A unifying model of variables and names

Marino Miculan[1] Kidane Yemane[2*]

[1] Dept. of Mathematics and Computing Science, University of Udine
Via delle Scienze 206, I-33100 Udine, Italy. miculan@dimi.uniud.it
[2] Dept. of Information Technology, Uppsala University
Box 337, S-751 05 Uppsala, Sweden. kidane.yemane@it.uu.se

Abstract. We investigate a category theoretic model where both "variables" and "names", usually viewed as separate notions, are particular cases of the more general notion of *distinction*. The key aspect of this model is to consider functors over the category of irreflexive, symmetric finite relations. The models previously proposed for the notions of "variables" and "names" embed faithfully in the new one, and initial algebra/final coalgebra constructions can be transferred from the formers to the latter. Moreover, the new model allows for defining *distinction-aware* simultaneous substitutions as clone-like structures. Finally, we apply this model to develop the first semantic interpretation of the $FO\lambda^{\nabla}$ logic.

1 Introduction

In recent years, many approaches have been proposed for modeling and reasoning about the pervasive notions of *(bound) variable*, *(fresh) name*, *reference*, *nonces*, etc. Although animated by different aims and purposes, most (if not all) of these models are based on some (sub)category of *(pre)sheaves*, i.e., functors from a suitable index category to *Set* [18, 6, 10, 8, 5, 17]. Objects of the index category represent different "stages of knowledge", about the dynamically allocable entities; a presheaf is a domain "stratified" according to these different degrees of knowledge. The kind of allocable entity is induced by the kind of the index category: different notions are represented by choosing different index categories.

Two common examples are $\mathbb{F}$, the category of finite sets and all functions, and $\mathbb{I}$, the category of finite sets and *injective* maps only. Objects of both categories can be seen as "stages of abstract symbols", or *atoms*, which can be dynamically allocated, and can occur in terms and semantic entities. Morphisms specify the kind of operations allowed on these abstract symbols, and hence ultimately the notion they represent. When we consider $\mathbb{F}$, atoms can be always unified: we can create a fresh atom, but it can be unified with others, later on, via some non-injective morphism. When we consider $\mathbb{I}$, atoms can never be unified. We can create fresh atoms, but they will be always distinct from all other atoms. The first case corresponds to that of "variables", in the sense of λ-calculus [6, 10, 13]; the second is the case of "names" or "nonces", and it is suited for π-calculus and similar cases [18, 5].

* Supported by EU project IST-2001-33100 "PROFUNDIS".

According to this view, *variables* and *names* are quite different concepts, and as such they are rendered by different index categories. This separation is a drawback when we have to model calculi or logics where *both aspects* are present and must be dealt with at once. Two examples are the fusion calculus and the open bisimulation of π-calculus. Even, a (still unknown) algebraic model for the Mobile Ambients is supposed to deal with both variables and names (which are declared as different entities in capabilities). Another example on this line, which we will examine in this paper, is the $FO\lambda^{\nabla}$ logic [14].

Why are $\mathbb{F}$ and $\mathbb{I}$ not sufficient for modeling these situations? Despite they model different notions of allocable entity, they share a common feature: we are forced to decide the behaviour of an atom *at the same time of its creation*. Variables will be always variables, names always names throughout.

We can generalize this approach by *separating* the action of *allocation* of a name, from that of *specification* of its behaviour. The behaviour of an atom is specified by its relations with other atoms, and these relations can change dynamically, *after* that the atom is introduced. So the stages of knowledge are not represented by sets of atoms with an *a priori* fixed behaviour (*either* variables *or* names), but by *symmetric, irreflexive* relation on finite sets, called *distinctions*, which form the objects of a new index category $\mathbb{D}$. The operations on the distinction relation are separated from the allocation actions: an atom may act as a variable for a while, and then become a name. This extra flexibility adds a degree of freedom in the expressive power of the model.

The aim of this paper is to investigate the model of (pre)sheaves over $\mathbb{D}$, and some of its applications. First proposed by Ghani, Yemane and Victor for capturing open bisimulation of π-calculus and hyperbisimulation in fusion calculus [9], this model turns out to be quite general, insomuch it can subsume both the notion of variables and that of names.

We proceed as follows: In Section 2, we introduce the category $\mathbb{D}$, its properties and relationships with $\mathbb{F}$ and $\mathbb{I}$. In Section 3 we investigate the structure of $Set^{\mathbb{D}}$, and its relationships with $Set^{\mathbb{F}}$, $Set^{\mathbb{I}}$. As a first application, we will study initial algebras and final coalgebras of polynomial functors over $Set^{\mathbb{D}}$, relating them to similar results in previous categories.

In Section 4, we study two key notions about allocable atoms, namely *support* and *apartness*. We give a general, abstract notions of both aspects, and then apply and compare their instances in the cases of $Set^{\mathbb{D}}$, $Set^{\mathbb{F}}$ and $Set^{\mathbb{I}}$. An application of apartness is in Section 5, where we present a clone-like definition of "apartness-preserving" simultaneous substitution.

In Section 6 we turn to the logical aspects of $Set^{\mathbb{D}}$. A self-dual quantifier, similar to Gabbay-Pitts' И, will be introduced; then, taking advantage of this quantifier and of the peculiar constructors of $Set^{\mathbb{D}}$, we give a model for Miller-Tiu's $FO\lambda^{\nabla}$. Final remarks and directions for future work are in Section 7.

2 Distinctions

Let us fix an infinite, countable set of *atoms* $\mathbb{A}$. Atoms are abstract elements with no structure, intended to act both as *variables* and as *names* symbols.

We denote finite subsets of $\mathbb{A}$ as $n, m, \ldots$. Functions among these finite sets are "atom substitutions". The category of all these finite sets, and any maps among them is $\mathbb{F}$. The subcategory of $\mathbb{F}$ with only *injective* maps is $\mathbb{I}$. Thus, while a morphism in $\mathbb{F}$ may map different atoms to the same target, this cannot happen in $\mathbb{I}$. This corresponds to the difference between variables and names, that is, the formers can be identified and replaced, while names cannot. In fact, we can see a name essentially as an atom which must be kept apart from the others. We can formalize this concept as follows:

Definition 1. (The category $\mathbb{D}$) *The category $\mathbb{D}$ of* distinctions relations *is the full subcategory of Rel whose objects are irreflexive, symmetric relations over $\mathbb{A}$ with a finite carrier set. (In this paper Rel is taken to be the category with relations as objects and monotone functions as morphisms.)*

Intuitively, a distinction relation is a finite set of atoms n and a relation $d \subseteq n \times n$ such that related elements are thought of as definitely distinct, and which must never be renamed to be the same.

As a syntactic shorthand, we will write a distinction (n, d) as $d^{(n)}$, possibly dropping the superscript when clear from the context. A morphisms $f : d^{(n)} \to e^{(m)}$ is any *monotonic* function $f : n \to m$, that is a function which preserves the distinction relation (if $(a, b) \in d$ then $(f(a), f(b)) \in e$). Since morphisms represent "atom substitutions", this formalism ensures that atoms which are intended to be "names" (i.e., apart from other names) are renamed injectively, while variables may be unified with other variables.

2.1 Structure of $\mathbb{D}$

The category $\mathbb{D}$ inherits from *Rel* products and coproducts. More explicitly, coproducts are given by disjoint unions, and products are defined on objects as

$$d_1^{(m)} \times d_2^{(n)} = (m \times n, \{((i_1, j_1), (i_2, j_2)) \mid (i_1, i_2) \in d_1 \text{ and } (j_1, j_2) \in d_2\}).$$

Notice that $\mathbb{D}$ has no terminal object, but it has initial object $(\emptyset, \emptyset)$. In fact, $\mathbb{D}$ inherits meets, joins and partial order from $\wp(\mathbb{A})$:

- $d^{(n)} \wedge e^{(m)} = (d \cap e)^{(m \cap n)}$, and $d^{(n)} \vee e^{(m)} = (d \cup e)^{(m \cup n)}$
- $d^{(n)} \leq e^{(m)}$ iff $d \wedge e = d$, that is, iff $d \subseteq e$.

For each n, let us denote $\mathbb{D}_n$ the full subcategory of $\mathbb{D}$ whose objects are all relations over n. Then, $\mathbb{D}_n$ is a complete Boolean algebra. Let $\perp^{(n)} \triangleq (n, \emptyset)$ and $\top^{(n)} \triangleq (n, n^2 \setminus \Delta_n)$ be the *empty* and *complete* distinction on n, respectively, where $\Delta : \mathbb{F} \to Rel$ is the *diagonal* functor defined as $\Delta_n = (n, \{(i, i) \mid i \in n\})$. For $d_i^{(n)}$ $(i \in J)$ a set of distinctions of $\mathbb{D}_n$, we define $\bigvee_{i \in J} d_i^{(n)} \triangleq \bigcup_{i \in J} d_i$ as sets; similarly for meets. Finally, $\neg d^{(n)} = (n, n^2 \setminus (d \cup \Delta_n))$, and as usual, $d \Rightarrow e \triangleq (\neg d) \vee e$.

$\mathbb{D}$ can be given another monoidal structure. Let us define $\oplus : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ as

$$d_1^{(m)} \oplus d_2^{(n)} = (m + n, d_1 \cup d_2 \cup \{(i, j), (j, i) \mid i \in m, j \in n\}).$$

Proposition 1. $(\mathbb{D}, \oplus, \bot^{(0)})$ *is a symmetric monoidal category.*

By applying coproduct and tensor to $\bot^{(1)}$ we get two distinguished *dynamic allocation* functors $\delta^-, \delta^+ : \mathbb{D} \to \mathbb{D}$, as $\delta^- \triangleq \bot^{(1)} + \_$ and $\delta^+ \triangleq \bot^{(1)} \oplus \_$. More explicitly, the action of δ^+ on objects is $\delta^+(d^{(n)}) = d_{+1}^{(n+1)}$ where $d_{+1} = d \cup \{(*, i), (i, *) \mid i \in n\}$. Thus both δ^- and δ^+ add an extra element to the carrier, but, as the superscript $^+$ is intended to suggest, δ^+ adds in *extra* distinctions. The extra element can be used to represent a bound variable; δ^+ asks that, in addition, this new element is made distinct from the other elements. The functor δ^+ will be used for the binding associated with restriction to ensure that the extruded name cannot be renamed to other name as in open semantics of π-calculus, while the δ^- functor is used for bound input where no such restrictions are necessary.

2.2 Embedding $\mathbb{I}$ and $\mathbb{F}$ in $\mathbb{D}$

Let $\mathbb{D}_e$ denote the full subcategory of $\mathbb{D}$ of empty distinctions $\bot^{(n)} = (n, \emptyset)$, and $\mathbb{D}_c$ the full subcategory of complete distinctions $\top^{(n)} = (n, n^2 \setminus \Delta_n)$. Notice that all morphisms in $\mathbb{D}_c$ are *mono* morphisms of $\mathbb{D}$—that is, injective maps.

Let us consider the forgetful functor $U : \mathbb{D} \to \mathbb{F}$, dropping the distinction relation. The functor $\mathsf{v} : \mathbb{F} \to \mathbb{D}_e$ mapping each n in $\mathbb{F}$ to $\bot^{(n)}$, and each $f : n \to m$ to itself, is inverse of the restriction of U to $\mathbb{D}_e$.

