Definition. (Constructive approach) A stochastic process W in $C_+(S)$ with constant $\omega_0 > 0$ is a simple Pareto process if $W(s) = YV(s)$, for all $s \in S$, for some Y and $V = \{V(s)\}_{s\in S}$ satisfying:

a) $V \in C_+(S)$ is a stochastic process satisfying $\sup_{s\in S} V(s) = \omega_0$ almost surely, $E[V(s)] > 0$ for all $s \in S$,

b) Y is a standard Pareto random variable, $P(Y > y) = y^{-1}, y > 1$,

c) Y and V are independent.

See Ferreira and de Haan (2014) for other variants of the definition.

Coefficient of asymptotic dependence. For random variables X_1 and X_2 having common marginal distribution F, let

$$\chi_{12} = \lim_{z\to z_+} P(X_1 > z | X_2 > z)$$

where z_+ is the (possibly infinite) right end-point.

For $s_1, s_2 \in S$ and $x > \omega_0$, then for $i = 1, 2$,

$$\begin{aligned}
P(W(s_i) > x) &= P(YV(s_i) > x)\\
&= P\left(Y > \frac{x}{V(s_i)}\right)\\
&= E_{Y,V(s_i)}\left[\mathbb{1}\left(Y > \frac{x}{V(s_i)}\right)\right]\\
&= E_{V(s_i)}\left\{E_{Y|V(s_i)}\left[\mathbb{1}\left(Y > \frac{x}{V(s_i)}\right)\middle|V(s_i)\right]\right\}\\
&= E_{V(s_i)}\left\{P\left(Y > \frac{x}{V(s_i)}\middle|V(s_i)\right)\right\}\\
&= E_{V(s_i)}\left\{\frac{V(s_i)}{x}\right\} = \frac{E[V(s_i)]}{x}
\end{aligned}$$

and

$$\begin{aligned}
P(W(s_1) > x, W(s_2) > x) &= P(YV(s_1) > x, YV(s_2) > x)\\
&= P\left(Y > \frac{x}{V(s_1)}, Y > \frac{x}{V(s_2)}\right)\\
&= P\left(Y > \frac{x}{V(s_1)} \vee \frac{x}{V(s_2)}\right)\\
&= P\left(Y > x\left(\frac{1}{V(s_1)} \vee \frac{1}{V(s_2)}\right)\right)\\
&= P\left(Y > x\left(\frac{1}{V(s_1) \wedge V(s_2)}\right)\right)
\end{aligned}$$

$$= \frac{E[V(s_1) \wedge V(s_2)]}{x}$$

by using arguments similar in the first set of equations. Then for a simple Pareto process at points $s_1, s_2 \in S$, we have

$$\begin{aligned}
\chi_{12}^W &= \lim_{x\to\infty} P(W(s_1) > x | W(s_2) > x) \\
&= \lim_{x\to\infty} \frac{xE[V(s_1) \wedge V(s_2)]}{xE[V(s_2)]} \\
&= \frac{E[V(s_1) \wedge V(s_2)]}{E[V(s_2)]}
\end{aligned}$$

This is from Ferreira and de Haan (2014), but with some clarity (for the dummies of the universe) as to how they got to the expectations.

NOTE: This has some issues since the marginals of $W(s_1)$ and $W(s_2)$ need not be the same. Unless conditioning on them both being above a certain value (i.e. 1 or ω_0), they have different marginals (except the degenerate case when $W(s_1) = W(s_2)$ a.s. where they are both standard paretos. We did not explicitly account for this conditioning in the above calculation.

Better χ. From before, we can find the distribution function for $W_i \equiv W(s_i)$, as

$$F_{W_i}(w) = 1 - \frac{E(V_i)}{w}$$

where $V_i \equiv V(s_i)$. Using this fact, we can standardize W_i to be uniform and compute χ in this way

