Barbell Portfolios

What do They Accidentally Optimize?

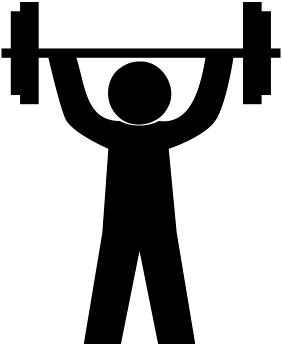
Peter Cotton

Intech Investments

Overview

Overview

The barbell portfolio has been offered as investment advice from time to time, but without rigorous analysis.



This short note presents a surprising analytical result that establishes what a barbell portfolio *accidentally* optimizes.

The original barbell

The original barbell strategy was advocated for bond investing only. A definitive source is not known to us, nor a consensus motivation. Some heuristic justifications have been given:

> The short end securities are turned over quickly, rolling them into new short term securities. Usually this leads to a higher increase in value. The purpose of this is that mid-range securities can often be mispriced for the risks involved; they have longish maturities yet often their coupon is not a lot above T-bills.[1]

This plausible *sounding* advice has not to our knowledge been the subject of a lot of empirical or theoretical study.

A simple model for interest rates

Assume a lattice of zero coupon bonds with prices

$$B^i(t) = B(t; t + \tau^i)$$

and integer time to maturities $\tau^i = i$ as i ranges from 1 to n years. We assume that all bonds are priced off the same piecewise constant forward curve with knot points also at integer years. We write

$$B^i(t) = \exp\left(- \int_t^{t+i} f(t,s) ds \right) \qquad (1)$$

and assume further that the changes in forward rates $f(t,s)$ at time t for different years are independent. We presume the forward rates are driven by standard Brownian motion with the same standard deviation η.

Bond price dynamics

Rates may have non-trivial drift but here it suffices to observe that the vector of bonds has dynamics given by

$$d \begin{bmatrix} logB^1(t) \\ logB^2(t) \\ \dots \\ logB^n(t) \end{bmatrix} = \begin{bmatrix} \gamma^1(t) \\ \gamma^2(t) \\ \vdots \\ \gamma^n(t) \end{bmatrix} dt + \eta \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} dW^1(t) \\ dW^2(t) \\ \dots \\ dW^n(t) \end{bmatrix}$$

or more succinctly

$$d(logB) = \gamma \; dt + \eta J \; dW \tag{2}$$

for scalar constant η, an n by n matrix J (implicitly defined by the above) and some drift coefficients γ that won't enter the subsequent calculations.

A mysterious quantity to optimize

We consider a portfolio of these bonds with weights π summing to unity. For any such portfolio we define a quantity

$$E(\pi) = \sum_{i=1}^{n} \pi_i \sigma_{ii} - 2 \sum_{i,j=1}^{n} \pi_i \pi_j \sigma_{ij} + \sum_{i=1}^{n} \pi_i^2 \sigma_{ii}$$

which we call the *Empirical Skepticism* of the portfolio. Here σ_{ij} is the log-asset covariance equal to η^2 multiplied by the i,j'th element of $JJ^\top$.

We shall choose the portfolio π that maximises $E(\pi)$ because, why not?

Barbell Portfolios

We begin by observing that $(JJ')_{i,j} = min(i,j)$ by inspection:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \dots & n \end{bmatrix}$$

and with this simplification of σ_{ij} we have

$$E(\pi) \propto \sum_{i=1}^{n} i\pi_i - 2 \sum_{i,j=1}^{n} min(i,j)\pi_i\pi_j + \sum_{i=1}^{n} i\pi_i^2$$

It is easier to see that this quantity is maximized by choosing a barbell portfolio where all π_i are zero except the first and last.

Claim: Barbell maximizes $E(\pi)$

We let

$$\pi^* = \begin{bmatrix} 1/2 \\ 0 \\ \vdots \\ 0 \\ 1/2 \end{bmatrix}$$

denote the so-called barbell portfolio. The claim is that $E(\pi^*) \geq E(\pi)$ for all portfolios π.