On the other hand, the restriction of U to $\mathbb{D}_c$ is a functor $U : \mathbb{D}_c \to \mathbb{I}$, because the only morphisms in $\mathbb{D}_c$ are the injective ones. The functor $\mathsf{t} : \mathbb{I} \to \mathbb{D}_c$ mapping each n in $\mathbb{I}$ to $\top^{(n)}$, and each $f : n \rightarrowtail m$ to itself, is inverse of U. Hence:

Proposition 2. $\mathbb{D}_e \cong \mathbb{F}$*, and* $\mathbb{D}_c \cong \mathbb{I}$*.*

Therefore, we can say that the category of $\mathbb{D}$ generalises both $\mathbb{I}$ and $\mathbb{F}$. In fact, it is easy to check that the forgetful functor $U : \mathbb{D} \to \mathbb{F}$ is the right adjoint of the inclusion functor $\mathsf{v} : \mathbb{F} \hookrightarrow \mathbb{D}$.

Remark 1. While we are on this subject, we define the functor $V : \mathbb{D} \to \mathbb{I}$ which singles out from each d the (atoms of the) largest complete distinction contained in d. More precisely, V is defined on objects as $V(d^{(n)}) = \max\{m \mid \top^{(m)} \leq d^{(n)}\}$ and on morphisms as the restriction. This defines a functor: if $f : d^{(n)} \to e^{(m)}$ is a morphism, then it preserves distinctions, and thus for $i \in V(d)$, since i is part of a complete subdistinction of d, it must be mapped in a complete subdistinction of e, and hence $f(i) \in V(e)$. However, V is not an adjoint of t. $\square$

We recall finally that both $\mathbb{F}$ and $\mathbb{I}$ have finite products (and hence also $\mathbb{D}_e$, $\mathbb{D}_c$). Disjoint unions are finite coproducts in $\mathbb{F}$, but not in $\mathbb{I}$. Actually, disjoint union $\uplus : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ is only a monoidal structure over $\mathbb{I}$, which quite clearly corresponds to the restriction of $\oplus$ to $\mathbb{D}_c$:

Proposition 3. $\oplus \circ \langle \mathsf{t}, \mathsf{t} \rangle = \mathsf{t} \circ \uplus$*, that is, for* $n, m \in \mathbb{I}$*:* $\top^{(n \uplus m)} = \top^{(n)} \oplus \top^{(m)}$*.*

As a consequence, for Proposition 2, we have $\uplus = U \circ \oplus \circ \langle \mathsf{t}, \mathsf{t} \rangle$. On the other hand, $\oplus$ restricted to $\mathbb{D}_e$ is *not* equivalent to the coproduct $+$ in $\mathbb{F}$.

3 Presheaves over $\mathbb{D}$

The category $\mathit{Set}^{\mathbb{D}}$ has all functors from $\mathbb{D}$ to *Set* as objects, and natural transformations as morphisms. Objects of $\mathit{Set}^{\mathbb{D}}$ are often called *presheaves (over $\mathbb{D}^{op}$)*. Note that the restriction to finite distinction relations means that there are no size problems when talking about the category of presheaves.

3.1 Structure of $\mathit{Set}^{\mathbb{D}}$

The structure exhibited in Section 2.1 lifts to the category of presheaves:[3]

Proposition 4 (Structure of $\mathit{Set}^{\mathbb{D}}$). *The category $\mathit{Set}^{\mathbb{D}}$ has:*

1. *Products and coproducts, which are computed pointwise (as with all limits and colimits in functor categories); e.g. $(P \times Q)_{d^{(n)}} = P_{d^{(n)}} \times Q_{d^{(n)}}$. The terminal object is the constant functor $\mathcal{K}_1 = \mathbf{y}(\bot^{(\emptyset)})$: $\mathcal{K}_1(d) = 1$.*
2. *A presheaf of* atoms *$Atom \in \mathit{Set}^{\mathbb{D}}$, $Atom = \mathbf{y}(\bot^{(1)}) = \mathbf{y}(\top^{(1)})$. The action on objects is $Atom(d^{(n)}) = n$.*
3. *Two dynamic allocation functors $\delta^-, \delta^+ : \mathit{Set}^{\mathbb{D}} \to \mathit{Set}^{\mathbb{D}}$, induced by each $\kappa \in \{\delta^+, \delta^-\}$ on $\mathbb{D}$ as $\_ \circ \kappa : \mathit{Set}^{\mathbb{D}} \to \mathit{Set}^{\mathbb{D}}$.*
4. *If $\wp_f$ is the finite powerset functor on Set, then $\wp_f \circ \_ : \mathit{Set}^{\mathbb{D}} \to \mathit{Set}^{\mathbb{D}}$ defines the finite powerset operator on $\mathbb{D}$-presheaves.*
5. *Exponentials are defined as usual in functor categories:*

$$(B^A)_d \triangleq \mathit{Set}^{\mathbb{D}}(A \times \mathbb{D}(d, \_), B)$$
$$(B^A)_f(m) \triangleq m \circ (id_A \times (\_ \circ f)) \quad \text{for } f : d \to e \text{ in } \mathbb{D}, m : A \times \mathbb{D}(d, \_) \longrightarrow B$$

In particular, exponentials of representable functors have a nice definition:

Proposition 5. *For all $d \in \mathbb{D}$, B in $\mathit{Set}^{\mathbb{D}}$: $B^{\mathbf{y}(d)} \cong B_{d+\_}$.*

Proof.
$$\begin{aligned} (B^{\mathbf{y}(d)})_e &= \mathit{Set}^{\mathbb{D}}(\mathbf{y}(d) \times \mathbf{y}(e), B) && \text{by definition of exponential} \\ &\cong \mathit{Set}^{\mathbb{D}}(\mathbf{y}(d+e), B) && \text{since } \mathbf{y} \text{ preserves coproducts} \\ &\cong B_{d+e} && \text{by Yoneda Lemma.} \end{aligned}$$ □

This allows us to highlight a strict relation between $Atom$ and δ^-:

Proposition 6. *$(\_)^{Atom} \cong \delta^-$, and hence $\_ \times Atom \dashv \delta^-$.*

Proof. Since $Atom = \mathbf{y}(\bot^{(1)})$, by Proposition 5 we have that $F^{Atom} \cong F_{\bot^{(1)}+\_} = F_{\delta^-(\_)} = \delta^-(F)$. The second part is an obvious consequence, because in CCC's it is always $\_ \times B \dashv (\_)^B$. □

[3] Rather than invent new symbols for the lifted structure, we shall use the same symbols but ensure the reader has enough information to deduce which category we are working in.

3.2 Embedding $Set^{\mathbb{I}}$ and $Set^{\mathbb{F}}$ into $Set^{\mathbb{D}}$

Let us consider first the functors $\mathsf{v} : \mathbb{F} \hookrightarrow \mathbb{D}$ and $U : \mathbb{D} \to \mathbb{F}$.

Proposition 7. *v induces an essential geometric morphism $\mathsf{v} : Set^{\mathbb{F}} \to Set^{\mathbb{D}}$, that is two adjunctions $\mathsf{v}_! \dashv \mathsf{v}^* \dashv \mathsf{v}_*$, where $\mathsf{v}_! \cong {\_}\circ U$, $\mathsf{v}^* = {\_}\circ\mathsf{v}$, and $\mathsf{v}_*(F)(d^{(n)}) = F_n$ if $d^{(n)} = \bot^{(n)}$, 1 otherwise.*

$$\begin{tikzcd} Set^{\mathbb{F}} \arrow[r, bend left, "\mathsf{v}_! \cong {\_}\circ U"] \arrow[r, bend right, "\mathsf{v}_*"'] & Set^{\mathbb{D}} \arrow[l, "\mathsf{v}^* = {\_}\circ\mathsf{v}"'] \end{tikzcd} \qquad \textit{where} \qquad \mathsf{v}_*(F)(d^{(n)}) = \begin{cases} F_n & \textit{if } d^{(n)} = \bot^{(n)} \\ 1 & \textit{otherwise} \end{cases}$$

Proof. The existence of the essential geometric morphism, and that the inverse image is ${\_} \circ \mathsf{v}$, is a direct application of [12, VII.2, Theorem 2].

Let us prove that the direct image v_* has the definition above. By [12, VII.2, Theorem 2], we know that

$$\mathsf{v}_* = \underline{\mathrm{Hom}}_{\mathbb{F}^{op}}({}^{\bullet}\mathbb{D}^{op}_{\mathsf{v}}, {\_}) \tag{1}$$

where ${}^{\bullet}\mathbb{D}^{op}_{\mathsf{v}} : \mathbb{D}^{op} \times \mathbb{F} \to Set$ is the bifunctor defined on objects as ${}^{\bullet}\mathbb{D}^{op}_{\mathsf{v}}(d, n) = \mathbb{D}^{op}(\mathsf{v}(n), d) = \mathbb{D}(d, \bot^{(n)})$. By expanding the equation (1), we have that for all $F : \mathbb{F} \to Set$ and $d^{(m)}$ in $\mathbb{D}$:

$$\mathsf{v}_*(F)(d^{(m)}) = Set^{\mathbb{F}}(\mathbb{D}(d, \mathsf{v}({\_})), F) : Set \tag{2}$$

Now, an element of the set $Set^{\mathbb{F}}(\mathbb{D}(d, \mathsf{v}({\_})), F)$ is a natural transformation $\phi : \mathbb{D}(d, \mathsf{v}({\_})) \longrightarrow F$, that is a family of functions $\phi_n : \mathbb{D}(d, \bot^{(n)}) \to F_n$.

If d is not an empty distinction, then the set $\mathbb{D}(d, \bot^{(n)})$ is always empty, because there is no monotone map from $d \neq \bot^{(m)}$ to $\bot^{(n)}$. Therefore ϕ_n can be only $? : \emptyset \to F_n$, and hence $Set^{\mathbb{F}}(\mathbb{D}(d, \mathsf{v}({\_})), F) = \{? : \mathcal{K}_{\emptyset} \longrightarrow F\} = 1$.

If d is the empty distinction $\bot^{(m)}$, then $\mathbb{D}(d, \bot^{(n)}) = \mathbb{D}(\bot^{(m)}, \bot^{(n)}) = \mathbb{F}(m, n)$ by Proposition 2. Hence we can write equation (2) as

$$\mathsf{v}_*(F)(\bot^{(m)}) = Set^{\mathbb{F}}(\mathbb{F}(m, {\_}), F) = F_m$$

by Yoneda lemma, hence the thesis.

Let us prove $\mathsf{v}_! \cong {\_} \circ U$. Again by [12, VII.2, Theorem 2], we have

$$\mathsf{v}_! = {\_} \otimes_{\mathbb{F}^{op}} {}_{\mathsf{v}}\mathbb{D}^{op\bullet} \tag{3}$$

where ${}_{\mathsf{v}}\mathbb{D}^{op\bullet} : \mathbb{F}^{op} \times \mathbb{D} \to Set$ is the bifunctor defined on objects as

$${}_{\mathsf{v}}\mathbb{D}^{op\bullet}(n, d^{(m)}) = \mathbb{D}^{op}(d, \mathsf{v}(n)) = \mathbb{D}(\bot^{(n)}, d).$$

By expanding the equation (3), we can give the following more elementary definition of $\mathsf{v}_!$ on objects $F : \mathbb{F} \to Set$, $d^{(m)} : \mathbb{D}$:

$$\begin{aligned} \mathsf{v}_!(F)(d) &= F \otimes_{\mathbb{F}^{op}} \mathbb{D}(\mathsf{v}({\_}), d) = (\amalg_{n\in\mathbb{N}} F_n \times \mathbb{D}(\mathsf{v}(n), d))_{/\sim} \\ &= (\amalg_{n\in\mathbb{N}} F_n \times \mathbb{F}(n, m))_{/\sim} = (\amalg_{n\in\mathbb{N}} F_n \times m^n)_{/\sim} \end{aligned}$$

since $\mathbb{D}(\mathsf{v}(n), d) = \mathbb{D}(\perp^{(n)}, d) = \mathbb{F}(n, m)$, and where $\sim$ is the equivalence relation on pairs defined as follows: for $n, n' \in \mathbb{N}$, $f : n \to n'$, $g : n' \to m$, $a \in F_n$:

$$F_{n'} \times m^{n'} \ni (a[f], g) \sim (a, g \circ f) \in F_n \times m^n$$

By definition of $\sim$, any pair $(a, f) \in F_n \times m^n$ is equivalent to $(a[f], id) \in F_m \times m^m$. On the other hand, each $a \in F_m$ identifies uniquely an equivalence class $[(a, id)]_\sim$. Therefore, each equivalence class $\mathsf{v}_!(F)(d)$ can be given a unique canonic representantive $a \in F_m$. This means that there is a bijective equivalence between $\mathsf{v}_!(F)(d)$ and F_m, and hence $\mathsf{v}_!(F) \cong F \circ U$.

