1. Prove the results about the smoothness of the members of the Matérn family.

We use the theorem that if

$$\frac{d^{2\nu}}{d\tau^{2\nu}}\rho(\tau)$$

exists and is finite at $\tau = 0$, then the random field having $\rho(\tau)$ as its correlation function is ν times differentiable at 0.

Without loss of generality, let $\phi = 1$. The Matérn correlation function is given by

$$\rho(\tau) = \frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}K_\nu(\tau), \qquad \tau \geq 0, \nu > 0.$$

For small τ and $\nu > 0$, $K_\nu(\tau) \approx \Gamma(\nu)2^{\nu-1}\tau^{-\nu}$. Also, $\frac{d}{d\tau}\tau^\nu K_\nu(\tau) = -\tau^\nu K_{\nu-1}(\tau)$ and $K_\nu(\tau) = K_{-\nu}(\tau)$. We will be taking $\tau \to 0$, so we use the approximation for $K_\nu(\tau)$. This leads to the derivative,

$$\begin{aligned}
\frac{d}{d\tau}\rho(\tau) &= -\frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}K_{\nu-1}(\tau) \\
&= \begin{cases} -\frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}K_{\nu-1}(\tau), & \nu - 1 \geq 0 \\ -\frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}K_{1-\nu}(\tau), & \nu - 1 < 0 \end{cases} \\
&\approx \begin{cases} -\frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}\Gamma(\nu-1)2^{\nu-2}\tau^{-\nu+1}, & \nu - 1 \geq 0 \\ -\frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}\Gamma(1-\nu)2^{-\nu}\tau^{\nu-1}, & \nu - 1 < 0 \end{cases} \\
&\approx \begin{cases} -\tau G_1(\nu), & \nu - 1 \geq 0 \\ -\tau^{2\nu-1}G_2(\nu), & \nu - 1 < 0 \end{cases}.
\end{aligned}$$

Therefore,

$$\rho'(0) \begin{cases} = 0, & \nu \geq 1 \\ \in (-\infty, 0), & 1/2 \leq \nu < 1 \\ = -\infty, & 0 < \nu < 1/2 \end{cases}.$$

The second derivative is given by

$$\rho''(\tau) = \begin{cases} \frac{-\tau^{\nu-1}K_{\nu-1}(\tau) + \tau^\nu K_{\nu-2}(\tau)}{2^{\nu-1}\Gamma(\nu)}, & \nu \geq 2 \\ \frac{-\tau^{\nu-1}K_{\nu-1}(\tau) + \tau^\nu K_{2-\nu}(\tau)}{2^{\nu-1}\Gamma(\nu)}, & 1 \leq \nu < 2 \\ \frac{-\tau^{\nu-1}K_{1-\nu}(\tau) + \tau^\nu K_{2-\nu}(\tau)}{2^{\nu-1}\Gamma(\nu)}, & 0 < \nu < 1 \end{cases},$$

and is evaluated at $\tau = 0$ to

$$\rho''(0) \begin{cases} = 0, & \nu \geq 2 \\ \in (-\infty, 0), & 1 \leq \nu < 2 \\ = -\infty, & 0 < \nu < 1 \end{cases}.$$

We have that the second derivative is finite when $\nu \geq 1$, leading to a random field that is one time mean square differentiable. To show this generalizes to $\nu \geq d$, we need to keep taking derivatives of $\rho(\tau)$. I suspect that on each even derivative, the orders of certain Bessel functions are negated when those orders are less than ν. When this happens, the approximation will lead to a term having τ raised to a negative exponent causing the derivative to evaluate to $-\infty$.

2. Use the spectral representation to show that the product of two valid correlation functions is a valid correlation function.

A valid correlation function is the characteristic function of some random variable,

$$\rho(\boldsymbol{\tau}) = E\left[e^{i\boldsymbol{\tau}^\top \mathbf{X}}\right].$$

Suppose we have two valid correlation functions $\rho_1(\tau)$ and $\rho_2(\tau)$ associated with independent random variables $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively. Then the product is written

$$\begin{aligned} \rho(\boldsymbol{\tau}) = \rho_1(\boldsymbol{\tau})\rho_2(\boldsymbol{\tau}) &= E\left[e^{i\boldsymbol{\tau}^\top \mathbf{X}_1}\right] E\left[e^{i\boldsymbol{\tau}^\top \mathbf{X}_2}\right] \\ &= E\left[e^{i\boldsymbol{\tau}^\top \mathbf{X}_1} e^{i\boldsymbol{\tau}^\top \mathbf{X}_2}\right] \\ &= E\left[e^{i\boldsymbol{\tau}^\top (\mathbf{X}_1 + \mathbf{X}_2)}\right], \end{aligned}$$

so ρ is the characteristic function of $\mathbf{X}_1 + \mathbf{X}_2$ and thus the product of two valid correlation functions is a valid correlation function. Note, our assumption of independence for $\mathbf{X}_1$ and $\mathbf{X}_2$ presents no issues. $\mathbf{X}_1$ and $\mathbf{X}_2$ may be dependent, but then we could simply define new independent random variables $\mathbf{Y}_1$ and $\mathbf{Y}_2$ with the same marginal distributions as $\mathbf{X}_1$ and $\mathbf{X}_2$, resulting in the same correlation functions in either case.

3. The spectral density of a correlation in the Matérn family has tails whose thickness depends on the smoothness parameter. Conjecture: the smoothness of the corresponding random field depends on the number of moments of the spectral density. What can you say about this conjecture?

For correlation function

$$\rho(\tau) \propto (a\tau)^\nu K_\nu(a\tau), \qquad \tau \geq 0, \nu > 0, a = 1/\phi > 0,$$

we have the corresponding spectral density

$$f(x) \propto \frac{1}{(1 + (x/a)^2)^{\nu+n/2}},$$

where n is the dimension τ (and x). This density has a form comparable to the t-distribution. Using integration by parts, we calculate the kth moment as

$$\begin{aligned}
E(X^k) &\propto \int x^k (1 + (x/a)^2)^{-(\nu+n/2)} dx \\
&= -\frac{a^2}{2\nu + n - 2} \frac{x^{k-1}}{(1 + (x/a)^2)^{(2\nu+n-2)/2}} \Bigg|_{-\infty}^{\infty} + \int \frac{(k-1)a^2}{2\nu + n - 2} \frac{x^{k-2}}{(1 + (x/a)^2)^{-(2\nu+n-2)/2}} dx.
\end{aligned}$$

The first term (and hence the second term also) will be finite when $2\nu + n - 2 \geq k - 1$, or $\nu \geq (k - n + 1)/2$. In one dimension, $n = 1$, we see that when $\nu \geq k/2$ the kth moment exists. This may be related to the theorem used in the first problem, that we need to have $2d$-differentiable correlation function to have a d-differentiable random field. Here, we need the $2d$th moment to exist, i.e. $\nu \geq d$, to have smoothness.