Presheaf categories for algebraic presentation of compositional graphical structures

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Abstract. Milner's supported (pre)categories have been introduced as the formal framework for the development of the theory of bigraphs and bigraphical reactive systems. We give a new presentation of supported precategories as particular presheaves inspired to profunctors. These presheaf categories are naturally endowed with two monoidal structures, representing composition and tensor product in supported (pre)categories. This new categorical setting allows for new results about compositional graphical structures (not only bigraphs), taking advantage of the well-established theory of presheaf categories. As an application, we show that bigraphs and similar graphical objects can be characterized as free algebras arising from the monoidal structures of the presheaf category. Moreover, we show how sorting disciplines can be expressed formally using the internal language of these topoi.

1 Introduction

Robin Milner introduced *supported (pre)categories* as a generalization of the notion of category, by relaxing the requirement that each pair of compatible arrows can be composed. Each morphism is associated with a finite set (its *support*); two morphisms compose if and only if their supports are disjoint. Supported precategories have been used especially in the theory of reactive systems [15], leading to important results. In particular, *bigraphs* (and similar compositional graphical structures) can be formalized as supported precategories [14], akin to the presentation of term algebras as Lawvere theories: objects are interfaces, morphisms are bigraphs and supports are internal names [21, Def. 2.17]; composition captures bigraph composition along a shared interface. This formalization has been the fertile soil for the wide range of results offered by the bigraphical framework [14, 19, 21], such as the so called *IPO construction* for the synthesis of minimal labeled transition systems for bigraphical reactive systems.

Nevertheless, this presentation is not well suited for studying several issues and constructions that we are often interested in, when dealing with process calculi and dynamic systems. Examples are the *initial semantics* and *bialgebraic semantics*, as in [27], where the syntax of systems has to be characterized as the initial algebra of a endofunctor induced by the signature. So the whole class of bigraphs has to be represented as a single object of a suitable category, instead of being a category on its own.

As Milner pointed out in [21, Cap. 6], another crucial notion is that of *sorting discipline*. Sortings should be seen as an integral part of the formalization of any language, since non-well formed agents must be ruled out in order to obtain a faithful formalization. Sortings are defined as predicates over supported morphisms, but we need a full fledged language, a logical setting, and a formal interpretation for these predicates.

Another issue concerns the *refinement* of signatures, i.e. what happens when we change the signature, e.g. by adding/removing/merging controls? How are the syntax and semantics affected? To investigate these issues we need a formal setting for representing *operations* on signatures, and studying how these operations are reflected to the corresponding theories of bigraphs.

In fact, one quickly realizes that bigraphs are just one example in a countless family of "pluri-graphical" structures. Similar but different structures are directed bigraphs [13] and bigraphs with sharing [5]. Trying to find the "most general" pluri-graphical structure is hopeless. More reasonably, we can aim to define a scenario where generic graphical structures can be combined and merged, in the way hypergraphs and trees combine to yield bigraphs.

In this paper, we propose such a general scenario for addressing these issues. Our main aim is to define a category where each object represents a whole class of supported terms or graphical structures (such as link/place graphs and bigraphs) generated by a given signature. Morphisms between these objects represent transformations between different graphical languages. Objects can be defined as initial algebras of suitable endofunctors, thus yielding naturally algebraic description for these graphical structures. This setting will allow to show how different graphical structures can be combined together (also beyond bigraphs), and will also accomodate the notions of sortings as predicates and subobject.

The main technical step towards this aim is to consider *presheaf categories*, i.e. categories of SET-valued functors over an index category of *types*. As for normal term theories, these types have to convey the informations exposed by graphical structures needed for defining composition and juxtaposition. As an example, a bigraph $G = (V, E, ctrl, link, prnt) : \langle n, X\rangle \to \langle m, Y\rangle$ exposes three distinct informations: the inner and the outer interfaces $\langle n, X\rangle$, $\langle m, Y\rangle$, and the *support*, i.e. the set of its nodes. The index category has to cover these informations, and the operations on them. The key point is to see presheaves as "indexed sets" of morphisms (e.g., bigraphs), whose domain, codomain and support are given by the index; then, we can take advantage of many standard constructions in presheaf categories for building new data structures from existing ones.

This setting turns out to be very flexible, as we can capture several notions of composition for arrows by changing the structure of the index category. This enables a great degree of freedom without the loss of important tools and results offered by the theory of presheaves and profunctors. Moreover, as any presheaf category is also a topos, we can take advantage of its internal language for defining predicates, and subobjects.

The rest of the paper is organized as follows. In Section 2 we present the formal general settings of "profunctors with support" as particular presheaves; we show how these correspond to supported (pre)categories, and some useful results. In Section 3 we put this theory at work, by giving a new presentation of bigraphs, starting with their components (*i.e.* signatures, link and place graphs), how these combine yielding bigraphical structures, and giving an algebraic presentation—all this, within the *same* environment category.

Then, in Section 4, we take into account the notion of sorting. We will see that we can represent sortings as suitable predicates (called *monoidal sortings*) using the internal language of the presheaf category. As an example, we show that sorted bigraphs form a subpresheaf (a subobject) closed under compositions.

Conclusions and final remarks are in Section 5. We assume the reader familiar with the basic definitions of bigraphs; see e.g. [21].

2 Supported precategories and presheaves

In this section we give the connection between supported precategories and presheaf categories. To this end, we first recall the notion of supported precategories (or s-categories [21]), then propose a new generalization that relaxes the requirement that supports must be subsets of a given set (hence the "s-" prefix) or objects of the category of finite sets, as in [23]. Finally, in the wake of the insights offered by profunctors, we propose a presheaf category as a setting for modeling and studying our generalization of supported (pre)categories.

Definition 1 (Precategory) *A* precategory $`\mathrm{C}$ *is a category except for the composition operator* $(\_ \circ \_)$ *being a partial function such that*

(1) *for any arrow* $f : A \to B$, $\mathrm{Id}_B \circ f$ *and* $f \circ \mathrm{Id}_A$ *are defined and equal to* f*;*
(2) *for any* $f : A \to B$, $g : B \to C$, $h : C \to D$, $(h \circ g) \circ f$ *is defined iff* $h \circ (g \circ f)$ *does and then* $(h \circ g) \circ f = h \circ (g \circ f)$*.*

Definition 2 (Well supported (pre)category) *Let* SET_f *be the category of finite sets. A* well supported category *is a pair* $(\mathrm{A}, |\_|)$*, where* A *is a category and* $|\_| : \mathrm{A}_1 \to \mathrm{SET}_f$ *is the* support function *such that for any* $g \circ f \in \mathrm{A}_1$, $|g \circ f| = |g| \cup |f|$*, and for any* $A \in \mathrm{A}_0$ $|\mathrm{Id}_A| = \emptyset$*. A* well supported precategory *is a pair* $(`\mathrm{A}, |\_|)$*, where* $`\mathrm{A}$ *is a precategory such that* $g \circ f$ *is defined iff* $|g| \cap |f| = \emptyset$*.*

Actually, in this definition what is essential of the category SET_f is the monoidal structure given by set union. In the case of supported precategories, composition is subject to the requirement of support disjointness that is a predicate over pair of supports. Therefore, one wonders about generalizing beyond finite sets; for instance, supports may be strings over a given alphabet, and composition is concatenation but disciplined by a grammar; this would relate the structure of a (pre)category to a language of traces. This observation leads us to introduce the notion of *monoidal supported (pre)categories* (where "monoidal" refers to the monoidal structure of supports).

