Covariance *Inflation*: A Geodesic Alternative to Shrinkage

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Abstract

Traditional covariance shrinkage estimates pull an empirical covariance matrix Σ towards a lower information anchor (often λI), reducing estimation error at the cost of *suppressing* correlation. This note proposes the opposite: geodesic **covariance *inflation***—a controlled movement of Σ towards a *perfectly-correlated* limit. Intended not as a consistent estimator but merely a device for use in the context of portfolio construction, we propose that instead of discounting correlation, we exaggerate it in a geometrically natural way, yielding a family of positive–definite matrices indexed by an inflation parameter $\gamma \in [0, 1]$.

1 Motivation

Consider d risk factors with sample covariance $\Sigma \in \mathbb{S}^d_{++}$. Shrinkage estimators

$$\Sigma_{\text{shrunk}} = (1 - \alpha)\Sigma + \alpha\lambda I, \qquad 0 \leq \alpha \leq 1,$$

attenuate off-diagonal entries. Inflation, by contrast, keeps the original variances but *boosts* correlation, moving towards the 'all-ones' correlation matrix $\mathbf{1}\mathbf{1}^\top$ (or a nearly perfect value $\rho \lesssim 1$ to avoid singularity). Such an estimator may be attractive when

- small samples *under*-estimate correlation strength;
- risk management prefers a conservative, higher-dependency view;[1]
- one wishes to bound diversification benefits from above.

2 The Perfectly Correlated Target

Given $\Sigma = \mathrm{Cov}(X)$, let

$$D = \mathrm{diag}(\Sigma), \qquad \sigma_i = \sqrt{D_{ii}}, \quad i = 1, \ldots, d.$$

For a chosen $\rho \in (0, 1)$, define the *inflation target*

$$\Sigma^\star_\rho \;=\; (\sigma_i\sigma_j\rho)_{1\leq i,j\leq d}, \qquad \Sigma^\star_\rho \in \mathbb{S}^d_{++}. \tag{1}$$

Diagonal elements are reset to the original variances (ρ drops out when $i = j$), so the target preserves scale while maximising (almost) mutual dependence.

[1] For instance, stress testing highly-leveraged portfolios.

3 Geodesic Interpolation

The SPD cone equipped with the affine-invariant Riemannian metric has a closed-form geodesic: for $C_0, C_1 \in \mathbb{S}^d_{++}$

$$C_\gamma \;=\; C_0^{1/2}\,(C_0^{-1/2} C_1 C_0^{-1/2})^\gamma\, C_0^{1/2}, \qquad 0 \le \gamma \le 1. \tag{2}$$

Setting $C_0 = \Sigma$ and $C_1 = \Sigma^\star_\rho$ produces the inflated covariance $\Sigma(\gamma) = C_\gamma$ that *monotonically* increases every pairwise correlation while keeping eigenvalues positive.

Interpretation.

- $\gamma = 0$ returns the sample covariance.
- $\gamma = 1$ yields the perfectly (or ρ-) correlated matrix.
- Intermediate γ constitutes "inflation".

4 Python Reference Implementation

Listing 1 reproduces the essential functions used to compute (2) and (1) in practice. The helper `nearest_positive_def` ensures numerical SPDness whenever rounding errors push eigenvalues negative. Code is provided in the `randomcov` Python package [2].

Listing 1: Minimal implementation of covariance inflation.

```python
import numpy as np
from scipy.linalg import eigh
from randomcov.covutil.nearestposdef import nearest_positive_def

def geodesic_interpolation(start_cov, end_cov, gamma):
    # Symmetrise and make SPD
    def make_spd(M):
        vals, vecs = np.linalg.eigh((M+M.T)/2)
        vals[vals < 1e-8] = 1e-8
        return vecs @ np.diag(vals) @ vecs.T
    C0 = make_spd(start_cov)
    C1 = make_spd(end_cov)

    # Geodesic under affine-invariant metric
    vals, vecs = eigh(C0)
    C0_sqrt   = vecs @ np.diag(np.sqrt(vals)) @ vecs.T
    C0_isqrt  = vecs @ np.diag(1/np.sqrt(vals)) @ vecs.T
    middle    = C0_isqrt @ C1 @ C0_isqrt
    m_vals, m_vecs = eigh(middle)
    middle_pow = m_vecs @ np.diag(m_vals**gamma) @ m_vecs.T
    return C0_sqrt @ middle_pow @ C0_sqrt

def covariance_with_perfect_correlation(cov, rho=0.99):
    sd = np.sqrt(np.diag(cov))
    inflated = np.outer(sd, sd) * rho
    np.fill_diagonal(inflated, np.diag(cov))
    return nearest_positive_def(inflated)
```

5 Status and research ideas

Covariance inflation preserves eigen-structure while steering estimates towards a high-dependency, risk-conservative regime. Its Riemannian construction maintains positive-definiteness and coordinate invariance, making it a practical counterpart to classical shrinkage when post-shrinkage *under*-estimation of correlation can be a concern.

Inflation is not suitable for all purposes and is evidently not a good estimator of true correlation - clearly the opposite as made clear in [8], [6]. Its potential use in asset management arises only from an accident of sorts: portfolios are approximately invariant to global increases in correlation among all assets. It was included, almost by accident, in some benchmarking of methods for minimum variance portfolio construction under uncertain covariance information and performed quite well, although this has not at the time of writing progressed to a convincing study (see also [1] if this direction is of interest). In asset management, the poor out-of-sample performance of optimized portfolios can sometimes be arrested quite simply with more careful selection of covariance, as discussed in [3] for example, so having additional tools at our disposal or a new direction to explore may help.

There are other uses, potentially, where stress-testing might demand we increase correlation between assets, as with [7]. Furthermore, it may find use when we desire a mixture of covariance matrices, say in the design of robust methods, as with [4]. Further reading should also include [5].

Keywords: covariance estimation, Riemannian geometry, stress testing, risk management.

References

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