# Sparsity 2017 - Homework 2

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**Problem 1** Let's slightly change the notation: let $\perp(v)$ be the $\sigma$-smallest vertex that is weakly $r$-reachable from $v$ so that $D = \{\perp(v) : v \in V(G)\}$ and let $D^*$ be any of the smallest $r$-dominating sets, i.e. $|D^*| = dom_r(G)$.

The proof that $D$ is $r$-dominating set is trivial: every vertex $v$ has its picked $\perp(v)$ in $D$ and $\perp(v) \in WReach_r[v] \subseteq N_r[v]$. Now let's take any vertex $v \in V(G)$, then there exists some $x \in D^*$ such that $v \in N_r[x]$. We can assume that $x \neq v$. Now let's try to prove the following

**Claim 1.** $\perp(v) \in WReach_{2r}[x]$

*Proof.* We can assume that $\perp(v) \neq x$ and let's consider the path $P$ in $G$ between $x$ and $v$: it has a length of at most $r$ and no matter if either $x \leq_\sigma v$ or $v \leq_\sigma x$, the $\sigma$-smallest vertex of $P$ cannot be smaller than $\perp(v)$ because otherwise it would contradict the way $\perp(v)$ was picked as such guy would be weakly $r$-reachable from $v$ and smaller than $\perp(v)$. Now if $\perp(v)$ is internal vertex of P then $\perp(v) \in WReach_{r-1}[x] \subseteq WReach_{2r}[x]$. If it is not, then consider the path from $x$ to $v$ and then to $\perp(v)$: its length is at most $2r$ with all the internal vertices $\sigma$-greater than $\perp(v)$ thus $\perp(v) \in WReach_{2r}[x]$. $\square$

So given $G$, we can fix the smallest dominating set and then every vertex $v \in V(G)$ is mapped into one of the $dom_r(G)$ balls of that dominating set. If then we fix $\sigma$, then each vertex $v$ can only pick its $\perp(v)$ from the set no bigger than $WReach_{2r}[x]$ meaning it has at most $wcol_{2r}(G, \sigma)$ options to do that. Hence $|D| \leq wcol_{2r}(G, \sigma) \cdot dom_r(G)$

**Problem 2** If $I_v = [a, b]$ then let $\kappa(v)$ denote the number of intervals that $a$ belongs to. For fixed $k$, if $G \in \mathcal{I}_k$, then $G$ excludes $K_{k+1}$ as its subgraph and if $\kappa(v) = k$ then $v$ belongs to some clique $K_k$ in $G$. Let $\eta(r, k) := wcol_r(\mathcal{I}_k)$ and let also recursively specify $\xi(r, k)$ for $k, r \in \mathbb{N}$ as follows: $\xi(r, 1) = 1, \xi(1, k) = k - 1$ and $\xi(r, k) = \xi(r, k-1) + \xi(r-1, k)$. Then $\xi(r, k) = \binom{r+k-1}{r}$ by Pascal triangle. So if we succeed in showing that $\eta(r, k) \leq \xi(r, k)$ we are done.

If $k = 1$ then $\mathcal{I}_1$ is the class of graphs with zero edges as no two intervals can overlap. Then $WReach_r[G, \sigma, v] = \{v\}$ for any $\sigma, v, r$ so $\eta(r, 1) = 1$. If $r = 1$ then $WReach_1[G, \sigma, v] \subseteq N_1[v]$ which is at most $k$-set by definition of $\mathcal{I}_k$ (EDIT: this is actially not true but unofrtunately no time to figure out if it can be fixed), so $\eta(1, k) \leq k$. We conclude the boundary conditions hold.

Now let's fix $G \in \mathcal{I}_k$ and $\sigma$ and let's try to prove the following

**Claim 2.** *If $u \in WReach_r[G, \sigma, v]$ for some $u, v \in V(G)$ then $u \in WReach_{r-1}[G, \sigma, v]$ or $u \in WReach_r[\mathcal{A}(G), \sigma, v]$, where $\mathcal{A}$ is an operation specified below.*

*Proof.* Let $P$ be a path witnessing $u \in WReach_r[G, \sigma, v]$. $P$ is either simple by definition or can be made simple by cuting out the cycles. If the length of $P$ is shorter than $r$ then $u \in WReach_{r-1}[G, \sigma, v]$.

Otherwise assume $P = \langle v = v_1 \rightsquigarrow v_2 \rightsquigarrow .. \rightsquigarrow v_r = u \rangle$. For any graph input $G \in \mathcal{I}_k$, algorithm $\mathcal{A}$ first marks the edges of $P$ in $G$, then it destroys all the cliques $K_t$ without touching $P$: this is always possible as $P$ can use only $k-1$ edges from each $K_t$ so $\mathcal{A}$ just cuts one of the unused edges. $\square$

We can conclude that $\eta(r, k) \leq \eta(r-1, k) + \eta(r, k-1)$ and hence $\eta(r, k) \leq \binom{r+k-1}{r}$