Definition 3 (Monoidal supported (pre)category) *Let* $(\mathrm{M}, \otimes, \epsilon)$ *be a locally small monoidal category. A* monoidal supported category (over M or just M-supported) *is a pair* $(\mathrm{A}, |\_|)$*, where* A *is a category and* $|\_| : \mathrm{A}_1 \to \mathrm{M}$ *is the support function such that for any* $g \circ f$ $|g \circ f| = |g| \otimes |f|$ *and for any* $A \in \mathrm{A}_0$

$|\mathrm{Id}_A| = \epsilon$. *A* monoidal supported precategory *is a triple* $(`\mathrm{A}, |\mathbf{.}|, \varphi)$*, where* $\varphi$ *is a predicate over* $\mathrm{M} \times \mathrm{M}$ *and* $`\mathrm{A}$ *is a precategory such that* $g \circ f$ *is defined iff* $\varphi(|g|, |f|)$*. If* φ *recovers* $(`\mathrm{A}, |\mathbf{.}|, \varphi)$ *from* $(\mathrm{A}, |\mathbf{.}|)$ *the latter is called* companion *of the first.*

As anticipated, every well supported precategory $(`\mathrm{A}, |\mathbf{.}|)$ is a monoidal supported precategory over $(\mathrm{SET}_f, \cup, \emptyset)$ whose predicate is disjointness check "$|\mathbf{.}| \cap |\mathbf{.}| = \emptyset$", which can be expressed in the internal language of the support category SET_f (and in the internal logic of the presheaf category over supports).

M-supported categories are "normal" categories since identities, composition and associativity are well defined thanks to the monoidal structure formed by supports. Moreover, Definition 3 naturally extends with 2-cells taking into account the structure of M (*e.g.* capturing support translations) and so on.

Profunctors As said above, we propose presheaf categories whose objects are profunctors as a setting for studying our generalization of supported (pre)categories. We refer the reader to [2, 7] for an introduction to profunctors (called also bimodules or distributors) and to [1, 7, 25, 28] for some recent applications.

Definition 4 (Profunctors) *A profunctor* $F : \mathrm{C} \nrightarrow \mathrm{D}$ *from the locally small category* C *to* D *is a functor* $\mathrm{D}^{op} \times \mathrm{C} \to \mathrm{SET}$*. These are arrows of the bicategory* PROF *which has locally small categories as objects and natural transformation as 2-cells. Given* $F : \mathrm{C} \nrightarrow \mathrm{D}$ *and* $G : \mathrm{D} \nrightarrow \mathrm{E}$*; their composite* $G \circ F : \mathrm{C} \nrightarrow \mathrm{E}$ *is* $Lan_{\mathcal{Y}_\mathrm{D}}\widehat{G}\widehat{F}$ *where* $\widehat{F} : \mathrm{C} \to \mathrm{SET}^{\mathrm{D}^{op}}$ *and* $\widehat{G} : \mathrm{D} \to \mathrm{SET}^{\mathrm{E}^{op}}$ *are given by the cartesian closure of* CAT*.*

Automorphisms over a locally small category C form the bicategory PROF(C). Profunctor composition induces a monoidal structure over $[\mathrm{C}^{op} \times \mathrm{C}, \mathrm{SET}]$, the hom-category of PROF(C). In particular, the multiplication is given by the bifunctor $(\mathbf{.} \circ \mathbf{.})$ and the unit is the identity profunctor Id_C *i.e.* the hom-functor of C. Let (A, μ, η) be a monoid of PROF(C), then for each stage $(X, Z) \in \mathrm{C}^{op} \times \mathrm{C}$

$$(A \circ A)_{(X,Z)} \cong \{(g, f) \mid g \in A(Y, Z) \wedge f \in A(X, Y) \wedge Y \in \mathrm{C}\}_{/\approx}$$

where $\approx$ is the equivalence relation of the unfolding of profunctor composition. This set can be seen as a description of all pairs of composable arrows of some category: if sets generated by A are thought as homsets, the monoidal unit $\eta : Id_C \to A$ determines which elements are identities and the monoidal multiplication $\mu : A \circ A \to A$ can be interpreted as some notion of composition up-to morphisms in C. Therefore, if C is discrete, every monoid in the hom-category of PROF(C) can be read as a category with the "same" objects as C.

In this setting, we can capture several notions of composition by suitably adapting the structure of the index category C. This enables a great degree of freedom without the loss of important tools and results offered by a categorical approach. The key point is to intend presheaves as "indexed sets" of morphisms (e.g., bigraphs), whose domain and codomain are given by the index; then, we can take advantage of many standard constructions in presheaf categories for building new presheaves from existing ones. In particular, constraints over morphisms expressed by commuting diagrams can be described by means of pullbacks, as in the following simple example.

Proposition 1 *Let (H, μ, η) be a monoid of $([\mathrm{C}^{op} \times \mathrm{C}, \textsc{Set}], \circ, \mathrm{Id}_{\mathrm{C}})$. The kernel pair of μ describes all commuting squares in the category modeled by H.*

Similar constructions allow to capture more complex and useful concepts such as *relative* and *idem pushouts*; due to lack of space, we refer the reader to [17].

Monoidal supported profunctors A precategory ‘A can be thought as a kind of refinement of a suitable category, say A, since the homsets of the first are subsets of the latter's. Let A be modeled by the presheaf monoid (H, μ, η). Then ‘A defines a subpresheaf of H whose characteristic map determines a predicate over the homsets of A such that it recovers ‘A from A.

This idea applies also to supported categories and precategories in order to formalize and study their relations within the same category. To capture the central rôle played by supports, profunctors have to be equipped with the monoid of supports; this leads to the definition of *monoidal supported profunctors*.

Definition 5 (Monoidal supported profunctors) *Let $(\mathrm{M}^{op}, \otimes, \epsilon)$ be a locally small monoidal category.* M-Prof*, the bicategory of* M-supported profunctors*, has locally small categories as objects and presheaves of $[\mathrm{M} \times \mathrm{D}^{op} \times \mathrm{C}, \textsc{Set}]$ as morphisms from* C *to* D*. Composition is defined combining the lifting of $(\mathrm{M}^{op}, \otimes, \epsilon)$ through Day's convolution [8] with profunctor composition (see diagram below). Specifically, given $F : \mathrm{C} \nrightarrow_{\mathrm{M}} \mathrm{D}$ and $G : \mathrm{D} \nrightarrow_{\mathrm{M}} \mathrm{E}$, their composite is defined as:*

$$(G \circ F) \triangleq Lan_{Lan_{\mathcal{Y}}\widehat{G}\widehat{F}}\mathcal{Y}(\bullet \otimes \bullet) \tag{1}$$

where $\widehat{F} : \mathrm{M} \times \mathrm{C} \to \textsc{Set}^{\mathrm{D}^{op}}$ and $\widehat{G} : \mathrm{D} \to \textsc{Set}^{\mathrm{M} \times \mathrm{E}^{op}}$ exist since Cat *is cartesian.*

$$\begin{tikzcd}
\mathrm{D} \arrow[r, "\mathcal{Y}"] \arrow[d, "\widehat{G}"'] & \textsc{Set}^{\mathrm{D}^{op}} \arrow[dl, "Lan_{\mathcal{Y}}\widehat{G}"] & \mathrm{M} \times \mathrm{C} \arrow[l, "\widehat{F}"'] \\
\textsc{Set}^{\mathrm{M} \times \mathrm{E}^{op}} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{M}^{op} \times \mathrm{M}^{op} \arrow[r, "Lan_{\mathcal{Y}}\widehat{G}\widehat{F}"] \arrow[d, "\mathcal{Y}(\bullet \otimes \bullet)"'] & \textsc{Set}^{\mathrm{E}^{op} \times \mathrm{C}} \arrow[dl, "Lan_{Lan_{\mathcal{Y}}\widehat{G}\widehat{F}}\mathcal{Y}(\bullet \otimes \bullet)"] \\
\textsc{Set}^{\mathrm{M}} &
\end{tikzcd}$$

