Characterizing the smoothness and differentiability of a random function is key when choosing a family of models that is most suited for a problem.

Since a random field is a collection of random variables, there are technical subtleties in the definition of continuity. Intuitively continuity corresponds to any realization of $X(s)$ being continuous as a function of $s$.

We will consider three different criteria for the continuity of a random field. Of those three, mean square continuity is the most tractable.

Definition: A random field $X$ has **continuous sample paths with probability one** in $B$ if, for every sequence $s_n$ such that $||s_n - s|| \to 0$ as $n \to \infty$, then

$$Pr(\omega : |X(s_n, \omega) - X(s, \omega)| \to 0, \text{ as } n \to \infty, \forall s \in B) = 1$$

This definition implies that there are no discontinuities, with probability one, in the whole domain $B$.

# Almost Sure Continuity

Definition: A random field $X$ is **almost surely continuous** in $B$ if for every sequence $s_n$ such that $||s_n - s|| \to 0$ as $n \to \infty$, then

$$Pr(\omega : |X(s_n, \omega) - X(s, \omega)| \to 0, \text{ as } n \to \infty) = 1 \quad \forall s \in B$$

This definition allows discontinuities in the domain $B$, but the probability of finding a discontinuity at a given location $s$ is zero. Sample path continuity is a stronger condition that almost sure continuity.

Definition: A random field $X$ is **mean square continuous** in $B$ if for every sequence $s_n$ such that $||s_n - s|| \to 0$ as $n \to \infty$, then

$$E(|X(s_n) - X(s)|^2) \to 0, \text{ as } n \to \infty \quad \forall s \in B$$

provided the expectation exists.

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provided the expectation exists.

Mean square continuity of Gaussian processes is controlled by the smoothness of the covariance function. For a stationary random field all we need is to look at one point.

Theorem: Assume that $E(X(s))$ is continuous. Then, a random field $X(s)$ is mean square continuous at $t$ if and only if its covariance function $C(s, s')$ is continuous at $s = s' = t$.

Corollary: A stationary random field $X(s)$ is mean square continuous at $s \in S$ if and only if its correlation function $\rho(h)$ is continuous at 0.

Theorem: Assume that $E(X(s))$ is continuous. Then, a random field $X(s)$ is mean square continuous at $t$ if and only if its covariance function $C(s, s')$ is continuous at $s = s' = t$.

Proof: If $C$ is continuous, then use the identity

$$E|X(s_n) - X(s)|^2 = C(s_n, s_n) - 2C(s_n, s) + C(s, s)$$

If $X$ is mean square integrable, then

$$0 = \lim_{n\to\infty} E|X(s_n) - X(s)|^2 = \lim_{n\to\infty} C(s_n, s_n) - 2C(s_n, s) + C(s, s)$$

after some manipulations and the use of Cauchy-Schwartz inequality, we get that $\lim_{\to\infty} C(s_n, s_n) = C(s, s)$.

