Trading Illiquid Goods
Analytic Results and Intuition

Peter Cotton and Andrew Papanicolaou

Disclaimer

Any views presented here are the authors' and may not represent those of their employers past or present.

This material is provided for information purposes only and does not bind J.P. Morgan, New York University or Roar Data in any way. It is not intended as a recommendation or an offer or solicitation for the purchase or sale of any security or financial instrument, or to enter into a transaction involving any financial instrument or trading strategy, or as an official confirmation or official valuation of any transaction mentioned herein.Nothing in this material should be construed as investment, tax, legal, accounting, regulatory or other advice (including within the meaning of Section 15B of the Securities Exchange Act of 1934) or as creating a fiduciary relationship.

Background and Motivation

Optimized market making in the context of repeated sealed-bid auctions.

For fungible goods not traded by open outcry the following sequence is sometimes an approximation of reality:

1. Customer enquiry is sent to multiple dealers
2. Dealers respond, not knowing others' responses
3. The customer takes the best price

Challenges

1. Significant time between trading opportunities for any given item implies material market risk and funding costs.

2. A trade-off exists between aggressive responses seeking faster turnover and more conservative (wider) pricing to ensure carrying costs don't erode profits entirely.

3. It is not obvious how to evaluate any one decision in isolation, even ex-post, and even if the "cover" price is known.

4. Illiquidity implies short time series for back-testing, challenges for reinforcement learning out of the box.

Business objectives

1. Understand the economics of the optimal policy (whether to invest more in accuracy, et cetera)
2. Distinguish inventory management skill from directional skill.
3. Move toward a better combination of human and algorithmic trading.
4. Quantify the correct price for internal trades.

Applications

1. Block trading in securities
2. Over the counter trading
3. Trading advertising blocks?
4. Trading most things (how often is there *not* a special class of market participant who invests time and expects to be compensated for it?)

Modeling Assumptions

Adverse selection

We assume a fixed loss on every trade arising from asymmetric information. When we markup a item by m we assume we only make $m - \epsilon$.

Steady state

We assume we play the game forever.

Independent enquiry arrival

We assume inquiries are Poisson with mean time τ between arrivals.

Enquiry imbalance

With probability p the inquiry is from a customer looking to sell a item (versus $1-p$ for buy).

Zero price volatility

The “fair price” is constant (trick!) and commonly discerned by all dealers.

Exponentially distributed inside price

The distribution of the most competitive markup (markdown) among all responding dealers who compete with us is exponential with mean w and hazard rate $h = 1/w$. We refer to w as the market width. The more competitive the other dealers are, the smaller the market width.

Direct costs

We pay a direct cost per unit time $c(x)$ when holding inventory. Typically

$$c(x) = c_+ x^+ + c_- |x^-| + c_2 x^2$$

perhaps with parameters like

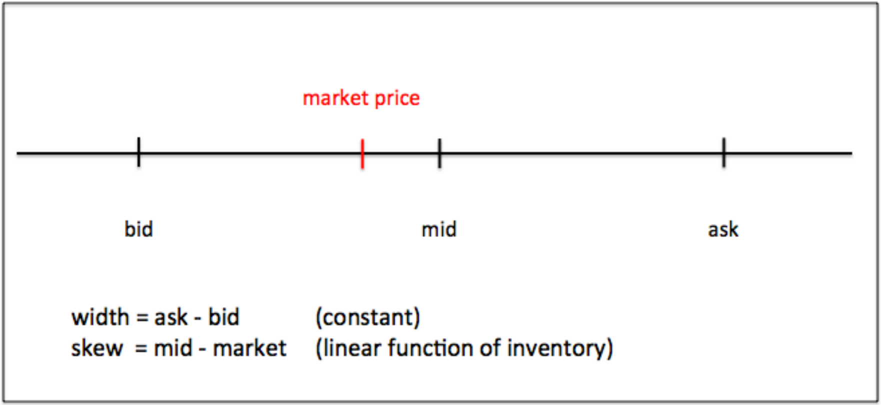
$$\begin{aligned} c_+ &= 0.045 \\ c_- &= 0.04 \\ c_2 &= \frac{0.1 \times 0.045}{10,000,000} = 4.51e^{-10} \end{aligned}$$

since we have both funding, risk costs and self-imposed equity return hurdles.

Remark: The quadratic term is what allows us to remove explicit dynamics from the price model and focus on inventory management.