In particular, ordinary profunctors are 1-supported. In the following we may refer to M-supported profunctors simply as "profunctors", when there is no risk of confusion. Notice that supported profunctor composition is associative only up to isomorphisms. In fact, unfolding (1) we have:

$$(G \circ F)_{(Q,E,C)} \cong \left(\coprod_{(M,N),D} \mathrm{M}(M \otimes N, Q) \times F_{(N,D,C)} \times G_{(M,E,D)} \right)_{/\approx}$$

where $\approx$ is the equivalence relation such that:

$$\begin{aligned}
&(q : M \otimes N \to Q, f, g) \approx (q' : M' \otimes N' \to Q, f', g') \iff \\
&\qquad \exists m \in \mathrm{M}(M, M') \exists n \in \mathrm{M}(N, N') \exists d \in \mathrm{D}(D, D') : \\
&\qquad f = F_{(\mathrm{Id}_N, d, \mathrm{Id}_C)}(f') \wedge g' = G_{(\mathrm{Id}_M, \mathrm{Id}_E, d)}(g) \wedge q' = \mathrm{M}(m \otimes n, Q)(q).
\end{aligned}$$

Lemma 1 *For any locally small category* C*, automorphisms over $\mathrm{C} \in$* M-Prof *form the bicategory* M-Prof(C) *whose hom-category $[\mathrm{M} \times \mathrm{C}^{op} \times \mathrm{C}, \textsc{Set}]$ is monoidal.*

Proof. The bifunctor $(\centerdot \circ \centerdot)$ defines a monoidal multiplication and together with the identity on C *i.e.* $\mathrm{Id_C} = \mathrm{M}(\centerdot, \epsilon) \times \mathrm{C}(\centerdot, \centerdot)$ endows $[\mathrm{M} \times \mathrm{C}^{op} \times \mathrm{C}, \textsc{Set}]$ with a monoidal structure. Associator, left and right unitors are defined by composition in M-$\textsc{Prof}$(C). Then coherence conditions are straightforward. □

Examples 1 *Let* M *be a grupoidal category (i.e. every morphism has an inverse) and* C *a discrete category. Let* (A, μ, η) *be a monoid of the hom-category of* M-$\textsc{Prof}$(C)*. Then the notion of composition induced by the monoidal multiplication is up to arrows in* M*, that are isomorphisms between supports. Specifically,* $\approx$ *equates* $(q : M \otimes N \to Q, f, g)$ *to every* $(q' : M' \otimes N' \to Q, f', g')$ *such that* $M \cong M'$ *and* $N \cong N'$*. If* C *is a grupoidal category too, then the resulting composition is also up to isomorphism in the middle object and the monoid* A *can be interpreted as a category with the objects of the skeleton of* C *(i.e. equivalence classes of objects of* C *closed under isomorphisms).*

As an immediate consequence of the notion of companion (Def. 3), also M-supported precategories can be modeled in a suitable hom-category of M-supported profunctors. This result reproduces the relation between an M-supported precategory and its companion leveraging the information of the predicate over supports associated to the precategory in order to "restrict" the monoid that describes its companion category characterizing a suitable partial monoid.

Proposition 2 *Every supported precategory can be modeled in a hom-category of* M-$\textsc{Prof}$ *as a partial monoid.*

Proof. Let $(`\mathrm{A}, |\centerdot|, \varphi)$, $(\mathrm{A}, |\centerdot|)$ and (A, μ, η) be a M-supported precategory, its companion and the monoid describing the latter respectively. The predicate φ over $\mathrm{M} \times \mathrm{M}$ readily extends to a predicate over $A \circ A$ *i.e.* a morphism to the subobject classifier of the presheaf topos. This arrow determines a subobject of $A \circ A$ isomorph, for each stage (M, X, Z), to the set of triples of the like of $(m : |g| \otimes |f| = M, g : Y \to Z, f : X \to Y)$ where $f, g \in \mathrm{A}$ and $\psi(|g|, |f|)$ – if M is not discrete, this set would be of equivalence classes closed under morphisms of M *e.g.* support translations. □

As a corollary of Proposition 2, the predicate over support pairs associated to each supported precategory has to be expressible in the internal language of the presheaf category (which is a topos). This fact naturally suggests to use this internal language as the formal mean for specifying these predicates. This issue was intentionally left out in Definition 3 for the sake of simplicity.

Product of supported profunctors Supported precategories have been introduced as a framework for the formalization of bigraphs. Besides composition, bigraphs can be also paired side-by-side, to represent juxtaposition of systems. This operation is defined only when operands have disjoint supports and it is modeled endowing the s-precategory with a partial tensor defined only on pairs of objects and pairs of arrows ensuring that interfaces (objects) and supports are disjoint. Where the tensor is defined the usual coherence conditions for a (symmetric) monoidal category are required to hold [21, Def. 2.10]. We can capture this notion in M-Prof(C) taking C to be a (symmetric) monoidal category.

Lemma 2 *Let* $(\mathrm{C}, \otimes, \epsilon)$ *be a locally small monoidal category. Then, this structure can be lifted by Day convolution to the hom-category* M-Prof(C) *yielding a monoidal category* $(\text{M-Prof}(\mathrm{C}), \otimes, \epsilon)$.

Proof. The monoidal structure of C is combined with the one of M rendering $M^{op} \times \mathrm{C} \times \mathrm{C}^{op}$ a monoidal category. This structure lifts to the hom-category of M-Prof(C) by straightforward application of Day convolution. $\square$

Let us consider a monoid (H, μ, η) for this additional monoidal structure, where H describes a supported category $(\mathrm{A}, |\bullet|)$; then $\eta : \mathcal{Y}\epsilon \to H$ and $\mu : H \otimes H \to H$ describe a monoidal structure over A (defining unit and multiplication respectively). By construction, A has the "same" objects of C (up to arrows in C) inheriting the data of $(\mathrm{C}, \otimes, \epsilon)$ for what concern objects. On the other hand, arrows depend only on information contained in H and are subject to the structure of $(\mathrm{M}, \otimes, \epsilon)$ which is lifted to the presheaf category by Day convolution.

The (symmetric) monoidal structure of an M-supported category $(\mathrm{A}, |\bullet|)$ offers the base for defining a partial (symmetric) monoidal structure on every precategory whose companion is A. The predicate over support pairs of $(`\mathrm{A}, |\bullet|, \varphi)$ restricts the monoidal multiplication of A on the base of the supports of the arrows involved. In [21, Def. 2.10] the tensor of a partial monoidal precategory can be partially defined on both arrows and objects and partiality on objects is independent from supports and vice versa. In order to define the partial monoidal structure over $`\mathrm{A}$ this information is encoded in a predicate over pairs of objects which (in combination with the one over supports) restricts the monoidal multiplication of A. Likewise of Proposition 2, partial monoidal M-supported precategories can be easily modeled in M-Prof.

Free algebras over monoidal structures An important application of presheaf categories is to allow to define as free algebras also languages with complex operators such as binders, non-interference parallels, substitutions, etc., by taking advantage of specific tensor products (and their adjoints) [6, 10, 11, 16, 18, 26]. A similar result holds in our settings: the two monoidal structures (say $\circ$ and $\bullet$) of the hom-category M-Prof(C) yield two orthogonal ways to compose presheaves offering a natural way to express "support-aware" operations on the elements described by these presheaves. Moreover, this setting can be used to define languages based on such operations as initial algebras for suitable endofunctors arising from these tensors (*cf.* Section 3). This class of $\bullet\circ$-terms is an instance of the general notion captured by the following definition.

Definition 6 (Free $\mathcal{M}$-algebras) *Let* C *be a locally small category with a collection of monoidal structures* $\mathcal{M}$. *For* X *an object of* C, *we define* $T_{\mathcal{M}}(X)$, *the* free $\mathcal{M}$-algebra *over* X, *as the free* $\Sigma_{\mathcal{M}}$*-algebra over* X, *where* $\Sigma_{\mathcal{M}}(Y) = \coprod_{\star \in \mathcal{M}} Y \star Y$.

