Sparsity 2017 - Homework 1

Michał Szafraniuk

22 października 2017

Problem 1 Let's set $|V(G)| = n$, $arb(G) = a$ and $deg(G) = k$ and let σ be the vertex ordering such that $deg(G, \sigma) = k$. By definition, each vertex in σ has at most k σ-smaller neighbours, i.e. at most k edges going out into, say, the left from it. At any vertex v we can label each edge going left from v by consecutive numbers from $\{1, 2, ..., k\}$. So let's do the labelling one by one starting from the σ-largest vertex. We end up with all the edges labelled by some $j \in \{1, 2, ..., k\}$.

We claim that for any label chosen, a set of all edges labelled by it is a forest. So let's suppose it is not a forest, i.e. it contains a cycle $C = u_1u_2..u_lu_1$, where $l \geqslant 2$ and u_1 is the σ-largest among the cycle vertices. Let's traverse C in σ-descending order ("left") starting from u_1: this is possible with except for the last edge u_lu_1 which needs to go right. But this implies that both u_1u_2 and u_1u_l were labelled with the same number which contradicts the way we constructed the labelling. Therefore we have all the edges in G labelled with at most k different labels and grouped into forests. This implies that $a \leqslant k$.

Moreover, each l-forest has at most $l - 1$ edges, so if $arb(G) = a$ then G has at most $a(n - 1)$ edges, which implies that G has a vertex of degree at most $2a - 1$ because otherwise if $d(v) \geqslant 2a$ for all $v \in V(G)$ then $|E(G)| = \frac{\sum_v d(v)}{2} \geqslant \frac{2an}{2} = an$ so contradiction. Therefore $k \leqslant 2a - 1$.