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Contraction of an Adapted Functional Calculus

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Abstract. We aim to show, using the example of a Riemannian symmetric pair $(G, K) = (\mathrm{SL}_2(\mathbb{R}), \mathrm{SO}(2))$, how contraction ideas may be applied to functional calculi constructed on coadjoint orbits of Lie groups. We construct such calculi on principal series orbits and generic orbits of the Cartan motion group $V \rtimes K$, and show how the two are related. Since the calculi are adapted to the representations traditionally attached to the orbits, we recover at the Lie algebra level the contraction results of Dooley and Rice [5].

1. Introduction

Let G be a real semisimple Lie group with finite centre and K a closed subgroup making (G, K) a Riemannian symmetric pair ([8] p. 209). We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + V$ where V is an $\mathrm{Ad}(K)$–invariant subspace and $\mathfrak{k}$ is the Lie algebra of K. The *Cartan motion group* associated with the pair (G, K) is the semidirect product $V \rtimes K$ formed with respect to the adjoint action of K on V.

The motion group is related to G by a family of contraction mappings $\pi_\lambda : V \rtimes K \to G$ defined by

$$\pi_\lambda(v, k) = \exp_G(\lambda v) \cdot k,$$

for $v \in V$ and $k \in K$, and indexed by $\lambda \in \mathbb{R}^+$. These maps were introduced by Dooley and Rice in [5] and are approximate group homomorphisms. In [5] the unitary irreducible representations of $V \rtimes K$ are obtained as limits of sequences of principal series representations of G composed with contraction mappings.

The derivatives $d\pi_\lambda : V + \mathfrak{k} \to \mathfrak{g}$ have in a sense been around for longer, albeit in a less desirable form. The idea of obtaining a new Lie algebra by deformation of the structure constants first appears in Inonu and Wigner [9]. It is clear however that Dooley's contraction mappings

$$d\pi_\lambda(v, A) = \lambda v + A$$

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for $v \in V, A \in \mathfrak{k}$ and $\lambda \in \mathbb{R}^+$ represent a more coordinate free approach.

The present work is stimulated by the above considerations and a recent PhD thesis by Benjamin Cahen of the University of Metz [3]. Cahen has shown that on certain coadjoint orbits of G we may construct a symbol calculus (a mapping from a class of functions or distributions on the orbit to operators in a Hilbert space) which is adapted to the orbit in the following sense : Let O be some such orbit and $d\sigma : \mathfrak{g} \to \mathcal{H}$ the representation of $\mathfrak{g}$ associated with O by the traditional orbit methods [10]. For each $X \in \mathfrak{g}$, define the function $\widetilde{X}$ on O by

$$\widetilde{X}(\xi) = \langle \xi, X \rangle$$

for $\xi \in O \subset \mathfrak{g}^*$. Then the symbol $\widetilde{X}$ is assigned to an operator $A_{\widetilde{X}}$ acting on $\mathcal{H}$, and

$$A_{\widetilde{X}} = \frac{1}{i} d\sigma(X)$$

holds on a dense subspace of $\mathcal{H}$.

Cahen's procedure is analogous to that of Arnal and Cortet [1] and Wildberger [17], establishing first suitable parametrisations of the coadjoint orbits of a nilpotent (connected simply connected) group and using the same parametrisations given by Duflo in [6]. If O is a principal series orbit then we have a diffeomorphism

$$\Psi : \overline{N} \times \overline{\mathfrak{n}} \times O_M \to \widetilde{O}$$

where $\overline{N}$ is a nilpotent Lie group with Lie algebra $\overline{\mathfrak{n}}$, O_M is a coadjoint orbit of a compact subgroup $M \subset K$ and $\widetilde{O}$ is an open dense subset of O. A combination of the Berezin calculus (Berezin [2], Rawnsley [12]) on O_M and a generalisation of the Weyl calculus (Weyl [16], Voros [14]) on $\overline{N} \times \overline{\mathfrak{n}}$ yields the desired calculus on O.

Cahen also provides a construction for a similarly adapted calculus on $V \rtimes K$–orbits which has prompted us to ask whether this second calculus may be obtained as a limit is some sense of the calculus on G–orbits. This paper is an attempt to answer this question, in the special case $(G, K) = (\mathrm{SL}_2(\mathbb{R}), \mathrm{SO}(2))$. Here the analysis is simplified by degeneracy of the orbits O_M mentioned in the previous paragraph. Operators in the Berezin calculus on O_M become complex numbers, and symbols in the calculus on O are precisely the symbols on the calculus on $\overline{N} \times \overline{\mathfrak{n}}$ in Cahen [3] Paragraph 7.2, Chapter 1.

In Section 3 we find explicit realisations of the principal series representations of G in $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$, where $\mathbb{T}$ is the torus with underlying set $(-\pi, \pi]$. In Sections 4 and 5 we define calculi adapted to these realisations. The first is Cahen's construction, defined by the Bruhat decomposition and referred to here as the Ψ–calculus. It maps symbols on a given orbit to operators in $C_c^\infty(\mathbb{R})$. The second, which we call the Υ–calculus, uses instead the Iwasawa decomposition. It maps the same symbols to operators in $C^\infty(\mathbb{T})$ and is defined on each orbit O using a diffeomorphism $\Upsilon : \mathbb{T} \times \mathbb{R} \to O$. The Υ–calculus is better suited to our purposes.

In Section 7 we find irreducible unitary representations of the motion group $V \rtimes K$ indexed in the same way as the principal series representations of G. We then establish the Γ–calculus using, for a given coadjoint orbit O, a diffeomorphism $\Gamma : \mathbb{T} \times \mathbb{R} \to O$. It is a slight modification of Cahen [3], and is adapted to the representations of $V \rtimes K$.

Section 9 summarises the results of Dooley and Rice [5] in the case $(G, K) = (\mathrm{SL}_2(\mathbb{R}), \mathrm{SO}(2))$. The representations of G are "contracted" onto the representations of $V \rtimes K$, thus introducing the point of this paper. We aim to contract the Υ–calculus on G–orbits onto the Γ–calculus on $V \rtimes K$–orbits.

We demonstrate two methods of doing this. The first scheme runs as follows. Suppose f is a symbol on a $V \rtimes K$–orbit C_ψ passing through $\psi \in \mathfrak{g}^*$ and let A_f^Γ denote its corresponding operator under the Γ–calculus, acting on $C^\infty(\mathbb{T})$. We take a family of G–orbits $O_{\psi/\lambda}$ passing through ψ/λ and a family of symbols f_λ on $O_{\psi/\lambda}$ related to f in a simple way. The operators $A_{f_\lambda}^\Upsilon$ corresponding to the symbols f_λ then also act on $C^\infty(\mathbb{T})$ and converge (in a sense defined later) to the original operator A_f^Γ. As the calculi are adapted, one recovers at the Lie algebra level the aforementioned results of Dooley and Rice [5] pertaining to the pair (G, K).

The second method is more direct. For a symbol f on a $V \rtimes K$–orbit with operator A_f^Γ we define $\widetilde{f} = f \circ \Gamma \circ \Upsilon^{-1}$. Then $\widetilde{f}$ is a symbol on a G–orbit O and if $A_{\widetilde{f}}^\Upsilon$ denotes the corresponding operator under the Υ–calculus we have $A_f^\Gamma = A_{\widetilde{f}}^\Upsilon$ on $C^\infty(\mathbb{T})$. Thus the Υ–calculus contracts nicely onto the Γ–calculus and we have a contractive relationship between the Lie group G and its motion group $V \rtimes K$ not involving limits. Interestingly there is an immediate converse. If $\widetilde{f}$ is a symbol on a G–orbit O then $f = \widetilde{f} \circ \Upsilon \circ \Gamma^{-1}$ is a symbol on a $V \rtimes K$–orbit and $A_{\widetilde{f}}^\Upsilon = A_f^\Gamma$.