A Benchmark and Related Literature

A Rule of Thumb



Literature Review

Market making and formation of the bid-ask spread is the subject of various papers including Luckock '03, and Hubbard, Paarsch, Wright '04 where markets are modeled as double auctions.

More general auctions literature with some application to finance includes Milgrom '82, Myerson '81, Roughfarden et al '16, Vickrey '61. , and also Smidt '79 where auctions are studied in the setting of negotiated deals and block trades.

Ho and Macris consider dealer markets under competition. And of particular mention is the formal justification for constant width linear skew in Avellaneda and Stoikov '08 where market making as a carefully formulated stochastic control problem in which constand width linear skew is provided as a solution.[1]

[1] This requires some approximation, however, which more or less ensures linear skew.

Properties of the optimal policy

A “financial” derivation of the Bellman equation

A useful fiction is the indifference price for inventory $\nu(x)$. At any time we imagine the market maker is free to liquidate her position, but will incur a cost of $\nu(x)$ for doing so (thus $\nu(x)$ might be 0.1 or so on $\$100$ notional).

To derive constraints on the form taken by $\nu(x)$ we can consider two courses of action.

1. The market maker liquidates her holding and pays $\nu(x)$ to do so. She then waits until the next trade opportunity, makes a trading decision, and then again liquidates her position if necessary.
2. She defers liquidation until after the trading opportunity.

Denote the costs for the first path as Δ_1. The market maker must decide on a markup ($m^{\uparrow}$ for buying, $m^{\downarrow}$ for selling). Let $\iota(q;m) = 1$ if $q > m$ be the indicator function for winning the trade. If a buying opportunity arrives first with probability p then the cost for the best policy is the original liquidation cost plus the net final liquidation cost (after deducting the immediate gain from that trade). Thus

$$
\begin{aligned}
\Delta_1 \quad = \quad & \overbrace{\nu(x)}^{first\ liquidation} \quad + p \inf_{m^{\uparrow}} \left\{ \overbrace{\left(\nu(s) - (m^{\uparrow} - \epsilon)s\right)}^{net\ cost} \overbrace{E^q\left[\iota(q^{\uparrow}, m^{bid})\right]}^{=P^{\uparrow}(m^{bid})} \right\} \\
& + (1-p) \inf_{m^{\downarrow}} \left\{ \left(\nu(-s) - (m^{\downarrow} - \epsilon)s\right) P^{\downarrow} \right\}
\end{aligned}
$$

On the other hand if the market maker defers liquidation she also pays a direct cost for carrying the inventory to the next trading opportunity, and must liquidate even if there is no trade at that time. Hence:

$$\begin{aligned}
\Delta_2 &= \overbrace{E[\delta t]c(x)}^{carry} + \\
&+p \inf_{m^\uparrow} \left\{ \left(\nu(x+s) - (m^\uparrow - \epsilon)s \right) P^\uparrow(m^\uparrow) + \nu(x) \left(1 - P^\uparrow(m^\uparrow) \right) \right\} \\
&+(1-p) \inf_{m^\downarrow} \left\{ \left(\nu(x-s) - (m^\downarrow - \epsilon)s \right) P^\downarrow(m^\downarrow) + \nu(x) \left(1 - P^\downarrow\left(m^\downarrow\right) \right) \right\}
\end{aligned}$$

where again $P^\uparrow(m^\uparrow)$ is the probability of winning the trade when the market maker marks up by $m^\uparrow$.

Break-even markups ("strikes")

We shall equate Δ_1 and Δ_2 but first we introduce terminology:

$$K^{\uparrow}(x;s) \quad = \quad \epsilon + \frac{\nu(x+s)-\nu(x)}{s} \tag{1}$$

which we call a strike. Similarly

$$K^{\downarrow}(x;s) \quad = \quad \epsilon + \frac{\nu(x-s)-\nu(x)}{s} \tag{2}$$

as the "strike" when selling, which may well be negative for positive inventory. We shall interpret these as the non-myopic break-even markups, as will be shortly apparent.