Alternative proof of $\mathsf{v}_! \cong \_ \circ U$: $\mathsf{v}_!$ can be defined as the left Kan extension along $\mathbf{y} : \mathbb{F}^{op} \hookrightarrow Set^{\mathbb{F}}$ of the functor $T : \mathbb{F}^{op} \to Set^{\mathbb{D}}$, $T(n) = \mathbb{D}(\perp^{(n)}, \_) = \mathbf{y} \circ \mathsf{v}^{op}$:

$$\begin{tikzcd}
 & \mathbb{F}^{op} \arrow[r, hook, "\mathbf{y}"] \arrow[dl, "\mathsf{v}^{op}"'] \arrow[d, "T"] & Set^{\mathbb{F}} \arrow[dl, "\mathsf{v}_! = \mathrm{Lan}_{\mathbf{y}}(T)"] \\
\mathbb{D}^{op} \arrow[r, hook, "\mathbf{y}"] & Set^{\mathbb{D}} &
\end{tikzcd}$$

. Hence:

$$\begin{aligned}
\mathsf{v}_!(F) = (\mathrm{Lan}_{\mathbf{y}}(T))(F) &= \int^{m \in \mathbb{F}} Set^{\mathbb{F}}(\mathbf{y}(m), F) \cdot \mathbb{D}(\perp^{(m)}, _) \\
&= \int^{m \in \mathbb{F}} F_m \cdot \mathbb{F}(m, U(_)) = \left(\int^{m \in \mathbb{F}} F_m \cdot \mathbb{F}(m, _) \right) \circ U = F \circ U \quad \square
\end{aligned}$$

Proposition 8. $\mathsf{v} : Set^{\mathbb{F}} \to Set^{\mathbb{D}}$ *is an embedding, that is:* $\mathsf{v}^* \circ \mathsf{v}_* \cong Id$.

Proof. For $F : \mathbb{F} \to Set$, we have to prove that $\mathsf{v}_*(F) \circ \mathsf{v} \cong F$. For $n \in \mathbb{F}$, we have $\mathsf{v}_*(F)_{\mathsf{v}(n)} = \mathsf{v}_*(F)_{\perp^{(n)}} = F_n$ by definition. Analogously, it is easy to prove that for $f : n \to m$ in $\mathbb{F}$, it is $\mathsf{v}_*(F)(\mathsf{t}(f)) = \mathsf{v}_*(F)(f) = F_f$. $\square$

As a consequence, by [12, VII.4, Lemma 1] we have also $\mathsf{v}^* \circ \mathsf{v}_! \cong Id$, and hence both v_* and $\mathsf{v}_!$ are full and faithful.

A similar result holds also for $\mathsf{t} : \mathbb{I} \hookrightarrow \mathbb{D}$, although the adjoints have not a neat description as in the previous case.

Proposition 9. t *induces an essential geometric morphism* $\mathsf{t} : Set^{\mathbb{I}} \to Set^{\mathbb{D}}$, *that is two adjunctions* $\mathsf{t}_! \dashv \mathsf{t}^* \dashv \mathsf{t}_*$,

$$\begin{tikzcd}
Set^{\mathbb{I}} \arrow[r, bend left, "\mathsf{t}_!"] \arrow[r, bend right, "\mathsf{t}_*"'] & Set^{\mathbb{D}} \arrow[l, "i^* = _ \circ \mathsf{t}"']
\end{tikzcd}$$

where for all $G : \mathbb{I} \to Set$, *and* $d \in \mathbb{D}$, *it is* $\mathsf{t}_*(G)(d) = Set^{\mathbb{I}}(\mathbb{D}(d, \mathsf{t}(\_)), G)$ *and* $\mathsf{t}_!(G)(d) = GV(d) \times \mathbb{D}(\top^{(V(d))}, d)$.

Proof. The definition of t_* is a direct application of [12, VII.2, Theorem 2]. Let us prove the definition of $\mathsf{t}_!$. We know that

$$\mathsf{t}_! = \_ \otimes_{\mathbb{I}^{op}} {}_{\mathsf{t}}\mathbb{D}^{op\bullet} \tag{4}$$

where ${}_{\mathsf{t}}\mathbb{D}^{op\bullet} : \mathbb{I}^{op} \times \mathbb{D} \to \mathit{Set}$ is the bifunctor defined on objects as

$$ {}_{\mathsf{t}}\mathbb{D}^{op\bullet}(n, d^{(m)}) = \mathbb{D}^{op}(d, \mathsf{t}(n)) = \mathbb{D}(\top^{(n)}, d). $$

By expanding the equation (4), we can give the following more elementary definition of $\mathsf{t}_!$ on objects $G : \mathbb{I} \to \mathit{Set}$, $d^{(m)} : \mathbb{D}$:

$$ \mathsf{t}_!(G)(d) = G \otimes_{\mathbb{I}^{op}} \mathbb{D}(\mathsf{t}(\_), d) = \Big(\amalg_{n \in \mathbb{N}} G_n \times \mathbb{D}(\top^{(n)}, d)\Big)_{/\sim} $$

where $\sim$ is the equivalence relation on pairs defined as follows: for $n, n' \in \mathbb{N}$, $f : n \rightarrowtail n'$, $g : \top^{(n')} \to d^{(m)}$, $a \in G_n$:

$$ G_{n'} \times \mathbb{D}(\top^{(n')}, d) \ni (a[f], g) \sim (a, g \circ f) \in G_n \times \mathbb{D}(\top^{(n)}, d) $$

Now, notice that for any $h \in \mathbb{D}(\top^{(n)}, d^{(m)})$ is a function $h : n \rightarrowtail m$ which can be factorized as $h = g \circ in$, where $in : n \hookrightarrow V(d)$ is the inclusion (and thus $in : \top^{(n)} \to \top^{(V(d))}$) and $g : \top^{(V(d))} \to d$ is a suitable monomorphism $g : V(d) \rightarrowtail m$. Therefore, for any pair $(a, h) \in G_n \times \mathbb{D}(\top^{(n)}, d)$ there is an equivalent one $(a[in], g) \in G_{V(d)} \times \mathbb{D}(\top^{(V(d))}, d)$, hence the thesis. $\square$

Proposition 10. $\mathsf{t} : \mathit{Set}^{\mathbb{I}} \to \mathit{Set}^{\mathbb{D}}$ *is an embedding, that is:* $\mathsf{t}^* \circ \mathsf{t}_* \cong Id$.

Proof. For $F : \mathbb{I} \to \mathit{Set}$, we have to prove that $\mathsf{t}_*(F) \circ \mathsf{t} \cong F$. For $n \in \mathbb{I}$, we have $\mathsf{t}_*(F)_{\mathsf{t}(n)} \cong F_n$, since $\mathsf{t}_*(F)_{\mathsf{t}(n)} = \mathit{Set}^{\mathbb{I}}(\mathbb{D}(\top^{(n)}, \mathsf{t}(\_)), F) \cong \mathit{Set}^{\mathbb{I}}(\mathbb{I}(n, \_), F) \cong F_n$.

It is similarly easy to prove that on morphisms, the action of $\mathsf{t}_*(F) \circ \mathsf{t}$ is isomorphic to that of F. Let $f : n \to m$ in $\mathbb{I}$; then, $\mathsf{t}_*(F)(\mathsf{t}(f))$ maps a natural transformation $\phi : \mathbb{I}(n, \_) \to F$ to the natural transformation $\phi : \mathbb{I}(m, \_) \to F$ whose components are $\psi_k = \phi_k(\_ \circ f) = F_f \circ \phi_k$, hence the thesis. $\square$

This means that also $\mathsf{t}^* \circ \mathsf{t}_! \cong Id$, and hence both t_* and $\mathsf{t}_!$ are full and faithful.

3.3 Algebras and coalgebras of polynomial functors

Polynomial functors play an important rôle in relation with initial algebras and final coalgebras. It is well-known that any polynomial functors over Set (i.e., defined only by constant functors, finite products/coproducts and finite powersets) has initial algebra. This result has been generalized to $\mathit{Set}^{\mathbb{F}}$ [6, 10] in order to deal with signatures with *variable bindings*; in this case, polynomials can contain also Var, the functor of *variables*, and a dynamic allocation functor $\delta_{\mathbb{F}} : \mathit{Set}^{\mathbb{F}} \to \mathit{Set}^{\mathbb{F}}$. For instance, the datatype of λ-terms up-to α-conversion can be conveniently defined as the initial algebra of the functor

$$ \Sigma_{\Lambda}(X) = Var + X \times X + \delta_{\mathbb{F}}(X) \qquad (5) $$

that is, for all $n \in \mathbb{F}$: $\Sigma_{\Lambda}(X)_n = n + X_n \times X_n + X_{n+1}$. A parallel generalization has been developed for dealing with signatures with *name generation*, using the category $\mathit{Set}^{\mathbb{I}}$ and its variants [10, 7, 5]. Here we are provided with N, the functor of *names*, and a dynamic allocation functor $\delta_{\mathbb{I}} : \mathit{Set}^{\mathbb{I}} \to \mathit{Set}^{\mathbb{I}}$. The datatype of

λ-terms where all bound variables are "fresh"[4] is defined as the initial algebra of the functor

$$\Sigma_\Lambda(X) = N + X \times X + \delta_{\mathbb{I}}(X) \tag{6}$$

that is, for all $n \in \mathbb{I}$: $\Sigma_\Lambda(X)_n = n + X_n \times X_n + X_{n+1}$.

The domain for late semantics of π-calculus [5] can be defined as the final coalgebra of the functor $B : Set^{\mathbb{I}} \to Set^{\mathbb{I}}$

$$BP \triangleq \wp_f(\overbrace{N \times P^N}^{\text{input}} + \overbrace{N \times N \times P}^{\text{output}} + \overbrace{N \times \delta_{\mathbb{I}} P}^{\text{bound output}} + \overbrace{P}^{\tau}) \tag{7}$$

$$(BP)_X = \wp_f(X \times (P_X)^X \times P_{X \uplus 1} + X \times X \times P_X + X \times P_{X \uplus 1} + P_X).$$

In $Set^{\mathbb{D}}$, we can generalize a step further. We say that a functor $F : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$ is *polynomial* if it be defined by using only $Atom$, constant functors, finite products/coproducts, dynamic allocations δ^+ and δ^- and finite powersets.

