AN ANALYTIC APPROACH TO ORNSTEIN-UHLENBECK PROCESSES WITH FLUCTUATING PARAMETERS AND APPLICATIONS IN THE MODELING OF FIXED INCOME SECURITIES

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Abstract

We generalize the Vasicek model for bond prices, in which the instantaneous rate of interest follows an Ornstein-Uhlenbeck process with positive mean, to accommodate randomly fluctuating mean and volatility. Fluctuations are introduced by allowing the Ornstein-Uhlenbeck parameters governing the long run mean and instantaneous variance of this primary interest rate process to depend on a secondary mean reverting stochastic process. The time scale over which oscillations in the secondary process occur is assumed to be short compared with the longer mean reversion time of the primary process and this separation of time scales is exploited in our asymptotic approach to the solution of certain partial differential equations relevant to pricing and estimation.

Both continuous and discretely valued secondary stochastic processes are treated. The former gives rise to a family of stochastic volatility models for which we supply explicit formulas correcting the original Vasicek model - the corrections being parsimonious and robust to the exact specification of volatility process. The latter finds interpretation in the regime switching literature and here too, for the first time in this context, accurate closed form bond prices are derived. The domain of applicability of our results, bolstered by numerical tests of the pricing formulas, is discussed with particular emphasis placed on the modeling of default intensities and credit derivatives. We supply evidence for bimodality and predictability in US corporate spreads consistent with regime switches, and evidence of rapid fluctuations in US short term interest rates consistent with our treatment of stochastic volatility.

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Contents

Chapter 1

Main Results and Applications

In Vasicek's seminal paper "An Equilibrium characterization of the term structure" 1977 [57], a stochastic evolution of the instantaneous rate of interest arises endogenously in an economy where investors share a common plausible preference for certainty over risk. This **short rate** follows an Ornstein-Uhlenbeck process

$$dx_t = \kappa \left(\theta - x_t\right) ds + \sigma dW_t. \tag{1.1}$$

with positive mean. Vasicek calculated quantities essential for the modeling of bonds and related securities, notably the expectation of the exponentiated negative integrated x_t process:

$$B(t, x; T) = \mathbb{E}^{x,t}\left[e^{-\int_t^T x_s ds}\right], \tag{1.2}$$

whose analytic solution implies a **term structure of interest rates**, or equivalently, prices for **discount bonds** (those paying no coupons) of all maturities. Given the mathematical similarity between such "discounting" and randomly killed diffusions it comes as no surprise that (1.2) is also useful in the modeling of financial products sensitive to credit events (such as the sudden default on obligations by the issuer of a corporate bond).

The primary focus of our work is a calculation of (1.2) performed in more general settings where the process x_t is similar to that used by Vasicek, but accommodates randomly fluctuating parameters θ and σ. We model fluctuations by introducing a process x_t satisfying

$$dx_t = \kappa(\theta(y_t) - x_t)ds + \sigma(y_t)dW_t \tag{1.3}$$

where the parameters $\theta(y_t)$ and $\sigma(y_t)$ are themselves stochastic and depend on a second process y_t. Our work branches at this point according to whether the y_t process is continuous or discretely valued.

In the first half of the thesis we treat a class of processes where y_t is Ornstein-Uhlenbeck and correlated with x_t. The joint process is described:

$$\begin{aligned} dx_t &= \kappa(\theta - x_t)dt + f(y_t)dW_t \\ dy_t &= \alpha(m - y_t)dt + \beta\left(\rho dW_t + \sqrt{1-\rho^2}\,dZ_t\right), \end{aligned}$$

We permit the diffusion parameter $\sigma(y_t) = f(y_t)$ to take almost any form, yielding a broad class of **stochastic volatility** term structure models.

In the latter half of our work we examine a process

$$dx_t = \kappa(\theta(y_t) - x_t)ds + \sigma(y_t)dW_t$$

where y_t is a Markov process taking two values with generator:

$$\lambda Q = -\lambda \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

The y_t process switches between the two states with instantaneous probability proportional to λ. The immediate application is a **regime switching** short rate model (generalizing Hansen and Poulsen, 2000 [37]) for which we supply accurate analytic bond pricing formulas. We also find empirical evidence supporting the use of regime switching models in credit derivative applications, and in this context illustrate a genuine practical advantage of analytic formulas over numerical methods.

