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H2SI – A New Perceptual Colour Space

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***Abstract*—In this paper we introduce a new colour space which is equipped with a metric that shares many properties with the human perception of colour, and we derive its the most important geometric properties. The new colour space is well suited for algorithmic purposes, as we demonstrate with the 'Eigen' colour decomposition of images.**

***Index Terms*—Human colour perception, primary colour matching functions, HSI colour space, opponent colour vision, brightness of colour, colour distance, quantum information, signal/sensor fusion, image segmentation, 'Eigen' colour.**

I. Introduction

Even today colour remains a miracle. Some 20,000 years ago our ancestors started to liven up their paintings on cave walls with colour. In the past whole industrial imperia developed around it to produce certain rare colours. Colour seems to be very special for beings that can sense it. In this article we try to shed some new light on it, which, if taken literally, does not harm our perception of colour too much, as should become clear later.

Colour, its nature, representation, perception, and geometric properties have been studied by scientists for centuries. The scientific history of work on colour dates back at least to Isaac Newton, who found amongst others, that light sent through a prism splits into most of the perceivable colour tones, which can be recombined to bright colourless light with a lens. Goethe, Young, Maxwell, Hering, Helmholtz, Munsell, Silberstein, and Schrödinger, to name but a few, worked on the topic, sometimes disagreeing with each other in their results.

Modern colour science cannot be seen as a unique field of research as it involves many research areas ranging from psychological, biological, and medical sciences to more technically oriented areas such as chemistry or computer vision. [1]–[5] give an overview of the different aspects in this area. An excellent and very readable sketch of the history of ideas about colour from Aristotle to modern colour science is given in [6] (pp. 624–636).

The more technically oriented work in the area is based on some of this preceding groundwork and deals mainly with the representation of colour and the geometrical and topological relations that colours have. The most widely used explicit definition of a colour representation in a mathematical space - the XYZ colour matching functions space defined by the CIE (Commission Internationale de l'Eclairage) in 1931 - might be seen as the starting point of a normative process in order to standardize colour for technical/industrial and algorithmic purposes. The original aim of the design of mathematical colour spaces was to develop an isotropic colour representation, sometimes called 'the ideal colour space'. Isotropic in the sense that perceived colour attributes map to a space where unit differences in one dimension are equal to unit differences in the other dimensions. In loose terms these attributes are given by the colour tone or hue, the chromaticity or saturation and the brightness or lightness or luminance of colour, but the exact definition of these attributes varies greatly in the literature. Other colour spaces were defined by Munsell [7] (1905) and several by the CIE (cf. [2], [3], [6]).

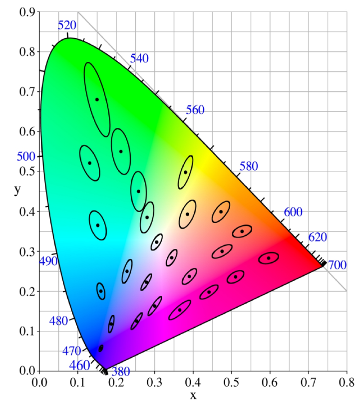

Fig. 1. MacAdam ellipses plotted on the CIE 1931 xy chromaticity diagram [9]. The ellipses show the location of colours that have a just noticable difference (JND) to the center colour. The ellipses are ten times their actual size, as depicted in MacAdam's paper [8].

MacAdam showed in the 1940s that the CIE XYZ colour space is not isotropic [8]. Instead of circles of perceived unit differences of colour it contains ellipses in the xy chromaticity diagram which are elongated in chroma with respect to hue, and ellipsoids when the brightness dimension is taken into account as well (see Figure 1). His results were strengthened by Judd [6], [10] (p. 646), Schrödinger [11]–[15] and Silberstein [16]–[20] who concluded that no ideal colour space, especially no three-dimensional Euclidean space supports geometrical properties where MacAdam ellipses, i.e. perceived unit differences of colour, can lie on unit circles. Judd termed this property of colour perception '*Super-importance of hue differences*' [10]. Judd noted [6] (p. 649):

> "...; but the implication that gray is both precisely between 5R and 10Y (Munsell units, authors note)

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> and precisely between 5R and 5BG is rather hard to take. Furthermore, by this model, gray lies precisely between any two hues differing by 25 or more and 50 or less Munsell hue steps." And
> "... There does not seem to be a geometrical model agreeing with Eq. 15; (colour difference formula, authors note) at least I have not been able to think of one."