Indifference to the choice of path means $\Delta_1 = \Delta_2$. This simplifies after we subtract $\nu(x)$ from both sides, write $\tau = E[\delta t]$ for the average waiting time, swap signs and replace $\inf$ with $\sup$:

$$\begin{aligned}
\frac{\tau c(x)}{s} \quad = \quad & p \sup_{m^\uparrow} \left\{ \overbrace{\left(m^\uparrow - K^\uparrow(x;s)\right)}^{net\ \ gain} P^\uparrow(m^\uparrow) \right\} \\
& + (1-p) \sup_{m^\downarrow} \left\{ \left(m^\downarrow - K^\downarrow(x;s)\right) P^\downarrow(m^\downarrow) \right\} \\
& - p \sup_{m^\uparrow} \left\{ \left(m^\uparrow - K^\uparrow(0;s)\right) P^\uparrow(m^\uparrow) \right\} \\
& - (1-p) \sup_{m^\downarrow} \left\{ \left(m^\downarrow - K^\downarrow(0;s)\right) P^\downarrow(m^\downarrow) \right\}
\end{aligned}$$

Now we start to appreciate the intuition. Differential "option" value of the trading opportunities at x versus zero inventory equates to the direct cost of carry.

Intuition: Relating direct and indirect inventory cost

Here is why direct costs $c(x)$ appearing on the left hand side are not synonymous with $\nu(x)$ appearing on the right.

- In deciding on the true cost of moving from inventory x to $x+s$, an optimal trader will weigh not only the differential in direct costs proportional to $c(x+s)-c(x)$ but also the *differential value of trading opportunities at each inventory level*.
- The larger the inventory, the greater the value to the trader of a trading opportunity because getting out (or rather, the potential to get out) helps more. *This counteracts direct inventory costs*

Optimal markups as a function of break-even markups

Since $K^{\downarrow}(x;s)$ is the zero profit markup for a market maker when offering to sell a quantity s to move inventory from x to $x-s$, evidently the *optimal* choice of markup lies precisely where the increase in profit exactly offsets the possibility of losing the trade. For say we increased the markup by Δm then:

$$\overbrace{\left(m^{ask} - K^{\downarrow}(x;s)\right)}^{existing\ benefit} \quad \overbrace{h(m^{ask})\Delta m}^{chance\ of\ losing\ it} \quad = \quad \overbrace{\Delta m}^{increase\ in\ profit} \quad .$$

where we have emphasized that in this "greed equation" we don't require that $h = 1/w$ be a constant.

Indeed we always have the first order condition

$$m^{ask}(x;s) = \frac{1}{h(m^{ask}(s))} + K^{\downarrow}(x;s)$$

for the optimal markup, independent of the distributional assumption for the inside market. And recalling the definition of $K^{\downarrow}(x;s)$ and also $1/h = w$ the optimal markup when offering to sell therefore breaks down as

$$m^{ask}(x;s) = w + \epsilon + \frac{\nu(x-s) - \nu(x)}{s} \tag{3}$$

This reads straightforwardly:

"markup = markup width + adverse selection + marginal inventory cost"

though of course $\nu(x)$ remains to be found.

Skew and width

We can now substitute the optimal choice of markup back into the (Bellman) functional relation relating $\nu(x)$ to $c(x)$.[2] However, it turns out to be a little more intuitive to do so after relating $\nu(x)$ to skew and width

Technical note: We will restrict attention here to the where the first order condition relating $\nu(x)$ and markups is relevant. This isn't the case for very large or very small inventories. See my notes for the extreme cases where the optimal choice is a markup of zero.

[2] I will refer to this as the Bellman equation, even though the usual convention for value function would use $-\nu(x)$.

Let p^{market} be the market mid that all markups are defined with respect to. Then our market maker's bid p^{bid} is at $p^{market} - m^{bid}$ and her ask p^{ask} is at $p^{market} + m^{ask}$. The arithmetic midpoint of her bid and offer is

$$p^{mid} = p^{market} + \frac{m^{ask} - m^{bid}}{2}$$

We can re-express the optimal bid and offer as

$$m^{bid}(x) = \frac{1}{h} + K^{\uparrow} = \overbrace{\epsilon + \frac{1}{h}}^{\Delta} + \overbrace{\frac{\nu(x+s) - \nu(x)}{s}}^{K^{\uparrow}-\epsilon} \tag{4}$$

$$m^{ask}(x) = \frac{1}{h} + K^{\downarrow} = \epsilon + \frac{1}{h} + \overbrace{\frac{\nu(x-s) - \nu(x)}{s}}^{K^{\downarrow}-\epsilon} \tag{5}$$

where $\Delta = \frac{1}{h} + \epsilon$. Substituting these into p^{mid} we see that our market maker is skewing the mid point of her quotes as follows:

$$p^{mid} = p^{market} - \frac{\nu(x+s) - \nu(x-s)}{2s}$$

We notice that the skew term is the secant slope. For brevity, we will simply call it the slope of the inventory cost.