1.1 Exploiting Differences in Time Scales

Our approach follows Sircar, Fouque, Papanicolaou [31], [28], [33], [32], [12], [29], [50] closely in chapter (2) and in spirit throughout. When we describe the parameters $\theta(y_t)$ and $\sigma(y_t)$ as *fluctuating* we envisage a functional of a stable *rapidly oscillating* ergodic process y_t. In the case where y_t is also an Ornstein-Uhlenbeck process this literally means that the mean reversion parameter α is large *compared with the mean reversion parameter κ of the x_t process.* In the case where y_t is a two state process we envisage state switching occurring relatively rapidly ($1/\lambda$ is small) *compared with the typical mean reversion speed* (proportional to $1/\kappa$) *of the x_t process.* These notions are not precise, and are complicated by the fact that we are interested in expectations of processes integrated over finite time horizons: a third time scale.

1.1.1 Limitations

As stated, the parameter restrictions, $1/\alpha << 1/\kappa$ and $1/\lambda << 1/\kappa$, may appear to represent, a priori, a rather harsh limitation on the scope of our results and the ability of the process (1.3) to match empirical time series. However, whilst these assumptions *do* restrict the scope of application, the limitations turn out to be far less severe than our mathematical ansatz would suggest. Of course, where the approach breaks down, existing well-developed methods of modeling should be used instead. For example, we may downgrade the primary process x_t and upgrade the secondary process y_t in keeping with the philosophy of a more egalitarian multifactor model.

Therefore, we introduce the separation of time scales as a very deliberate modeling and pricing technique. We are well aware that the burden of demonstrating a significant domain of applicability within finance lies squarely on our shoulders and this is particularly true for the family of stochastic volatility models we introduce. In this stochastic volatility context our corrections capture the first order effects of introducing fluctuations in volatility (or more precisely $\sigma(y_t)$) which are correlated with movements in the primary process x_t. To demonstrate the relevance of the rapid fluctuation analysis use variogram techniques in section 2.3.2 to demonstrate, in a very direct way, that there is indeed a fast mean reverting component in the volatility of time series proxying the short rate. This work was assisted by Knut Solna and draws on techniques in Fouque, Papanicolaou, Sircar, Solna [33].

In contrast to fast mean reverting volatility, regime switches in financial time series (in particular interest rates) have attracted considerable attention (for example [36],[5], [4], [48], [37]) and represent a well established modeling tool. In section 3.5 we add to this literature by examining a data set consisting of thousands of time series of US corporate spreads. We find new indirect evidence for regime switches using tests for multimodality seldom applied in the financial literature. We do not claim that these regime switches occur rapidly in nature.

The possibility of rapid regime switches in nature or as a modeling device (discussion in section 3.1) is interesting, but, as we now explain, by no means crucial to establishing a broad domain of application for the results. The fluctuations appearing in the regime switching process are treated somewhat differently from the fluctuations in the stochastic volatility model. We are able to push the bond price formulas we derive well beyond the limitations implied by the assumption $1/\lambda << 1/\kappa$ (corresponding to relatively fast regime switches) and show that the rapid oscillation asymptotics are useful even when $1/\lambda \approx 1/\kappa$ (corresponding to moderately frequent regime switches). For example, in section 3.4.5 we test numerically a model where the mean short rate θ switches dramatically between five per cent and fifteen percent, the diffusive parameter is also stochastic and a large short rate mean reversion parameter $\kappa = 2$ is used in contrast to a relatively small regime switching intensity $\lambda = 0.5$. The regime switches occur every two years on average, *the same or slightly longer* time scale than the typical mean reversion time of the short rate process. Our analytic results stand up quite well nonetheless, accurately estimating two key quantities. The first quantity is the difference between the simulated bond price one obtains when starting y_t in the high interest rate regime and the bond price one obtains when starting y_t in the low interest regime. The second is the regime independent correction to bond prices accounting for the error one would make if one used the standard Vasicek model, an average θ of ten percent and a root mean square average σ.

1.1.2 Advantages

To summarize, our enthusiasm for the asymptotic techniques exploiting differences of time scales comes from two sources. The first source is direct empirical evidence for rapid mean reversion in short rate volatility supplied in section 2.3.2. The second is the success of a collection of numerical tests of the regime switching model demonstrating a broader domain of applicability than than might be initially supposed.

