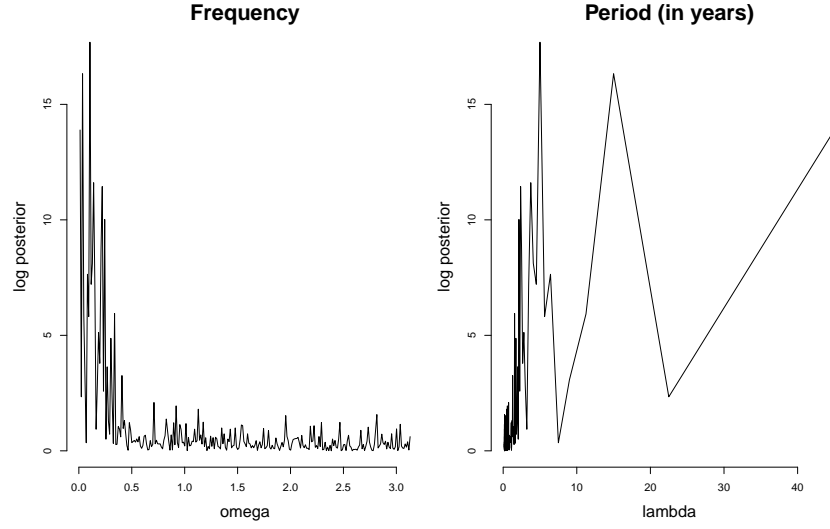


Figure 2: Left: log marginal posterior the angular frequency $p(\omega|\mathbf{y})$. Right: log marginal posterior of the period/wavelength $p(\lambda = 2\pi/\omega|\mathbf{y})$.

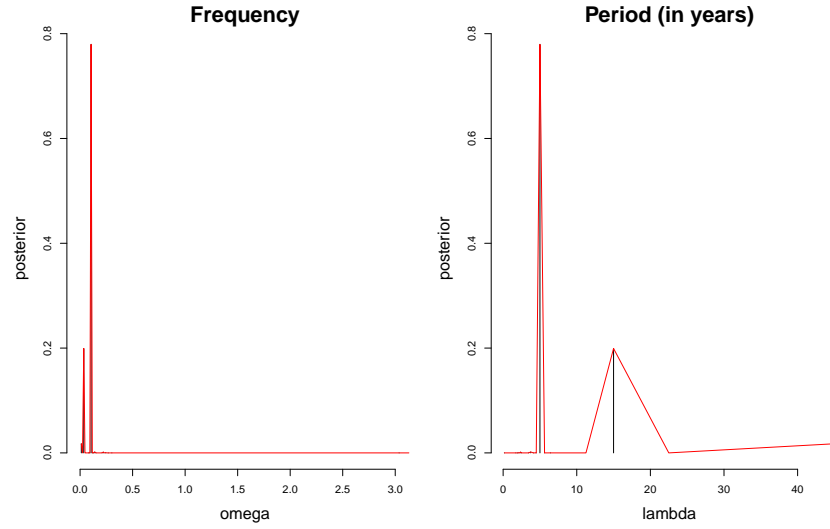


Figure 3: Left: posterior distribution based on samples of $\omega|\mathbf{y}$. Right: posterior distribution the period based on samples. The red lines are the functional value of the marginal distribution.

P&W 3.4 – Luteinizing hormone in blood samples

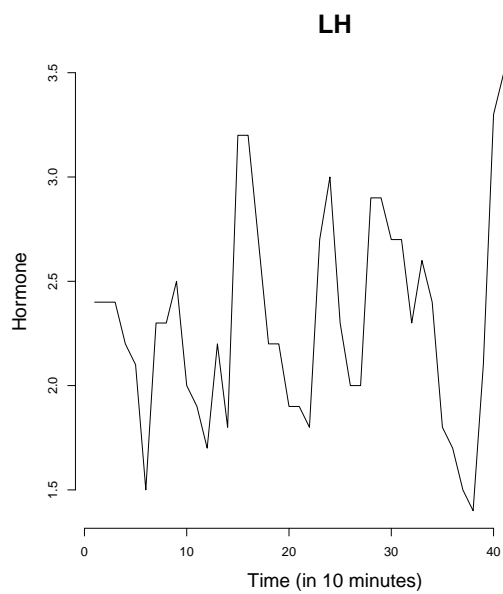


Figure 4: Plot of the lh data.

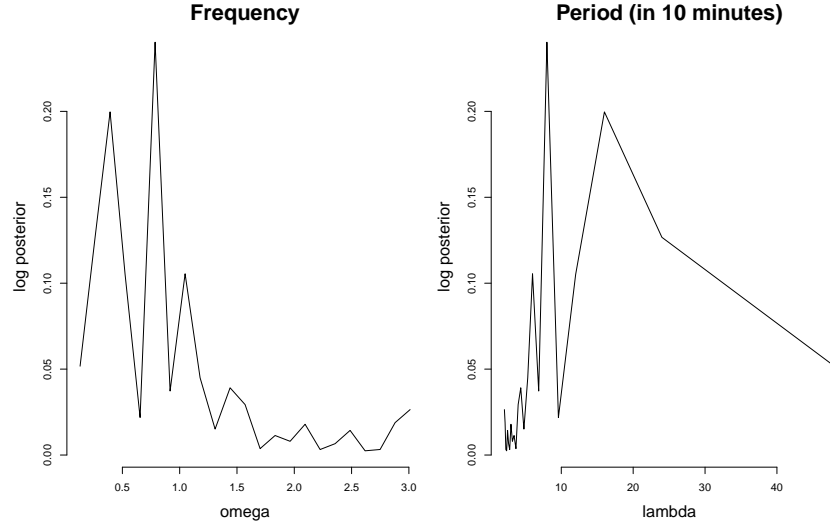


Figure 5: Left: log marginal posterior the angular frequency $p(\omega|\mathbf{y})$. Right: log marginal posterior of the period/wavelength $p(\lambda = 2\pi/\omega|\mathbf{y})$.

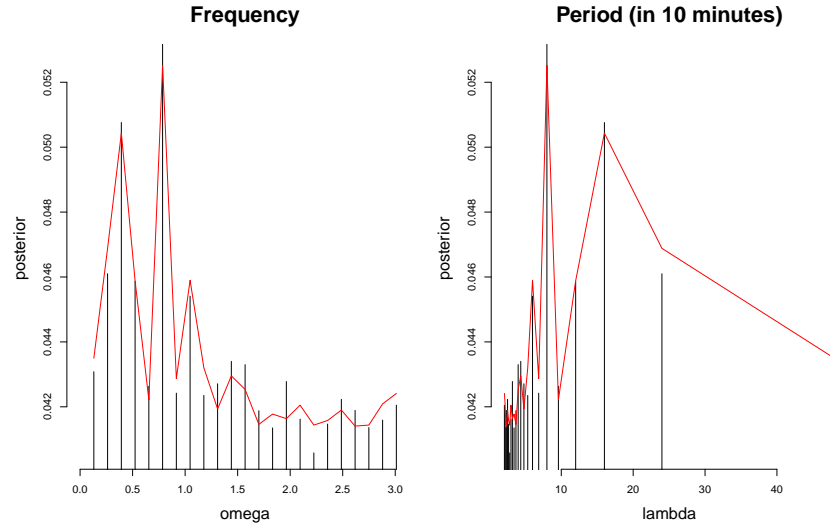


Figure 6: Left: posterior distribution based on samples of $\omega|\mathbf{y}$. Right: posterior distribution the period based on samples. The red lines are the functional value of the marginal distribution.