The existence of this phenomenon, at least in human colour perception, has been validated in many psychological experiments over the last century with remarkably small differences with respect to the super-importance of hue differences property from one person to another.

There is a simple consequence for any mathematical model containig this property. 'Super-importance of hue', as pointed out by Judd [6] (p. 636), requires a space for colour representations that enables unit circles to have a circumference to radius ratio of roughly 720°, i.e. twice the circumference of a unit circle in Euclidean space. He visualized this as a 'fan crinkled surface' and explained some consequences as cited above.

Newer normalization developments such as the CIE colour spaces and distance formulae such as ΔE take a different approach and try to map geometrical properties of colour in a non-linear way onto three-dimensional Euclidean spaces. There have been many attempts to overcome the non-linear behaviour of colour distances, such as ΔE, by developing approximations in Euclidean space [21]–[24]. But although ΔE allows us to measure local distances of colour under controlled illumination and surrounding conditions reasonably well, it seems to fail when bigger colour differences have to be judged [23].

Recently Bengtsson and Zyczkowski [25] pointed out that a space in which the 'super-importance of hue' property is represented by its geometry would be given by the well-studied mathematical spaces of quantum mechanics and quantum information theory, but without giving an explicit formula. The apparent discrepancy between this and Judd's claim is resolved when the complex-valued nature of these spaces is taken into account.

Later developments mainly took two directions: an industrial one and a computational one, or, to phrase it differently albeit not entirely correctly, the direction of absolute and relative colour. Absolute colour refers to the reproduction process of colour in modern technical devices. This is limited, depending on the device. The task is to find good substitutes if a concrete colour (or colorant) is not available. Nowadays this is done via look up tables the so-called ICC profiles that map colour seen by one device such as a camera onto colours producible by another device which could be a printer or monitor. It is not pleasing to see the blue sea on the latest holiday pictures turn purple on the printouts. Therefore the colour spaces that have been devised and the error criteria that have been derived try to model the end users and consumers, i.e. human perception more accurately in newly created colour spaces. The revised colour mappings of the CIE, such as the Lab or Luv colour space, and the ΔE distance function on absolute colour errors give a good history of this development. The state of the art is quite impressive, as can be seen in technical processes of colour reproduction such as printing, TV, etc. or in modern computer-generated movies.

In other words, the technical processes seem to be handled reasonably well by the techniques we have, and for the purposes they were created for.

Colour as an information source for computational purposes is a different matter. The ease with which humans, and most likely other beings that are able to enjoy a colourful environment, are able to separate scenes into 'important' or dominant parts has puzzled scientists from the beginning. The question of whether this is mainly a product of cognitive skills such as learning and context, or whether other helpful mechanisms exist is unanswered and this article will not provide an answer. But with respect to colour, we would like to put a more 'syntactical', or rather, mathematical regime into place that coincides far more closely with our perception than the ones developed before and has the benefit of being helpful for algorithmic purposes. It might be surprising that something seemingly as simple as a colour edge still has no precise definition, and consequently no algorithm to compute it can exist.[1]

The rest of this paper is organized as follows. In section II we will give a new co-ordinate mapping into a three dimensional complex space which is equivalent to a six dimensional real space. The complex extension can be seen as a way to fuse the colour channels and to compensate the redundancy of the primary colour matching functions. Colours will be uniquely represented as unit vectors lying within manifolds of a complex hypersphere. In section III we will discuss the geometric properties of the space. Section IV will demonstrate some applications.

II. The H2SI colour space

To keep our argument simple and avoid a major discussion on primary colour matching functions (but see [1], [26]) we use the HSI-colour space, given by a double cone as introduced by Hering, as our basic colour space and refer to the literature on colour space conversions, e.g. [4], [6], [26]. Colour is defined in terms of three variables namely hue, $0 \leq H < 2\pi$, saturation (see Eq. ((1) and 11)), $0 \leq S \leq 1$, and intensity, $0 \leq I \leq 1$. If the 'rgb' space is the starting point for colour representations our preferred choice of transformation is:

$$\begin{aligned} x &= min\{r,g,b\}\,, y = max\{r,g,b\}\,, \\ I &= (x+y)/2\,, S = y-x\,, \\ c_1 &= r-(g+b)/2\,, c_2 = \frac{\sqrt{3}(b-g)}{2}\,, \\ H &= \arctan\left(c_2/c_1\right)\,. \end{aligned} \tag{1}$$

[1] Of course there are many algorithms that give results closely related to our perception of a colour edge.