The assumed separation of scales enables us to peer into the pricing partial differential equations and solve them (or approximately solve them) in an asymptotic sense. In particular we are able to derive what we believe to be the first analytic formula for bond prices in the regime switching literature. The representation for the bond price given in section 3.4.3 isolates the effect of regime switches into two informative components. The "symmetric" adjustment to bond prices accounts for the effect of fluctuations of the parameters about their means and is *regime independent.* The "antisymmetric" correction to bond prices can be interpreted as a regime delta, or the jump in bond prices which occurs as we move from one regime to another.

The ability to isolate the effects of fluctuations in x_t independent of the initial value of y_t is a key advantage of the asymptotic approach and suggests methods for stable pricing and estimation. In chapter 2, we illustrate that the first order effects of stochastic volatility can be captured in a parsimonious manner and robust to the exact specification of both the y_t process and the functional relationship between y_t and the volatility of x_t. The advantage of the asymptotic approach is not so much that we can approximately solve the PDE analytically - which is in itself interesting - but that we can, as far as first order effects are concerned, eliminate the unobservable y_t process from direct consideration in pricing and estimation. Stable and rapid estimation of the primary process x_t, avoiding the need to estimate jointly the observable x_t and unobservable y_t, is a promising possibility for which we provide tools in section 2.2.7.

In this manner a two factor model (x_t, y_t) can be considered to be a one factor model x_t corrected for the effect of y_t fluctuations, perhaps the closest thing in pricing and estimation with diffusion processes to a free lunch. Put another way, we demonstrate a class of one dimensional diffusions which are, in the context of pricing and estimation of fixed income securities, tractable enough to complement candidates such as Ornstein-Uhlenbeck, CIR and more generally affine square root processes in an "almost affine setting" for which there is much existing technology.

1.2 Theoretical Results

Our pricing formulas build on those for the original Vasicek model, discussed in more detail in appendix A. Equation (1.2)

$$B_{vas}(t,x;T) = \mathbb{E}^{x,t}\left[e^{-\int_t^T x_s ds}\right],$$

with an appropriate interpretation of parameters, represents the price of a bond and has analytic solution

$$B_{vas}(x;t,T) = \exp\left\{-\left[\tilde{\theta}(T-t) - (\tilde{\theta}-x)\frac{1-e^{-\kappa(T-t)}}{\kappa} + \frac{\sigma^2}{4\kappa^3}\left(1-e^{-\kappa(T-t)}\right)^2\right]\right\}, \tag{1.4}$$

where $\tilde{\theta} = \theta - \sigma^2/2\kappa^2$. The bond price is the exponential of a constant plus linear function of the **state variable** x_t, placing it in the **affine** term structure models as first classified by Brown and Schwartz [10], developed by Duffie, Kan, Singleton and others (including [19],[20]) and recently characterized cleanly by Filipovic[1]. Bond option prices can also be computed analytically, as in appendix A.

Our bond pricing results for Vasicek models with fluctuating parameters take the form of asymptotic series in a parameter which corresponds to the rapidity of fluctuations of the secondary process y_t about its mean. The first term in the expansion has the same functional form as the bond price solution for the original Vasicek model. This term represents a somewhat naive attempt to approximate the bond price by averaging y_t-dependent parameters over the invariant distribution of the secondary process y_t.

We treat higher order terms in the bond price asymptotic series as corrections to the standard constant parameter Vasicek model. These corrections arise from a

[1]Paper shortly to be released: http://www.math.ethz.ch/ filipo/

variety of complex interactions and convexity effects, many of which are ignored by averaging θ and σ^2.

1.2.1 Processes with Fluctuating Volatility, General Results

In chapter 2 we assume a separation of time scales in the pricing PDE

$$\begin{aligned}\tfrac{\partial B}{\partial t} + \tfrac{1}{2}f(y)^2\tfrac{\partial^2 B}{\partial x^2} + (\kappa(x_\infty - x) - \lambda_1(y)f(y))\tfrac{\partial B}{\partial x} - xB \\ + \beta\rho f(y)\tfrac{\partial^2 B}{\partial x \partial y} + \tfrac{1}{2}\beta^2\tfrac{\partial^2 B}{\partial y^2} \\ + \left(\alpha(m-y) - \beta\left[\rho\lambda_1(y) - \lambda_2(y)\sqrt{1-\rho^2}\right]\right)\tfrac{\partial B}{\partial y} = 0,\end{aligned}$$

for the price at time t of a T maturity discount bond $B(t,x,y;T)$ when $x_t = x$ and $y_t = y$. Namely, we assume $\epsilon := 1/\alpha$ is small. The pricing PDE is well understood and, with different boundary conditions, applies to securities other than bonds. We sketch a derivation of the bond pricing PDE in appendix B.2, including a plausibility argument for the appearance of functions $\lambda_1(y_t)$ and $\lambda_2(y_t)$ representing market prices of risk.