2. $\mathrm{SL}_2(\mathbb{R})$ and its motion group $\widetilde{M(2)}$

Hereafter $G = \mathrm{SL}_2(\mathbb{R})$ is the matrix Lie group consisting of two by two real matrices with determinant unity. Its Lie algebra $\mathfrak{g} = \mathrm{sl}_2(\mathbb{R})$ consists of traceless two by two real matrices. Fix a basis

$$e_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}$$

for $\mathfrak{g} = \mathrm{sl}_2(\mathbb{R})$. Letting β denote the Killing form we take $\{\hat{e_i} = \beta(e_i, \cdot)\}$, $i = 1, .., 3$, as a basis for $\mathfrak{g}^*$. All subspaces of $\mathfrak{g}$ and their duals may thus be identified with subspaces of $\mathbb{R}^3$.

We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + V$ where $k = \mathrm{sp}\,\{e_3\}$ and $V = \mathrm{sp}\,\{e_1, e_2\}$ are the eigenspaces of $\Theta(X) = -X^T$. The restriction of the adjoint representation

$$\mathrm{Ad}(g)X = gXg^{-1}, \; g \in G, \; X \in \mathfrak{g}$$

to $K = \exp(\mathfrak{k})$ preserves both $\mathfrak{k}$ and V, allowing us to define the semi-direct product $\widetilde{M(2)} = V \rtimes K$ which is the Cartan motion group associated with the pair (G, K). That is, we equip $V \rtimes K$ with the group multiplication

$$(v, k) \cdot (v', k') = (kv' + v, kk')$$

for $v, v' \in V$ and $k, k' \in K$. The Lie algebra $\widetilde{\mathbf{m(2)}}$ of $\widetilde{M(2)}$ shares the same underlying vector space $V + \mathfrak{k}$ as $\mathfrak{g}$, but has Lie bracket

$$[(\omega, A), (\omega', A')]_{\widetilde{\mathbf{m(2)}}} = \left([A, \omega']_{\mathfrak{g}} - [A', \omega]_{\mathfrak{g}}\,, 0\right)$$

for $\omega, \omega' \in V$ and $A, A' \in \mathfrak{k}$. The Lie subalgebra $\mathfrak{a} = \mathrm{sp}\,\{e_1\}$ is maximal abelian in $V \subset \mathfrak{g}$. Let $A = \exp(\mathfrak{a})$, $\mathfrak{a}^+ = \{a\ e_1 : a \in \mathbb{R}^+\}$, and $A^+ = \exp(\mathfrak{a}^+)$. Let M be the centraliser of A in K, and $\mathfrak{m}$ its Lie algebra. Put $Z = K/M$. Define also $\mathfrak{n} = \mathrm{sp}\,\{e_2 + e_3\}$, $\overline{\mathfrak{n}} = \mathrm{sp}\,\{e_2 - e_3\}$, $N = \exp(\mathfrak{n})$ and $\overline{N} = \exp(\overline{\mathfrak{n}})$. Take bases for $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ which are orthonormal with respect to the inner product $(X, Y) = \beta\left((X, \Theta(Y)\right)$ - namely $\{(e_2 + e_3)/2\}$ and $\{(e_2 - e_3)/2\}$. This fixes parametrisations by $\mathbb{R}$ of $\mathfrak{n}$ and $\overline{\mathfrak{n}}$, and also, in a manner consistent with the exponential map, N and $\overline{N}$.

As $G = \mathrm{SL}_2(\mathbb{R})$ we find $M = \{\pm\mathbf{1}\}$ and $\mathfrak{m} = \{0\}$. Letting $\mathbb{T}$ and $2\mathbb{T}$ denote the tori with respective underlying sets $(-\pi, \pi]$ and $(-2\pi, 2\pi]$ we parametrise K by

$$\theta \in 2\mathbb{T} \mapsto \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \in K.$$

Similarly parametrise $Z \cong \mathbb{T}$. The parametrisations of $\mathfrak{n}$,$\overline{\mathfrak{n}}$, N and $\overline{N}$ given above are

$$\begin{aligned}
b \in \mathbb{R} \quad &\mapsto \quad \begin{pmatrix} 0 & b/2 \\ 0 & 0 \end{pmatrix} \in \mathfrak{n}, \\
y \in \mathbb{R} \quad &\mapsto \quad \begin{pmatrix} 0 & 0 \\ y/2 & 0 \end{pmatrix} \in \overline{\mathfrak{n}}, \\
b \in \mathbb{R} \quad &\mapsto \quad \begin{pmatrix} 1 & b/2 \\ 0 & 1 \end{pmatrix} \in N, \\
y \in \mathbb{R} \quad &\mapsto \quad \begin{pmatrix} 1 & 0 \\ y/2 & 1 \end{pmatrix} \in \overline{N}.
\end{aligned}$$

We shall be making use of both the Iwasawa decomposition of G and a corollary of the Bruhat lemma found in Wallach [15]. For any $g \in G$ we have unique elements $\mathfrak{k}_{\mathbf{I}}(g) \in K$, $\mathfrak{a}_{\mathbf{I}}(g) \in A^+$ and $\mathfrak{n}_{\mathbf{I}}(g) \in N$ such that $g = \mathfrak{k}_{\mathbf{I}}(g)\mathfrak{a}_{\mathbf{I}}(g)\mathfrak{n}_{\mathbf{I}}(g)$. Similarly if $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $g \in G \backslash \{\pm\mathbf{J}\}$ then there are unique elements $\overline{\mathfrak{n}}_{\mathbf{B}}(g) \in \overline{N}$, $\mathfrak{m}_{\mathbf{B}}(g) \in M$, $\mathfrak{a}_{\mathbf{B}}(g) \in A^+$ and $\mathfrak{n}_{\mathbf{B}}(g) \in N$ such that $g = \overline{\mathfrak{n}}_{\mathbf{B}}(g)\mathfrak{m}_{\mathbf{B}}(g)\mathfrak{a}_{\mathbf{B}}(g)\mathfrak{n}_{\mathbf{B}}(g)$.

We mention in passing that $\widetilde{M(2)}$ is the twofold cover of $M(2) = V \rtimes Z$, the Euclidean motion group in two dimensions, the covering being given by the epimorphism $(v, k) \mapsto (v, kM)$.

3. Principal Series Representations

To certain coadjoint orbits of $\mathrm{SL}_2(\mathbb{R})$ may associate a pair of unitary irreducible representations. We find realisations of these representations in $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$, and explicit formulas for the operators which intertwine them.

The Killing form on G is

$$\beta\left(\sum_{i=1}^{3} x_i e_i, \sum_{i=1}^{3} y_i e_i\right) = 2(x_1 y_1 + x_2 y_2 - x_3 y_3).$$

It is preserved by the adjoint action, and thus intertwines the adjoint and coadjoint representations since if $\xi = \beta(Y, \cdot) \in \mathfrak{g}^*$ then

$$\begin{aligned} \langle \mathrm{Ad}^*(g)\xi, X\rangle &= \left\langle \xi, \mathrm{Ad}(g^{-1})X \right\rangle \\ &= \beta\left(Y, \mathrm{Ad}(g^{-1})X\right) \\ &= \beta\left(\mathrm{Ad}(g)Y, X\right). \end{aligned}$$

Therefore the coadjoint orbits are precisely the images of the adjoint orbits under the duality $e_i \mapsto \hat{e}_i$. Fix $\psi = R\hat{e_1} \in \mathfrak{a}^{+*}$ for some $R > 0$. The coadjoint orbit passing through ψ is a hyperboloid of revolution about the $\hat{e}_3$ axis intersecting the $\hat{e}_1\hat{e}_2$–plane in a circle radius R :

$$O_\psi = \left\{ \xi = \sum_{i=1}^{3} \xi_i \hat{e}_i \in \mathfrak{g}^* : \xi_1^2 + \xi_2^2 - \xi_3^2 = R^2 \right\}.$$