It is recommended to use the IEEE $atan2$ function for the calculations. The result, which is in the open interval $-\pi \leq H < \pi$, has to be shifted by π. Other choices of primary colour matching functions and/or transformations into the HSI space are possible and we will leave this open for discussion.

The following set of equations maps the HSI space into a three dimensional complex space, called the $H2SI$ colour space. The acronym stands for:

'Hilbert square, super-importance':

$$\begin{aligned} x_1 &= \sqrt{I\left(1-\frac{S}{2}\right)} \\ x_2 &= \sqrt{\frac{S}{2}}\cos(2H)e^{-iH} \\ x_3 &= \sqrt{\frac{S}{2}}\sin(2H)e^{iH} \\ x_4 &= \sqrt{(1-I)\left(1-\frac{S}{2}\right)} \\ \sum_{i=1}^{4}|x_i|^2 &= 1 \end{aligned} \tag{2}$$

$x_1$and x_4 can be combined into a complex variable

$$\begin{aligned} X_1 &= x_1 + ix_4 = \sqrt{\left(1-\frac{S}{2}\right)}e^{i\arctan\sqrt{\frac{1-I}{I}}} \\ X_2 &= x_2 \\ X_3 &= x_3\ . \end{aligned} \tag{3}$$

Eq. (2) ensures that all colour vectors are normalized and within manifolds of a complex unit hypershere. X_1, X_2, X_3 form the $H2SI$ colour space. Equivalently they can be split up into their real and complex parts to build a six-dimensional real colour space. Formally, colour vectors can be seen as triplet states of a three-dimensional quantum system or a three-dimensional normalized Hilbert space $\mathcal{H} = \mathcal{C}^3$, which means that concepts developed for these spaces apply. A brief introduction to quantum information theory can be found in [27] and the literature mentioned therein. At the same time the colour co-ordinates are an instance of a curved Riemannian space. We note that the colour co-ordinates as defined in Eq. (2) are the complex analogon of homogenous co-ordinates and in their four-dimensional representation may be interpreted as unit bi-quaternions, although we will not make use of these facts. However they indicate that the $H2SI$ space is endowed with a rich mathematical structure.

The inverse mapping is given by:

$$\begin{aligned} S &= 2(|X_2|^2+|X_3|^2)\ , \\ I &= \frac{Re(X_1)^2}{(1-S/2)}\ , \\ H &= \arctan\{\frac{Re(X_3)+Im(X_2)}{Re(X_2)+Im(X_3)}\}\ . \end{aligned} \tag{4}$$

It is recommended to use the IEEE $atan2$ function for the calculations. The result, which is in the open interval $-\pi \leq H < \pi$, has to be shifted by π. In the next section we will analyse the geometric properties of the space. As their calculation is straightforward and due to space limitations we will leave the derivation to the reader.

III. Geometric properties

In this section we will analyse the geometric properties of the space defined by the complex coordinates X_i, $i = 1\ldots, 3$. In order to determine the geometric properties of the $H2SI$ space we need to evaluate the line element which is given by the partial derivatives in each co-ordinate direction. For the calculation it is convenient to use the four dimensional representation as given in Equation 2:

$$ds^2 = \sum_{i=1}^{4}|dx_i|^2 \tag{5}$$

which (omitting the details) computes to:

$$ds^2 = a\,dS^2 + b\,dI^2 + c\,dH^2 \tag{6}$$

with a, b and c defined by:

$$a = \frac{1}{4S(2-S)}\ ,\ b = \frac{2-S}{8I(1-I)}\ ,\ c = \frac{5}{2}S\ . \tag{7}$$

If we identify: $x^1 = S$, $x^2 = I$ and $x^3 = H$ (here the numbers 1, 2 and 3 denote upper indices), we can write down the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ via the relation (with the implied summation convention):

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{8}$$

and therefore:

$$g_{\mu\nu} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\ ,\quad g^{\mu\nu} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix} \tag{9}$$

As the line element is a function of S and I but independent of H (hue invariant) we may visualize it as shown in Figure 2.