In English:

$$Slope\ of\ inventory\ cost = Skew \tag{6}$$

Trading is simple! Why do they pay those guys so much?

To be formal, define both slope and convexity of the inventory cost as follows:

$$
\begin{aligned}
S(x) &:= \frac{\nu(x+s) - \nu(x-s)}{2s} \left(= \frac{K^{\uparrow} - K^{\downarrow}}{2} \right) \\
C(x) &:= \frac{\nu(x+s) - 2\nu(x) + \nu(x-s)}{2s} = \left(\frac{K^{\uparrow} + K^{\downarrow}}{2} + \epsilon \right)
\end{aligned}
$$

so that, as we have observed, slope = skew. To interpret the role of convexity $C(x)$ we note that

$$C(x) + S(x) = K^{\uparrow} - \epsilon$$

and

$$C(x) - S(x) = K^{\downarrow} - \epsilon$$

But now we realize that

$$\begin{aligned} m^{ask} &= \frac{1}{h} + K^{\downarrow} = \overbrace{C(x) + \Delta}^{width} + \overbrace{S(x)}^{skew} \\ m^{bid} &= \frac{1}{h} + K^{\uparrow} = C(x) + \Delta - S(x) \end{aligned}$$

from which it is clear that $C(x)$ is playing the role of discretionary “width” over and above a minimum constant width Δ.

That is,

$$Convexity\ of\ inventory\ cost = (discretionary)\ Width. \tag{7}$$

An intuitive characterization of $\nu(x)$

The optimal policy is characterized by an inventory indifference price $\nu(x)$ whose slope is the skew and whose convexity is, within a constant, the width.

Off-policy usage

These relationships can be used to infer implied inventory cost from actual trading behavior, and analyze how that changes in consistent or inconsistent ways between items, on entry and exit, in special circumstances and so forth.

More on that later.

Consistent skew and width

We are now in a position to recast the Bellman equation in terms of optimal skew and width. We can also generalize slightly, since there was nothing in the original argument that required the second inventory level to be zero.

$$\begin{aligned}\frac{\tau h}{s}\left(c(x)-c(x')\right) &= p\,\overbrace{e^{-hC(x)+hS(x)}}^{P^{buy}(m^{bid})}+(1-p)\,\overbrace{e^{-hC(x)-hS(x)}}^{P^{sell}(m^{ask})} \\ &\quad -pe^{-hC(x')+hS(x')}-(1-p)e^{-hC(x')-hS(x')} \qquad (8)\end{aligned}$$

for all x and x', provided we are in the region where our assumptions are valid.

Tackling enquiry imbalance

If there are more buying opportunities for our market maker than selling opportunities, or vice versa, we will have the analytic inconvenience $p \neq \frac{1}{2}$. But this turns out to be a superficial difficulty. Indeed *we can derive analytically the impact of order imbalance on the optimal skew and width* as follows, even before solving $\nu(x)$.

Define

$$\delta = \frac{\log(1-p) - \log(p)}{2h}$$

and

$$\gamma = \frac{\log\left(\frac{1}{2}\right) - \frac{\log(1-p)+\log(p)}{2}}{h}$$

It is clear from concavity of $\log$ that $\gamma > 0$.

For reasons that will shortly be clear we allow δ to move the skew, defining

$$S_\delta(x) = S(x) - \delta.$$

In similar fashion we allow γ to increase the adverse selection $\epsilon \to \epsilon + \gamma$ resulting in an increase to the non-discretionary component of width:

$$\Delta_\gamma = \Delta + \gamma.$$

Claim: Order imbalance requires the market maker to skew by δ and widen by γ.