There is a precise relation among initial algebras of polynomial functors on $Set^{\mathbb{F}}$ and $Set^{\mathbb{D}}$. Let us recall a general result (see e.g. [10]):

Proposition 11. *Let $\mathcal{C}, \mathcal{D}$ be two categories and $f : \mathcal{C} \longrightarrow \mathcal{D}$, $T : \mathcal{C} \longrightarrow \mathcal{C}$ and $T' : \mathcal{D} \longrightarrow \mathcal{D}$ be three functors such that $T' \circ f \cong f \circ T$ for some natural isomorphism $\phi : T' \circ f \longrightarrow f \circ T$.*

1. *If f has a right adjoint f^*, and $(A, \alpha : TA \to A)$ is an initial T-algebra in $\mathcal{C}$, then $(f(A), f(\alpha) \circ \phi_A : T'(f(A)) \to f(A))$ is an initial T'-algebra in $\mathcal{D}$.*
2. *If f has a left adjoint f^*, and $(A, \alpha : A \to TA)$ is a final T-coalgebra in $\mathcal{C}$, then $(f(A), \phi_A^{-1} \circ f(\alpha) : f(A) \to T'(f(A)))$ is a final T'-coalgebra in $\mathcal{D}$.*

Proof. 1. The adjoint pair $f \dashv f^*$ can be lifted to a pair of adjoint functors between the categories of T- and T'- algebras. Since any functor with a right adjoint preserves colimits and the initial object is a colimit, then the initial object of the former category is preserved in the latter.
2. Like in the previous case, the adjoint $f^* \dashv f$ can be lifted to the categories of coalgebras, and functors with a left adjoint preserve limits. $\square$

For a polynomial functor $T : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$, let us denote $\bar{T} : Set^{\mathbb{F}} \to Set^{\mathbb{F}}$ the functor obtained by replacing $Atom$ with Var and δ^+, δ^- with $\delta_{\mathbb{F}}$ in T.

Theorem 1. *The polynomial functor $T : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$ has initial algebra, which is (isomorphic to) $F \circ U$, where (F, α) is the initial $\bar{T}$-algebra in $Set^{\mathbb{F}}$.*

Proof. The functor $\bar{T}$ has initial algebra (see e.g. [6, 10]); let us denote it by (F, α). In order to prove the result, we apply Proposition 11(1), where $f : \mathcal{C} \longrightarrow \mathcal{D}$ is the functor $\mathsf{v}_! = \_ \circ U : Set^{\mathbb{F}} \to Set^{\mathbb{D}}$ of Proposition 7, whose right adjoint is v^*. Then $\mathsf{v}_!(F) = F \circ U$. We have only to prove that $T \circ \mathsf{v}_! \cong \mathsf{v}_! \circ \bar{T}$. It is easy to see that this holds for products, coproducts, constant functors and finite powersets. It is also trivial to see that $Atom \cong Var \circ U$.

[4] This is what Barendregt called "hygienic convention".

It remains to prove that $\kappa \circ \mathsf{v}_! \cong \mathsf{v}_! \circ \delta_{\mathbb{F}}$, for $\kappa = \delta^+, \delta^-$. For F a functor in $\mathit{Set}^{\mathbb{F}}$, we prove that there is a natural isomorphism $\phi : \kappa(\mathsf{v}_!(F)) = \kappa(F \circ U) \longrightarrow \mathsf{v}_!(\delta_{\mathbb{F}}(F)) = \delta_{\mathbb{F}}(F) \circ U$. This is trivial, because for $d^{(n)}$ a distinction in $\mathbb{D}$, it is $\kappa(F \circ U)_d = (F \circ U)_{\kappa d} = F_{U(\kappa d)} = F_{n+1} = \delta_{\mathbb{F}}(F)_n = (\delta_{\mathbb{F}}(F) \circ U)_d$. $\square$

Therefore, initial algebras of polynomial functors in $\mathit{Set}^{\mathbb{D}}$ are nothing else but initial algebras of the corresponding polynomial functors in $\mathit{Set}^{\mathbb{F}}$. This means that $\mathit{Set}^{\mathbb{D}}$ can be used in place of $\mathit{Set}^{\mathbb{F}}$ for defining datatypes of syntax with variable binding, as in e.g. [9].

There is a similar connection between $\mathit{Set}^{\mathbb{I}}$ and $\mathit{Set}^{\mathbb{D}}$, about final coalgebras.

Lemma 1. $\delta^+ \circ \mathsf{t}_* \cong \mathsf{t}_* \circ \delta_{\mathbb{I}}$.

Proof. Let $F : \mathbb{I} \to \mathit{Set}$ be a functor, and $d^{(n)} \in \mathbb{D}$; we have to prove that

$$\mathit{Set}^{\mathbb{I}}(\mathbb{D}(\delta^+ d, \mathsf{t}(\_)), F) \cong \mathit{Set}^{\mathbb{I}}(\mathbb{D}(d, \mathsf{t}(\_)), \delta_{\mathbb{I}} F)$$

natural in d and F.

For $\phi : \mathbb{D}(\delta^+ d, \mathsf{t}(\_)) \longrightarrow F$, the corresponding natural transformation $\psi : \mathbb{D}(d, \mathsf{t}(\_)) \longrightarrow \delta_{\mathbb{I}} F$ has components $\psi_n \triangleq \phi_{n+1} \circ \delta^+$. More explicitly, for $f : d \to \top^{(n)}$, we have $\delta^+(f) : \delta^+ d \to \top^{(n+1)}$, thus $\phi_{n+1}(\delta^+(f)) \in F_{n+1} = (\delta_{\mathbb{I}}(F))_n$.

On the other hand, for $\psi : \mathbb{D}(d, \mathsf{t}(\_)) \longrightarrow \delta_{\mathbb{I}} F$, the components of the corresponding natural transformation $\phi : \mathbb{D}(\delta^+ d, \mathsf{t}(\_)) \longrightarrow F$ are defined as follows. Trivially, $\phi_0 = ? : \emptyset \to F_0$, because $\mathbb{D}(\delta^+ d, \top^{(0)}) = \emptyset$. Let us consider $n \neq 0$, and $f : \delta^+ d \to \top^{(n)}$, we have to define $\psi_n(f) \in F_n$. Now, let $n = m + 1$, where the element in 1 is the image along f of the element added by δ^+ to d. The restrict of f to d is a morphism $f_{|d} : d \to \top^{(m)}$. Thus, we define $\phi_n(f) \triangleq \psi_m(f_{|d})$.

It is easy to check that these two mappings are inverse of each other. $\square$

Lemma 2. $\delta^- \circ \mathsf{t}_* \cong \mathsf{t}_* \circ (\_)^N$.

Proof.
$$\begin{aligned}
\mathsf{t}_*(F^N)_d &= \mathit{Set}^{\mathbb{I}}(\mathbb{D}(d, \mathsf{t}(_)), F^{\mathbf{y}(1)}) && \text{by definition} \\
&\cong \mathit{Set}^{\mathbb{I}}(\mathbb{D}(d, \mathsf{t}(_)) \times \mathbf{y}(1), F) && \text{since we are in a CCC} \\
&= \mathit{Set}^{\mathbb{I}}(\mathbb{D}(d, \mathsf{t}(_)) \times \mathbb{D}(\top^{(1)}, \mathsf{t}(_)), F) && \\
&\cong \mathit{Set}^{\mathbb{I}}(\mathbb{D}(d + \top^{(1)}, \mathsf{t}(_)), F) && \mathbb{D}(_, e) \text{ preserves products} \\
&= \mathit{Set}^{\mathbb{I}}(\mathbb{D}(\delta^- d, \mathsf{t}(_)), F) && \text{by definition} \\
&= \mathsf{t}_*(F)_{\delta^- d} = \delta^-(\mathsf{t}_*(F))_d &&
\end{aligned}$$
$\square$

Let $T : \mathit{Set}^{\mathbb{I}} \to \mathit{Set}^{\mathbb{I}}$ be a polynomial functor. Let us denote by $\tilde{T} : \mathit{Set}^{\mathbb{D}} \to \mathit{Set}^{\mathbb{D}}$ the functor obtained by replacing in (the polynomial) of T, every occurrence of N with $\mathsf{t}_*(N)$, δ with δ^+, $(\_)^N$ with δ^-. Then, we have the following:

Theorem 2. *The functor $\tilde{T} : \mathit{Set}^{\mathbb{D}} \to \mathit{Set}^{\mathbb{D}}$ has final coalgebra, which is (isomorphic to) $\mathsf{t}_*(F)$, where (F, β) is the final T-coalgebra in $\mathit{Set}^{\mathbb{I}}$.*

Proof. Follows from previous lemmas and Proposition 11(2). $\square$

Therefore, in $\mathit{Set}^{\mathbb{D}}$ we can define coalgebrically all the objects definable by polynomial behavioural functors in $\mathit{Set}^{\mathbb{I}}$, like that for late bisimulation [5]. But moreover, $\mathit{Set}^{\mathbb{D}}$ provides other constructors, such as Atom, which do not have a natural counterpart in $\mathit{Set}^{\mathbb{I}}$. For instance, following [9], the domain for open semantics of π-calculus is the final coalgebra of the functor $B_o : \mathit{Set}^{\mathbb{D}} \to \mathit{Set}^{\mathbb{D}}$

$$B_o P \triangleq \wp_f(\overbrace{\mathit{Atom} \times \delta^- P}^{\text{input}} + \overbrace{\mathit{Atom} \times \mathit{Atom} \times P}^{\text{output}} + \overbrace{\mathit{Atom} \times \delta^+ P}^{\text{bound output}} + \overbrace{P}^{\tau}) \tag{8}$$

Notice that, although similar in shape, B_o is not the lifting of the functor B of strong late bisimulation in $\mathit{Set}^{\mathbb{I}}$ (Equation 7). More precisely, open bisimulation is closed under all name substitutions keeping apart extruded names. Thus, names are actually atoms, which can be unified if the distinctions allow so. A bound output adds a new atom to the distinction, which must be kept apart from any other previously known atom—hence the usage of δ^+. On the other hand, an input action introduces an atom which can be unified with any other name—hence the usage of δ^-.

4 Support and apartness

A key features of $\mathit{Set}^{\mathbb{I}}$ and similar functor categories for names is to provide some notion of *support* of terms/elements, and of *non-interference*, or "apartness", between terms [18, 8]. In this section, we first introduce a general definition of *support* and *apartness*, and then we examine these notions in the case of $\mathit{Set}^{\mathbb{D}}$, and related categories.

4.1 Support

Definition 2 (support). *Let $\mathcal{C}$ be a category, $F : \mathcal{C} \to \mathit{Set}$ be a functor. Let C be an object of $\mathcal{C}$, and $a \in F_C$. A subobject $i : D \rightarrowtail C$ of C* supports a (at C) *if there exists a (not necessarily unique) $b \in F_D$ such that $a = F_i(b)$.*

A support is called proper *iff it is a proper subobject.*

The intuition is that D supports $a \in F_C$ if D is "enough" for defining a. It is clear that the definition does not depend on the particular subobject representative. As a consequence, a is affected by what happens to elements in D only:

Proposition 12. *For all $D \in \mathrm{Supp}_{F,C}(a)$, and for all $h, k : C \to C'$: if $h_{|D} = k_{|D}$ then $F_h(a) = F_k(a)$.*

Notice that in general, the converse of Proposition 12 does not hold.

We denote by $\mathrm{Supp}_{F,C}(a)$ the set of subobjects of C supporting a. $\mathrm{Supp}_{F,C}(a)$ is a poset, inheriting its order from $\mathrm{Sub}(a)$, and it has always a top element, namely C itself. However, it may be that an element has no proper supports.

Remark 2. When $\mathcal{C} = \mathbb{F}, \mathbb{I}$, the supports of $a \in F_n$ can be seen as *approximations at stage n* of the free variables/names of a—that is, the free variables/names which are observable from n. For instance, let us consider $t \in \Lambda_n$, where Λ is the algebraic definition of untyped λ-calculus in equation 5. It is easy to prove by induction on t that for all $m \subseteq n$: $m \in \mathrm{Supp}_{\Lambda,n}(t) \iff FV(t) \subseteq m$.