It will be useful to introduce the notation $u_i = \sum_{j=i+1}^{n} \pi_j$ as the sum of portfolio weights leaving out the first i. In the calculations to follow we exploit the tautology $u_0 = 1$.

Observe that $E(\pi)$ can be re-written as follows

$$
\begin{aligned}
E(\pi) &= -2\sum_{i,j=1}^{n} min(i,j)\pi_i\pi_j + \sum_{i=1}^{n} i\pi_i^2 + \sum_{i=1}^{n} i\pi_i \\
&= -(\pi_1 + \pi_2 + \ldots + \pi_n)^2 \\
&\quad +(\pi_1 + \pi_2 + \ldots + \pi_n) \\
&\quad -(\pi_2 + \ldots + \pi_n)^2 + (\pi_2 + \ldots + \pi_n) \\
&\quad \vdots \\
&\quad -(\pi_{n-1} + \pi_n)^2 + (\pi_{n-1} + \pi_n) \\
&\quad -(\pi_n)^2 + \pi_n \\
&= \sum_{i=0}^{n-1}(-u_i^2 + u_i) \\
&= \sum_{i=1}^{n-1}(-u_i^2 + u_i) \\
&= \sum_{i=1}^{n-1}\left(-(u_i - 1/2)^2 + 1/4\right) \\
&= \frac{n-1}{4} - \sum_{i=1}^{n-1}(u_i - 1/2)^2
\end{aligned}
$$

Steps are verified by counting the number of times each π_i and $\pi_i \pi_j$ occurs. The expression

$$\frac{n-1}{4} - \sum_{i=1}^{n-1} (u_i - 1/2)^2$$

is clearly maximized by setting $u_1 ... u_n$ equal to $1/2$. By back substitution beginning with π_n this implies $\pi = \pi^*$ as claimed.

What does this tell us?

We have established that under a stylized interest rate model, a barbell portfolio comprising an equal weighting of the lowest maturity bond and longest maturity bond maximizes Empirical Skepticism $E(\pi)$.

Might this help us understand heuristic motivations for barbell portfolios, properties of their returns or potential shortcomings?

Sacrificing long run return

We know from Stochastic Portfolio Theory[2] that

$$R(\pi) = \sum_{i=1}^{n} \pi_i \sigma_{ii} - \sum_{i,j=1}^{n} \pi_i \pi_j \sigma_{ij}$$

is the *excess return* for the portfolio π. Maximising this quantity will maximize long run portfolio return. On the other hand Empirical Skeptics maximize

$$E(\pi) = \sum_{i=1}^{n} \pi_i \sigma_{ii} + \sum_{i=1}^{n} \pi_i^2 \sigma_{ii} - 2 \sum_{i,j=1}^{n} \pi_i \pi_j \sigma_{ij} \sigma_{ii}$$

where we observe that the covariance term is given a weight of 2 instead of 1.

One could *interpret* a desire to maximize $E(\pi)$ as a tradeoff between long term growth and reduced portfolio variance (though interestingly the difference picks up the between-asset terms only, not the variances).

To be continued...perhaps...

Some things that could be covered...

1. Equity bar-bell portfolios and investment advice extending beyond fixed income
2. Simplified proof that all reasonable porfolios contain *every* asset - thus countering an entire class of investment strategies of which barbell is one example
3. Simple explanation of why barbell advice is not coordinate independent and thus cannot be entirely well motivated.
4. Are there surprising uses for portfolio objective functions - even those not obviously well motivated?

References

[1] Robert Farrington, “The Barbell Investment Strategy,” 2019. [Online]. Available: https://thecollegeinvestor.com/713/the-barbell-investment-strategy/

[2] R. Fernholz, “Stochastic Portfolio Theory: an Overview,” Tech. Rep., 2008.

Disclaimer

This material is for general informational purposes only and should not be construed as investment advice, as an offer or solicitation of an offer to sell or buy, or as an endorsement, recommendation, or sponsorship of any company, security, advisory service, or fund nor does it purport to address the financial objectives or specific investment needs of any individual reader, investor, or organization.