Fix a unitary irreducible representation η of M. There are only two such representations, indexed by $\eta' \in \{0,1\}$ such that $\eta(-\mathbf{1}) = -1^{\eta'}$. Let $\mu \in \mathfrak{a}^*$ be half the sum of the positive roots; namely $\hat{e}_1/4$. Define $\log : A^+ \to \mathbf{a}$ by $\exp\left(\log(a)\right) = a$ for all a in A^+. The pair (ψ, η) gives us a unitary irreducible representation of MAN :

$$\eta \otimes e^\gamma \otimes 1(g) = \eta\left(\mathfrak{m}_{\mathbf{I}}(g)\right) e^{\langle \mu + i\psi, \log \mathfrak{a}_{\mathbf{I}}(g)\rangle}$$

where we consider $\gamma = \mu + i\psi$ an element of the complexification of $\mathfrak{a}^*$. We take some liberties with notation, writing the second factor above for example as $e^\gamma\left(\mathfrak{a}_{\mathbf{I}}(g)\right)$. Inducing up to the whole group yields a unitary irreducible representation of G :

$$\sigma_{\gamma,\eta} = \mathrm{Ind}_{MAN}^G\, \eta \otimes e^\gamma \otimes 1.$$

More precisely, let

$$H = \left\{ f \in C^\infty(G) : f(gg_0) = \eta \otimes e^\gamma \otimes 1(g_0^{-1}) f(g)\ \forall g_0 \in MAN, g \in G \right\}.$$

Discard those f in H whose restrictions to K are not square integrable with respect to Haar measure on K. Complete with respect to the semi-norm

$$||f|| := ||f_{|K}||_{L^2(K)}$$

to obtain a Hilbert space $\mathcal{H}^G_{\gamma,\eta}$. Then the operation of left translation,

$$\sigma_{\gamma,\eta}(g) f(g_0) = f(g^{-1}g_0),$$

on $\mathcal{H}^G_{\gamma,\eta}$ provides a unitary irreducible representation. The representations thus formed comprise the *principal series*. For $f \in \mathcal{H}^G_{\gamma,\eta}$, $g = k \in K$ and $g_0 = an \in AN$ we have

$$\begin{aligned} f(kan) &= f(gg_0) \\ &= \eta \otimes e^\gamma \otimes 1(n^{-1}a^{-1}) f(k) \\ &= \eta \otimes e^\gamma \otimes 1\left(a^{-1}(an^{-1}a^{-1})\right) f(k) \\ &= e^\gamma(a^{-1}) f(k) \end{aligned}$$

since $an^{-1}a^{-1}$ is in N. Thus elements of $\mathcal{H}^G_{\gamma,\eta}$ are determined by their restrictions to K, and there is a bijection from $\mathcal{H}^G_{\gamma,\eta}$ to a subspace H^K of square integrable functions on $f : K \to \mathbb{C}$ satisfying

$$f(km) = \eta(m^{-1})f(k) \quad \forall k \in K, m \in M.$$

Clearly elements of H^K are in one to one correspondence with a collection H^Z of functions on $\mathbb{T} \cong Z$ square integrable with respect to Lebesgue measure. Indeed, transfering the action of G over to H^Z and completing yields a unitary representation of G in $\mathcal{H}^Z_{\gamma,\eta} = L^2(\mathbb{T})$.

By a similar process replacing K with $\overline{N}$, or by directly relating $\mathbb{R}$ and $\mathbb{T}$, we obtain a representation of G in a Hilbert space $\mathcal{H}^{\overline{N}}_{\gamma,\eta} = L^2(\mathbb{R})$. We now exhibit both realisations.

For $f \in L^2(\mathbb{T})$, $F \in L^2(\mathbb{R})$, $y \in \mathbb{R} \cong \overline{N}$, $\theta \in \mathbb{T} \cong Z$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $\widetilde{\gamma} = (1/4 + iR)$ we calculate

$$\begin{aligned}
\sigma^Z_{\gamma,\eta}(g)f(\theta) &= \Big[(d\cos(\theta/2) + b\sin(\theta/2))^2 + (c\cos(\theta/2) + a\sin(\theta/2))^2\Big]^{-2\widetilde{\gamma}} \\
&\quad \times \mathrm{signum}(d\cos(\theta/2) + b\sin(\theta/2))^{\eta'} \\
&\quad \times f\left(2\tan^{-1}\left(\frac{c\cos(\theta/2) + a\sin(\theta/2)}{d\cos(\theta/2) + b\sin(\theta/2)}\right)\right) \\
\sigma^{\overline{N}}_{\gamma,\eta}(g)F(y) &= (d - by/2)^{-4\widetilde{\gamma}}\mathrm{signum}(d - by/2)^{\eta'} F\left(2\frac{ay/2 - c}{d - by/2}\right).
\end{aligned}$$

Next we calculate the representations of $\mathfrak{g}$ derived from the representations of G above. For $X = \sum_{i=1}^3 x_i e_i \in \mathfrak{g}$ we have

$$\begin{aligned}
d\sigma^Z_{\gamma,\eta}(X)f(\theta) &= 2\widetilde{\gamma}\,(x_1\cos\theta - x_2\sin\theta)\,f(\theta) \\
&\quad + (x_2\cos\theta + x_1\sin\theta - x_3)\,f'(\theta) && (3.1) \\
d\sigma^{\overline{N}}_{\gamma,\eta}(X)F(y) &= 2\widetilde{\gamma}\,(x_1 + x_2 y/2 + x_3 y/2)\,F(y) \\
&\quad + \Big(x_2(y^2/4 - 1) + x_3(y^2/4 + 1) + yx_1\Big)\,F'(y) && (3.2)
\end{aligned}$$

where primes indicate differentiation with respect to θ and y respectively.

Proposition 3.1. *The representations of G and $\mathfrak{g}$ on $L^2(\mathbb{T})$ and $L^2(\mathbb{R})$ are intertwined by $I : D \to C^\infty_c(\mathbb{R})$ where D is a dense subspace of $C^\infty(\mathbb{T})$ and*

$$I(f)(y) = \left(4 + y^2\right)^{-2\widetilde{\gamma}} f\left(-2\tan^{-1}(y/2)\right),$$

for all $f \in D$.

Proof. We may take $D = \{f \in C^\infty(\mathbb{T}) : \pi \notin \mathrm{supp} f\}$. Now I has inverse

$$I^{-1}(F)(\theta) = \left(4 + \tan^2\theta/2\right)^{2\widetilde{\gamma}} F\left(-2\tan(\theta/2)\right)$$

for $F \in C_c^\infty(\mathbb{R})$. Since $I^{-1}(C_c^\infty(\mathbb{R})) \subset D$, $I(D) = C_c^\infty(\mathbb{R})$. Thus D and $I(D)$ are dense subspaces of $L^2(\mathbb{T})$ and $L^2(\mathbb{R})$. As we are primarily interested in representations of $\mathfrak{g}$, we check directly that I respects the $\mathfrak{g}$ actions on these dense subspaces using the following convenient definitions. The remainder of the proof is left to the reader. Let

$$\begin{aligned}
p(y) &= -2\tan^{-1}(y/2)\\
\kappa_0(\theta) &= 2\widetilde{\gamma}\,(x_1\cos\theta - x_2\sin\theta)\\
\kappa_1(\theta) &= x_2\cos\theta + x_1\sin\theta - x_3\\
\tau_0(y) &= 2\widetilde{\gamma}\,(x_1 + x_2 y/2 + x_3 y/2)\\
\tau_1(y) &= x_2(y^2/4 - 1) + x_3(y^2/4+1) + yx_1\\
\varphi(y) &= \left(4+y^2\right)^{-2\widetilde{\gamma}}
\end{aligned}$$

and for $f \in C_c^\infty(\mathbb{R})$, $y \in \mathbb{R}$ put $\theta = p(y)$, $F = I(f)$. Writing $X \cdot f$ for $\sigma^Z_{\gamma,\eta}(X)f$, $X \cdot F$ for $\sigma^{\overline{N}}_{\gamma,\eta}(X)F$ and $t = \tan(\theta/2)$ we have