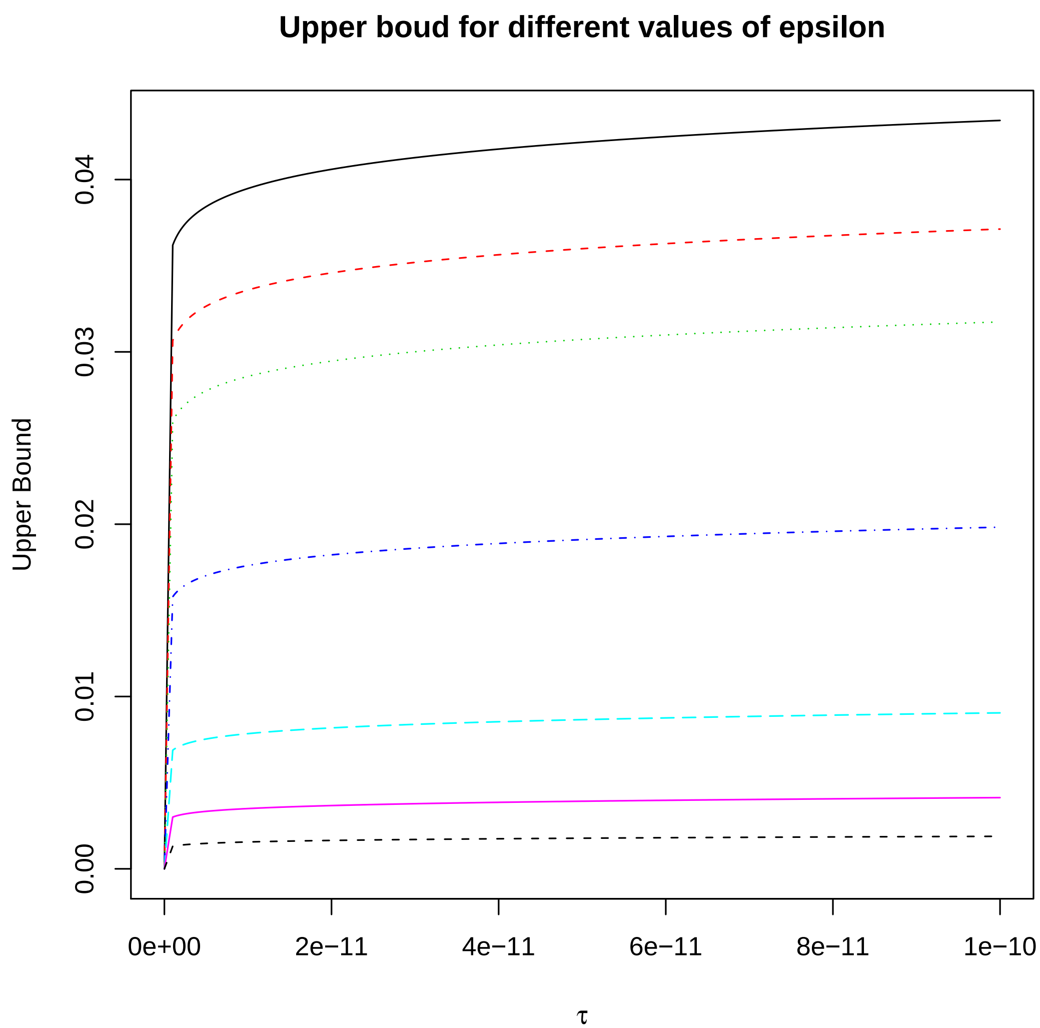
Corollary: A stationary random field $X(s)$ is mean square continuous at $s \in S$ if and only if its correlation function $\rho(h)$ is continuous at 0.

Theorem: Let $X(s)$ be a stationary Gaussian random field with a continuous correlation function. Then, if for some finite $c > 0$ and some $\varepsilon > 0$,

$$1 - \rho(\tau) \leq \frac{c}{|\log \tau|^{1+\varepsilon}}$$

for all $\tau < 1$, then the random field $X(s)$ will have continuous sample paths with probability one.

For different values of $\varepsilon$ we observe that the upper bound has pretty large values, even for very small values of $\tau$. So we can expect the bound to hold for most continuous correlation functions.



Consider a Gaussian random field $X(s)$. Then the associated gradient field is given by

$$\frac{\partial X(s)}{\partial s_i} = \lim_{\Delta \to 0} \frac{X(s - \Delta e_i, \omega) - X(s, \omega)}{\Delta}$$

provided the limit exists. Here $e_i$ is a unit vector in the $i$-th direction.

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The resulting gradient field is a vector of Gaussian processes, as the differential operator is linear. Furthermore:

$$\text{If} \quad E(X(s)) = m(s), \quad \text{then} \quad E\left(\frac{\partial X(s)}{\partial s_i}\right) = \frac{\partial m(s)}{\partial s_i}$$

provided $m(s)$ is differentiable.

