1. Prove the results about the smoothness of the members of the Matérn family.

We use the theorem that if

$$\frac{d^{2\nu}}{d\tau^{2\nu}}\rho(\tau)$$

exists and is finite at $\tau = 0$, then the random field having $\rho(\tau)$ as its correlation function is ν times differentiable at 0.

Without loss of generality, let $\phi = 1$. The Matérn correlation function is given by

$$\rho(\tau) = \frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}K_\nu(\tau), \qquad \tau \geq 0, \nu > 0.$$

For small τ and $\nu > 0$, $K_\nu(\tau) \approx \Gamma(\nu)2^{\nu-1}\tau^{-\nu}$. Also, $\frac{d}{d\tau}\tau^\nu K_\nu(\tau) = -\tau^\nu K_{\nu-1}(\tau)$ and $K_\nu(\tau) = K_{-\nu}(\tau)$. We will be taking $\tau \to 0$, so we use the approximation for $K_\nu(\tau)$. This leads to the derivative,

$$\begin{aligned}
\frac{d}{d\tau}\rho(\tau) &= -\frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}K_{\nu-1}(\tau) \\
&= \begin{cases} -\frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}K_{\nu-1}(\tau), & \nu-1 \geq 0 \\ -\frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}K_{1-\nu}(\tau), & \nu-1 < 0 \end{cases} \\
&\approx \begin{cases} -\frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}\Gamma(\nu-1)2^{\nu-2}\tau^{-\nu+1}, & \nu-1 \geq 0 \\ -\frac{\tau^\nu}{2^{\nu-1}\Gamma(\nu)}\Gamma(1-\nu)2^{-\nu}\tau^{\nu-1}, & \nu-1 < 0 \end{cases} \\
&\approx \begin{cases} -\tau G_1(\nu), & \nu-1 \geq 0 \\ -\tau^{2\nu-1}G_2(\nu), & \nu-1 < 0 \end{cases}.
\end{aligned}$$

Therefore,

$$\rho'(0) \begin{cases} = 0, & \nu \geq 1 \\ \in (-\infty, 0), & 1/2 \leq \nu < 1 \\ = -\infty, & 0 < \nu < 1/2 \end{cases}.$$

$$\rho''(\tau) = \begin{cases} \frac{-\tau^{\nu-1}K_{\nu-1}(\tau)+\tau^\nu K_{\nu-2}(\tau)}{2^{\nu-1}\Gamma(\nu)}, & \nu \geq 2 \\ \frac{-\tau^{\nu-1}K_{\nu-1}(\tau)+\tau^\nu K_{2-\nu}(\tau)}{2^{\nu-1}\Gamma(\nu)}, & 1 \leq \nu < 2 \\ \frac{-\tau^{\nu-1}K_{1-\nu}(\tau)+\tau^\nu K_{2-\nu}(\tau)}{2^{\nu-1}\Gamma(\nu)}, & 0 < \nu < 1 \end{cases}$$

2. Use the spectral representation to show that the product of two valid correlation functions is a valid correlation function.

A valid correlation function is the characteristic function of some random variable,

$$\rho(\boldsymbol{\tau}) = E\left[e^{i\boldsymbol{\tau}^\top \mathbf{X}}\right].$$

Suppose we have two valid correlation functions $\rho_1(\tau)$ and $\rho_2(\tau)$ associated with independent random variables $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively. Then the product is written

$$\begin{aligned}\rho(\boldsymbol{\tau}) = \rho_1(\boldsymbol{\tau})\rho_2(\boldsymbol{\tau}) &= E\left[e^{i\boldsymbol{\tau}^\top \mathbf{X}_1}\right] E\left[e^{i\boldsymbol{\tau}^\top \mathbf{X}_2}\right] \\ &= E\left[e^{i\boldsymbol{\tau}^\top \mathbf{X}_1} e^{i\boldsymbol{\tau}^\top \mathbf{X}_2}\right] \\ &= E\left[e^{i\boldsymbol{\tau}^\top (\mathbf{X}_1+\mathbf{X}_2)}\right],\end{aligned}$$

so ρ is the characteristic function of $\mathbf{X}_1 + \mathbf{X}_2$ and thus the product of two valid correlation functions is a valid correlation function. Note, our assumption of independence for $\mathbf{X}_1$ and $\mathbf{X}_2$ presents no issues. $\mathbf{X}_1$ and $\mathbf{X}_2$ may be dependent, but then we could simply define new independent random variables $\mathbf{Y}_1$ and $\mathbf{Y}_2$ with the same marginal distributions as $\mathbf{X}_1$ and $\mathbf{X}_2$, resulting in the same correlation functions in either case.

3. The spectral density of a correlation in the Matérn family has tails whose thickness depends on the smoothness parameter. Conjecture: the smoothness of the corresponding random field depends on the number of moments of the spectral density. What can you say about this conjecture?

For correlation function

$$\rho(\tau) \propto (a\tau)^\nu K_\nu(a\tau), \qquad \tau \geq 0, \nu > 0, a = 1/\phi > 0,$$

we have the corresponding spectral density

$$f(x) \propto \frac{1}{(1 + (x/a)^2)^{\nu+n/2}},$$

where n is the dimension τ (and x). This density has a form comparable to the t-distribution. We calculate the kth moment as

$$\begin{aligned}E(X^k) &\propto \int x^k (1 + (x/a)^2)^{-(\nu+n/2)} dx \\ &= -\frac{a^2}{2\nu+n-2} \frac{x^{k-1}}{(1+(x/a)^2)^{(2\nu+n-2)/2}}\Bigg|_{-\infty}^{\infty} + \int \frac{(k-1)a^2}{2\nu+n-2} \frac{x^{k-2}}{(1+(x/a)^2)^{-(2\nu+n-2)/2}} dx.\end{aligned}$$

The first term (and hence the second term as well) will be finite when $2\nu + n - 2 > k - 1$, or $\nu > (k - n + 1)/2$. In one dimension, $n = 1$, we see that when $\nu > k/2$, the kth moment exists. This may be related to the theorem used in the first problem, that we need to have $2d$-differentiable correlation function to have a d-differentiable random field. Here, we need the $2d$th moment to exist, so $\nu > d$, to have smoothness.

4. Use the K-L representation to approximate the exponential correlation for range parameter equal to 1. Plot the approximation for several orders and compare to the actual correlation.

5. Repeat for the approximation given on Page 13 of the fifth set of slides.

6. Generate 100 realizations of a univariate Gaussian process with exponential correlation with range parameter 1. Compare the empirically estimated eigenvalues and eigenfunctions to the ones given by the K-L and the approximation on Page 12.