Let Y be a presheaf and $T_{\bullet\circ}(Y)$ be the free $\bullet\circ$-algebra, and let H be a monoid for both $\bullet$ and $\circ$. Then, the two multiplications induce an algebraic structure over H, and hence by initiality there exists $\nu : T_{\bullet\circ}(Y) \to H$. This defines an interpretation of $\bullet\circ$-terms over Y into elements of H, and ultimately in the supported category A modeled by H. If ν is epic, then the language defined by $T_{\bullet\circ}(Y)$ is a complete representation for A. On the other hand, every mono $H \to T_{\bullet\circ}(Y)$ can be seen as the definition of a *normal form.* These constructions readily extend to morphisms between the monoids induced by $\circ$ and $\bullet$, lifting normal forms and interpretations to the category of monoids.

3 Application: An algebraic presentation of bigraphs

In this section we apply the results presented above to provide a new presentation of bigraphs, starting with their components (*i.e.* signatures, link and place graphs), how these combine yielding bigraphical structures, and finally relating the terms generated by the monoidal structures with the algebraic presentation [20]—all this, within the *same* environment category.

For the sake of simplicity, we focus on *bigraphs with abstract names*, i.e., whose composition and tensor product can rename clashing names and nodes. In this way, these operations are total, allowing bigraphs to form a category instead of a precategory. In a sense, bigraphs with abstract names lie between Milner's concrete bigraphs (which form a precategory) and abstract bigraphs (which do not have a notion of support). Our presentation relies on the category of finite ordinals and bijective permutations $\mathbb{B}$ to model abstract names and renaming but (as will be shown at the end of the section) Milner's concrete bigraphs are recovered by means of SET_b – the category of finite sets and bijective functions.

3.1 Signatures and control maps

Bigraphs are parametric on a given *signature* which is a set of controls. Each control define a finite number called *arity* [21, Def. 1.1]. Therefore, a signature K is a list $(K_1, \ldots, K_n)$ – where K_i is the arity of the i-th control – which is an object of the category of signatures $\mathcal{K}$. For the aims of this section let us define $\mathcal{K}$ as the monoidal closure of the discrete category of natural numbers $\mathbb{N}^*$ (which is discrete too) but richer definition are available and allows to model operation on signatures [17]. Moreover, this notion of signature can be easily adapted to more expressive settings, as needed, *e.g.* $(2 \times \mathbb{N})^*$, $(\mathbb{N} \times \mathbb{B})^*$ and $(\mathbb{N} \times \mathbb{N})^*$ model active, binding and directed signatures.

Given a signature K and a graph with V nodes, a *control map* labels each node with a control from K. V is a set of names and can be represented by an object of $\mathbb{B}$. Then, we define the category of *control maps over a signature* K as the comma category $\mathcal{C}_K \triangleq I_{\mathbb{B}} \downarrow U_{\mathcal{K}}K$ where $I_{\mathbb{B}} : \mathbb{B} \to \mathbb{F}$ is the inclusion functor from $\mathbb{B}$ to $\mathbb{F}$, the category of finite ordinals and functions; $U_{\mathcal{K}} : \mathcal{K} \to \mathbb{F}$ is the "length" forgetful functor, and $K : 1 \to \mathcal{K}$ is the constant functor returning the signature K. Objects of $\mathcal{C}_K$ are (isomorphic to) control maps $c : V \to K$, and an arrow $\pi : c \to c'$ amounts to a bijection $\pi : \mathrm{dom}(c) \to \mathrm{dom}(c')$ respecting controls; capturing the notion of *support translation* [21, Def. 2.13]. Finite ordinal renders $(\mathcal{C}_K, \oplus, \varnothing)$ a symmetric monoidal category where multiplication and unit represent disjoint union of maps (because of names being abstract there is an implicit rename) and the empty control map $\varnothing : 0 \to K$, respectively.

3.2 Place graphs

Definition 7 ([21, Def. 2.1]) *A* concrete place graph $P = (V_P, \mathrm{ctrl}_P, \mathrm{prnt}_P) : m \to n$ *over the signature K [21, Def. 2.1] is a triple having an inner interface m and an outer n, both finite ordinals. These index respectively* sites *and* roots *of the place graph; P has a finite set of internal nodes V_P labeled by the control map* $\mathrm{ctrl}_P : V_P \to K$*. Nodes, sites and roots are organized in a forest described by the parent map* prnt_P *and such that sites are leaves and roots are exactly n.*

A concrete place graph, and in general a place graph, is an object exposing three distinct informations: the inner interface, the outer one, and the support. The support is usually intended as the set of nodes of a place graph (resp. link graph or bigraph) but support translations for place graphs (resp. link graph or bigraph) are defined in [21, Def. 2.4] to be renaming of the internal names that respect controls. Therefore, not only names have to be observable, but also controls have to be so. This suggests to take the control map as the support. These three informations are exactly what is needed for defining composition and juxtaposition of place graphs and thus can be seen as their *type*.

A *place graph with abstract names* is a concrete place graph except for the set nodes being a finite ordinal drawn from $\mathbb{B}$ and the control map which is a suitable object of $\mathcal{C}_K$. Likewise concrete ones, place graphs with abstract names can be seen as supported morphism between interfaces *i.e.* finite ordinals forming the *category of place graphs with abstract names* $\textsc{Pga}_K$ which is a $\mathcal{C}_K$-supported category and is therefore modeled by a monoid in the hom-category of $\mathcal{C}_K$-$\textsc{Prof}(\mathbb{N})$ *i.e.* $[\mathcal{P}_K, \textsc{Set}]$ where $\mathcal{P}_K \triangleq \mathcal{C}_K \times \mathbb{N}^{op} \times \mathbb{N}$. Finite ordinals sum endows $\mathbb{N}$ with a symmetric monoidal structure which is lifted to presheaves applying Lemma 2. This renders $([\mathcal{P}_K, \textsc{Set}], \oplus, \mathcal{Y}\varepsilon)$ a symmetric monoidal category (and the Yoneda embedding a monoidal functor) where the unit is $(\varnothing, 0, 0)$ (*i.e.* the type of the empty place graph), and the tensor obtained through the Day's convolution is:

$$(A \oplus B)_P = \int^{P_1, P_2 \in \mathcal{P}_K} A_{P_1} \times B_{P_2} \times \mathcal{P}_K(P_1 \oplus P_2, P). \tag{2}$$

For any stage $P \in \mathcal{P}_K$, $(A \oplus B)_P$ is isomorph to the set of classes of triples closed under arrows of $\mathcal{C}_K$ *i.e.* support translations. These triples are of the like of (a, b, p) where $a \in A_{P_1}$, $b \in B_{P_2}$ and $p : P_1 \otimes P_2 \cong P$ is a renaming defining P as the sum of P_1 and P_2. Therefore this monoidal structure would express “support-aware” juxtaposition. In particular, it renders the presheaf describing the arrows of $\textsc{Pga}_K$ a monoid whose multiplication define a tensor over the category $\textsc{Pga}_K$ and captures exactly juxtaposition of place graphs with abstract names as a total operation thanks to the built-in rename.

Composition of place graphs with abstract names is modeled in $\textsc{Pga}_K$ as arrow composition. This is captured (up-to support translation) by the monoidal structure induced by profunctor composition over $\mathcal{C}_K$-$\textsc{Prof}(\mathbb{N})$. Seeking adherence with the usual ordering for arrow composition, we introduce the tensor:

$$(A\odot B)_{(C,M,N)} \triangleq \int^{C_1,\,C_2\in\mathcal{C}_K} \mathcal{C}_K(C_1\oplus C_2, C)\times\int^{Q\in\mathbb{N}} A_{(C_2,Q,N)}\times B_{(C_1,M,Q)} \quad (3)$$

whose arguments are swapped respect to $\circ$ induced by profunctor composition. The identity $\mathcal{C}_K(\cdot,\varnothing)\times\mathbb{N}(\cdot,\cdot)$ is the unit completing the monoidal structure $([\mathcal{P}_K,\textsc{Set}],\odot,\mathrm{Id}_{\mathbb{N}})$ which would express composition up-to support translations.