Bond Pricing

We show that in the rapid oscillation limit (as $\epsilon \to 0$ and fluctuations in the secondary process y_t become feverish) bond prices $B(t,x;T)$ take the same form as $(A.8)$,

$$B(t,x;T) \to B_{vas}(t,x;T;\kappa;\overline{\theta},\sqrt{\overline{\sigma^2}})$$

with averaged parameters

$$\begin{aligned}\overline{\sigma^2} &= \langle f^2(y)\rangle_{OU} \\ \overline{\theta} &= x_\infty - \frac{1}{\kappa}\langle\lambda_1(y)f(y)\rangle_{OU}\end{aligned}$$

where the notation $\langle\cdot\rangle_{OU}$ indicates an average with respect to the invariant distribution of the y_t process. For example

$$\langle f^2(y)\rangle_{OU} = \frac{1}{\sqrt{2\pi\nu^2}}\int_{-\infty}^{\infty} f^2(y)e^{-y^2/2\nu}dy$$

where $\nu^2 = \beta^2/2\alpha$. This limiting formula is a plausible approximation to the bond price but fails to take into account the interplay between volatility uncertainty, short rate uncertainty, correlation between the short rate and volatility, and market premia demanded for the taking of associated risks. The real effects of fluctuating volatility are considerably more complicated than those predicted by naive averaging.

Exploiting the assumption that volatility reverts rapidly to its mean we derive explicit formulas which capture those first order effects of stochastic volatility that the "averaging" limiting case misses. We show that in fact the bond price is better approximated by multiplying the fast oscillation limit $B_{vas}(t,x,T;\kappa,\theta,\sigma)$ by a maturity dependent correction factor:

$$B(t,x,y,T) \approx B_{vas}(t,x,T;\kappa,\theta,\sigma)\,(1+\epsilon V(T-t))$$

where

$$\begin{aligned} V(u) &= \frac{V_3}{\kappa^3}\left(u-\tau(u;\kappa)-\frac{1}{2}\kappa\tau(u;\kappa)^2-\frac{1}{3}\kappa^2\tau(u;\kappa)^3\right) \\ &- \frac{V_2}{\kappa^2}\left(u-\tau(u;\kappa)-\frac{1}{2}\kappa\tau(u;\kappa)^2\right)+\frac{V_1}{\kappa}\left(u-\tau(u;\kappa)\right) \\ \tau(u;\kappa) &= \frac{1-e^{-\kappa u}}{\kappa}, \end{aligned}$$

and the constants V_1, V_2, and V_3 are small. We supply formulas for calculating these numbers when the function $f(y)$ is known.

Bond Option Pricing

Similar results are obtained for the rapid fluctuation limit when we analyze bond option prices. The correction to account for stochastic volatility for a European call on a discount bond can be expressed

$$\begin{aligned} C(t,x;T,T_0) &= C_{vas}(t,x;T;T_0) + V(T-T_0)B_{vas}(t,x;T)N(h_1) \\ &\quad - \int_t^{T_0} \mathbb{E}\left[e^{-\int_t^u \bar{x}_s ds} \mathcal{A} C_{vas}(u,\bar{x}_u) | \bar{x}_t = x\right] du \,, \end{aligned}$$

where all notation carries over from section 1.2.1 and the operator $\mathcal{A}$ can be explicitly calculated once the function $f(y_t)$ is known. Here $C_{vas}(t,x;T,T_0)$ is the price of a European call on a discount bond as calculated using the constant parameter Vasicek model. The bond matures at time T. The owner of the option may call at time T_0.