Proof: Substitute S_δ and multiply by $e^{h\Delta}$:

$$\begin{aligned}\frac{\tau h}{s}\left(c(x)-c(x')\right)e^{h\Delta} &= pe^{-hC(x)+h(S_\delta(x)+\delta)}+(1-p)e^{-hC(x)-h(S_\delta(x)+\delta)}\\ &\quad -pe^{-hM(x')+h(S_\delta(x')+\delta)}-(1-p)e^{-hC(x')-h(S_\delta(x')+\delta)}\end{aligned}$$

Then multiply by $e^{h\gamma}$:

$$\begin{aligned}\frac{\tau h}{s}\left(c(x)-c(x')\right)e^{h\Delta_\gamma} &= e^{h\gamma}pe^{-hC(x)+h(S_\delta(x)-\delta)}+(1-p)e^{h\gamma}e^{-hC(x)-h(S_\delta(x')}\\ &\quad -pe^{h\gamma}e^{-hC(x')+h(S_\delta(x')+\delta)}-(1-p)e^{h\gamma}e^{-hC(x')-h(S}\\ &= \frac{1}{2}e^{-hC(x)+hS_\delta}+\frac{1}{2}e^{-hC(x)-hS_\delta(x')}\\ &\quad -\frac{1}{2}e^{-hC(x')+hS_\delta(x'))}-\frac{1}{2}e^{-hC(x')-hS_\delta(x')}\end{aligned}$$

where we have used $e^{h\delta+h\gamma}=\frac{1}{2p}$ and $e^{h\gamma-h\delta}=\frac{1}{2(1-p)}$.

We have recovered the symmetric relation with $S_\delta(x)$ in place of $S(x)$ and $\Delta + \gamma$ in place of Δ. Thus our market maker will have to be more defensive and use a wider market, as claimed. ■

Claim: Order imbalance increases the effective carrying cost by

$$e^{\log\left(\frac{1}{2}\right) - \frac{\log(1-p)+\log(p)}{2}} > 1$$

Proof: The second claim follows from inspection of the left hand side of (10), from which it is clear that the solution could also be achieved by solving the symmetric problem where the direct carrying cost $c(x)$ has been multiplied by

$$e^{h\gamma} = e^{\log\left(\frac{1}{2}\right) - \frac{\log(1-p)+\log(p)}{2}}$$

■

Approximate and exact solutions

The reader will notice that $\nu(x)$ is still hanging out there in the wind, even though we have an intuitive grasp of it's role in the optimal strategy. To summarize:

- Slope of $\nu(x)$ is skew, convexity of $\nu(x)$ is discretionary width.
- markup $=$ slope of $\nu(x)$ $+$ market width $+$ adverse selection

Next we show that the optimal policy can be viewed as a perturbation of "constant width linear skew" (CWLS) policy.

Backk to the constant width, linear skew (CWLS) policy

As noted previously, Avellaneda and Stoikov derive an approximately optimal policy for a market maker in a different but related model. They show that the solution is approximately given by constant width and skew that is linear in inventory. However they had to throw out some terms.

Choosing width or skew (but not both)

With the distraction of order imbalance behind us we clean up the consistency relation (10) by defining the modified gain

$$G_\delta(x) = e^{-hC(x)} \cosh\left(hS_\delta(x)\right)$$

and cost coefficient

$$\Omega = \frac{\tau h}{s} e^{h\Delta_\gamma} = \frac{\tau h}{s} e^{1+h\epsilon+h\gamma}$$

Then the Bellman equation can be written tersely:

$$G_\delta(x) - G_\delta(x') = \Omega\left(c(x) - c(x')\right) \qquad (11)$$

which holds for all x, x'.

The relation (11) defines family of self-consistent solutions for $S(x)$ and $C(x)$. We can relate the former to the latter readily if we know that $C(x)$ and $S_\delta(x')$ are specified, by inverting (11):

$$S_\delta(x) = \frac{1}{h}\cosh^{-1}\left(\Omega e^{hC(x)}c(x) + e^{h(C(x)-C(x'))}\cosh\left(hS_\delta(x')\right) - e^{hC(x)}\Omega c(x')\right)$$

Returning to the special case $x' = 0$ we have some simplification. We have $c(0) = 0$ and by symmetry, $S_\delta(0) = 0$. Consistency implies that the gain is an affine function of carrying cost.

$$G_\delta(x) = \Omega c(x) + e^{-hC_0}$$

If convexity $C(x)$ is known,

$$S_\delta(x) = \frac{1}{h}\cosh^{-1}\left(e^{hC(x)}\left\{\Omega c(x) + e^{-hC_0)}\right\}\right)$$

A constant width but not linear skew model

Suppose convexity $C(x)$ equals the constant C_0. Since

$$\frac{\nu(x+s) - 2\nu(x) + \nu(x-s)}{2s} = C_0$$

has solution

$$\nu(x) = \frac{C_0}{s} x^2$$

this corresponds to a quadratic inventory cost function.[3]

Obviously the slope is linear too, so we have CWLS.