Supports are viewed as "approximations" because elements may have not any proper support, at any stage. For example, consider the presheaf *Stream* : $\mathbb{F} \to \mathit{Set}$ constantly equal to the set of all *infinite* lists of variables. The stream $s = (x_1, x_2, x_3, \dots)$, which has infinite free variables, belongs to $s \in \mathit{Stream}_n$ for all n, but also $\mathrm{Supp}_{\mathit{Stream},n}(s) = \{n\}$. $\square$

Even in the case that an element has some finite (even proper) support, still it may be that it does not have a *least* support. (Consider, e.g., $G : \mathbb{F} \to \mathit{Set}$ such that $G_n = \emptyset$ for $|n| < 2$, and $= \{x\}$ otherwise; then $x \in G_{\{x,y,z\}}$ is supported by $\{x, y\}$ and $\{x, z\}$ but not by $\{x\}$ alone.) However, we can prove the following

Proposition 13. *Let $F : \mathcal{C} \to \mathit{Set}$ be a pullback-preserving functor, C be in $\mathcal{C}$, and $x \in F_C$. If both C_1, C_2 support x at C, then also $C_1 \wedge C_2$ supports x.*

Proof. $C_1 \wedge C_2$ is the pullback of the inclusions $j_1 : C_1 \rightarrowtail C$, $j_2 : C_2 \rightarrowtail C$; hence, by hypothesis the square in the diagram below is a pullback in *Set*:

$$\begin{tikzcd}
1 \arrow[dr, dotted, "y"] \arrow[drr, bend left, "y_1"] \arrow[ddr, bend right, "y_2"'] & & \\
& F_{C_1 \wedge C_2} \arrow[r, "F_{i_1}"] \arrow[d, "F_{i_2}"] & F_{C_1} \arrow[d, "F_{j_1}"] \\
& F_{C_2} \arrow[r, "F_{j_2}"] & F_C
\end{tikzcd}$$

Let $y_1 \in F_{C_1}$ and $y_2 \in F_{C_2}$ be the witnesses of x at stages C_1, C_2 by the definition of support. Due to the pullback there exists a (unique) $y \in F_{C_1 \wedge C_2}$ such that $F_{j_1 \circ i_1}(b) = F_{j_2 \circ i_2}(b) = a$, hence the thesis. $\square$

Remark 3. In the case that $\mathcal{C} = \mathbb{I}$, pullback-preserving functors correspond to sheaves with respect to the atomic topology, that is the Schanuel topos [12]. This subcategory of $\mathit{Set}^{\mathbb{I}}$ has been extensively used in previous work for modeling names and nominal calculi; see [10, 2, 4] among others, and ultimately also the FM techniques by Gabbay and Pitts [8, 16], since the category of nominal sets is essentially equivalent to the Schanuel topos [8, Section 7].

We will use pullback-preserving functors over $\mathbb{D}$ in Section 6 below. $\square$

Along the same line of Definition 2, we can introduce an abstract general notion of "closed element":

Definition 3. *Let $\mathcal{C}$ be a category with initial object* 0. *For $A : \mathcal{C} \to \mathit{Set}$, an element $a \in A_C$ is* closed *if* $0 \in \mathrm{Supp}_{A,C}(a)$.

Closed elements are not affected by any action on atoms whatsoever:

Proposition 14. *Let $\mathcal{C}$ be a category with initial object* 0. *For all $A : \mathcal{C} \to \mathit{Set}$, $C \in \mathcal{C}$, $a \in A_C$, if a is closed then for all $h, k : C \to D$ in $\mathcal{C}$: $A_h(a) = A_k(a)$.*

Proof. Follows from Proposition 12, noticing that $h_{|0} = k_{|0}$ always. $\square$

In the rest of the paper, we focus on the case when $\mathcal{C}$ is one of $\mathbb{F}$, $\mathbb{I}$, $\mathbb{D}$, which do have pullbacks and initial object ($\emptyset$, $\emptyset$ and $\perp^{(\emptyset)}$ respectively). As one may expect, the support in $\mathbb{D}$ is a conservative generalization of those in $\mathbb{F}$ and $\mathbb{I}$:

Proposition 15. *1. Let $n, m \in \mathbb{F}$, and $F : \mathbb{F} \to Set$. For all $a \in F_n$: $m \in \mathrm{Supp}_{F,n}(a) \iff \mathsf{v}(m) \in \mathrm{Supp}_{\mathsf{v}_!(F),\mathsf{v}(n)}(a)$.* [5]
2. Let $n, m \in \mathbb{I}$, and $F : \mathbb{I} \to Set$. For all $a \in F_n$: $m \in \mathrm{Supp}_{F,n}(a) \iff \mathsf{t}(m) \in \mathrm{Supp}_{\mathsf{t}_*(F),\mathsf{t}(n)}(a)$.

4.2 Apartness

We can now give the following general key definition, generalizing that used sometimes in $Set^{\mathbb{I}}$ (see e.g. [18]).

Definition 4 (Apartness). *Let $\mathcal{C}$ be a category with pullbacks and initial object. For $A, B : \mathcal{C} \to Set$, the functor $A \#_{\mathcal{C}} B : \mathcal{C} \to Set$ ("A apart from B") is defined on objects as follows:*

$$\begin{aligned}(A \#_{\mathcal{C}} B)_C = \{&(a,b) \in A_C \times B_C \mid \textit{for all } f : C \to D : \\ &\textit{there exist } s_1 \in \mathrm{Supp}_{A,D}(A_f(a)), s_2 \in \mathrm{Supp}_{B,D}(B_f(b)) \textit{ s.t. } s_1 \wedge s_2 = 0\}\end{aligned} \quad (9)$$

For $f : C \to D$, it is $(A \#_{\mathcal{C}} B)_f \triangleq A_f \times B_f$.

As a syntactic shorthand, we will write pairs $(a, b) \in (A \#_{\mathcal{C}} B)_c$ as $a \# b$. In the following, we will drop the index $_{\mathcal{C}}$ when clear from the context.

Let us now apply this definition to the three categories $Set^{\mathbb{I}}$, $Set^{\mathbb{F}}$, and $Set^{\mathbb{D}}$.

$\mathcal{C} = \mathbb{F}$ In this case we have that $a \# b$ iff at least one of a, b is closed, i.e., it is supported by the empty set: if both a and b have only non-empty supports, then some variable can be always unified by a suitable morphism. So the definition above simplifies as follows:

$$(A \#_{\mathbb{F}} B)_n = \{(a,b) \in A_n \times B_n \mid \emptyset \in \mathrm{Supp}_{A,n}(a) \text{ or } \emptyset \in \mathrm{Supp}_{B,n}(b)\} \quad (10)$$

$\mathcal{C} = \mathbb{I}$ In this case, names are subject only to injective renamings, and therefore can be never unified. So it is sufficient to look at the present stage, that is, the definition above simplifies as follows:

$$\begin{aligned}(A \#_{\mathbb{I}} B)_n = \{&(a,b) \in A_n \times B_n \mid \\ &\text{there exist } n_1 \in \mathrm{Supp}_{A,n}(a), n_2 \in \mathrm{Supp}_{B,n}(b) \text{ s.t. } n_1 \cap n_2 = \emptyset\}\end{aligned} \quad (11)$$

which corresponds to say that $a \# b$ iff a, b do not share any free name.

[5] Recall that $\mathsf{v}_!(F)_{\mathsf{v}(n)} \cong F_n$, and hence it is consistent to consider $a \in \mathsf{v}_!(F)_{\mathsf{v}(n)}$.

$\mathcal{C} = \mathbb{D}$ This case subsumes both previous cases: informally, $(a, b) \in (A \# B)_d$ means that if i is an atom appearing free in a, then any j occurring free in b can never be unified with i, that is $(i, j) \in d$:

$$\begin{aligned}(A \#_{\mathbb{D}} B)_{d^{(n)}} = \{&(a, b) \in A_d \times B_d \mid \\ &\text{there exist } s_1 \in \mathrm{Supp}_{A,d}(a), s_2 \in \mathrm{Supp}_{B,d}(b) \text{ s.t. } s_1 \oplus s_2 \leq d\}\end{aligned} \tag{12}$$

Actually, all these tensors arise from the monoidal structures $\oplus$ and $\uplus$ of the categories $\mathbb{I}$ and $\mathbb{D}$, via the following general construction due to Day [3]:

Proposition 16. *Let $(\mathcal{C}, \star, I)$ be a (symmetric) monoidal category. Then, $(Set^{\mathcal{C}}, \star_{\mathcal{C}}, \mathbf{y}(I))$ is a (symmetric) closed monoidal category, where*

$$(A \star_{\mathcal{C}} B)_C = \int^{C_1} A_{C_1} \times \int^{C_2} B_{C_2} \times \mathcal{C}(C_1 \star C_2, C) \tag{13}$$

Theorem 3. *The monoidal structure $(\mathbb{D}, \oplus, \bot^{(\emptyset)})$ induces, via equation 13, the monoidal structure $(Set^{\mathbb{D}}, \#_{\mathbb{D}}, \mathbf{y}(\bot^{(0)}) = \mathcal{K}_1 = 1)$ of equation 12.*

Proof. Let $A, B : \mathbb{D} \to Set$, and $d^{(n)} \in \mathbb{D}$; by applying Proposition 16 and since products preserves coends, we have

$$\begin{aligned}(A \star_{\mathbb{D}} B)_d &= \iint^{d_1, d_2} A_{d_1} \times B_{d_2} \times \mathbb{D}(d_1 \oplus d_2, d) \\ &= \left(\coprod_{d_1, d_2 \in \mathbb{D}} A_{d_1} \times B_{d_2} \times \mathbb{D}(d_1 \oplus d_2, d) \right)_{/\approx}\end{aligned} \tag{14}$$

where the equivalence $\approx$ is defined on triples as follows

$$\begin{aligned}(a, b, f : d_1 \oplus d_2 \to d) &\approx (a', b', g : d'_1 \oplus d'_2 \to d) \\ &\iff A_{f \circ inl}(a) = A_{g \circ inl}(a') \text{ and } B_{f \circ inr}(b) = B_{g \circ inr}(b')\end{aligned}$$

For each class $[(a, b, f : d_1 \oplus d_2 \to d)] \in (A \star_{\mathbb{D}} B)_d$ we can associate a unique pair $(A_{f \circ inl}(a), B_{f \circ inr}(b)) \in (A \#_{\mathbb{D}} B)_d$; the definition does not depend on the particular representative we choose.

On the converse, let us consider a pair $(a, b) \in (A \#_{\mathbb{D}} B)_d$; this means that

- there exists $f_1 : s_1 \rightarrowtail d$, $a' \in A_{s_1}$ such that $a = A_{f_1}(a')$
- there exists $f_2 : s_2 \rightarrowtail d$, $b' \in B_{s_2}$ such that $b = B_{f_2}(b')$

and such that $[f_1, f_2] : s_1 \oplus s_2 \rightarrowtail d$. We can associate this pair (a, b) to the equivalence class of the triple $(a', b', [f_1, f_2])$ in the coend 14. The class defined in this way does not depend on the particular a' and b' we choose.

It is easy to check that these two mappings are inverse of each other. □

A similar constructions applies also to $Set^{\mathbb{I}}$, as observed e.g. in [18]:

Proposition 17. *The monoidal structure* $(\mathbb{I}, \uplus, 0)$ *induces, via equation 13, the monoidal structure* $(Set^{\mathbb{I}}, \#_{\mathbb{I}}, \mathbf{y}(0) = 1)$ *of equation 11.*

Using Theorem 3, we can show that $\#_{\mathbb{F}}$ is a particular case of $\#_{\mathbb{D}}$:

Proposition 18. $\#_{\mathbb{F}} = \mathsf{v}^* \circ \#_{\mathbb{D}} \circ \langle \mathsf{v}_*, \mathsf{v}_* \rangle$.