1.2.2 Results for a Particular Stochastic Volatility Model

We apply the theory to a specific model with risk neutral short rate x_t and volatility driving process y_t obeying

$$\begin{aligned} dx_t &= \left(\kappa(x_\infty - x_t) - \tilde{\lambda}_1 e^{y_t}\right) dt + e^{y_t} dW_t \\ dy_t &= \left(\frac{1}{\tilde{\epsilon}}(m - y_t) - \tilde{\lambda}_1 \frac{\nu}{2}\rho - \tilde{\lambda}_2 \frac{\nu}{2}\sqrt{1-\rho^2}\right) dt + \frac{\nu}{\sqrt{2\tilde{\epsilon}}}\left(\rho dW_t + \sqrt{1-\rho^2} dZ_t\right) \end{aligned}$$

The parameters with tildes arise from a specification of risk preferences and preempt a transformation of parameters in section 2.2.5 which cleans up the analysis somewhat. The choice of the exponential function, taking the role of $f(y_t)$ in the general theory, is made for tractability and to exaggerate excursions of the y_t process. The scale separation is characterized by the size of $\tilde{\epsilon}$, assumed small. We call this the exponential Ornstein-Uhlenbeck stochastic volatility (**ExpOU SV**) model. The bond price correction can now be given very explicitly because the averaged parameters

$$\begin{aligned} \sigma &= e^{m+\nu^2} \\ \theta &= x_\infty + \frac{\sigma\tilde{\lambda}_1}{\kappa} e^{-\nu^2/2} \end{aligned}$$

are easily calculated from integrals with respect to the (gaussian) invariant distribution of y_t. The group parameters

$$\begin{aligned}
\tilde{V}_1 &= -\frac{\tilde{\lambda}_1}{\sqrt{2}}\left(\rho\tilde{\lambda}_1 + \sqrt{1-\rho^2}\tilde{\lambda}_2\right) e^{m+\frac{1}{2}\nu^2} \\
\tilde{V}_2 &= -\frac{1}{\sqrt{2}}\left(\rho\tilde{\lambda}_1 + \sqrt{1-\rho^2}\tilde{\lambda}_2\right)\sigma^2 - \frac{e^{3m}\rho\tilde{\lambda}_1\sqrt{2}}{\nu}\left(e^{\nu^2} - e^{\frac{1}{2}\nu^2}\right) \\
\tilde{V}_3 &= -\frac{e^{3m}\rho}{\sqrt{2}}\left(e^{9\nu^2/2} - e^{5\nu^2/2}\right).
\end{aligned}$$

are also available in closed form and enable us to write down the maturity dependent function

$$\tilde{V}(t) = \tilde{V}^{(1)}\int_0^t \tau(s;\kappa)ds - \tilde{V}^{(2)}\int_0^t \tau^2(s;\kappa)ds + \tilde{V}^{(3)}\int_0^t \tau^3(s;\kappa)ds$$

as an analytic formula. Again $\tau(s;\kappa) = \frac{1-e^{-\kappa t}}{\kappa}$ and the integrals $\int_0^t \tau^n(s;\kappa)ds$ present no difficulty. The corrected bond price

$$B(t,x,T) = \left(1 - \sqrt{\bar{\epsilon}}\nu\tilde{V}(T-t)\right) B_{vas}(x,t,T;\kappa,\theta,\sigma).$$

is thus obtained.

Comparison with Jump Diffusions

In section 2.2.6 we illustrate one use of the new analytic bond prices. We add exponentially distributed jumps to the Exponential Ornstein-Uhlenbeck Stochastic Volatility model. We hypothesize that excursions in y_t, exaggerated by the exponential function and governing the short time horizon variance of x_t, would have similar effects on pricing as the addition of small jumps to the x_t process. A clean expansion of the yield

$$y(t,x;T) = -\frac{\log B(t,x;T)}{T-t}$$

on discount bonds maturing at time T can be achieved using the convenient function $\tau(u;\kappa) = \frac{1-e^{-\kappa u}}{\kappa}$:

$$y(t,x;T) \approx y_0 + y_1\frac{\tau(T-t)}{T-t} + y_2\frac{\tau^2(T-t)}{T-t} + y_3\frac{\tau^3(T-t)}{T-t} + y_4\frac{\tau^4(T-t)}{T-t}.$$

