# Biostat 730 discussion note on 9/20

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## 1 Walk through of class note 4 page 5

Two rules we are applying for doing this algebra:

- 1. $exp(x) * exp(y) = exp(x + y)$
- 2. constant can be added or removed freely under exponential, $exp(x) = exp(x+a) * \frac{1}{exp(a)} \propto exp(x+a)$ where a is a constant

On slide 5 we have:

$$
\begin{aligned}
p(\mu|\boldsymbol{y}) &\propto p(\mu)p(\boldsymbol{y}|\mu) && (1)\\
&\propto \quad exp\Big(\frac{-1}{2s_{\mu_0}^2}(\mu - m_0)^2\Big) * exp\Big(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\Big) && (2)\\
&\propto \quad exp\Big(\frac{-1}{2s_{\mu_0}^2}(\mu - m_0)^2\Big) * exp\Big(\frac{-n}{2\sigma^2}(\bar{y} - \mu)^2\Big) && (3)\\
&\propto \quad exp\Big(\frac{1}{2}f(\mu)\Big) && (4)
\end{aligned}
$$

Question is how to get from equation (2) to equation (3)? The following shows the steps:

$$
\begin{aligned}
exp\Big(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\Big) &= exp\Big(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i^2 - 2\mu y_i + \mu^2)\Big) && (5)\\
&= exp\Big(\frac{-1}{2\sigma^2}(\sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\mu^2)\Big) && (6)\\
&= exp\Big(\frac{-1}{2\sigma^2}(n\mu^2 - 2\mu * n\bar{y})\Big)(as \sum_{i=1}^{n} y_i = n\bar{y}; \sum_{i=1}^{n} y_i^2 \text{ is constant with respect to } \mu, \text{ see rule 2}) && (7)\\
&\propto exp\Big(\frac{-n}{2\sigma^2}(\mu^2 - 2\mu\bar{y} + \bar{y}^2)\Big)(\bar{y}^2 \text{ is constant, applying rule 2}) && (8)\\
&= exp\Big(\frac{-n}{2\sigma^2}(\bar{y} - \mu)^2\Big)
\end{aligned}
$$

To move from (3) to (4):

$$
\begin{aligned}
(3) \qquad &= exp\Big( - \tfrac{1}{2}(\tfrac{1}{s_{\mu_0}^2}(\mu^2 - 2m_0\mu + m_0^2) + \tfrac{n}{\sigma^2}(\bar{y}^2 - 2\mu\bar{y} + \mu^2))\Big)(\text{rule 1}) && (9)\\
&= exp\Big( - \tfrac{1}{2}(\tfrac{1}{s_{\mu_0}^2}(\mu^2 - 2m_0\mu) + \tfrac{n}{\sigma^2}(\mu^2 - 2\bar{y}\mu))\Big)(m_0^2 \text{ and } \bar{y}^2 \text{ are constants}) && (10)
\end{aligned}
$$

so we have $f(\mu) = \frac{1}{s^2_{\mu_0}}(\mu^2 - 2m_0\mu) + \frac{n}{\sigma^2}(\mu^2 - 2\bar{y}\mu)$

## 2 Q1: how to get the distribution of Z given $z_i = y_i - r_i$

The most direct approach is to replace $z_i = y_i - r_i$ with $y_i$. Given $y_i|\mu \sim N(\mu + r_i, \sigma^2)$ and $r_i$ being fixed (meaning subtracting $r_i$ from $y_i$ doesn't change the variance of the distribution),

$$z_i|\mu = y_i - r_i|\mu \sim N(\mu + r_i - r_i, \sigma^2) = N(\mu, \sigma^2) \tag{11}$$

- ***Note***: The following steps show how to derive the distribution of $z_i$ theoretically. This is beyond our focus of this course, and you **DON'T** need to show this in the homework or exam. This is for if you are really interested in the details about variable transformation.

  We are given:

$$p(y_i|\mu, \sigma^2) \propto exp(-\frac{1}{2\sigma^2}(y_i - \theta_i)^2) = exp(-\frac{1}{2\sigma^2}(y_i - (\mu + r_i))^2) \tag{12}$$

$$\text{Let } z_i = y_i - r_i, \text{ then } y_i = g^{-1}(z_i) = z_i + r_i \tag{13}$$

  where $r_i$ is fixed.

  according to the rule on transformation of random variables;

$$\begin{aligned} p(z_i|\mu, \sigma^2) &= p_y(g^{-1}(z_i))|\tfrac{d}{dz_i}g^{-1}(z_i)| && (14)\\ &\propto exp(-\tfrac{1}{2\sigma^2}(g^{-1}(z_i) - (\mu + r_i))^2) * |\tfrac{d}{dz_i}(z_i - r_i)| && (15)\\ &= exp(-\tfrac{1}{2\sigma^2}((z_i + r_i) - (\mu + r_i))^2) * 1 && (16)\\ &= exp(-\tfrac{1}{2\sigma^2}(z_i - \mu)^2) && (17) \end{aligned}$$

  so $z_i|\mu, \sigma^2 \sim N(\mu, \sigma^2)$

## 3 Q2 extra credit part: question about how to get sampling distribution of MLE and Bayes estimators

You don't need a real sample to generate the sampling distributions. Instead, we get the parametric form of the distribution (normal) from considering the expression for the estimators.