Place graphs with abstract names can be seen as freely generated by *placings* and *ions*. Therefore we can define a term language for describing place graph as the free $\oplus\odot$-algebra over a presheaf describing palcings and ions thanks to Definition 6. Placings are characterized by the presheaf $plc_{K(c,M,N)} \triangleq \mathcal{C}_K(c,\varnothing)\times\mathbb{F}(M,N)$ and ions by $ion_{K(c,M,N)} \triangleq \coprod_{k:1\to K}\mathcal{C}_K(c,k)\times(\mathbb{N}(M,1)+\mathbb{N}(1,N))$. When $c\cong\varnothing$, the first yields precisely all functions from the inner interface M to the outer one N *i.e.* forests without controls. When c is a map which sends the only node in its domain to the control k, the second yields all pairs of functions in $\mathbb{N}$ from the inner face to the only node and from this node to the outer face, that is the parent map of a ion with control k. Then, $Plc_K \triangleq T_{\oplus\odot}(plc_K + ion_K)$ is the presheaf of terms describing place graphs on K. Grafting defines an epi to $\textsc{Pga}_K$ and every mono on the opposite direction is a normal form.

Example 2 (Place graph with sharing) *Bigraphs* with sharing *[5] (here with abstract names) are defined by merging pure link graphs with place graphs with sharings which rely for their underling structure on DAGs instead of forests. Hence placings are set-valued functions never returning the empty set which are described by* $s\text{-}plc_{K(C,M,N)} \triangleq \mathcal{C}_K(C,\varnothing)\times\textsc{Set}(J_M,\wp^+(J_N))$ *where* $J:\mathbb{N}\rightarrowtail\textsc{Set}$ *is the obvious injection functor. Ions are the same of pure place graphs.* $T_{\oplus\odot}(s\text{-}plc_K + ion_K)$ *defines the presheaf of terms.*

3.3 Link graphs

Definition 8 ([21, Def. 2.2]) *A* concrete link graph $L=(V_L,E_L,\mathrm{ctrl}_L,\mathrm{link}_L): X\to Y$ *over K is a quadruple having an inner interface X and an outer Y, both finite sets of names. L has a finite sets of nodes V_L and edges E_L, a control map* $\mathrm{ctrl}_L : V_L\to K$ *and a link map* link_L *from X and ports (i.e. elements of the finite ordinal associated to each node by its control) to* $E_L\uplus Y$.

Link graph with abstract names are basically concrete link graph except for sets of names which are replaced by objects of $\mathbb{B}$. These form the $\mathcal{C}_K$-supported[1] category $\textsc{Lka}_K$ which is modeled by monoid of the hom-category of $\mathcal{C}_K$-$\textsc{Prof}(\mathbb{B})$ *i.e.* $[\mathcal{L}_K,\textsc{Set}]$ where $\mathcal{L}_K \triangleq \mathcal{C}_K\times\mathbb{B}^{op}\times\mathbb{B}$. This category presents two monoidal structures. The first, $([\mathcal{L}_K,\textsc{Set}],\odot,\mathrm{Id}_{\mathbb{B}})$, is induced by arrow composition of $\mathcal{C}_K$-$\textsc{Prof}(\mathbb{B})$ and the second, $([\mathcal{L}_K,\textsc{Set}],\oplus,\mathcal{Y}\varepsilon)$, arises, by application of Lemma

[1] Consistently with the algebra for link graphs proposed by Milner's in [20] we safely omit edges since they are accessory to the definition of the link map of concrete link graphs rather than the definition of operations over them.

2 on $(\mathbb{B}, +, 0)$. These structures – whose multiplications are

$$(A \odot B)_{(C,M,N)} \triangleq \int^{C_1, C_2 \in \mathcal{C}_K} \mathcal{C}_K(C_1 \oplus C_2, C) \times \int^{Q \in \mathbb{B}} A_{(C_2,Q,N)} \times B_{(C_1,M,Q)} \quad (4)$$

$$(A \oplus B)_L \triangleq \int^{L_1, L_2 \in \mathcal{L}_K} A_{L_1} \times B_{L_2} \times \mathcal{L}_K(L_1 \oplus L_2, L) \quad (5)$$

respectively – allow to model composition and juxtaposition up-to isomorphisms in the index category *i.e.* renaming.

Link graphs with abstract names are freely generated from wirings and ions which are described by the presheaves $wir_{K(c,X,Y)} \triangleq \mathcal{C}_K(C, \varnothing) \times \mathbb{F}(X, Y+1)$ and $ion_{K(c,X,Y)} \triangleq \coprod_{k:1 \to K} \mathcal{C}_K(c,k) \times \mathbb{B}(X,0) \times \mathbb{B}(c(*), Y)$. The first, yields precisely all partial functions from X to Y which are link graphs without nodes, *i.e.* wirings, when $c = \varnothing$. The latter yields all bijection from ports to outer names, *i.e.* a k-ion, when c sends the only node of its domain to the control k. Lnk_K, the presheaf of $\oplus\odot$-terms describing link graphs with abstract names over K, is defined as $T_{\oplus\odot}(wir_K + ion_K)$.

3.4 Bigraphs

Likewise *concrete bigraphs* on a given signature K [21, Def. 2.3] are obtained merging concrete place graphs and link graphs ensuring that these structures share their control maps, *bigraph with abstract names* are composed by place graph and link graph with abstract names sharing control maps.

The category BGA_K can be modeled in the hom-category of $\mathcal{C}_K$-$\mathrm{PROF}(\mathbb{N} \times \mathbb{B})$ *i.e.* $[\mathcal{B}_K, \mathrm{SET}]$ where $\mathcal{B}_K \triangleq \mathcal{C}_K \times (\mathbb{N} \times \mathbb{B})^{op} \times (\mathbb{N} \times \mathbb{B})$ is defined by the pullback of the forgetful functors from $\mathcal{P}_K$ and $\mathcal{L}_K$ to $\mathcal{C}_K$ as shown in (6). $[\mathcal{B}_K, \mathrm{SET}]$ presents two monoidal structures: $([\mathcal{B}_K, \mathrm{SET}], \odot, \mathrm{Id}_{\mathbb{N}\times\mathbb{B}})$ and $([\mathcal{B}_K, \mathrm{SET}], \oplus, \mathcal{Y}\varepsilon)$. The first arise from profunctor composition and the second from finite ordinal sum. Both can be projected – consistently with the definition of bigraphs and their category – through (6) to recover their counterpart: (3) and (2) in $[\mathcal{P}_K, \mathrm{SET}]$ and (4) and (5) in $[\mathcal{L}_K, \mathrm{SET}]$ respectively. Moreover, (6) offers an alternative and consistent definition of the symmetric monoidal structure on $\mathcal{B}_K$ which results by "merging" the monoidal structures of $\mathcal{P}_K$ and $\mathcal{L}_K$: the tensor $\oplus$ is the unique map given by the universal property of $\mathcal{B}_K$.

$$\begin{tikzcd} \mathcal{B}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}_K \arrow[d] \\ \mathcal{L}_K \arrow[r] & \mathcal{C}_K \end{tikzcd} \quad (6)$$

Terms for describing bigraphs with abstract names are freely generated from placings, wirings and ions. The firsts are basically their counterparts[2] in $[\mathcal{P}_K, \mathrm{SET}]$ and $[\mathcal{L}_K, \mathrm{SET}]$; the latter are the contact surface between the orthogonal structures composing bigraphs (since nodes have to be shared) and are described by $ion_{K(c,(M,X),(N,Y))} \triangleq \coprod_{k:1\to K} \mathcal{C}_K(c,k) \times \mathbb{N}(M,1) \times \mathbb{N}(1,N) \times \mathbb{B}(X,0) \times \mathbb{B}(c(*),Y)$. This yields all bigraphical k-ions when c is $* \mapsto k$, M, N are 1 and X is empty. This presheaf can be projected to the ion_K of $[\mathcal{L}_K, \mathrm{SET}]$ and to the ion_K of $[\mathcal{P}_K, \mathrm{SET}]$. Then we define Big_K, the presheaf of $\oplus\odot$-terms describing bigraphs with abstract names over K, as $T_{\oplus\odot}(wir_K + plc_K + ion_K)$.