For small maturities $\tau(T-t;\kappa) \approx T-t$. For large $T-t$, $\tau(T-t;\kappa) \to 1/\kappa$. The expansion becomes less informative as $T-t$ tends to infinity, but for maturities less than several multiples of the characteristic mean reversion time $1/\kappa$ the series illuminates the contributions from jumps and stochastic volatility. The coefficients y_k can be written in terms of transformed parameters

$$\Omega = \{\kappa, \theta, \sigma, \xi, \lambda_1, \lambda_2, \epsilon, \nu, \mu\}$$

where ν, μ parameterize jumps and $\kappa, \theta, \sigma, \xi, \lambda_1, \lambda_2$ are related to the original model parameters for the process (1.5) in section 2.2.5. The coefficients in the yield expansion are

$$\begin{aligned}
y_0 &= \frac{\kappa\,\theta - \epsilon\,\lambda_1\,(\lambda_1+\lambda_2)\,\sigma}{\kappa} - \left(1/2\,\sigma^2 + \epsilon\,\left(-\left(\lambda_1+\lambda_2\right)\sigma^2 - 2\,\lambda_1\,\sigma^3\left(\xi^{-4}-\xi^{-1}\right)\right)\right)\kappa^{-2} \\
&\quad -\epsilon\,\sigma^3\sqrt{2}\left(\xi^3-\xi^{-1}\right)\kappa^{-3} + \frac{\eta}{1+\kappa} \\
y_1 &= -\frac{\kappa\,\theta - \epsilon\,\lambda_1\,(\lambda_1+\lambda_2)\,\sigma}{\kappa} + \left(1/2\,\sigma^2 + \epsilon\,\left(-\left(\lambda_1+\lambda_2\right)\sigma^2 - 2\,\lambda_1\,\sigma^3\left(\xi^{-4}-\xi^{-1}\right)\right)\right)\kappa^{-2} \\
&\quad +\epsilon\,\sigma^3\sqrt{2}\left(\xi^3-\xi^{-1}\right)\kappa^{-3} - \frac{\eta\,\kappa\,\mu}{\kappa+\mu} \\
y_2 &= 2\,\left(1/2\,\sigma^2 + \epsilon\,\left(-\left(\lambda_1+\lambda_2\right)\sigma^2 - 2\,\lambda_1\,\sigma^3\left(\xi^{-4}-\xi^{-1}\right)\right)\right)\kappa^{-1} \\
&\quad +1/2\,\epsilon\,\sigma^3\sqrt{2}\left(\xi^3-\xi^{-1}\right)\kappa^{-2} + \frac{\eta\,\kappa\,\mu^2}{2\,\kappa+2\,\mu} \\
y_3 &= 1/3\,\epsilon\,\sigma^3\sqrt{2}\left(\xi^3-\xi^{-1}\right)\kappa^{-1} - 1/3\,\frac{\eta\,\kappa\,\mu^3}{\kappa+\mu} \\
y_4 &= 1/4\,\frac{\eta\,\kappa\,\mu^4}{\kappa+\mu}
\end{aligned}$$

Though not terribly intuitive, these expressions indicate, under the assumption of small jumps (small μ), to what degree jumps and bursts of volatility impact the yield curve and moreover to what degree they have similar or overlapping effects. The stochastic volatility corrections and jumps modify the first four coefficients, with the coefficient y_4 being of order μ^3 at most. Put another way, there is significantly more freedom in the parameters when we use *both* small jumps and bursty volatility than we really need. If stochastic volatility corrections are used then the marginal increase in yield curve flexibility obtained by adding small jumps of mean size μ is only of order μ^3. If stochastic volatility corrections are not used then the marginal increase in yield curve flexibility obtained by adding small jumps of mean size μ is of order μ^2.

Likelihood Estimation

In section 2.2.7 we consider the transition probability $\tilde{p}(z_1, y_1, t_1; z_2, y_2, t_2)$ for the process

$$\begin{aligned} dz_t^\varepsilon &= -\kappa z^\varepsilon dt + \sigma e^{y_t^\varepsilon} dZ_t \\ dy_t^\varepsilon &= -\frac{1}{\varepsilon} y_t^\varepsilon dt + \sqrt{2}\frac{\nu}{\sqrt{\varepsilon}}\left(\rho dZ_t + \sqrt{1-\rho^2} dB_t\right) \end{aligned}$$

This is desired for likelihood estimation of the ExpOU SV short rate model. Fixing terminal values z_2, y_2, t_2, reversing time and writing $p(z, y, t) = p(z, y, 0; z_2, y_2, t)$ we examine the backward Kolmogorov equation

$$\left(\frac{\partial}{\partial t} + \mathcal{L}^\varepsilon\right) p(z, y, t) = \left\{\frac{\partial}{\partial t} + \frac{1}{\varepsilon} Q + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right\} p(z, y, t) = 0 \tag{1.5}$$

where

$$\begin{aligned} Q &= \frac{1}{2} 2\nu^2 \frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}, \\ \mathcal{L}_1 &= \sqrt{2}\nu\rho e^{kt+y}\frac{\partial^2}{\partial y \partial z} \\ \mathcal{L}_2 &= \frac{1}{2} e^{2kt+2y}\frac{\partial^2}{\partial z^2}. \end{aligned}$$