However, something is fishy here ...

[3] Quadratic inventory cost is sometimes imposed as an ansatz.

... because we should solve for skew directly from the Bellman equation:

$$S_\delta(x) \quad = \quad \frac{1}{h}\cosh^{-1}\left(1 + e^{hC_0}\Omega c(x)\right) \tag{12}$$

and this yields a constant width, non-linear skew model.

Only in the special case of small x and quadratic cost do we recover linear skew - since the right hand side is approximately proportional to $\sqrt{c(x)}$. This ceases to be the case, however, if funding costs or equity hurdles are material.

A better linear skew model?

We can similarly motivate a linear skew model, noting that for $\nu(x) = \frac{C_0}{s}x^2$ as before we have, by direct calculation,

$$S^*_\delta(x) = \frac{\nu(x+s) - \nu(x-s)}{2s} = \frac{1}{2}\frac{C_0}{s}x \tag{13}$$

This is, of course, a different skew than before and if we seek a consistent width $C(x)$ we'll have

$$C^*(x) = \frac{1}{h}\log\left(\frac{\cosh\left(\frac{C_0}{2s}hx\right)}{\Omega c(x) + e^{-hC_0}}\right) \tag{14}$$

The fact that C, S and C^*, S^* don't coincide exactly merely reflects the fact that $\nu(x) \propto x^2$ is not an internally consistent inventory cost function.

Obvious moral: Solve for $\nu(x)$ rather than pretend $\nu(x) \propto x^2$ works.

Just solve it already

In practice it is straightforward to solve $\nu(x)$ by discretizing x.

Alternatively, one can try parametric forms and then optimize the fit between the left and right hand sides of the Bellman equation. For instance:

$$\nu(x) = \begin{cases} c_{l_0} + c_{l_1}\frac{x}{s} + c_{l_2}\left(\frac{x}{s}\right)^2 + c_{l_3}\left(\frac{x}{s}\right)^3 & x < 0 \\ c_{r_0} + c_{r_1}\frac{x}{s} + c_{r_2}\left(\frac{x}{s}\right)^2 + c_{r_3}\left(\frac{x}{s}\right)^3 & x \geq 0 \end{cases}$$

was suggested by Nick West.[4]

[4] Who we also thank for some important debugging.

Interpreting the results

A quick review.

We've chosen a convention that the inventory indifference price $\nu(x)$ is the biggest markdown a *rational trader* will accept if offered the chance to get out of her inventory x immediately.

We've shown that order imbalance (i.e. $p \neq 1/2$) presents no analytic difficulty, reducing to the case $p = 1/2$ with simple adjustments to skew and width.

For "moderate" inventory, the optimal policy can be viewed as a perturbation of constant width linear skew.

Inventory cost $\nu(x)$ is an internal price

Because $\nu(x)$ is an indifference price for an optimal trader, it can also be used as an internal transfer price.

Our profit optimizing market maker should have two different bid-offer pairs. One for the outside world and one for the firm.[5]

[5] In contrast to the simpler pronouncement that internal trades be done "at mid", which is unfair to the market maker and sub-optimal from the firm perspective.

What's wrong with the common sense approach using direct costs $c(x)$ in place of $\nu(x)$?

One often hears straight-forward arguments for the cost of taking on a item, such as mean holding time multiplied by cost and so forth.

These sound reasonable, but ignore the counteracting benefit of larger inventory. So they implicitly overstate the real cost of increasing inventory and, therefore, tend to be too defensive.

And here's why, returning to the Bellman equation. Fixed costs (LHS) relate to differential expected gains from trading (RHS)

$$
\begin{aligned}
\frac{\tau c(x)}{s} \quad = \quad & p \sup_{m^{\uparrow}} \left\{ \overbrace{\left(m^{\uparrow} - K^{\uparrow}(x;s)\right)}^{net\ gain} P^{\uparrow}(m^{\uparrow}) \right\} \\
& +(1-p) \sup_{m^{\downarrow}} \left\{ \left(m^{\downarrow} - K^{\downarrow}(x;s)\right) P^{\downarrow}(m^{\downarrow}) \right\} \\
& -p \sup_{m^{\uparrow}} \left\{ \left(m^{\uparrow} - K^{\uparrow}(0;s)\right) P^{\uparrow}(m^{\uparrow}) \right\} \\
& -(1-p) \sup_{m^{\downarrow}} \left\{ \left(m^{\downarrow} - K^{\downarrow}(0;s)\right) P^{\downarrow}(m^{\downarrow}) \right\}
\end{aligned}
$$

(And while we are here, notice that the net gain is independent of inventory when we choose the optimal markup $m^{\uparrow} = w + \epsilon + K$).