$$\begin{aligned}
\chi_1 &= \lim_{u\to 1} P(F_{W_1}(W_1) > u | F_{W_2}(W_2) > u) \\
&= \lim_{u\to 1} P\left(1 - \frac{E(V_1)}{W_1} > u \middle| 1 - \frac{E(V_2)}{W_2} > u\right) \\
&= \lim_{u\to 1} P\left(W_1 > \frac{E(V_1)}{1-u} \middle| W_2 > \frac{E(V_2)}{1-u}\right) \\
&= \lim_{u\to 1} \frac{P\left(W_1 > \frac{E(V_1)}{1-u}, W_2 > \frac{E(V_2)}{1-u}\right)}{P\left(W_2 > \frac{E(V_2)}{1-u}\right)} \\
&= \lim_{u\to 1} \frac{P\left(YV_1 > \frac{E(V_1)}{1-u}, YV_2 > \frac{E(V_2)}{1-u}\right)}{1-u} \\
&= \lim_{u\to 1} \frac{1}{1-u} P\left(Y > \frac{E(V_1)}{(1-u)V_1}, Y > \frac{E(V_2)}{(1-u)V_2}\right) \\
&= \lim_{u\to 1} \frac{1}{1-u} P\left(Y > \frac{E(V_1)}{(1-u)V_1} \vee \frac{E(V_2)}{(1-u)V_2}\right) \\
&= \lim_{u\to 1} \frac{1}{1-u} P\left(Y > \frac{1}{1-u}\left(\frac{E(V_1)}{V_1} \vee \frac{E(V_2)}{V_2}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{u \to 1} \frac{1}{1-u}(1-u) E\left[\left(\frac{E(V_1)}{V_1} \vee \frac{E(V_2)}{V_2}\right)^{-1}\right] \\
&= E\left(\frac{V_1}{E(V_1)} \wedge \frac{V_2}{E(V_2)}\right)
\end{aligned}
$$

which is nice and symmetric. Concerns: Are we correctly using the fact that Y is standard Pareto? If we don't explicitly state that $x > 1$:

$$
P(Y > x) = \min(1, 1/x)
$$

Another concern goes back to solving the distribution function for W_i. Since V_i is a random variable, we required conditioning on it using iterated expectations to get the desired result. However, it is possible that $P(V_i == 0) > 0$, and so dividing by V_i may pose an issue. This may be circumvented perhaps by conditioning on $V_i > 0$, but how would this affect the calculation?

For $x > 0$,

$$
\begin{aligned}
P(W_i > x) &= P(YV_i > x) \\
&= P(YV_i > x | V_i = 0) P(V_i = 0) + P(YV_i > x | V_i > 0) P(V_i > 0) \\
&= P(0 > x | V_i = 0) P(V_i = 0) + P(YV_i > x | V_i > 0) P(V_i > 0) \\
&= P(Y > x/V_i | V_i = 0) P(V_i = 0) + P(YV_i > x | V_i > 0) P(V_i > 0) \\
&= P(Y > \infty) P(V_i = 0) + P(YV_i > x | V_i > 0) P(V_i > 0) \\
&= 0 \times P(V_i = 0) + P(YV_i > x | V_i > 0) P(V_i > 0) \\
&= P(YV_i > x | V_i > 0) P(V_i > 0) \\
&\stackrel{?}{=} P(V_i > 0) \times \frac{E(V_i)}{x}
\end{aligned}
$$

It is certainly possible in practice to observe a $V_i = 0$. For the joint distribution we may need to condition on $V_1 \wedge V_2 > 0$.