[2] They are related by composition with the injection or forgetful functors between the index categories.

Concrete bigraphs Well supported (pre)categories, as in the case of Milner's concrete bigraphs [21, Def. 2.17], have their supports as objects of $\textsc{Set}_f$. However, this category presents a very rich structure which is not used by supports and operations over them (*i.e.* union, translation and check for disjointness). As formalized [23, 24], support translation needs to form a grupoidal structure over supports. Therefore, supports are modeled in the category of finite sets and bijective functions $\textsc{Set}_b$ whose skeleton is $\mathbb{B}$. Set union endows this category with a symmetric monoidal structure whose unit is the empty set.

The precategory of Milner's concrete bigraphs $({}^{\backprime}\textsc{Bg}_K, |\boldsymbol{.}|, \psi)$, where $\psi \triangleq |\boldsymbol{.}| \cap |\boldsymbol{.}| = \emptyset$, is modeled in a suitable hom-category $\textsc{Set}_b$-$\textsc{Prof}$ like any well-supported (pre)category. Their companion category defines *concrete bigraphs with renaming*. The proposed setting allows to easily formulate characterizations for several variations on pluri-graphical languages such as, but not limited to, local bigraphs [4], directed bigraphs [12], bigraphs with sharing [5] *etc.*

4 Monoidal decomposable predicates and sortings

A crucial part of the formalization of a reactive system is the definition of a *sorting discipline*, ruling out spurious (i.e. non well-formed) states. In Milner's formulation [21, Cap. 6], sortings are judgments over supported morphisms. The notion was refined by Debois introducing *predicate sorting* which are based on the concept of *decomposable predicates* [3]. Despite these are a subclass of Milner's, predicate sortings allow to capture the vast majority of practical scenarios [9, Cap. 6]. Both notion are expressible in our setting as suitable predicates over (partial) monoids modeling (pre)categories, using the internal language of the presheaf topos, *i.e.* as arrows to the subobject classifier.

A predicate P over morphisms of a (pre)category is said to be decomposable if, and only if, for every morphisms f, g, $P(g \circ f) \Rightarrow P(f) \wedge P(g)$. Predicates of this kind are strongly connected with the concept of (de)composition which we model as monoidal structures. This notion readily generalizes to predicates decomposable with respect to the monoidal structure of any given monoidal category $(\mathrm{C}, \otimes, I)$ which is a topos; thus suggesting a generalization of Debois's results (in particular the ability to transfer RPOs). Decomposable predicates are not required to define sub-(pre)categories (e.g., nothing forbids to use a predicate to "ban" an identity). Reworded in our settings, they are not required to preserve the monoidal structure fo the (partial) monoids used to model (pre)categories.

Before giving the definition of *monoidal decomposable predicates* we need to "lift" the logical and to the monoidal structure of C. Mimicking the definition of $\wedge : \Omega \times \Omega \to \Omega$, we define $\owedge : \Omega \otimes \Omega \to \Omega$ as the characteristic map of the monomorphism $m : A \rightarrowtail \Omega$ uniquely determined by the unique and minimal epi-mono factorization of $\top \otimes \top : 1 \otimes 1 \to \Omega \otimes \Omega$ as illustrated by the diagram aside.

$$\begin{tikzcd}
1\otimes 1 \arrow[dr, two heads, dashed, "e"] \arrow[drr, bend left, "!"] \arrow[ddr, bend right, "\top\otimes\top"'] & & \\
& A \arrow[r, "!"] \arrow[d, tail, "m"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
& \Omega\otimes\Omega \arrow[r, "\owedge"] & \Omega
\end{tikzcd}$$

Definition 9 (Monoidal decomposable predicate) *Let* C *be a topos with a monoidal structure* $(\mathrm{C}, \otimes, I)$*. An arrow* $\varphi : M \to \Omega$ *is said to be a* monoidal decomposable predicate (MDP) *if* (M, μ, η) *is a monoid and the subobject characterized by* $\varphi \circ \mu$ *is a subobject of that characterized by* $\varphi \oslash \varphi$ *(cf. the diagram aside).*

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, tail] \arrow[rr, dashed, tail, bend left] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] & Y \arrow[l] \arrow[d, tail] \arrow[dl, phantom, "\llcorner", very near start] \\ M \otimes M \arrow[r, "\varphi \circ \mu"] & \Omega & M \otimes M \arrow[l, "\varphi \oslash \varphi"'] \end{tikzcd}$$

Let φ be a monoidal decomposable predicate for M. The morphisms φ, $\varphi \circ \mu$ and $\varphi \oslash \varphi$ describe the subobjects $P \rightarrowtail M$,$X \rightarrowtail M \otimes M$ and $Y \rightarrowtail M \otimes M$. The unique arrow ν defined by the pullback P can be seen as the restriction of the monoidal multiplication μ to the part of M that satisfies φ. If we take M to be an object of Set, and $\mu(a, b) = a \circ b$, then X determines all the pairs of operands $a, b \in M$ such that $a \circ b$ satisfies φ and Y determines all the pairs $(a, b) \in M \otimes M$ such that both its components satisfy φ. Because $X \rightarrowtail Y$ (or w.l.o.g., $X \subseteq Y$) if $a \circ b$ satisfies φ then also both a and b do ($a \circ b \in P \Rightarrow a, b \in P$).

Proposition 3 (Decomposable predicates) *Let* P *be a decomposable predicate on the arrows of a locally small category* C *and let the monoid* (H, μ_H, η_H) *describe* C *(cf. Example 1). There exists a MDP* $\varphi_P : H \to \Omega$*, whose subobject is isomorphic, for any stage* $(C, D) \in \mathrm{C}_0 \times \mathrm{C}_0$*, to* $\{f \in \mathrm{C}(C, D) \mid P(f)\}$*.*

Proof. Let the profunctor W be defined as $W_{(C,D)} \triangleq \{f \in \mathrm{C}(C, D) \mid P(f)\}$ and define φ_P to be its characteristic map $\chi_W : H \to \Omega$. Let X and Y be the subobjects of $H \odot H$ characterized by $\varphi_P \circ \mu_H$ and $\varphi_P \oslash \varphi_P$ respectively. Given $f : C \to D$ and $g : D \to E$, $g \circ f \in W_{(C,E)}$ iff $P(g \circ f)$, which implies $P(f)$ and $P(g)$. By construction, $[g, f] \in X_{(C,E)}$ since $P(g \circ f)$ and $[g, f] \in Y_{(C,E)}$ since $P(g) \wedge P(f)$. Therefore, for any stage (C, E), $X_{(C,E)} \subseteq Y_{(C,E)}$. □

Because of Proposition 3, monoidal decomposable predicates capture a class of objects strictly larger than those used to model supported categories. Moreover, this result offers a context for a logic for decomposable predicates which is currently missing (albeit some results in this direction are presented in [22]).

In [9, Ch. 5] the notion of decomposable predicates is extended to well supported precategories to capture supported reactive systems (*e.g.* bigraphical ones). Proposition 3 extends to monoidal supported precategories (with just few technicalities for dealing with partial monoids). In fact, Proposition 2 offers a way to model a supported precategory $(\grave{}\mathrm{A}, |{\bf .}|, \psi)$ taking advantage of the monoid (H, μ, η) for its companion category. Arrow composition is described by a sort of restricted monoidal multiplication $\nu : X \to H$ whose domain is the subobject of $H \circ H$ induced by ψ. Then, MDPs are extended to partial monoids defining a class of predicates that generalizes decomposable ones for supported precategories.