Fixed costs translate directly to fill ratios

Substituting in the optimal choice of markup and markdowns, and writing the consequent fill probabilities as functions of inventory x, we have

$$\begin{aligned} \tau(c(x) - c(x')) &= sw\left(pP^{\uparrow}(x) + (1-p)P^{\downarrow}(x)\right. \\ &\quad \left. - pP^{\uparrow}(x') - (1-p)P^{\downarrow}(x')\right) \end{aligned} \tag{15}$$

Interpretation: An optimal trader will accept a differential direct cost $c(x+s)-c(x)$ for the typical time between trading opportunities if this offsets the differential probability of executing a trade multiplied by the typical gain from trading (... and this last quantity, by accident, is invariant to inventory and always equals sw).

For the optimal policy,

$$\textit{change in direct cost} = \textit{change in trading probability} \times \textit{size} \times \textit{width} \tag{16}$$

Here the inventory indifferent cost $\nu(x)$ is buried inside the change in trading probability - but the latter is an observable quantity in its own right.[6]

What if we deviate from the optimal policy?

[6] To be clear, width is the market width, not our market maker's width.

Trading outcomes without $\nu(x)$

Without $\nu(x)$ it is unclear how to assess decisions ex-post.

	Algo trades	Algo misses
Trader trades	size((Trader markup) - (Algo markup))	?
Trader misses	?	0

We can resort to longitudinal averages (i.e. backtesting)... sometimes.

Trading outcomes viewed from algo perspective

Comparing the optimal policy outcome for a given decision to any other:

	Algo trades	Algo misses
Trader trades	$s(m - m^*)$	$s(m - m^*) + sw$
Trader misses	$-sw$	0

Table 1: Economic impact of different responses from algo perspective

Related results and future work

One benefit of the analytic solution is that sensitivity to parameters can be computed (details omitted).

The take-home intution: *there are no diminishing returns on accuracy*

Is this universally believed?

For this to hold, the trader must take their own uncertainty into account - unless it is small compared to the market width. Here's a picture...

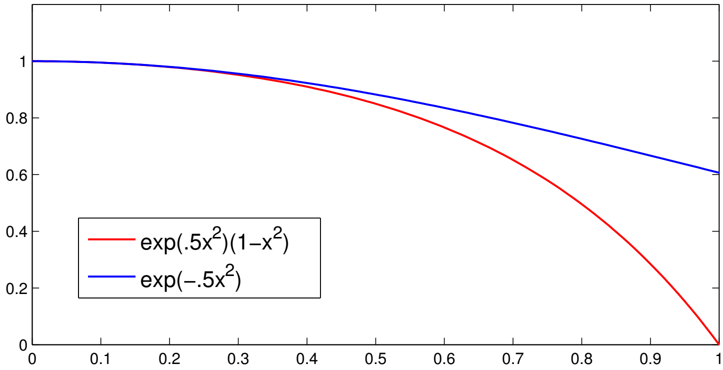


Figure 1: Relative profitability of a humble market maker (blue) and one who assumes their estimates are precisely correct (red)

And one final observation...

The estimate of the mid doesn't matter.[7]

Say whaaattttt ??!!!!

Our disseminated price is equal to the mean best inside response $\pm$ adverse selection and inventory cost.

[7] At least directly. Mids might help locate the best inside price, however.

Summary

1. A financial argument for constraints on inventory cost $\nu(x)$.
2. Clarity around how width and skew relate to inventory cost.

In the special case of exponentially distributed competitor markups:

3. A simple way to deal with client enquiry imbalance
4. Analytic expression for width as a function of inventory assuming constant width (and vice versa). Just a simple as CWLS!
5. The suggestion that numerically computed$\nu(x)$ improves this, is also quite convenient, and independently useful.
6. Analytic market making efficiency based on variance in parameter estimates

Questions?