Proof. Let us prove that for $F, G : \mathbb{F} \to Set$, it is $(\mathsf{v}_*(F) \#_{\mathbb{D}} \mathsf{v}_*(G))_{\perp^{(n)}} \cong (F \#_{\mathbb{F}} G)_n$. By applying Theorem 3, we have

$$\begin{aligned}(\mathsf{v}_*(F) \#_{\mathbb{D}} \mathsf{v}_*(G))_{\perp^{(n)}} &= \left(\coprod_{d_1^{(n_1)}, d_2^{(n_2)} \in \mathbb{D}} \mathsf{v}_*(F)_{d_1} \times \mathsf{v}_*(G)_{d_2} \times \mathbb{D}(d_1 \oplus d_2, \perp^{(n)}) \right)_{/\approx} \\ &= \left(\coprod_{d_1^{(n_1)}, d_2^{(n_2)} \in \mathbb{D}} F_{n_1} \times G_{n_2} \times \mathbb{D}(d_1 \oplus d_2, \perp^{(n)}) \right)_{/\approx}\end{aligned}$$

Let us consider the set $\mathbb{D}(d_1 \oplus d_2, \perp^{(n)})$. If $d_1 \oplus d_2 = \perp^{(m)}$ for some m, then $\mathbb{D}(d_1 \oplus d_2, \perp^{(n)}) = \mathbb{F}(m, n)$. Otherwise, $\mathbb{D}(d_1 \oplus d_2, \perp^{(n)}) = \emptyset$.

Now, the only way for having $d_1 \oplus d_2 = \perp^{(m)}$ is that both d_1 and d_2 are empty relations $\perp^{(n_1)}, \perp^{(n_2)}$, and at least one of them has no atoms at all (otherwise the $\oplus$ would add a distinction in any case). Therefore, the equivalence above can be continued as follows:

$$\ldots = \left(\left(\coprod_{n_1 \in \mathbb{F}} F_{n_1} \times G_{\emptyset} \times \mathbb{F}(n_1, n) \right) + \left(\coprod_{n_2 \in \mathbb{F}} F_{\emptyset} \times G_{n_2} \times \mathbb{F}(n_2, n) \right) \right)_{/\approx}$$

This means that the triples are either of the form $(a \in F_{\emptyset}, b \in G_{n_2}, f : n_2 \to n)$, or of the form $(a \in F_{n_1}, b \in G_{\emptyset}, f : n_1 \to n)$. The first is equivalent to the pair $(F_?(a), G_f(b))$, the second to the pair $(F_f(a), G_?(b))$, both in $(F \#_{\mathbb{F}} G)_n$. $\square$

The next corollary is a consequence of Theorem 3 and Proposition 16:

Corollary 1. *The functor* $A \# \_ : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$ *has a right adjoint* $[A]\_$*, defined on objects by* $([A]B)_d = Set^{\mathbb{D}}(A, B_{d\oplus \_})$.

Proof. By the general construction in [3], the right adjoint of $A \star \_$ is $[A]\_$, defined as $([A]B)_d = \int_{e^{(m)}} Set(A_e, B_{d\oplus e})$, which yields the thesis. $\square$

Remark 4. Let us consider the counit $ev_{A,B} : A \# [A]B \longrightarrow B$ of this adjunction. For $d \in \mathbb{D}$, the component $ev_d : (A \# [A]B)_d \longrightarrow B_d$ maps an element $a \in A_d$ and a natural transformation $\phi : A \to B_{d\oplus \_}$, *apart from each other*, to an element in B_d, which can be described as follows. Let $s_1, s_2 \in \mathrm{Sub}(d)$ supporting ϕ and a, respectively, and such that $s_1 \oplus s_2 \leq d$. By the definition of support, let $\phi' : A \to B_{s_1 \oplus \_}$ and $a' \in A_{s_2}$ be the witnesses of ϕ and a at s_1 and s_2, respectively. Then, $\phi'_{s_2}(a') \in B_{s_1 \oplus s_2}$, which can be mapped to an element in B_d by the inclusion $s_1 \oplus s_2 \leq d$. $\square$

Finally, for $A = Atom$ we have the counterpart of Proposition 6:

Proposition 19. *$[Atom]_{\_} \cong \delta^+$, and hence $\_ \# Atom \dashv \delta^+$.*

Proof. Since $Atom = \mathbf{y}(\perp^{(1)})$, we have $([Atom]B)_d = Set^{\mathbb{D}}(\mathbf{y}(\perp^{(1)}), B_{d\oplus\_}) = B_{d\oplus\perp^{(1)}} = (\delta^+(B))_d$ by Yoneda. $\square$

5 Substitution

In this section we apply the structure of $Set^{\mathbb{D}}$ presented in the previous sections for developing a notion of apartness-preserving substitution.

Let us define a tensor product $\bullet : Set^{\mathbb{D}} \times Set^{\mathbb{D}} \to Set^{\mathbb{D}}$ as follows:

$$\text{for } A, B \in Set^{\mathbb{D}} : \qquad A \bullet B \triangleq \int^{e\in\mathbb{D}} A_e \cdot B^e$$

$$\text{that is, for } d \in \mathbb{D} : \qquad (A \bullet B)_d = \int^{e\in\mathbb{D}} A_e \times (B^e)_d$$

where $B^{e^{(n)}} : \mathbb{D} \to Set$ is the functor defined by

$$(B^{e^{(n)}})_d = \{(b_1, \dots, b_n) \in (B_d)^n \mid \text{if } (i,j) \in e \text{ then } (b_i, b_j) \in (B \# B)_d\}$$

$$(B^{e^{(n)}})_f = (B_f)^n \qquad \text{for } f : d^{(m)} \to d'^{(m')}$$

Unfolding the coend, we obtain the following explicit description of $A \bullet B$:

$$(A \bullet B)_d = \left(\coprod_{e\in\mathbb{D}} A_e \times (B^e)_d \right)_{/\approx}$$

where $\approx$ is the equivalence relation defined by

$$(a; b_{\rho(1)}, \dots, b_{\rho(n)}) \approx (A_\rho(a); b_1, \dots, b_{n'}) \quad \text{for } \rho : e^{(n)} \to e'^{(n')}.$$

Actually, $B^{(\text{-})}$ can seen as a functor $B^{(\text{-})} : \mathbb{D}^{op} \to Set^{\mathbb{D}}$, adding the "reindexing" action on morphisms: for $\rho : e^{(n)} \to e'^{(n')}$, define $B^f : B^{e'} \longrightarrow B^e$ as the natural transformation with components $B^f_d : (B^{e'})_d \longrightarrow (B^e)_d$, $B^f_d(b_1, \dots, b_{n'}) = (b_{f(1)}, \dots, b_{f(n)})$. It is easy to check that B^f is well defined: if $(i,j) \in e'^{(n')}$, then $(f(i), f(j)) \in e^{(n)}$ and hence $(b_{f(i)}, b_{f(j)}) \in (B \# B)_d$.

The functor $B^{(\text{-})}$ is a generalization of Cartesian extension; for instance, $B^{\perp^{(2)}} = B \times B$, while $B^{\top^{(2)}} = B \# B$.

We can give now a more abstract definition of $\_ \bullet B : Set^{\mathbb{D}} \to Set^{\mathbb{D}}$, for all $B \in Set^{\mathbb{D}}$. In fact, $\_ \bullet B$ arises as the left Kan extension of the functor $B^{(\text{-})}$:

$$\begin{tikzcd}
1 \arrow[r, "\perp^{(1)}"] \arrow[dr, "B"'] & \mathbb{D}^{op} \arrow[r, hook, "\mathbf{y}"] \arrow[d, "B^{(\text{-})}"'] \arrow[dr, phantom, "\overset{\mathrm{Lan}}{\cong}"] & Set^{\mathbb{D}} \arrow[dl, bend left, "_\bullet B"] \\
& Set^{\mathbb{D}} \arrow[ur, bend right, "\langle B, _\rangle"'] &
\end{tikzcd} \tag{15}$$

where $\langle B, \_\rangle$ is the right adjoint of $\_ \bullet B$, defined as $\langle B, A\rangle_d = Set^{\mathbb{D}}(B^d, A)$.

Proposition 20. *($Set^{\mathbb{D}}, \bullet, Atom$) is a (non-symmetric) monoidal category.*

Proof. Since $Atom = \mathbf{y}(\perp^{(1)})$, the equivalence $A \bullet Atom \cong A$ follows from Diagram 15. The equivalence $Atom \bullet A \cong A$ is a simple calculation:

$$(Atom \bullet A)_d = \int^{e^{(m)}} Atom_{e^{(m)}} \times (A^{e^{(m)}})_d = \left(\coprod_{e^{(m)}} m \times (A^{e^{(m)}})_d \right)_{/\approx} \cong A_d$$

where the last equivalence holds because the class of a tuple $(i; a_1, \dots, a_m) \in m \times (A^{e^{(m)}})_d$ corresponds uniquely to $a_i \in A_d$.

We prove now associativity of $\bullet$:

$$\begin{aligned}
((A \bullet B) \bullet C)_d &= \int^{e} (A \bullet B)_e \times (C^e)_d && \text{by definition,} \\
&= \int^{e} \left(\int^{e'} A_{e'} \times (B^{e'})_e \right) \times (C^e)_d && \text{by definition,} \\
&= \int^{e} \int^{e'} A_{e'} \times (B^{e'})_e \times (C^e)_d && \text{product preserves coends,} \\
&= \int^{e'} A_{e'} \times \int^{e} (B^{e'})_e \times (C^e)_d && \text{by Fubini result,} \\
&= \int^{e'} A_{e'} \times ((B^{e'}) \bullet C)_d && \text{by definition,} \\
&\cong \int^{e'} A_{e'} \times ((B \bullet C)^{e'})_d \\
&= (A \bullet (B \bullet C))_d
\end{aligned}$$

□

Monoids of this monoidal category satisfy the usual properties of clones. In particular, the multiplication $\sigma : A \bullet A \to A$ of a monoid $(A, \sigma, v : Atom \to A)$ can be seen as a *distinction-preserving* simultaneous substitution: for every $d^{(n)} \in \mathbb{D}$, σ_d maps (the class of) a tuple $(a; a_1, \dots, a_m) \in A_{e^{(m)}} \times (B^e)_d$ to the corresponding element in A_d, making sure that distinct atoms are "replaced by" elements which are apart (if $(i, j) \in e$, then $(a_i, a_j) \in (A \# A)_d$).

6 Self-dual quantifier

As for any topos, $Set^{\mathbb{D}}$ can be used for modeling (higher-order) intuitionistic logic. However, like in $Set^{\mathbb{I}}$, the Schanuel topos, and FM-set theory, the extra structure given by apartness product brings in other, peculiar logical constructors. In this section we define a *self-dual* quantifier, *a la* И by Gabbay and Pitts [8], in (a suitable subcategory of) $Set^{\mathbb{D}}$; then, we apply this structure for giving a semantic interpretation of Miller-Tiu's $FO\lambda^{\nabla}$ logic.

6.1 Construction of a self-dual quantifier

We begin with a standard construction of categorical logic. For $A, B \in \mathit{Set}^{\mathbb{D}}$, let us consider the morphism $\theta : A \# B \hookrightarrow A \times B \xrightarrow{\pi} B$, given by inclusion in the cartesian product. We can define the *inverse image* of θ, $\theta^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A \# B)$: for $U \in \mathrm{Sub}(A)$, the subobject $\theta^*(U) \in \mathrm{Sub}(A \# B)$ is the pullback of $U \rightarrowtail B$ along θ: $\quad \theta^*(U)_d = \{(x, y) \in (A \# B)_d \mid y \in U_d\}$.