If $C(s, s') = \text{cov}(X(s), X(s'))$, then, if $C(s, s')$ is differentiable in $s$ and $s'$,

$$\text{cov}\left(X(s), \frac{\partial X(s')}{\partial s'_i}\right) = \frac{\partial C(s, s')}{\partial s'_i}$$

and

$$\text{cov}\left(\frac{\partial X(s)}{\partial s_i}, \frac{\partial X(s')}{\partial s'_j}\right) = \frac{\partial^2 C(s, s')}{\partial s_i \partial s'_j}$$

We can obtain a new "average" random field by integrating an existing random field as

$$Y(t) = \int_B X(s)w(t,s)ds$$

where $B \subset S$ and $w(t,s)$ is a weight function. When $\dim(S) = 1$, this integral is defined as the following limit in the mean square sense.

$$\lim_{\max|s_k - s_{k-1}| \to 0} \sum_{k=1}^{n} X(s'_k)w(t, s'_k)(s_k - s_{k-1})$$

where $B = [a,b]$, $s_i$ defines a partition of size $n$ of $B$, $s_{k-1} \leq s'_k \leq s_k$, and $s_0 = a,\ s_n = b$.

The expectation of $Y$ is

$$E(Y(t)) = \int_B m(s)w(t,s)ds$$

The covariance is

$$\operatorname{cov}(Y(t), Y(s)) = \int_B \int_B C(v,u)w(t,u)w(s,v)dvdu$$

Also

$$\frac{\partial Y(t)}{\partial t_i} = \int_B X(s)\frac{\partial w(t,s)}{\partial t_i}ds$$

So that the smoothness of the integrated process can be controlled by the weight function $w$.

Additional smoothness of the random field depends on the differentiability of the covariance function. $C(\cdot,\cdot)$ needs to be twice differentiable for $X(s)$ to be differentiable.

Theorem: Let $\nu = \sum_i \nu_i$, then, if the derivative

$$\frac{\partial^{2\nu} C(s,t)}{\partial s_1^{\nu_1} \cdots \partial s_n^{\nu_n} \partial t_1^{\nu_1} \cdots \partial t_n^{\nu_n}} \tag{1}$$

exists and is finite for all $i = 1, \ldots n$ at $(s,s)$, $X(s)$ is $\nu$ times differentiable at $s$. Moreover, the covariance function of

$$\frac{\partial^{\nu} X(s)}{\partial s_1^{\nu_1} \cdots \partial s_n^{\nu_n}}$$

is given by (1).

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Consider the power exponential correlation $\rho(\tau) = \exp\{-\tau^\nu\}$, with $0 < \nu \leq 2$. Then $\rho'(\tau) = -\nu\tau^{\nu-1}\exp\{-\tau^\nu\}$. So that

$$\rho'(0) = \begin{cases} -\infty & 0 < \nu < 1 \\ -1 & \nu = 1 \\ 0 & 1 < \nu \leq 2 \end{cases}$$

So there is no differentiability for $0 < \nu < 1$.

The second derivative is $\rho''(\tau) = \nu\tau^{\nu-2}(1-\nu+\nu\tau^{\nu})\exp\{-\tau^{\nu}\}$ and we have that

$$\lim_{\tau\to 0}\rho''(0) = \begin{cases} -\infty & 1<\nu<2 \\ -2 & \nu = 2 \end{cases}$$

which implies that the only case where the resulting process is differentiable is $\nu = 2$. In such case the process is infinitely smooth. This lack of continuity in the smoothness of the family of power exponential correlation is undesirable.

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For small values of $\tau$ we have that $K_\nu(\tau) \approx \Gamma(\nu) 2^{\nu-1} \tau^{-\nu}$. Thus

$$\lim_{\tau \to 0} \frac{1}{\Gamma(\nu) 2^{\nu-1}} \tau^\nu K_\nu(\tau) = 1, \quad \nu > 0$$

so that continuity holds. For the derivatives we have that

$$\frac{d}{d\tau}(\tau^\nu K_\nu(\tau)) = -\tau^\nu K_{\nu-1}(\tau)$$

Using the results on the previous slide we have that:

- for $0 < \nu < 1/2$, $\rho'(0) = -\infty$. So these cases correspond to extremely erratic processes.
- for $1/2 \leq \nu < 1$ $\rho'(0) \in (-\infty, 0)$. Which produces a range of erratic processes.
- for $\nu \geq d$ we have that $\rho^{(2d-1)}(0) = 0$ and $\rho^{(2d)}(0) \in (-\infty, 0)$. This implies that the process is $d$ times mean square differentiable.