$$\begin{aligned}
(X \cdot I(f))\,(y) &= \tau_0(y)F(y) + \tau_1(y)F'(y)\\
&= \tau_0(y)\varphi(y) f\circ p(y) + \tau_1(y)\left[\varphi'(y) f\circ p(y) + \varphi(y)p'(y) f'\circ p(y)\right]\\
&= \varphi(y)\tau_0\circ p^{-1}(\theta) + \varphi(y)\tau_1\circ p^{-1}(\theta) f(\theta)\\
&\quad + \tau_1\circ p^{-1}(\theta) f'(\theta) p'\circ p^{-1}(\theta)\\
&= 2\widetilde{\gamma}\varphi(y)\left\{\left[\tau_0(-2t) + \frac{t}{1+t^2}\tau_1(-2t)\right] f(\theta)\right.\\
&\quad \left. + \left[\frac{-1}{1+t^2}\tau_1(-2t)\right] f'(\theta)\right\}\\
&= 2\widetilde{\gamma}\varphi(y)\left\{\left[x_1 - x_2 t - x_3 t + \frac{t}{1+t^2}\left(x_2(t^2-1) + x_3(t^2+1)\right.\right.\right.\\
&\quad \left.\left. -2tx_1)\right] f(\theta) - \frac{1}{1+t^2}\left[x_2(t^2-1) + x_3(t^2+1) - 2tx_1\right] f'(\theta)\right\}\\
&= 2\widetilde{\gamma}\varphi(y)\left\{\left[x_1\frac{1-t^2}{1+t^2} - x_2\frac{-2t}{1+t^2}\right] f(\theta)\right.\\
&\quad \left. - \left[x_1\frac{2t}{1+t^2} + x_2\frac{1-t^2}{1+t^2} - x_3\right] f'(\theta)\right\}\\
&= 2\widetilde{\gamma}\varphi(y)\left\{\left[x_1\cos\theta - x_2\sin\theta\right] f(\theta)\right.\\
&\quad \left. - \left[x_1\sin\theta + x_2\cos\theta - x_3\right] f'(\theta)\right\}\\
&= \varphi(y)\left\{\kappa_0(\theta) f(\theta) + \kappa_1(\theta) f'(\theta)\right\}\\
&= \varphi(y)\,(X\cdot f)\,(p(y))\\
&= I\,(X\cdot f)\,(y), \qquad (3.3)
\end{aligned}$$

showing that I respects the $\mathfrak{g}$ actions on (dense subspaces of) $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$. ■

Next we give two examples of symbol calculi on the coadjoint orbits of G, both adapted to the orbits in the sense given in the introduction. The first is a special case of the construction given in Cahen [3] for real semisimple Lie groups with finite centre. We aim to compare this with a second calculus constructed using the Iwasawa decomposition and better suited to contractions.

4. The Ψ–Calculus on G–Orbits

The prescription for a symbol calculus on the adjoint orbits of $\mathrm{SL}_2(\mathbb{R})$ given in [3] amounts to a transferral of the classical Weyl correspondence on $\mathbb{R}^2$ ([16], [11], [14], [7]) to the orbit O_ψ by the parametrisation

$$\Psi(y,z) = (R + yz/2)\hat{e}_1 + (-z/2 + Ry/2 + zy^2/8)\hat{e}_2 + (-z/2 - Ry/2 - zy^2/8)\hat{e}_3$$

which is a diffeomorphism from $\mathbb{R}^2$ to $\widetilde{O}_\psi = O_\psi \backslash \{-R\hat{e}_1\}$. For this reason we shall refer to it as the Ψ–calculus. We now describe this calculus, essentially following Voros [14].

We say that $f : \widetilde{O}_\psi \to \mathbb{C}$ is *polynomial in* z, or simply *polynomial*, if $f \circ \Psi(y,z)$ is a polynomial in z whose coefficients are smooth functions of y. For such an f we define a differential operator on $C_c^\infty(\mathbb{R})$ by the formula

$$\left(A_f^\Psi \phi\right)(y) = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} e^{-itz} f\circ\Psi\left(y + t/2, z\right)\phi(y+t)\ dt\ dz,$$

for $\phi \in C_c^\infty(\mathbb{R})$. This is precisely Formula 5.1 of Cahen [3] in the special case $G = \mathrm{SL}_2(\mathbb{R})$ with the identifications of Section 2. It is the correspondence introduced by Weyl in [16] applied to $f \circ \Psi$. To see how the definition works, let $f \circ \Psi = u(y)z^a$ for some $a \in \mathbb{N}$, $u \in \mathbb{C}^\infty(\mathbb{R})$. For $\phi \in C_c^\infty(\mathbb{R})$ define $\vartheta_y(t) = u(y + t/2)\phi(y+t)$. Integrating by parts and applying the Fourier inversion theorem we have

$$\begin{aligned}
\left(A_f^\Psi \phi\right)(y) &= \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} e^{-itz} u(y+t/2) z^a \phi(y+t)\ dtdz \\
&= \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} e^{-itz} z^a \vartheta_y(t)\ dtdz \\
&= \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} \left(-i\frac{d}{dt}\right)^a \vartheta_y(t)\ e^{-itz}\ dtdz \\
&= \left[\left(-i\frac{d}{dt}\right)^a \vartheta_y\right]_{t=0} \\
&= \left[\left(-i\frac{d}{dt}\right)^a \left(u(y+t/2)\phi(y+t)\right)\right]_{t=0} \\
&= \sum_{k=0}^{a} c_k \frac{d^k}{dy^k}\ \phi(y)
\end{aligned}$$

where

$$c_k = (-i)^a \binom{a}{k} 2^{(k-a)} \frac{d^{(a-k)}}{dy^{(a-k)}} u(y).$$

This permits an easy calculation of A_f^Ψ for polynomial f. As demanded by [3], the calculus is adapted to the representation (3.2). If $X = \sum_{i=1}^3 x_i e_i \in \mathfrak{g}$ and $\widetilde{X}$ defined by $\widetilde{X}(\xi) = \xi(X)$ for $\xi \in \mathfrak{g}^*$ we have

$$\widetilde{X} \circ \Psi(y,z) = g(y) + h(y)z$$

where $g(y) = 2R(x_1 + x_3 y/2 + x_2 y/2)$ and $h(y) = x_2(y^2/4 - 1) + x_3(y^2/4+1) + x_1 y$. Denoting derivatives with respect to y by primes we therefore have

$$\left(iA_{\widetilde{X}}^\Psi \phi\right)(y) \ = \ i\left\{g(y)\phi(y) - \frac{i}{2}h'(y)\phi(y) - \frac{1}{i}h(y)\phi'(y)\right\}$$

$$\begin{aligned}
&= 2\,(1/4+iR)\,(x_1+x_2y/2+x_3y/2)\,\phi(y) \\
&\quad + \Big(x_1y+x_2(y^2/4-1)+x_3(y^2/4+1)\Big)\,\phi'(y) \\
&= \tau_0(y)\phi(y)+\tau_1(y)\phi'(y) \\
&= (X\cdot\phi)\,(y).
\end{aligned}$$

for all $\phi \in C_c^\infty(\mathbb{R})$. That is,

$$A^{\Psi}_{\widetilde{X}} = \frac{1}{i} d\sigma^{\overline{N}}_{\gamma,\eta}(X) \tag{4.4}$$

on $C_c^\infty(\mathbb{R})$.

5. The Υ–Calculus on G–Orbits

Define $\Upsilon : \mathbb{T}\times\mathbb{R} \to O_\psi$ by

$$\Upsilon(\theta,z) = \left(R\cos\theta - \frac{z}{2}\sin\theta\right)\hat{e}_1 - \left(R\sin\theta + \frac{z}{2}\cos\theta\right)\hat{e}_2 - \left(\frac{z}{2}\right)\hat{e}_3. \tag{5.5}$$

This map is a slight modification of Ψ, where we take $O_\psi = \mathrm{Ad}(KAN)\psi$ in place of $\widetilde{O}_\psi = \mathrm{Ad}(\overline{N}MAN)\psi$. It is a diffeomorphism from $\mathbb{T}\times\mathbb{R}$ to O_ψ.