By general and well-known results [15, 12], θ^* has both left and right adjoints, denoted by $\exists_\theta, \forall_\theta : \mathrm{Sub}(A \# B) \to \mathrm{Sub}(B)$, respectively. (If $\#$ is replaced by $\times$, these are the usual existential and universal quantifiers $\exists, \forall : \mathrm{Sub}(A \times B) \to \mathrm{Sub}(B)$.) Our aim is to prove that, under some conditions, it is $\exists_\theta = \forall_\theta$.

The condition is suggested by the following result, stating that if a property of a "well-behaved" type holds for a fresh atom, then it holds for *all* fresh atoms:

Proposition 21. *Let $B : \mathbb{D} \to \mathit{Set}$ be a pullback preserving functor, and let U a subobject of $\mathit{Atom} \# B$. Let $d \in \mathbb{D}$, and $(a, x) \in U_d$. Then for all $b \in \mathit{Atom}_d$ such that $b \# x$: $(b, x) \in U_d$.*

Proof. It suffices to define $f : d \to d$ in $\mathbb{D}$ such that $(\mathit{Atom} \# B)_f(a, x) = (b, x)$; that is, we have to find an $f : d \to d$ such that $f(a) = b$ and $B_f(x) = x$. By functoriality of U, this means that $U_f(a, x) = (b, B_f(x)) \in U_d$.

Since $(a, x) \in U$, it is $a \# x$; hence, let s_1, r_1 be the two subdistinctions supporting a and x at d, such that $s_1 \oplus r_1 \leq d$ (equation 12). Similarly, for $b \# x$, let s_2, r_2 be the two subdistinctions supporting b and x at d, such that $s_2 \oplus r_2 \leq d$. Both r_1 and r_2 support x; hence, by Proposition 13, also $r_1 \wedge r_2$ supports x. Thus we can define the map $f : d \to d$ as $f(a) = b$, $f(b) = a$, and $f(i) = i$ otherwise. f is well defined, and moreover $f_{|r_1 \wedge r_2} = id_{|r_1 \wedge r_2}$ because both $a, b \notin \mathrm{Im}(r_1 \wedge r_2)$ ($a \notin \mathrm{Im}(r_1)$ and $b \notin \mathrm{Im}(r_2)$). By Proposition 12, this means that $B_f(x) = x$, hence the thesis. $\square$

Then, we have to restrict our attention to a particular class of subobjects, that is of properties we can consider:

Definition 5. *Let $A : \mathbb{D} \to \mathit{Set}$ be an object of $\mathit{Set}^{\mathbb{D}}$. A subobject $U \leq A$ is* closed *if for all $d \in \mathbb{D}$, $f : d \to e$, $x \in A_d$: if $A_f(x) \in U_e$ then $x \in U_d$.*

The lattice of closed subobjects of A is denoted by $\mathrm{ClSub}(A)$.

However, pullback-preserving subobjects of pullback-preserving functors are automatically closed, so this requirement is implied by the first one:

Proposition 22. *Let $A : \mathbb{D} \to \mathit{Set}$ be a pullback preserving functor, and $U \leq A$ be a subobject of A. If also U is pullback preserving, then it is closed.*

Proof. Let $f : d \to e$ be a morphism in $\mathbb{D}$, and $x \in A_d$ such that $A_f(x) \in U_e$. Take any object d' and $g_1, g_2 : d \to d'$ such that d is the pullback of g_1 along g_2:

$$
\begin{tikzcd}
d \arrow[r, "f"] \arrow[d, "f"'] & e \arrow[d, "g_1"] \\
e \arrow[r, "g_2"] & d'
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[dr, dotted, "y"] \arrow[drr, bend left, "U_f(x)"] \arrow[ddr, bend right, "U_f(x)"'] & & \\
& U_d \arrow[r, "U_f"] \arrow[d, "U_f"'] & U_e \arrow[d, "U_{g_1}"] \\
& U_e \arrow[r, "U_{g_2}"] & U_{d'}
\end{tikzcd}
$$

Then, the square of the diagram on the right is a pullback, and hence there exists a unique $y \in U_d$ such that $U_f(y) = A_f(x)$. It must be $y = x$, because there must be exactly one x satisfying a similar pullback diagram for A. $\square$

Let us denote by $\mathcal{D}$ the full subcategory of $Set^{\mathbb{D}}$ of pullback preserving functors. By above, for all $A \in \mathcal{D}$, the lattice $\mathrm{Sub}(A)$ of pullback-preserving subobjects is $\mathrm{ClSub}(A)$, but we will keep writing $\mathrm{ClSub}(A)$ for avoiding confusions.

For "well-behaved" types, θ^* restricts to closed subobjects:

Proposition 23. *For all $A, B \in \mathcal{D}$ and $U \in \mathrm{ClSub}(A) : \theta^*(U) \in \mathrm{ClSub}(A \# B)$.*

Its left and right adjoints $\exists_\theta, \forall_\theta : \mathrm{ClSub}(A \# B) \to \mathrm{ClSub}(A)$ have the following explicit descriptions: for $U \leq A \# B$:

$$
\begin{aligned}
\exists_\theta(U)_d = \{y \in B_d \mid{} & \text{there exist } f : d \to e, x \in A_e, \\
& \text{such that } x \# B_f(y) \text{ and } (x, B_f(y)) \in U_e\} && (16) \\
\forall_\theta(U)_d = \{y \in B_d \mid{} & \text{for all } f : d \to e, x \in A_e, \\
& \text{if } x \# B_f(y) \text{ then } (x, B_f(y)) \in U_e\} && (17)
\end{aligned}
$$

Proposition 24. *For all $B \in \mathcal{D}$: $\theta^* \circ \exists_\theta = id_{\mathrm{ClSub}(Atom \# B)}$*

Proof. For $U \in \mathrm{ClSub}(Atom \# B)$, we have to prove that $\theta^*(\exists_\theta(U)) = U$. Inclusion $\supseteq$ is trivial. Let us prove $\subseteq$. If $(a, y) \in \theta^*(\exists_\theta(U))_d$, then $a \# y$, and by definition of $\exists_\theta$ there exist $f : d \to e$, $b \in Atom_e$ such that $(b, B_f(y)) \in U_e$ (and hence $b \# B_f(y)$). But also $f(a) \# A_f(y)$, and therefore by Proposition 21, this means that also $(f(a), A_f(y)) \in U_e$. By closure of U, it must be $(a, y) \in U_d$. $\square$

Proposition 25. *Let $B \in \mathcal{D}$, and $U \in \mathrm{ClSub}(B)$; then, for all $x \in U_d$, there exist $f : d \to e$ and $a \in Atom_e$ such that $a \# B_f(x)$.*

Proof. We can "lift" this result from the subcategory of pullback preserving functors of $Set^{\mathbb{I}}$, i.e. the Schanuel topos, where this property is known to hold [8]. It is easy to check that if $F : \mathbb{D} \to Set$ is pullback-preserving, then also $F \circ t : \mathbb{I} \to Set$ is pullback preserving. As a consequence, if $x \in U_{d\langle n \rangle}$, then $x \in U_{\top\langle n \rangle} = (U \circ \mathsf{t})_n$, and hence there exist $f : n \rightarrowtail m$, $a \in N_m = Atom_{\top\langle m \rangle}$ such that $a \#_{\mathbb{I}} (B \circ \mathsf{t})_f(x)$, and thus $a \# B_f(x)$. $\square$

Proposition 26. *Let $B : \mathbb{D} \to Set$ be a pullback-preserving functor. Then, $\exists_\theta \circ \theta^* = id : \mathrm{ClSub}(B) \to \mathrm{ClSub}(B)$.*

Proof. Let $U \in \mathrm{ClSub}(B)$ be a closed subobject. For any $d \in \mathbb{D}$, we have

$$\begin{aligned}\exists_\theta(\theta^*(U))_d &= \{x \in B_d \mid \text{there exist } f : d \to e, a \in Atom_e,\\ &\qquad\qquad \text{s.t. } a \# B_f(x) \text{ and } (a, B_f(x)) \in \theta^*(U)_e\}\\ &= \{x \in B_d \mid \text{there exist } f : d \to e, a \in Atom_e,\\ &\qquad\qquad \text{s.t. } a \# B_f(x) \text{ and } B_f(x) \in U_e\}\\ &= \{x \in U_d \mid \text{there exist } f : d \to e, a \in Atom_e,\ \text{s.t. } a \# B_f(x)\}\end{aligned}$$

For Proposition 25 above, this is exactly equal to U_d, hence the thesis. $\square$

Corollary 2. *For $A \in \mathcal{D}$, the inverse image $\theta^* : \mathrm{ClSub}(A) \to \mathrm{ClSub}(Atom \# A)$ is an isomorphism, and hence $\theta^* \dashv \exists_\theta = \forall_\theta \dashv \theta^*$*

Let us denote by $\mathsf{И} : \mathrm{ClSub}(Atom \# A) \to \mathrm{ClSub}(A)$ any of $\exists_\theta$ and $\forall_\theta$. As the notation suggests, there is a close connection between this quantifier and Gabbay-Pitts'. In fact, both quantifiers enjoy the following inclusions:

Proposition 27. *Let $i : A \# B \hookrightarrow A \times B$ be the inclusion map, and $i^* : \mathrm{ClSub}(A \times B) \to \mathrm{ClSub}(A \# B)$ its inverse image. Then: $\forall \le \mathsf{И} \circ i^* \le \exists$, that is, for all $U \in \mathrm{ClSub}(A \times B)$: $\forall U \le \mathsf{И}(i^*(U)) \le \exists U$.*

Proof. A direct check $\square$

6.2 Interpretation of $FO\lambda^\nabla$

In this section we apply the structure of $\mathcal{D}$ for giving a semantic interpretation of the logic $FO\lambda^\nabla$ [14]. As we will see, the dynamic allocation functor δ^-, the apartness tensor (right adjoint to δ^+) and the $\mathsf{И}$ quantifier will come into play.

$FO\lambda^\nabla$ has been introduced as a proof theory of *generic judgments*. Typing judgments have the usual form $\Sigma \vdash t : \tau$, while sequents have the form

$$\Sigma : \sigma_1 \triangleright B_1, \ldots, \sigma_n \triangleright B_n \longrightarrow \sigma_0 \triangleright B_0$$

where Σ is the *global* signature (i.e., a set of "global" variables $x_1{:}\tau_1, \ldots, x_m{:}\tau_m$), and each σ_i is a *local* signature. A judgment $\sigma \triangleright B$ is called *generic*; variables appearing in each B may come both from the global signature and the local signature σ (formally: $\Sigma, \sigma \vdash B : o$). The intended meaning of the symbols declared in the local signatures is that of "locally scoped constants". We refer to [14] for derivation rules and further details.

We give an interpretation of this logic in $\mathcal{D}$, using functors for interpreting types, natural transformations for terms, and closed subobjects for predicates.

The interpretation of types and terms is standard: each type τ is interpreted as a functor $[\![\tau]\!]$ in $\mathcal{D}$; the interpretation is extended to global signatures using the cartesian product. A well-typed term $\Sigma \vdash t : \gamma$ is interpreted as a morphism (i.e., a natural transformation) $[\![t]\!] : [\![\Sigma]\!] \longrightarrow [\![\gamma]\!]$ in $\mathcal{D}$. Notice that here, "local" signatures do not have any special rôle, so that terms are simply typed λ-terms without any peculiar "freshness" or "scoping" constructor.[6]

On the other hand, in the interpretation of generic judgments we consider variables in local signatures as *distinguished* atoms. A declaration y appearing in a local signature σ, is intended as a "fresh, local" atom.