$$
\begin{aligned}
\chi_2 &= \lim_{u \to 1} P(F_{W_1}(W_1) > u | F_{W_2}(W_2) > u) \\
&= \lim_{u \to 1} P\left(1 - \frac{E(V_1)}{W_1} > u \middle| 1 - \frac{E(V_2)}{W_2} > u\right) \\
&= \lim_{u \to 1} P\left(W_1 > \frac{E(V_1)}{1-u} \middle| W_2 > \frac{E(V_2)}{1-u}\right) \\
&= \lim_{u \to 1} \frac{P\left(W_1 > \frac{E(V_1)}{1-u}, W_2 > \frac{E(V_2)}{1-u}\right)}{P\left(W_2 > \frac{E(V_2)}{1-u}\right)} \\
&= \lim_{u \to 1} \frac{P\left(YV_1 > \frac{E(V_1)}{1-u}, YV_2 > \frac{E(V_2)}{1-u}\right)}{1-u}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{u\to 1}\left[\frac{P\left(YV_1 > \frac{E(V_1)}{1-u}, YV_2 > \frac{E(V_2)}{1-u}\middle| V_1 \wedge V_2 = 0\right) P\left(V_1 \wedge V_2 = 0\right)}{1-u}\right.\\
&\qquad \left. + \frac{P\left(YV_1 > \frac{E(V_1)}{1-u}, YV_2 > \frac{E(V_2)}{1-u}\middle| V_1 \wedge V_2 > 0\right) P\left(V_1 \wedge V_2 > 0\right)}{1-u}\right]\\
&= \lim_{u\to 1}\left[\frac{0 \times P\left(V_1 \wedge V_2 = 0\right)}{1-u}\right.\\
&\qquad \left. + \frac{P\left(YV_1 > \frac{E(V_1)}{1-u}, YV_2 > \frac{E(V_2)}{1-u}\middle| V_1 \wedge V_2 > 0\right) P\left(V_1 \wedge V_2 > 0\right)}{1-u}\right]\\
&= \lim_{u\to 1}\frac{P\left(YV_1 > \frac{E(V_1)}{1-u}, YV_2 > \frac{E(V_2)}{1-u}\middle| V_1 \wedge V_2 > 0\right) P\left(V_1 \wedge V_2 > 0\right)}{1-u}\\
&= \lim_{u\to 1}\frac{1}{1-u}P\left(Y > \frac{E(V_1)}{(1-u)V_1}, Y > \frac{E(V_2)}{(1-u)V_2}\middle| V_1 \wedge V_2 > 0\right) P\left(V_1 \wedge V_2 > 0\right)\\
&= \lim_{u\to 1}\frac{1}{1-u}P\left(Y > \frac{E(V_1)}{(1-u)V_1} \vee \frac{E(V_2)}{(1-u)V_2}\middle| V_1 \wedge V_2 > 0\right) P\left(V_1 \wedge V_2 > 0\right)\\
&= \lim_{u\to 1}\frac{1}{1-u}P\left(Y > \frac{1}{1-u}\left(\frac{E(V_1)}{V_1} \vee \frac{E(V_2)}{V_2}\right)\middle| V_1 \wedge V_2 > 0\right) P\left(V_1 \wedge V_2 > 0\right)\\
&= \lim_{u\to 1}\frac{1}{1-u}(1-u)E\left[\left(\frac{E(V_1)}{V_1} \vee \frac{E(V_2)}{V_2}\right)^{-1}\middle| V_1 \wedge V_2 > 0\right] P\left(V_1 \wedge V_2 > 0\right)\\
&= E\left(\frac{V_1}{E(V_1)} \wedge \frac{V_2}{E(V_2)}\middle| V_1 \wedge V_2 > 0\right) P\left(V_1 \wedge V_2 > 0\right)
\end{aligned}
$$

(Note the expectations within the larger expectation (i.e. $E(V_1)$ and $E(V_2)$) are not condition on $V_1 \wedge V_2 > 0$.) When conditioning $V_1 \wedge V_2 > 0$, there was no difference when computing estimates for χ_1 and χ_2. This is perhaps due to the following.

Suppose we have samples $X_1, \ldots, X_n$ having p.d.f. $f(x) = p\delta_0(x) + (1-p)g(x)$ where $g(x)$ is defined on $(0, \infty)$. Then

$$
\begin{aligned}
E(X) &= \int_0^\infty xp\delta_0(x)dx + \int_0^\infty x(1-p)g(x)dx\\
&= 0 + (1-p)E_g(X)\\
&= P(X > 0)E_g(X)
\end{aligned}
$$

We could compute the sample mean in two ways:

$$
\bar{X}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i
$$

and

$$\bar{X}_2 = \left[\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(x_i > 0)\right]\left[\frac{1}{|A|}\sum_{i \in A} x_i\right]$$

where $A = \{x : x > 0\}$ and $|A|$ is the size of A. The two multiplicands in $\bar{X}_2$ are estimates for $P(X > 0)$ and $E_g(X)$, respectively. Of course, $\bar{X}_1$ and $\bar{X}_2$ are equal, they just use the data differently:

$$\begin{aligned}
\bar{X}_2 &= \left[\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(x_i > 0)\right]\left[\frac{1}{|A|}\sum_{i \in A} x_i\right] \\
&= \left[\frac{1}{|A|}\sum_{i=1}^{n} \mathbb{1}(x_i > 0)\right]\left[\frac{1}{n}\sum_{i \in A} x_i\right] \\
&= \left[\frac{1}{|A|}\sum_{i=1}^{n} \mathbb{1}(x_i > 0)\right]\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] \\
&= [1] \times \left[\bar{X}_1\right] = \bar{X}_1
\end{aligned}$$

I suspect we have a similar situation with χ_1 and χ_2.