Say $f : O_\psi \to \mathbb{C}$ is *polynomial in* z if $f\circ\Upsilon(\theta,z)$ is a polynomial in z whose coefficients are smooth functions of θ. For f supported in $\widetilde{O}_\psi$ this agrees with the definition of the previous section since

$$\begin{aligned}
f\circ\Upsilon(\theta,z) &= f\circ\Psi\circ\Psi^{-1}\circ\Upsilon(\theta,z) \\
&= f\circ\Psi\left(-2\tan(\theta/2), R\sin\theta + \frac{z}{2}(\cos\theta+1)\right)
\end{aligned}$$

and

$$\begin{aligned}
f\circ\Psi(y,z) &= f\circ\Upsilon\circ\Upsilon^{-1}\circ\Psi(y,z) \\
&= f\circ\Upsilon\left(-2\tan^{-1}(y/2), \left(1+y^2/4\right)z + Ry\right)
\end{aligned}$$

so $f\circ\Psi$ is polynomial in z with smooth coefficients in y if and only if $f\circ\Upsilon$ is polynomial in z with smooth coefficients in θ. For such an f define an operator A^Υ_f acting on $\phi \in C^\infty(\mathbb{T})$ by

$$\left(A^\Upsilon_f\phi\right)(\theta) = \frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{R}} e^{itz} f\circ\Upsilon\,(\theta+t/2,z)\,\phi(\theta+t)\;dt\;dz. \tag{5.6}$$

If $f\circ\Upsilon(\theta,z) = v(\theta)z^a$ for some $v\in C^\infty(\mathbb{T})$ and $a\in\mathbb{N}$ then by similar manipulations to those of Section 4. we obtain

$$\left(A^\Upsilon_f\phi\right)(\theta) = \sum_{k=0}^{a}\widetilde{c}_k\frac{d^k}{d\theta^k}\;\phi(\theta)$$

where

$$\widetilde{c}_k = (i)^a\binom{a}{k}2^{(k-a)}\frac{d^{(a-k)}}{d\theta^{(a-k)}}v(\theta).$$

This calculus is also adapted to the orbit O_ψ. With $\widetilde{X}$ as before

$$\widetilde{X} \circ \Upsilon(\theta, z) = \widetilde{g}(\theta) + \widetilde{h}(\theta) z$$

with $\widetilde{g}(\theta) = 2R(x_1 \cos\theta - x_2 \sin\theta)$ and $\widetilde{h}(\theta) = -x_1 \sin\theta - x_2 \cos\theta + x_3$. Thus, denoting derivatives with respect to θ by primes,

$$\begin{aligned} \left(iA^{\Upsilon}_{\widetilde{X}}\phi\right)(\theta) &= i\left\{\widetilde{g}(y)\phi(y) + \frac{i}{2}\widetilde{h}'(\theta)\phi(\theta) + i\widetilde{h}(\theta)\phi'(\theta)\right\} \\ &= \left(i\frac{2R}{2\gamma'}\kappa_0(\theta) + \frac{1}{2}\frac{1}{2\gamma'}\kappa_0(\theta)\right)\phi(\theta) + \kappa_1(\theta)\phi'(\theta) \\ &= \kappa_0(\theta)\phi(\theta) + \kappa_1(\theta)\phi'(\theta) \\ &= (X \cdot \phi)(\theta). \end{aligned}$$

That is,

$$A^{\Upsilon}_{\widetilde{X}} = \frac{1}{i} d\sigma^{Z}_{\gamma,\eta}(X). \tag{5.7}$$

on $C^\infty(\mathbb{T})$, showing that the Υ–calculus is adapted to the representation (3.1).

6. Non-equivalence of the Calculi

The operators of the Ψ-calculus act on $C_c^\infty(\mathbb{R})$, whereas the operators of the Υ-calculus act on $C^\infty(\mathbb{T})$. It is natural to ask whether, for every fixed symbol f on O_ψ, the operators A^{Ψ}_f and A^{Υ}_f are intertwined by the maps I and I^{-1} of Section 3. The two calculi would then be equivalent.

We conclude this section by showing that this is not the case. If A and B are operators on $C_c^\infty(\mathbb{R})$ and $C^\infty(\mathbb{T})$ respectively we say that A is equivalent to B and write $A \sim B$ if $A = I \circ B \circ I^{-1}$. Equations (4.4), (5.7) and (3.3) show that

$$A^{\Psi}_{\widetilde{X}} \sim A^{\Upsilon}_{\widetilde{X}}$$

for all $X \in \mathfrak{g}$. No such equivalence holds for arbitrary symbols however. Let $f = \hat{e}_3$. As a special case of the above equation we have $A^{\Psi}_f \sim A^{\Upsilon}_f$, and it is easily seen that $A^{\Upsilon}_{f^2} = A^{\Upsilon}_f \circ A^{\Upsilon}_f$. Suppose $A^{\Psi}_{f^2} \sim A^{\Upsilon}_{f^2}$. This immediately gives

$$\begin{aligned} \left(A^{\Psi}_{f^2} - A^{\Psi}_f \circ A^{\Psi}_f\right)\phi(y) &= \left(I \circ A^{\Upsilon}_{f^2} \circ I^{-1} - I \circ A^{\Upsilon}_f \circ I^{-1} \circ I \circ A^{\Upsilon}_f \circ I^{-1}\right)\phi(y) \\ &= I \circ \left(A^{\Upsilon}_{f^2} - A^{\Upsilon}_f \circ A^{\Upsilon}_f\right) \circ I^{-1}\phi(y) \\ &= 0. \end{aligned}$$

which (we omit the calculation) is false for any ϕ in $C_c^\infty(\mathbb{R})$ not identically zero at any point $y \neq 0$. We conclude that $A^{\Psi}_{f^2}$ is not equivalent to $A^{\Upsilon}_{f^2}$.

7. Representations of the Motion Group

To certain coadjoint orbits of $\widetilde{M(2)}$ we associate a pair of unitary irreducible representations. This collection of representations is parametrised in the same way as the principal series representations of $\mathrm{SL}_2(\mathbb{R})$. They are limits, in a precise sense, of these representations.

The coadjoint representation of $\widetilde{M(2)}$ is neatly expressed as

$$(v, k) \cdot (p, f) = (k \cdot p, k \cdot f + v \wedge k \cdot p)$$

for $v \in V$, $k \in K$, $p \in V^*$ and $f \in \mathfrak{k}^*$. Here $k \cdot f = \mathrm{Ad}_K^*(k) f$, $k \cdot p = \mathrm{Ad}_G(k) p$ and $v \wedge p \in \mathfrak{k}^*$ is defined by

$$\langle v \wedge p, A \rangle = \langle p, A \cdot v \rangle$$

for $A \in \mathfrak{k}$ and $A \cdot v = [A, v]_{\mathfrak{g}}$. The wedge product for $\mathrm{SL}_2(\mathbb{R})$ is

$$(v_1 e_1 + v_2 e_2) \wedge (p_1 \hat{e}_1 + p_2 \hat{e}_2) = (p_1 v_2 - p_2 v_1) \, \hat{e}_3.$$

and the coadjoint orbits of M are cylinders centred along the $\hat{e}_3$ axis. As in Section 3. we fix an orbit C_ψ passing through an element $\psi = R\hat{e}_1 \in \mathfrak{a}^{+*}$ and a unitary irreducible representation η of M. Just as was the case for the principal series representations of $\mathrm{SL}_2(\mathbb{R})$, the pair (ψ, η) determines an irreducible unitary representation of $\widetilde{M(2)}$. We begin with the unitary irreducible representation $e^{i\psi} \otimes \eta$ of $V \rtimes M$ and induce up to a representation

$$\rho_{\psi,\eta} = \mathrm{Ind}_{V \times M}^{V \rtimes K} \, e^{i\psi} \otimes \eta$$

of the whole group $\widetilde{M(2)}$. For details see Dooley [5].