A. Length of the line segment along the dimensions H, S, I

1) If we assume the quantities H and I to be constant, the length of the line element reads

$$s_1(I) = \int_0^{S(I)} \frac{dS'}{2\sqrt{S'(2-S')}} = \frac{1}{2}\left(\arcsin\sqrt{S(I)\,[\,2-S(I)\,]}\right)\ . \tag{10}$$

$s_1(i)$ is the distance of any colour of saturation $S(I)$ to the (neutral) grey-point, where $0 \leq s_1(I) \leq \pi/4$. The saturation $S(I)$ can be specified through (see Eq. (1))

$$\begin{aligned} S(I) &= 2\sigma I \quad\quad ,\ 0 \leq I \leq \frac{1}{2}\ ,\ 0 \leq \sigma \leq 1 \\ S(I) &= 2\sigma(1-I)\ ,\ \frac{1}{2} \leq I \leq 1\ . \end{aligned} \tag{11}$$

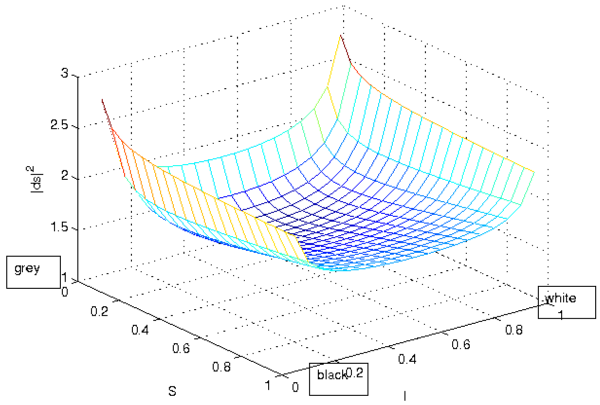

Fig. 2. Sensitivity of the line element ($\equiv ds^2(dS = dI = dH = 1)$). The line element is a function of S and I but independent of H. The graph indicates that the metric is most and equally sensitive around black and white, has an enlarged sensitivity towards gray and is allmost flat with respect to S and I, i.e. almost Euclidean, when colour is involved.

σ is the parameter for saturation. $\sigma = 1$ means full saturation for $I = \frac{1}{2}$. For those parameters we obtain from Eq. (10) and Eq. (11)

$$s_1(I = \frac{1}{2}, \sigma = 1) \equiv s_1(S = 1) = \frac{\pi}{4} \quad . \tag{12}$$

2) Assuming S and I to be constant, the length of the line element can be calculated from Eq. (6) and Eq. (7):

$$\begin{aligned} s_2[S(I)] &= \sqrt{\frac{5}{2}S(I)} \int_{\gamma_1}^{\gamma_2} dH \\ &= \frac{\gamma_2 - \gamma_1}{2}\sqrt{10S(I)} \; . \end{aligned} \tag{13}$$

For $\gamma_2 - \gamma_1 = 2\pi$ we obtain the circumference of a colour circle with saturation S. The ratio of the circumference of a fully saturated colour circle $s_2(S = 1)$ to its diameter s_1 amounts to

$$\begin{aligned} \frac{s_2(S = 1)}{s_1(S = 1)} &= 4\sqrt{10} = 12.65 \\ &\approx 4\pi = 12.56 \equiv 720^o \end{aligned} \tag{14}$$

$$\text{and} \quad \frac{s_2(S \to 0)}{s_1(S \to 0)} = 2\pi\sqrt{5} = 14.10 \, , \tag{15}$$

which establishes the *super-importance of hue differences* feature of the geometry (Eq. (14)). The ratio increases for smaller circles towards the neutral axes where it reaches its maximum (Eq. (15)).

3) Assuming S and H to be constant, the length of the line element $s_3(S)$ depends on saturation S (Eq. (6) and Eq. (7)):

$$\begin{aligned} s_3(S) &= \frac{\sqrt{2 - S}}{2\sqrt{2}} \int_{I_1(S)}^{I_2(S)} \frac{dI}{\sqrt{I(1 - I)}} \\ &= \frac{1}{2}\sqrt{1 - \frac{S}{2}} \; (\arcsin[-2I_1(S) + 1] \\ &\quad - \arcsin[-2I_2(S) + 1] \;) \quad . \end{aligned} \tag{16}$$

If $I_1 = 0$ and $I_2 = 1$ (hence $S = 0$) the length of this line element is $s_3(I_1 = 0, I_2 = 1, S = 0) = \frac{\pi}{2}$ which is the distance from black to white (double the radius or the diameter of the sphere).