Remark 5. A correct model for $FO\lambda^{\nabla}$ would require a distinguished functor of atoms for each type (which can occur in local signatures) of the term language. Although it is technically possible to develop a typed version of the theory of $Set^{\mathbb{D}}$ (along the lines of [13] for $Set^{\mathbb{F}}$), it does not add anything substantial to our presentation; so in the following we assume variables of local signatures, or bound by ∇, can be only of one type (denoted by α). Hence, local signatures σ are of the form $(y_1{:}\alpha, \ldots, y_n{:}\alpha)$, or better $(y_1, \ldots, y_n)$ leaving α's implicit. $\square$

The distinguished type of propositions, o, is interpreted as the classifier of *(closed) subobjects*: $[\![o]\!]_d = \mathrm{ClSub}(\mathbf{y}(d)) = \mathrm{ClSub}(\mathbb{D}(d, \_))$. A generic judgment $(y_1, \ldots, y_n) \rhd B$ in Σ (i.e., $\Sigma, y_1 : \alpha, \ldots y_n : \alpha \vdash B : o$) is interpreted as a closed subobject $[\![(y_1, \ldots, y_n) \rhd B]\!]_{[\![\Sigma]\!]} \leq [\![\Sigma]\!]$. More precisely, $[\![\sigma \rhd B]\!]_A \in \mathrm{ClSub}(A)$ is defined first by induction on the length of the local context σ, and then by structural induction on B. Local declarations and the ∇ quantifier are rendered by the functor $\text{И} : \mathrm{ClSub}(A \# Atom) \to \mathrm{ClSub}(A)$ above. Some interesting cases:

$$[\![(y, \sigma) \rhd B]\!]_A \triangleq \text{И}([\![\sigma \rhd B]\!]_{A\#Atom}) \qquad [\![\rhd B_1 \wedge B_2]\!]_A \triangleq [\![\rhd B_1]\!]_A \wedge [\![\rhd B_2]\!]_A$$
$$[\![\rhd \nabla y.B]\!]_A \triangleq \text{И}([\![\rhd B]\!]_{A\#Atom}) \qquad [\![\rhd \forall_\gamma x.B]\!]_A \triangleq \forall([\![\rhd B]\!]_{A \times [\![\gamma]\!]})$$

It is easy to prove by induction on σ that

Proposition 28. $[\![(\sigma, y) \rhd B]\!]_A = [\![\sigma \rhd \nabla y.B]\!]_A$.

Finally, a sequent $\Sigma : \mathcal{B}_1, \ldots, \mathcal{B}_n \longrightarrow \mathcal{B}_0$ is *valid* if $\bigwedge_{i=1}^n [\![\mathcal{B}_i]\!]_{[\![\Sigma]\!]} \leq [\![\mathcal{B}_0]\!]_{[\![\Sigma]\!]}$. A rule $\frac{\mathcal{S}_1 \ldots \mathcal{S}_n}{\mathcal{S}}$ is *sound* if, whenever all $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are valid, also $\mathcal{S}$ is valid.

Using this interpretation, one can check that the rules of $FO\lambda^{\nabla}$ are sound. In particular, the rules $\nabla\mathcal{L}$ and $\nabla\mathcal{R}$ are trivial consequence of above. The verification of $\forall\mathcal{R}$, and $\exists\mathcal{L}$ requires some work. Here, we have to give a categorical account of a particular encoding technique, called *raising*, used to "gain access" to local constants from "outside" their scope. E.g.:

$$\frac{\Sigma, h{:}\sigma \to \gamma : \Gamma \longrightarrow \sigma \rhd B[(h\ \sigma)/x]}{\Sigma : \Gamma \longrightarrow \sigma \rhd \forall_\gamma x.B}\forall\mathcal{R}$$

A simpler (i.e., monadic) application of raising occurs, in the following equivalence, which is provable in $FO\lambda^{\nabla}$:

$$\nabla x \forall_\gamma y.B \equiv \forall_{\alpha\to\gamma} h \nabla x.B[(h\ x)/y] \qquad \text{where } \Sigma, x : \alpha, y : \gamma \vdash B : o \tag{18}$$

[6] As Miller and Tiu say, this is a precise choice in the design of $FO\lambda^{\nabla}$, motivated by the fact that standard unification algorithms still work unchanged.

We show first how to represent (monadic) raising as in the equation 18; interestingly, it is here where the δ^- comes into play. Referring to equation 18, let us denote $A = [\![\Sigma]\!]$ and $C = [\![\gamma]\!]$. By the definition above, the interpretation of B is a subobject of $(A \# \mathit{Atom}) \times C$, while $B[(h\ x)/y]$ corresponds to a subobject of $(A \times C^{\mathit{Atom}}) \# \mathit{Atom}$. Now, notice that $C^{\mathit{Atom}} = \delta^- C$ (Proposition 6); thus, $h : \alpha \to \gamma$ is actually a term $[\![h]\!] \in \delta^- C$, that is a term which can make use of a locally declared variable. We can define the *raising* morphism

$$\begin{aligned} \mathsf{r} : (A \times \delta^- C) \# \mathit{Atom} &\to (A \# \mathit{Atom}) \times C \\ (x, h, a) &\mapsto (x, a, h(a)) \end{aligned}$$

The inverse image of r is $\mathsf{r}^* : \mathrm{ClSub}((A \# \mathit{Atom}) \times C) \to \mathrm{ClSub}((A \times \delta^- C) \# \mathit{Atom})$, defined by the following pullback:

$$\begin{tikzcd} \mathsf{r}^*(U) \arrow[r] \arrow[d, tail] & U \arrow[d, tail] \\ (A \times \delta^- C) \# \mathit{Atom} \arrow[r, "\mathsf{r}"] & (A \# \mathit{Atom}) \times C \end{tikzcd}$$

This morphism r^* is the categorical counterpart of the syntactic raising:

Proposition 29. *Let $\Sigma, x{:}\alpha, y{:}\gamma \vdash B : o$. Let us denote $A = [\![\Sigma]\!]$, $C = [\![\gamma]\!]$. Then, $\mathsf{r}^*([\![y \rhd B]\!]_C) = [\![y \rhd B[(h\ y)/x]]\!]_{A \times \delta^- C}$.*

Then, quite obviously, the equation 18 states that $\text{И} \circ \forall_\gamma = \forall_{\alpha\to\gamma} \circ \text{И} \circ \mathsf{r}^*$, that is, the following diagram commutes:

$$\begin{tikzcd} \mathrm{ClSub}((A \# \mathit{Atom}) \times C) \arrow[r, "\mathsf{r}^*"] \arrow[d, "\forall_\gamma"] & \mathrm{ClSub}((A \times \delta^- C) \# \mathit{Atom}) \arrow[r, "\text{И}"] & \mathrm{ClSub}(A \times \delta^- C) \arrow[d, "\forall_{\alpha\to\gamma}"] \\ \mathrm{ClSub}(A \# \mathit{Atom}) \arrow[rr, "\text{И}"] & & \mathrm{ClSub}(A) \end{tikzcd}$$

which can be checked by calculation. The raising morphism can be easily generalized to the polyadic case (recall that $B^{\top(n)} = B \# \cdots \# B$, n times):

$$\begin{aligned} \mathsf{r} : (A \times \delta^{-n} C) \# \mathit{Atom}^{\top(n)} &\to (A \# \mathit{Atom}^{\top(n)}) \times C \\ (x, h, a_1, \ldots, a_n) &\mapsto (x, a_1, \ldots, a_n, h(a_1, \ldots, a_n)) \end{aligned}$$

Then, the soundness of the rule $\forall\mathcal{R}$ is equivalent to the following:

Proposition 30. *Let $A, C \in \mathcal{D}$ be functors, and $n \in \mathbb{N}$. Let $\pi : A \times \delta^{-n} C \to A$ be the projection, and $\mathsf{r} : (A \times \delta^{-n} C) \# \mathit{Atom}^{\top(n)} \longrightarrow (A \# \mathit{Atom}^{\top(n)}) \times C$ the raising morphism. For all $G \in \mathrm{ClSub}(A)$, and $U \in \mathrm{ClSub}((A \# \mathit{Atom}^{\top(n)}) \times C)$, if $\pi^*(G) \leq \text{И}^n(\mathsf{r}^*(U))$ then $G \leq \text{И}^n(\forall_\gamma(U))$.*

7 Conclusions

In this paper, we have presented and investigated a new model for dynamically allocable entities, based on the notion of *distinction*. Previous models for variables and for names can be embedded faithfully in this model, and also results about initial algebras/final coalgebras and about simultaneous substitutions are extended to the more general setting. In a suitable subcategory of the model, it is possible to define also a self-dual quantifier, with properties similar to that of Gabbay-Pitts' "И". We have used this quantifier for giving the first (up-to our knowledge) model for Miller-Tiu's $FO\lambda^{\nabla}$.

Future work. A possible application of $Set^{\mathbb{D}}$, yet to be investigated, is to model other calculi featuring both variables and names at once, e.g., mobile ambients. Apartness-preserving substitution can be applied as the basic computational step of a interference-aware functional language.

$FO\lambda^{\Delta\nabla}$ is not complete with respect to the model presented in this paper; one reason is that the И quantifier enjoys properties which are not derivable (e.g., $\forall_\alpha x.B \supset \nabla x.B$). The construction of a model for a self-dual quantifier which is not implied by $\forall$ and does not imply $\exists$, is still an open problem.

Acknowledgments. The authors wish to thank Dale Miller and Alwen Tiu for useful discussions about $FO\lambda^{\nabla}$, and Neil Ghani for hints about Kan extensions.

References

1. J. Adamek, editor. *Coalgebraic Methods in Computer Science*, ENTCS. 2004.
2. A. Bucalo, M. Hofmann, F. Honsell, M. Miculan, and I. Scagnetto. Consistency of the theory of contexts. Submitted, 2001.
3. B. J. Day. On closed categories of functors. In *Reports of the Midwest Category Seminar*, volume 137 of *Lecture Notes in Mathematics*, pages 1–38. Springer, 1970.
4. M. Fiore and S. Staton. Comparing operational models of name-passing process calculi. In Adamek [1].
5. M. Fiore and D. Turi. Semantics of name and value passing. In H. Mairson, editor, *Proc. 16th LICS*, pages 93–104, 2001.IEEE.
6. M. P. Fiore, G. D. Plotkin, and D. Turi. Abstract syntax and variable binding. In Longo [11], pages 193–202.
7. M. J. Gabbay and A. M. Pitts. A new approach to abstract syntax involving binders. In [11], pages 214–224.
8. M. J. Gabbay and A. M. Pitts. A new approach to abstract syntax with variable binding. *Formal Aspects of Computing*, 13:341–363, 2002.
9. N. Ghani, K. Yemane, and B. Victor. Relationally staged computation in calculi of mobile processes. In Adamek [1].
10. M. Hofmann. Semantical analysis of higher-order abstract syntax. In Longo [11].
11. G. Longo, editor. *Proceedings, Fourteenth Annual IEEE Symposium on Logic in Computer Science*, Trento, Italy, 1999. IEEE.
12. S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic: a First Introduction to Topos Theory.* Universitext. Springer-Verlag, 1994.

13. M. Miculan and I. Scagnetto. A framework for typed HOAS and semantics. In *Proc. PPDP'03*, pages 184–194. ACM Press, 2003.
14. D. Miller and A. F. Tiu. A proof theory for generic judgments: An extended abstract. In *LICS 2003*, pages 118–127, 2003. IEEE.
15. A. M. Pitts. Categorical logic. In *Handbook of Logic in Computer Science*, volume 5. Oxford University Press, 2000.
16. A. M. Pitts. Nominal logic, a first order theory of names and binding. *Information and Computation*, 186:165–193, 2003.
17. J. Power. A unified category-theoretic approach to variable binding. In *Proc. 2nd MERλIN*, ACM Digital Library. ACM, 2003.
18. I. Stark. A fully abstract domain model for the π-calculus. In *Proc. LICS'96*.