In a manner analogous to Section 3. we realise the representation in the Hilbert space $\mathcal{H}^Z_{\gamma,\eta} = L^2(\mathbb{T})$. Again, details are found in [5]. The realisation on $L^2(\mathbb{T})$ is given by

$$\left(\rho^Z_{\psi,\eta}(v_0, \theta_0)\phi\right)(\theta) = e^{2iR(v_1 \cos\theta + v_2 \sin\theta)} \phi\,(\theta - \theta_0)$$

for $\phi \in L^2(\mathbb{T})$, $\theta, \theta_0 \in \mathbb{T}$ and $v_0 = v_1 e_1 + v_2 e_2 \in V$. Its derivative is

$$d\rho^Z_\psi(\omega, A)\phi(\theta) = 2iR\,(\omega_1 \cos\theta - \omega_2 \sin\theta)\,\phi(\theta) - \mathcal{A}\phi'(\theta) \tag{7.8}$$

for $\omega = \omega_1 e_1 + \omega_2 e_2 \in V$, $A = \mathcal{A} e_3 \in \mathfrak{k}$, $\theta \in Z$ and $\phi \in L^2(\mathbb{T})$.

8. The Γ–Calculus on $V \rtimes K$ Orbits

Define the diffeomorphism $\Gamma : \mathbb{T} \times \mathbb{R} \to C_\psi$ by

$$\Gamma(\theta, z) = (R\cos\theta)\,\hat{e}_1 - (R\sin\theta)\,\hat{e}_2 - (z/2)\,\hat{e}_3. \tag{8.9}$$

Say $f : C_\psi \to \mathbb{C}$ *is polynomial in* z if $f \circ \Gamma$ is a polynomial in z whose coefficients are smooth functions of θ. For such an f define an operator A^Γ_f on $C^\infty(\mathbb{T})$ by

$$A^\Gamma_f \phi(\theta) = \frac{1}{2\pi} \int_{\mathbb{T}\times\mathbb{R}} e^{itz} f \circ \Gamma(\theta + t/2, z)\phi(\theta + t) \; dz \; dt. \tag{8.10}$$

This is almost equivalent to Cahen's procedure. The first difference is a factor of R in our parametrisation in the $\hat{e}_3$ direction, which is compensated for by elimination of a factor of R in the exponent itz. The second difference is that we have replaced $f \circ \Gamma(\theta, z)$ with $f \circ \Gamma(\theta + t/2, z)$ in the integration.

We proceed in a similar fashion to Section 5. Suppose $f(\theta, z) = w(\theta)z^a$ for some $w \in \mathbb{C}^\infty(\mathbb{T})$, $a \in \mathbb{N}$. For $\phi \in C^\infty(\mathbb{T})$ define $\vartheta_\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ by $\vartheta_\theta(t, z) = w(\theta + t/2)\phi(\theta + t/2)\chi_{(-\pi,\pi]}(t)$ where $\chi_{(-\pi,\pi]}$ is a characteristic function. We have

$$\begin{aligned} A_f^\Gamma \phi(\theta) &= \frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{T}} e^{itz} z^a w(\theta + t/2)\phi(\theta + t/2)\; dt\; dz \\ &= \frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{R}} e^{itz} z^a \vartheta_\theta(t)\; dt\; dz \\ &= \frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{R}} \left(i\frac{d}{dt}\right)^a \vartheta_\theta(t)\; e^{itz}\; dt\; dz \\ &= \left[\left(i\frac{d}{dt}\right)^a \vartheta_\theta\right]_{t=0} \\ &= \left[\left(i\frac{d}{dt}\right)^a (w(\theta + t/2)\phi(\theta + t))\right]_{t=0} \\ &= \sum_{k=0}^{a} \tilde{\tilde{c}}_k \frac{d^k}{d\theta^k}\phi(\theta) \end{aligned}$$

where

$$\tilde{\tilde{c}}_k = (i)^a \binom{a}{k} 2^{(k-a)} \frac{d^{(a-k)}}{d\theta^{(a-k)}} w(\theta).$$

In particular, if $f = \widetilde{X}$ where $X = (\omega, A) = (\omega_1 e_1 + \omega_2 e_2, \mathcal{A}e_3) \in V + \mathfrak{k}$ then

$$\widetilde{X} \circ \Gamma(\theta, z) = 2R(\omega_1 \cos\theta - \omega_2 \sin\theta) + \mathcal{A}z$$

and we calculate

$$iA^\Gamma_{\widetilde{X}}\phi(\theta) = 2iR(\omega_1 \cos\theta - \omega_2 \sin\theta)\phi(\theta) - \mathcal{A}\phi'(\theta).$$

That is,

$$A^\Gamma_{\widetilde{X}} = \frac{1}{i} d\rho^Z_\psi(X), \tag{8.11}$$

on $C^\infty(\mathbb{T})$ showing that the calculus is adapted to the representation (7.8).

9. Contraction of G to $V \rtimes K$

In Section 1 we mentioned the contraction maps of Dooley defined by

$$\begin{aligned} \pi_\lambda(v, k) &= \exp_G(\lambda v) \cdot k \\ d\pi_\lambda(\omega, A) &= \lambda\omega + A \end{aligned}$$

for $v \in V$, $k \in K$, $\omega \in V$, $A \in \mathfrak{k}$ and $\lambda \in \mathbb{R}^+$. Here we briefly illustrate how these may be used to contract the Lie algebra structure and representations of $\mathrm{SL}_2(\mathbb{R})$ to those of its motion group $\widetilde{M(2)}$. These results are special cases of theorems in Dooley and Rice [5].

Proposition 9.1. *For all* $X, Y \in \widetilde{\mathbf{m(2)}}$*,*

$$\lim_{\lambda\to 0} d\pi_\lambda^{-1}\left[d\pi_\lambda(X), d\pi_\lambda(Y)\right]_{\mathfrak{g}} = [X,Y]_{\widetilde{\mathbf{m(2)}}}$$

Proposition 9.2. *For all* $\phi \in C^\infty(\mathbb{T})$*,*

$$\sup_{\theta\in\mathbb{T}}\left|d\sigma^Z_{\psi/\lambda,\eta}\circ d\pi_\lambda(\omega,A)\phi(\theta) - d\rho^Z_\psi(\omega,A)\phi(\theta)\right| \to 0$$

as $\lambda \to 0$ *for all* $(\omega, A) \in \widetilde{\mathbf{m(2)}}$*. Moreover the limit is obtained uniformly on compact subsets of* $\widetilde{\mathbf{m(2)}}$*.*

Proposition 9.3. *For all* $\phi \in C^\infty(\mathbb{T})$

$$\sup_{\theta\in\mathbb{T}}\left|\sigma^Z_{\psi/\lambda,\eta}\circ d\pi_\lambda(v,\theta')\phi(\theta) - \rho^Z_\psi(v,\theta')\phi(\theta)\right| \to 0$$

as $\lambda \to 0$ *uniformly on compact subsets of* $\widetilde{M(2)}$*.*

Proof. These properties are routinely verified. We check only the second claim, which will be of greatest interest to us. Suppose $\phi \in C^\infty(\mathbb{T})$. If $\omega = \omega_1 e_1 + \omega_2 e_2 \in V$ and $A = \mathcal{A}e_3 \in \mathfrak{k}$ then

$$\begin{aligned} d\sigma^Z_{\psi/\lambda,\eta}\left(d\pi_\lambda(\omega,\mathcal{A})\right)\phi(\theta) &= d\sigma^Z_{\psi/\lambda,\eta}\left(\lambda\omega,\mathcal{A}\right)\phi(\theta) \\ &= 2\left(1/4 + iR/\lambda\right)\left(\lambda\omega_1\cos\theta - \lambda\omega_2\sin\theta\right)\phi(\theta) \\ &\quad + \left(\lambda\omega_2\cos\theta + \lambda\omega_1\sin\theta - \mathcal{A}\right)\phi'(\theta) \\ &= 2iR\left(\omega_1\cos\theta - \omega_2\sin\theta\right)\phi(\theta) - \mathcal{A}\phi'(\theta) + O(\lambda) \\ &= d\rho^Z_\psi(\omega,A)\phi(\theta) + O(\lambda) \end{aligned}$$

where

$$O(\lambda) = \lambda\left\{\frac{1}{2}\left(\omega_1\cos\theta - \omega_2\sin\theta\right)\phi(\theta) + \left(\omega_1\sin\theta + \omega_2\cos\theta\right)\phi'(\theta)\right\}.$$