B. Geodesic line

From differential geometry we have the following results: The differential equations for a parametric curve which minimizes the distance measured along this curve is given by (with the implied summation convention):

$$\frac{d^2x^\nu}{dt^2} + \Gamma^\nu_{\sigma\rho}\frac{dx^\sigma}{dt}\frac{dx^\rho}{dt} = 0 \tag{17}$$

The Christoffel symbols $\Gamma^\nu_{\sigma\rho}$ are defined through

$$\Gamma^\nu_{\sigma\rho} = g^{\nu\mu}\frac{1}{2}\left(\frac{\partial g_{\sigma\mu}}{\partial x^\rho} + \frac{\partial g_{\rho\mu}}{\partial x^\sigma} - \frac{\partial g_{\sigma\rho}}{\partial x^\mu}\right) \tag{18}$$

The Christoffel symbols follow from Eq. (9) and Eq. (7):

$$\begin{aligned} \Gamma^1_{11} &= \frac{S-1}{S(2-S)} \;\; , \;\; \Gamma^1_{22} = \frac{S(2-S)}{4I(1-I)} \;\; , \\ \Gamma^1_{33} &= 5S(S-2) \;\; , \;\; \Gamma^2_{22} = \frac{2I-1}{2I(1-I)} \;\; , \\ & \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2(S-2)} \;\; , \\ & \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2S} \;\; . \end{aligned} \tag{19}$$

All other Christoffel symbols are zero. Using

$$\begin{aligned} u &= \frac{dx^1}{dt} = S' \;\; , \;\; v = \frac{dx^2}{dt} = I' \;\; , \\ w &= \frac{dx^3}{dt} = H' \;\; , \end{aligned} \tag{20}$$

we obtain a set of 3 differential equations from Eq. (17):

$$\begin{aligned} u' + \Gamma^1_{11}u^2 + \Gamma^1_{22}v^2 + \Gamma^1_{33}w^2 &= 0 \;\; , \\ v' + \Gamma^2_{22}v^2 + 2\Gamma^2_{12}uv &= 0 \;\; , \\ w' + 2\Gamma^3_{13}uw &= 0 \;\; . \end{aligned} \tag{21}$$

The length s of the line element ranging from t_1 to t_2 can be written as (see Eq. (6))

$$s = \int_{t_1}^{t_2} \sqrt{a\,u^2 + b\,v^2 + c\,w^2}\, dt \;\; . \tag{22}$$

Due to space limitations we omit further discussion of the solutions of Eq. (21) which can be found analytically and will be published elsewhere. Figure 3 shows geodesic lines in the equatorial plane ($I = 1/2$), confirming Judd's conclusion [6] (p. 649).

IV. Some Applications

The new colour space enables a plethora of new applications in the area of colour and colour image processing. One of the most interesting applications might be that it enables us to calculate the 'Eigen' colours of an image, see Eq. (23 and 25). There is a long-standing discussion about how colours should be mapped to brightness values. Colour spaces like the

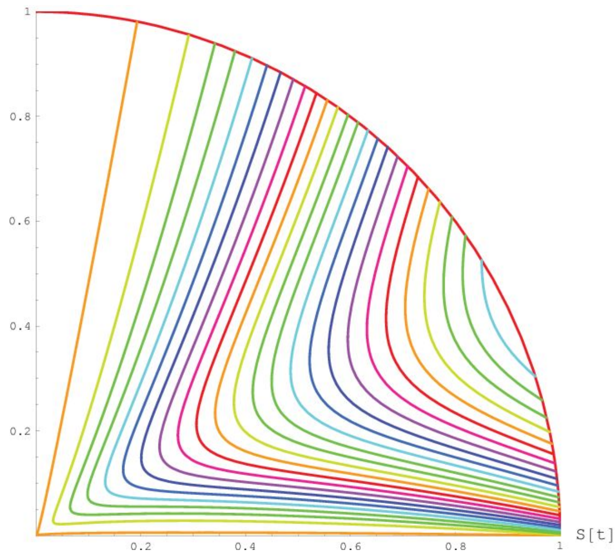

Fig. 3. Geodesic lines in the equatorial plane ($I = 1/2$) for $0 \leq H \leq \pi/2$. For hue differences greater than $2/3\pi$ (e.g. red-yellow) the shortest path leaves the plane. This differs slightly from Judds expectation (25 Munsell units $\equiv \pi/4$). Similar geodesic lines exist for all constant I planes.