We consider the marginal $\bar{p}(z_1, y_1, t_1; z_2, t_2)$ where y_2 is integrated out. In other words, we wish to find the probability density of terminal x_t states given initial knowledge of both x_t and y_t. We show that under the assumption of fast mean reversion this transition probability is, up to first order in ϵ, independent of the initial state of the y_t process. This is an intuitive result. The initial value of y_t is quickly forgotten amidst the rapid fluctuations and therefore has little impact on the probability of arriving at a given x_t value.

Two terms in an asymptotic approximation of the transition density are derived. The first term takes on the same simple gaussian form as the well known transition probability for an Ornstein-Uhlenbeck process with constant parameters. This is easily understood. The rapid fluctuations of y_t are in the limit $\epsilon \to 0$ a device for replacing a constant diffusion parameter σ with one drawn each instant from the unconditional distribution of $\sigma(y_t)$. However this is equal in law to an analogous Ornstein-Uhlenbeck process with constant σ equal to the square root of the average of the squared diffusion parameter $\sigma^2(y_t) = e^{-2y_t}$ over the unconditional distribution of y_t.

The second term in the expansion takes into account effects missed by this averaging trick. Correlation between the x_t and y_t processes becomes important when ϵ is small (but not infinitesimal). We show that the correction required can be deduced from the solution $p_1(z,t)$ to a nonhomogeneous PDE

$$\left\{\frac{\partial}{\partial t} + \langle\mathcal{L}_2\rangle_{OU}\right\} p_1(z,t) \quad = \quad -ae^{3kt}\frac{\partial^3}{\partial z^3}p_0(z,t). \tag{1.6}$$

where $\langle\mathcal{L}_2\rangle_{OU}$ is a y_t averaged operator

$$\langle\mathcal{L}_2\rangle_{OU} = \frac{1}{2}e^{2kt}\langle e^{2y}\rangle_{OU}\frac{\partial^2}{\partial z^2}$$

and

$$p_0(z,t) = \frac{1}{\sqrt{2\pi\frac{1-e^{-2\kappa t}}{\kappa}\bar{\sigma}^2}}\exp\left(\frac{-z^2}{2\frac{1-e^{-2\kappa t}}{\kappa}\bar{\sigma}^2}\right).$$

We show that equation (1.6) can be solved by Duhamel's principle and the expression for $p_1(z,t)$ reduced to a one dimensional quadrature. In this way, the likelihood function for the original ExpOU SV x_t process can be rapidly approximated. We hope to illustrate how this may be used in future work.

1.2.3 Processes with Regime Switching Mean and Volatility

In the latter half of the thesis we move from a continuous y_t process to one with discrete jumps, aligning us with some regime switching literature. Here both θ and σ are allowed to depend on y_t which follows a two state Markov process with Poisson switching times independent of the x_t process. Models in this genre have been treated in the literature and perform well in empirical tests such as those carried out by Ang and Bekaert [4] and Basal and Zhou [5]. Our contribution is to demonstrate, under some restrictive conditions, the existence of accurate analytic approximations to the bond price, and to isolate the terms which depend on knowledge of the initial regime.

These formulas apply only to a certain simple regime switching models, and not to the full variety in the literature. However, the short rate process we consider does generalize the one considered by Hansen and Poulsen 2000 [37]. We assume a risk neutral short rate process

$$dx_t = \kappa\left(\theta(y) - x\right)dt + \sigma(y)dW.$$

whose qualitative features are discussed in section 3.1. Bimodal unconditional distributions, predictability, trending and jump like phenomena can be accommodated depending on the choice of parameters. We also illustrate in 3.1.4 how the likelihood of negative x_t values can, if so desired, be severely reduced without unduly curtailing volatility. This may be interpreted as a delayed response to an objection which has haunted the Vasicek model since its publication. The problem of negative interest rates (and negative spreads) is tempered by the introduction of parameter fluctuations which *increase* parameter uncertainty - a surprising turn of events perhaps but one whose mystery evaporates rather rapidly upon inspection.