Therefore

$$\left|d\sigma^Z_{\psi/\lambda,\eta}\circ d\pi_\lambda(\omega,A)\phi(\theta) - \rho^Z_\psi(\omega,A)\phi(\theta)\right| \le \lambda|\omega|\left(|\phi(\theta)| + |\phi'(\theta)|\right)$$

for all $\phi \in C^\infty(\mathbb{T})$,$\theta \in \mathbb{T}$. Property 2 follows. ■

10. Contraction of the Υ–Calculus

Fix a $V \rtimes K$–orbit C_ψ passing through $\psi \in \mathfrak{a}^{+*}$ and suppose $f : C_\psi \to \mathbb{C}$ is polynomial in z of degree d. That is,

$$f\circ\Gamma(\theta,z) = \sum_{a=0}^{d} u_a(\theta)z^a$$

say, where each $u_a \in C^\infty(\mathbb{T})$. For $\lambda > 0$ the G–orbit $O_{\psi/\lambda}$ passing through ψ/λ is parametrised by

$$\Upsilon_\lambda(\theta, z) = \left(\frac{R}{\lambda}\cos\theta - \frac{z}{2}\sin\theta\right)\hat{e}_1 - \left(\frac{z}{2}\sin\theta + \frac{R}{\lambda}\cos\theta\right)\hat{e}_2 - \left(\frac{z}{2}\right)\hat{e}_3.$$

Suppose also that $\mathcal{F} = \left\{f_\lambda : O_{\psi/\lambda} \to \mathbb{C}\right\}_{\lambda\in\mathbb{R}^+}$ is a family of polynomials in z of degree equal to the degree of f :

$$f_\lambda \circ \Upsilon_\lambda(\theta, z) = \sum_{a=0}^{d} v_a^\lambda(\theta) z^a$$

for $v_a^\lambda \in C^\infty(\mathbb{T})$. If $(v_a^\lambda - u_a)$ and all its derivatives of degree less than or equal to d tend uniformly to zero on $\mathbb{T}$ as λ tends to zero then we say that the family $\mathcal{F}$ *approximates* f.

Proposition 10.1. *Let f be a polynomial symbol on a $V \rtimes K$–orbit C_ψ and suppose $\mathcal{F} = \{f_\lambda\}_{\lambda\in\mathbb{R}}$ is an approximating family of polynomial symbols on G–orbits $O_{\psi/\lambda}$ as above. Let A_f^Γ (respectively $A_{f_\lambda}^\Upsilon$) denote the operator on $C^\infty(\mathbb{T})$ corresponding to f (respectively f_λ) under the Γ–calculus on C_ψ (respectively Υ–calculus on $O_{\psi/\lambda}$). Then for all $\phi \in C^\infty(\mathbb{T})$*

$$\sup_{\theta\in\mathbb{T}} \left|A_{f_\lambda}^\Upsilon\phi(\theta) - A_f^\Gamma\phi(\theta)\right| \to 0$$

as $\lambda \to 0$.

Proof. With the same notation as above and recalling the results of sections 5. and 7. we have

$$A_{f_\lambda}^\Upsilon\phi(\theta) - A_f^\Gamma\phi(\theta) = \sum_{a=0}^{d}\sum_{k=0}^{d}(i)^a\binom{a}{k}2^{(k-a)}\left(\frac{d}{d\theta}\right)^{(a-k)}(v_a^\lambda - u_a)(\theta)\left(\frac{d}{d\theta}\right)^k(\phi)(\theta)$$

for $\phi \in C^\infty(\mathbb{T})$,$\theta \in \mathbb{T}$. As all derivatives of ϕ appearing above are uniformly bounded in $\mathbb{T}$ and all derivatives of $(v_a^\lambda - u_a)$ tend uniformly to zero in $\mathbb{T}$, each term tends uniformly to zero on $\mathbb{T}$. ∎

Corollary 10.2. *For all $\phi \in C^\infty(\mathbb{T})$ and $(\omega, A) \in \widetilde{\mathbf{m(2)}}$*

$$\sup_{\theta\in\mathbb{T}} \left|d\sigma_{\psi/\lambda,\eta}^Z \circ d\pi_\lambda(\omega, A)\phi(\theta) - d\rho_\psi^Z(\omega, A)\phi(\theta)\right| \to 0$$

as $\lambda \to 0$.

Proof. For $X = \omega_1 e_1 + \omega_2 e_2 + \mathcal{A}e_3 \in \widetilde{\mathbf{m(2)}}$,$\lambda \in \mathbb{R}^+$ let $f = \widetilde{X}$ and define the family $\mathcal{F} = d\left\{f_\lambda = \widetilde{\pi_\lambda(X)}\right\}_{\lambda\in\mathbb{R}^+}$. Then

$$\begin{aligned}
f_\lambda \circ \Upsilon_\lambda(\theta, z) &= d\widetilde{\pi_\lambda(X)} \circ \Upsilon_\lambda(\theta, z) \\
&= \beta\Big(\lambda\omega_1 e_1 + \lambda\omega_1 e_2 + \mathcal{A}e_3, \\
&\qquad \left(\frac{R}{\lambda}\cos\theta - \frac{z}{2}\sin\theta\right)e_1 - \left(\frac{R}{\lambda}\sin\theta + \frac{z}{2}\cos\theta\right)e_2 - \left(\frac{z}{2}\right)e_3\Big) \\
&= 2(R\omega_1\cos\theta - R\omega_2\sin\theta) + (\mathcal{A} - \lambda\omega_1\sin\theta - \lambda\omega_2\cos\theta)z,
\end{aligned}$$

whereas

$$\begin{aligned} f \circ \Gamma(\theta, z) &= \widetilde{X} \circ \Gamma(\theta, z) \\ &= 2(R\omega_1 \cos\theta - R\omega_2 \sin\theta) + \mathcal{A}z. \end{aligned}$$

so $\mathcal{F}$ approximates f. By Proposition 1,

$$\sup_{\theta \in \mathbb{T}} \left| A^{\Upsilon}_{f_\lambda} \phi(\theta) - A^{\Gamma}_{f} \phi(\theta) \right| \to 0$$

as $\lambda \to 0$ for all $\phi \in C^\infty(\mathbb{T})$. Since $A^{\Upsilon}_{f_\lambda} = A^{\Upsilon}_{\widetilde{\pi_\lambda(X)}} = -i\ d\sigma^{Z}_{\gamma,\eta} \circ d\pi_\lambda(X)$ and $A^{\Gamma}_{f} = A^{\Gamma}_{\widetilde{X}} = -i\ d\rho^{Z}_{\psi}(X)$ the result follows. ■

We have thus essentially recovered Proposition 9.2. Uniform convergence on compact subsets on $\widetilde{\mathbf{m(2)}}$ can be demonstrated by a slight generalisation of Proposition 1, which we leave to the reader.

The form of the above proposition above was chosen purely to illustrate how the contraction results of Dooley and Rice [5] at the Lie algebra level follow from the contraction of adapted calculi. We have however, a more direct approach.