CIE Lab assume this mapping to be fixed with respect to a given white point.

But Silberstein has shown [19] that a brightness plane must necessarily be curved in any colour space, which means that we cannot single out brightness as one independent colour co-ordinate. In fact, phenomena like colour constancy [4] and colour assimilation by a local opponent perception mechanism [1] exist, which indicates that colour and brightness perception depend on the context.

If we are given a number of colour samples $C_1 \dots C_n$ then their $H2SI$ co-ordinates form a $3 \times n$ complex matrix $\mathbf{C}$. Using the singular value decomposition gives:

$$\frac{1}{\sqrt{n}}\mathbf{C} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\dagger}\,. \tag{23}$$

The columns of the (3×3) matrix $\mathbf{U}$ and the $(n \times n)$ matrix $\mathbf{V}$ contain the left and right singular vectors of $\mathbf{C}$, respectively and they are unitary. The diagonal matrix $\mathbf{\Sigma}$ contains the singular values which are unique and are the positive square roots of the 'Eigen' values and the columns of $\mathbf{U}$ are the 'Eigen' vectors of $\mathbf{C}\mathbf{C}^{\dagger}$. The row vectors of $\mathbf{V}$ are the 'Eigen' vectors of $\mathbf{C}^{\dagger}\mathbf{C}$ with the same 'Eigen' values. As all colour vectors have unit length it is easy to see that $\mathbf{\Sigma}$ has unit length. Eq. (23) has a nice interpretation in terms of quantum information theory. If we form the (1x3n) vector

$$\mathbf{c} = \frac{1}{\sqrt{n}}\left[\begin{array}{ccc} c_{1,1} \dots c_{1,3} & \dots & c_{3,n} \end{array}\right]\,, \tag{24}$$

then $\mathbf{c}$ is a unit vector, i.e. the representation of a high dimensional quantum state. Let $\mathbf{u_i}$ be the column vectors of $\mathbf{U}$ and $\mathbf{v_i}$ the row vectors of $\mathbf{V}$. In this case Eq. (23) can be reformulated as the Schmidt decomposition of $\mathbf{c}$ into a three dimensional and into a n dimensional subsystem [28] (p. 109), which is given by:

$$\mathbf{c} = \sum_{i=1}^{3} \sigma_i \mathbf{u}_i{}^{\dagger} \otimes \mathbf{v}_i \tag{25}$$

where $\otimes$ denotes the Kronecker or tensor product and

$$\mathbf{A} = \sum_{i=1}^{3} \sigma_i^2 \mathbf{u_i}\mathbf{u_i}^{\dagger}\ ,\ \mathbf{B} = \sum_{i=1}^{3} \sigma_i^2 \mathbf{v_i}^{\dagger}\mathbf{v_i} \tag{26}$$

are density operators, i.e. they are hermitian and their trace is one, of the left and right 'Eigen' or singular space. The density operators contain statistical information about the distribution of the given colour samples, like the (unbiased) variance-co-variance matrix. We can interpret $\mathbf{u_i}\mathbf{v_i}$, $i = 1\dots3$, as 'Eigen' colours of the sample distribution. The diagonal of $\mathbf{A}$ contains the frequencies of the 'Eigen' colours. The biggest 'Eigen' value corresponds to the most frequent 'Eigen' colour and usually points in the intensity direction[2], and we may interpret it as brightness of a colour in the given context. The other two 'Eigen' colours point into opponent colour directions depending on the given colour distribution.

V. Conclusion and Future Work

In this paper we have developed a new colour space that corresponds to many aspects of human colour perception. We have derived important features of the resulting geometry and showed some new applications. The $H2SI$ space is well suited for algorithmic purposes. Being able to interpret colours as quantum states opens a new perspective on several open questions in colour science, such as colour constancy and assimilation and allows us to exploit results from quantum information theory.

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