Example 3 *Consider the presheaves S-Big*$_K$ *and Big*$_K$ *describing* $\oplus\odot$*-terms for bigraphs with and without sharing respectively. Let* φ : *S-Big*$_K \to \Omega$ *be a predicate checking whatever a term describes a bigraph without sharing (just leverage plc*$_K \hookrightarrow$ *s-plc*$_K$*). Then* φ *defines Big*$_K \hookrightarrow$ *S-Big*$_K$*.*
Moreover, φ *is MDP (in particular a monoidal morphism)* w.r.t $\odot$ *and* $\oplus$*. A similar results holds for the presheaves describing bigraphs and between presheaves for terms and bigraphs, etc.*

$$\begin{tikzcd} \text{S-B}{\rm GA}_K & \mathrm{B}{\rm GA}_K \arrow[l, hook'] \\ \textit{S-Big}_K \arrow[u, hook] & \textit{Big}_K \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

Bigraphical monoidal sorting: encoding the π-calculus. Milner proposed in [14] an encoding of the π-calculus into binding (or local) bigraphs, an extension of pure bigraphs where names can be associated to localities. Binding bigraphs can be recovered from pure bigraphs by means of a suitable sorting discipline that constrains the use of bounded resources and this allows to encode faithfully the π into pure bigraphs. This encoding resembles the one for the CCS given in [21, Def. 6.5] except for controls send and get which present an additional port, say the second one, to represent actual and formal parameters of the communication *i.e.* the exchanged name. Therefore the signature to be used is $\Sigma_\pi = (\mathsf{alt}{:}0, \mathsf{send}{:}2, \mathsf{get}{:}2)$. In order to rule out bad formed bigraphs, we define – by means of the internal language of the boolean topos $[\mathcal{B}_{\Sigma_\pi}, \mathrm{SET}]$ – a predicate $\varphi = \varphi' \wedge \varphi''$ on the presheaf of bigraphs such that:

$$\begin{aligned}
\varphi'_{(c,\langle m,X\rangle,\langle n,Y\rangle)}(G) &= \textstyle\bigwedge_{v\in \mathrm{dom}(c)} \mathrm{prnt}_G(v) \in \mathrm{dom}(c) \to (c(v) \in A \leftrightarrow \mathrm{prnt}_G(v) \notin A)\\
\varphi''_{(c,\langle m,X\rangle,\langle n,Y\rangle)}(G) &= \textstyle\bigwedge_{v\in \mathrm{dom}(c)} c(v) = \mathsf{recv} \to \mathrm{link}_G(2@v) \notin Y \wedge \forall u \in \mathrm{dom}(c).\forall i \in\\
&\qquad c(u).(\mathrm{link}_G(i@u) = \mathrm{link}_G(2@v) \to v \in \mathrm{prnt}^*_G(u) \setminus \{u\})
\end{aligned}$$

where $A = \{\mathsf{send}, \mathsf{get}\}$, $(c, \langle m, X\rangle, \langle n, Y\rangle) \in \mathcal{B}_{\Sigma_\pi}$, $G \in \mathrm{BGA}_{\Sigma_\pi(c,\langle m,X\rangle,\langle n,Y\rangle)}$ and $i@u$ denotes the i-th port of the node u. φ' ensures alternation between alt and other controls; φ'' forces the second port of a recv node v (*i.e.* the parameter of a receive) to be connected only to inner names or ports of descendants of v. Since this property is preserved by decomposition φ is a MDP for $\odot$ on a presheaf describing bigraphs, hence a bigraphical monoidal sorting.

5 Conclusions and future works

In this paper we have given a new categorical presentation of precategories and s-categories as particular presheaves, i.e., profunctors. This allows to study these objects and their relations by leveraging the rich theory of presheaf categories. In particular, we have shown how graphical structures can be composed to yield (pluri-)graphical structures, like link and place graphs can be merged to yield bigraphs. Within this presheaf category, we have shown that a language for describing bigraphs arises naturally as a free algebra over basic elements (ions, places and wirings); finally, we have introduced a class of *monoidal decomposable predicates* for specifying sorting disciplines. Remarkably, these predicates are definable in the internal language of the same presheaf category (which is a topos), pointing out that this can be the right context for the interpretation of logics for decomposable predicates, like the one introduced in [22].

Our categorical presentation allows also for the characterization of RPOs, IPOs and other results about rewriting systems, thus covering also the dynamic part of bigraphs. We have omitted these results due to lack of space; see [17].

A future work worth investigating is the representation of *vertical refinements* in presheaf categories. In fact, taking $\mathcal{K}$ to be the free category with finite coproducts over $\mathbb{N}$ makes $\mathrm{cod} : \mathcal{C} \to \mathcal{K}$ (where $\mathcal{C} = I_{\mathbb{B}} \downarrow U_{\mathcal{K}}$) an opfibration. This offers a formal way to propagate operation over signatures (*e.g.* unification of controls) to supports and eventually presheaves, thus paving the way for formalizing refinements for systems and languages modeled in these categories.

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A Appendix

A bicategory is a particular algebraic notion of weak 2-category. The idea is that a bicategory is a category weakly enriched over $\mathrm{C{\scriptsize AT}}$: the hom-objects of a bicategory are hom-categories, but the associativity and unity laws of enriched categories hold only up to coherent isomorphism.

Definition 10 (Bicategory) *A bicategory* B *consists of*

(1) *a collection of objects, also called* 0-cells*;*
(2) *for each pair of 0-cells* X *and* Y*, a category* $\mathrm{B}(X, Y)$*, whose objects are called morphisms or* 1-cells *and whose morphisms are called 2-morphisms or* 2-cells*;*
(3) *for each 0-cell* X*, a distinguished 1-cell* $\mathrm{Id}_X \in \mathrm{B}(X, X)$ *called the identity morphism or identity 1-cell at* X*;*
(4) *for each triple of 0-cells* X*,* Y *and* Z*, a functor* $\circ : \mathrm{B}(Y, Z) \times \mathrm{B}(X, Y) \to \mathrm{B}(X, Z)$ *called* horizontal composition*;*
(5) *for each pair of 0-cells* X *and* Y*, natural isomorphisms called* unitors*:* $\mathrm{Id}_{\mathrm{B}(X,Y)} \circ \mathcal{K}_{\mathrm{Id}_X} \cong \mathrm{Id}_{\mathrm{B}(X,Y)} \cong \mathcal{K}_{\mathrm{Id}_Y} \circ \mathrm{Id}_{\mathrm{B}(X,Y)} : \mathrm{B}(X, Y) \to \mathrm{B}(X, Y)$*;*
(6) *for each quadruple of 0-cells* W*,* X*,* Y *and* Z*, a natural isomorphism called the* associator *between the two functors from* $\mathrm{B}(Y, Z) \times \mathrm{B}(X, Y) \times \mathrm{B}(W, X)$ *to* $\mathrm{B}(W, Z)$ *built out of* $(- \circ -)$ *such that the same axioms as the constraint isomorphisms in a monoidal category (which we do not write out in full here) are satisfied.*

Definition 11 (The bicategory of profunctors) *A profunctor* $F : \mathrm{C} \nrightarrow \mathrm{D}$ *from the locally small category* C *to* D *is a functor* $\mathrm{D}^{op} \times \mathrm{C} \to \mathrm{S{\scriptsize ET}}$*. Profunctors are morphisms of the bicategory* $\mathrm{P{\scriptsize ROF}}$ *which has locally small categories as objects and natural transformation as 2-cells. Given two profunctors* $F : \mathrm{C} \nrightarrow \mathrm{D}$ *and* $G : \mathrm{D} \nrightarrow \mathrm{E}$*; their composite* $G \circ F : \mathrm{C} \nrightarrow \mathrm{E}$ *is defined as* $Lan_{\mathcal{Y}_{\mathrm{D}}} \widehat{G}\widehat{F}$ *where* $\widehat{F}$ *and* $\widehat{G}$ *are given by the cartesian closure of* $\mathrm{C{\scriptsize AT}}$*, the category of all locally small categories.*

Profunctors are composed by using a coend to "trace out" the middle variable. Specifically given F and G as above $G \circ F$ is defined by the diagram

$$\begin{tikzcd}
\mathrm{D} \arrow[r, "\mathcal{Y}_{\mathrm{D}}"] \arrow[d, "\widehat{G}"'] & \mathrm{SET}^{\mathrm{D}^{op}} \arrow[dl, "Lan_{\mathcal{Y}_{\mathrm{D}}}\widehat{G}"] & \mathrm{C} \arrow[l, "\widehat{F}"'] \\
\mathrm{SET}^{\mathrm{E}^{op}} & &
\end{tikzcd}$$

and, unfolding the left Kan extension, the composite can be formulated as:

$$G \circ F \triangleq \int^{D \in \mathrm{D}} G(-, D) \times F(D, -) \cong \left(\coprod_{D \in \mathrm{D}} G_{(E,D)} \times F_{(D,C)} \right)_{/\approx}$$

where $\approx$ is the equivalence relation defined as

$$(g, f) \approx (g', f') \iff \exists d \in \mathrm{D}(D, D') : F_{(d,\mathrm{Id}_C)}(f') = f \wedge G_{(\mathrm{Id}_E,d)}(g) = g'.$$

Composition of profunctors is associative only up to isomorphism because of the product not being strictly associative in SET.

