Sparsity 2017 - Homework 2

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Problem 1 Let $H_{\langle r\rangle}$ and $H_{\langle !r\rangle}$ denote the r-subdivision and the exact r-subdivision of graph H, respectively and let $\subseteq$ be a subgraph relation. Let's also define for a vertex v of degree 2 its "crushing" as the operation of removing v and adding edge between its two neighbours.

($\Rightarrow$) Let's assume that $\mathcal{C}$ has bounded expansion. From the lecture we know that via topological minors this means that there is a function $f \in \mathbb{N}^{\mathbb{N}}$ such that for all $r \in \mathbb{N}$ and all $G \in \mathcal{C}$ we have $\widetilde{\nabla}_r(G) \leqslant f(r)$. If $H_{\langle !r\rangle}$ is a subgraph of some graph $G \in \mathcal{C}$ then trivial observation yields that $H \preccurlyeq^{top}_{\lceil \frac{r}{2}\rceil} G$ as every subgraph is 0-depth topological minor, every graph is a topological minor of its subdivision and $\preccurlyeq^{top}$ is transitive (or simply by crushing - and counting - the vertices). So let's fix r. If $H_{\langle !r\rangle} \subseteq G$ then $H \preccurlyeq^{top}_{\lceil \frac{r}{2}\rceil} G$ and because $\mathcal{C}$ has bounded expansion then average density of H is bounded by $f(\lceil \frac{r}{2}\rceil)$ so its average degree is at most $c_r := 2f(\lceil \frac{r}{2}\rceil)$.

($\Leftarrow$) Let's assume the condition holds for $\mathcal{C}$ and $H \preccurlyeq^{top}_r G$ for some H and $G \in \mathcal{C}$, which is equivalent to $H_{\langle 2r\rangle} = (H_{\langle 2r\rangle})_{\langle !0\rangle} \subseteq G$. Let's denote $m := |E(H_{\langle 2r\rangle})|$ and $n := |V(H_{\langle 2r\rangle})|$. Hence we have $\frac{m}{n} \leqslant c_0$ from the condition. From this we want to derive the bound for density of the minor H: we simply crush all the "extra" vertices that the subdivision added. Any such crushing decreases both m and n by one so at the time we get to original H we have $|V(H)| = n-r$ and $|E(H)| = m-r$. Now we can bound the density of H as follows: $\frac{m-r}{n-r} = \frac{nc_0-r}{n-r} = c_0 + \frac{r(c_0-1)}{n-r} \leqslant c_0 + r(c_0-1)$. As H and G have been chosen arbitrary, this gives a bound for density of any r-depth topological minor which implies $\mathcal{C}$ has a bounded expansion.

Problem 2 We want to prove that $\mathcal{C}$ is nowhere dense iff for every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a constant $c_{r,\varepsilon}$ such that every graph $G \in \mathcal{C}\nabla r$ contains at most $c_{r,\varepsilon}|V(G)|^{1+\varepsilon}$ distinct cliques.

($\Rightarrow$) Let's assume that $\mathcal{C}$ is nowhere dense and let's fix r. Then there is t_r such that $K_{t_r} \in \mathcal{C}\nabla r$ and $K_{t_r+1} \notin \mathcal{C}\nabla r$. The class $\mathcal{C}\nabla r$ can be partitioned into the subclasses of graphs of certain number of edges. Among each such subclass there are some graphs (i.e. at least one) with the largest number of distinct cliques possible - so let's think how those graphs are built. Let n and m be the number of - respectively - vertices and edges of the graph we build. It seems pretty clear that to arrive at the largest number of distinct cliques we need to "cluster" the edges as much as possible (and allowed): if $m = t_r$ then clearly forming H as K_{t_r} yields the largest number of cliques equal to 2^{t_r}. Then one can add one vertex and keeps adding edges connecting the new vertex with the K_{t_r} as long as K_{t_r+1} is not formed but I do not claim that this is optimal strategy. I claim that whatever the optimal clique-boosting strategy is, one can not boost the clique number more than linearly on the number of edges added, specifically by more than $\beta_r 2^{t_r} m^{'}$ where $m^{'}$ is the additional number of edges used and β_r is some constant added for safeguard. That's because each edge cannot contribute more than 2^{t_r} new distinct cliques as otherwise there would be a larger clique than K_{t_r} in the graph. So let H be the graph and κ_H the number of distinct cliques it contains. Then $\kappa_H \leqslant \beta_r 2^{t_r} m = \xi_r m$. But from the lecture on edge density in nowhere dense classes we know that for every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a function $f(r, \varepsilon)$ such that every graph $G \in \mathcal{C}\nabla r$ has at most $f(r, \varepsilon)|V(G)|^{1+\varepsilon}$ edges ("almost linearity" of nowhere dense classes). Hence $\kappa_H \leqslant \xi_r f(r, \varepsilon) n_H^{1+\varepsilon} = c_{r,\varepsilon} n_H^{1+\varepsilon}$.

($\Leftarrow$) Indirect proof: let's assume $\mathcal{C}$ is somewhere dense. Then there exists such r that $K_s \in \mathcal{C}\nabla r$ for all $s \in \mathbb{N}$. But $\kappa_{K_s} = 2^s$ so that contradicts the stated condition that it contains at most $c_{r,\varepsilon} s^{1+\varepsilon}$ distinct cliques.