Proposition 10.3. *Let f be a function on C_ψ, polynomial in z. Let $\widetilde{f} = f \circ \Gamma \circ \Upsilon^{-1}$. Then $\widetilde{f}$ is a function of O_ψ, polynomial in z and*

$$A^{\Gamma}_{f} = A^{\Upsilon}_{\widetilde{f}}$$

on $C^\infty(\mathbb{T})$. Conversely, if $\widetilde{f}$ is a function on O_ψ, polynomial in z and $f = \widetilde{f} \circ \Upsilon \circ \Gamma^{-1}$ then f is a function on C_ψ polynomial in z and

$$A^{\Upsilon}_{\widetilde{f}} = A^{\Gamma}_{f}.$$

Proof. As $\widetilde{f} \circ \Upsilon = f \circ \Gamma$, $\widetilde{f}$ is polynomial in z if and only if f is polynomial in z. The equality of operators follows immediately from the defining equations (5.6) and (8.10). ■

11. Non-polynomial Symbols

In Section 5 we said that $f : O_\psi \to \mathbb{C}$ was polynomial in z if $f \circ \Upsilon$ was polynomial in z. In the same manner [3] we define L^p functions, rapidly decreasing functions and distributions on O_ψ. Of interest to us here will be the Hilbert spaces $L^2(O_\psi)$ consisting of those functions f on O_ψ for which $f \circ \Upsilon \in L^2(\mathbb{T} \times \mathbb{R})$. The aim of this section is to extend the Υ calculus to these functions. We first recall a few facts from [13].

The family $\mathcal{L}_2(\mathcal{H})$ of (compact) operators A on a Hilbert space $\mathcal{H}$ satisfying $\mathrm{Tr} A^* A < \infty$ is a Hilbert space with respect to the inner product

$$(A, B) = \mathrm{Tr}(A^* B).$$

Such operators are called *Hilbert-Schmidt*. In the case $\mathcal{H} = L^2(\mathcal{M}, d\mu)$ for $\langle \mathcal{M}, \mu \rangle$ a measure space the Hilbert-Schmidt operators are precisely those with kernels K_A in $L^2(\mathcal{M} \times \mathcal{M}, d\mu \otimes d\mu)$, the map $A \mapsto K_A$ being a Hilbert space isometry from $\mathcal{L}_2(L^2(\mathcal{M}, d\mu))$ to $L^2(\mathcal{M} \times \mathcal{M}, d\mu \otimes d\mu)$ [13].

Lemma 11.1. *Equation* (5.6) *of* Section 5 *defines a map from* $L^2(O_\psi)$ *to* $\mathcal{L}_2(L^2(\mathbb{T}))$. *Further,*

$$\|A_f^\Upsilon\|_{\mathcal{L}_2(L^2(\mathbb{T}))} \leq \|f\|_{L^2(O_\psi)}$$

holds for all $f \in L^2(O_\psi)$.

Proof. (Similar to [3], [14] and [11]). Define the partial Fourier transform $\mathcal{F}_2 : L^2(\mathbb{T}\times\mathbb{R}) \to L^2(\mathbb{T}\times\mathbb{R})$ by

$$\mathcal{F}_2(f)(\theta, z) = \frac{1}{2\pi}\int_{\mathbb{R}} f(\theta, z)e^{-ixz}dz.$$

Then $\mathcal{F}_2$ is an isometry. Define also $U : L^2(\mathbb{T}\times\mathbb{R}) \to L^2(\mathbb{T}\times\mathbb{T})$ by

$$U(f)(\theta,\theta') = f\left((\theta+\theta')/2, \theta-\theta'\right).$$

As is easily checked, $\|U(f)\|_{L^2(\mathbb{T}\times\mathbb{T})} \leq \|f\|_{L^2(\mathbb{T}\times\mathbb{R})}$. Finally let $K = U \circ \mathcal{F}_2$. Then for $f \in L^2(O_\psi)$ we have

$$\|K(f\circ\Upsilon)\|_{L^2(\mathbb{T}\times\mathbb{T})} \leq \|\mathcal{F}_2(f\circ\Upsilon)\|_{L^2(\mathbb{T}\times\mathbb{R})} = \|f\circ\Upsilon\|_{L^2(\mathbb{T}\times\mathbb{R})} =: \|f\|_{L^2(O_\psi)}.$$

So $K(f\circ\Upsilon)$ is in $L^2(\mathbb{T}\times\mathbb{T})$ and is thus the kernel of a Hilbert-Schmidt operator on $L^2(\mathbb{T})$. Denoting this operator by $\widetilde{A}_f$ we then have

$$\|\widetilde{A}_f\|_{\mathcal{L}_2(L^2(\mathbb{T}))} = \|K(f\circ\Upsilon)\|_{L^2(\mathbb{T}\times\mathbb{T})} \leq \|f\|_{L^2(O_\psi)},$$

so if we can show that $\widetilde{A}_f = A_f^\Upsilon$ we are done. Using Fubini's theorem and the substitution $t = \theta' - \theta$ we have

$$\begin{aligned}
\widetilde{A}_f\phi(\theta) &= \int_{\mathbb{T}} K(f\circ\Upsilon)(\theta,\theta')\phi(\theta')\, d\theta' \\
&= \int_{\mathbb{T}} \frac{1}{2\pi}\int_{\mathbb{R}} f\circ\Upsilon\left((\theta+\theta')/2, z\right) e^{-iz(\theta-\theta')}\, dz\; \phi(\theta')\, d\theta' \\
&= \frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{R}} f\circ\Upsilon(\theta+t/2)\phi(\theta+t)e^{izt}\, dtdz \\
&= A_f^\Upsilon\phi(\theta)
\end{aligned}$$

for all $\phi \in C^\infty(\mathbb{T})$, $\theta \in \mathbb{T}$. ■

We may similarly extend the Ψ–calculus (see [3], [11] or [14]) and the Γ–calculus to functions in $L^2(O_\psi)$ and $L^2(C_\psi)$ respectively. Taking this as done, we continue in the same vein as Section 10.

Proposition 11.2. *Suppose* $f \in L^2(C_\psi)$ *and* $\mathcal{F} = \left\{f_\lambda : O_{\psi/\lambda} \to \mathbb{C}\right\}_{\lambda\in\mathbb{C}}$ *such that* $f_\lambda \in L^2(O_{\psi/\lambda})$. *We say that* $\mathcal{F}$ *approximates* f *in* L^2 *sense if*

$$\|f_\lambda\circ\Upsilon_\lambda - f\circ\Gamma\|_{L^2(\mathbb{T}\times\mathbb{R})} \to 0$$

as $\lambda \to 0$. *In this case* $A_{f_\lambda}^\Upsilon$ *tends to* A_f^Γ *in Hilbert-Schmidt norm.*

Proof. By Lemma 11.1 the operators $A^{\Upsilon}_{f_\lambda}$ tend in Hilbert-Schmidt norm to the operator $A^{\Upsilon}_{f\circ\Gamma\circ\Upsilon^{-1}}$. By inspection of the defining equations (5.6) and (8.10), this operator is precisely A^{Γ}_{f}. ■

Proposition 11.3. *If $f \in L^2(C_\psi)$ then $\widetilde{f} := f \circ \Gamma \circ \Upsilon^{-1}$ is in $L^2(O_\psi)$ and*

$$A^{\Gamma}_{f} = A^{\Upsilon}_{\widetilde{f}}.$$

Conversely if $\widetilde{f} \in L^2(O_\psi)$ then $f = \widetilde{f} \circ \Upsilon \circ \Gamma^{-1} \in L^2(C_\psi)$ and

$$A^{\Upsilon}_{\widetilde{f}} = A^{\Gamma}_{f}.$$

Proof. As $\widetilde{f}\circ\Upsilon = f\circ\Gamma$, $\widetilde{f}$ is an element of $L^2(O_\psi)$ if and only if f is an element of $L^2(C_\psi)$. Equality of operators follows directly from the defining equations (5.6) and (8.10). ■

12. Concluding Remarks

The authors expect the results of this paper to apply to all pairs (G, K) discussed in the introduction. For G a general semisimple Lie group with finite centre the subgroup M is of course no longer discrete (we recall that M was the centraliser of A in K, where K, A and N provide an Iwasawa decomposition). Whilst this introduces a fair amount of work in construction of the adapted calculi, it should not complicate the contraction process at all, since M may be treated in an identical fashion in both cases (as in Cahen [3]).

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