M-PROF is a bicategory since composition is associative only up to isomorphisms. In fact, two M-profunctors $F : \mathrm{C} \nrightarrow_{\mathrm{M}} \mathrm{D}$ and $G : \mathrm{D} \nrightarrow_{\mathrm{M}} \mathrm{E}$, their composition expands as follows:

$$(G \circ F)_{(Q,E,C)} \triangleq \int^{(M,N) \in \mathrm{M}^2} \mathrm{M}^{op}(Q, M \otimes N) \times \int^{D \in \mathrm{D}} F_{(N,D,C)} \times G_{(M,E,D)}$$

$$(G \circ F)_{(Q,E,C)} \cong \left(\coprod_{(M,N) \in \mathrm{M}^2} \mathrm{M}(M \otimes N, Q) \times \left(\coprod_{D \in \mathrm{D}} F_{(N,D,C)} \times G_{(M,E,D)} \right)_{/\approx_1} \right)_{/\approx_2}$$

where $\approx_1$ and $\approx_2$ are the equivalence relations defined as

$$(f, g) \approx_1 (f', g') \iff \exists d \in \mathrm{D}(D, D') : F_{(\mathrm{Id}_N,d,\mathrm{Id}_C)}(f') = f \wedge G_{(\mathrm{Id}_M,\mathrm{Id}_E,d)}(g) = g'$$

and as

$$\begin{aligned}(q : M \otimes N \to Q, g, f) \approx_2 (q' : M' \otimes N' \to Q, g', f') \iff \exists m \in \mathrm{M}(M, M')\\ \exists n \in \mathrm{M}(N, N') : \mathrm{M}(m \otimes n, \mathrm{Id}_Q)(q') = q \wedge Lan_{\mathcal{Y}}\widehat{G}\widehat{F}(m, n)(g, f) = (g', f')\end{aligned}$$

respectively. Then (1) can be unfolded in

$$(G \circ F)_{(Q,E,C)} \cong \left(\coprod_{(M,N),D} \mathrm{M}(M \otimes N, Q) \times F_{(N,D,C)} \times G_{(M,E,D)} \right)_{/\approx}$$

where $\approx$ is the equivalence relation such that:

$$\begin{aligned}(q : M \otimes N \to Q, f, g) \approx (q' : M' \otimes N' \to Q, f', g') \iff \\ \exists m \in \mathrm{M}(M, M') \exists n \in \mathrm{M}(N, N') \exists d \in \mathrm{D}(D, D') : \\ f = F_{(\mathrm{Id}_N,d,\mathrm{Id}_C)}(f') \wedge g' = G_{(\mathrm{Id}_M,\mathrm{Id}_E,d)}(g) \wedge q' = \mathrm{M}(m \otimes n, Q)(q).\end{aligned}$$

Therefore, composition is associative only up to isomorphisms as in the case of ordinary profunctors. In fact, $(J \circ (K \circ L))_{(Q,F,C)}$ unfolds into the following:

$$
\begin{aligned}
&=(J \circ (K \circ L))_{(Q,F,C)} \\
&= \int^{(M,N)\in \mathrm{M}^2} \mathrm{M}(M \otimes N, Q) \times \int^{E \in \mathrm{E}} (K \circ L)_{(N,E,C)} \times J_{(M,F,E)} \\
&= \int^{(M,N),(O,P),D,E} \mathrm{M}(M \otimes N, Q) \times \mathrm{M}(O \otimes P, N) \times L_{(P,D,C)} \times K_{(O,D,E)} \times J_{(M,F,E)} \\
&= \int^{M,O,P,D,E} \mathrm{M}(M \otimes (O \otimes P), Q) \times (L_{(P,D,C)} \times K_{(O,D,E)}) \times J_{(M,F,E)} \\
&\cong \int^{M,O,P,D,E} \mathrm{M}((M \otimes O) \otimes P, Q) \times L_{(P,D,C)} \times (J_{(M,F,E)} \times K_{(O,D,E)}) \\
&= \int^{M,N,O,P,D,E} \mathrm{M}(N \otimes P, Q) \times \mathrm{M}(M \otimes O, N) \times L_{(P,D,C)} \times K_{(O,D,E)} \times J_{(M,F,E)} \\
&= \int^{N,P} \mathrm{M}(N \otimes P, Q) \times L_{(P,D,C)} \times \int^{D \in \mathrm{D}} (J \circ K)_{(N,F,D)} \\
&=((J \circ K) \circ L)_{(Q,F,C)}
\end{aligned}
$$

Proposition 4 (Day's convolution [8]) *Given a (symmetric) monoidal category* $(\mathrm{M}, \otimes, \epsilon)$*, the Yoneda embedding* $\mathcal{Y}$ *is a strong monoidal functor defining the (symmetric) closed monoidal category* $(\mathrm{SET}^{\mathrm{M}^{op}}, \star, \mathcal{Y}\epsilon)$

Proof. The tensor $\star$ is defined as the left Kan extension along $\mathcal{Y} \times \mathcal{Y}$ of $\mathcal{Y}(- \otimes -)$ illustrated by the diagram below.

$$
\begin{tikzcd}
\mathrm{M} \times \mathrm{M} \arrow[r, "\mathcal{Y} \times \mathcal{Y}"] \arrow[d, "\mathcal{Y}(- \otimes -)"'] & \mathrm{SET}^{\mathrm{M}^{op}} \times \mathrm{SET}^{\mathrm{M}^{op}} \arrow[dl, "Lan_{\mathcal{Y}\times\mathcal{Y}}\mathcal{Y}(- \otimes -)"] \\
\mathrm{SET}^{\mathrm{M}^{op}} &
\end{tikzcd}
$$

Then, the definition of $\star$ can be unfolded as:

$$
X \star Y \triangleq \int^{M,N} (X \times Y)(M,N) \times \mathrm{M}(-, M \otimes N)
$$

$$
X \star Y \cong \left(\coprod_{M,N} (X \times Y)(M,N) \times \mathrm{M}(-, M \otimes N) \right)_{/\approx}
$$

where $\approx$ is the relation defined as:

$$
\begin{aligned}
&(x, y, q : Q \to M \otimes N) \approx (x', y', q' : Q \to M' \otimes N') \iff \exists m \in \mathrm{M}(M, M'). \\
&\qquad \exists n \in \mathrm{M}(N, N').(X \times Y)(m \times n)(x', y') = (x, y) \wedge \mathrm{M}(Q, m \otimes n)(q) = q'.
\end{aligned}
$$

By application of the co-Yoneda lemma on the two coends above $\mathcal{Y}\epsilon$ is the monoidal unit. Then, coherence conditions, closure and the natural isomorphism $\phi_{M,N} : \mathcal{Y}M \star \mathcal{Y}N \cong \mathcal{Y}(M \otimes N)$ readily follow. □