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AN EXACT ANALYTICAL METHOD OF FLUTTER ANALYSIS USING SYMBOLIC COMPUTATION

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Abstract

The problem of the flutter of a uniform cantilever wing is solved analytically in an exact sense by rigorous application of the symbolic computing package REDUCE. The wing is idealised as a bending-torsion coupled beam for which the frequency equation and mode shape in free natural vibration are presented in closed analytical form. For a given number of selected normal modes, the expressions for generalised mass, generalised stiffness and generalised aerodynamic force are derived in analytical form by making extensive use of symbolic algebra. This was greatly assisted by the package REDUCE. Finally the flutter problem is formulated by summing algebraically the analytical expressions for generalised mass, generalised stiffness and generalised aerodynamic force terms. This enabled a complete reduction of a universally accepted numerical problem of flutter analysis to an analytical one. From the final expression containing all the above terms the flutter speed and flutter frequency are determined by using a standard root finding procedure, thus avoiding any numerical matrix manipulation. As a consequence, the proposed analytical method is found to be computationally more efficient than numerical methods, and therefore, it holds out the prospect of aeroelastic optimisation. An illustrative example confirming the accuracy and computational efficiency of the method when predicting the flutter speed and flutter frequency of a cantilever wing is provided.

Introduction

The flutter analysis of an aircraft wing is customarily based on numerical methods even for the simple case of a uniform cantilever wing[1-5]. This is because the complexities involved in dealing analytically with various parameters (particularly those which include aerodynamic terms) are of enormous magnitude, and may assume unmanageable proportions. Flutter literature also reveals that a popular method of flutter analysis of an aircraft wing uses some selected normal modes of the wing together with the associated generalised coordinates[4,6]. Furthermore, for simplicity and for ease of computation, flutter analysts have often relied on the use of strip theory based on Theodorsen-type unsteady aerodynamics[7]. This is particularly relevant when carrying out the flutter analysis of high aspect ratio aircraft wings. A standard numerical procedure of flutter analysis is to formulate the flutter matrix by summing numerically the generalised mass, stiffness and aerodynamic matrices. Once the flutter matrix is formulated, further use of numerical methods become inevitable for the solution of the complex flutter determinant. Such numerical procedures eventually yield the flutter speed and the flutter frequency. There will be no-doubt serious difficulties to solve the flutter problem analytically rather than numerically because of the very nature of the problem involving integral equations. It is probably for this reason that the flutter literature shows that no one has apparently made such an analytical attempt of flutter analysis despite the fact that the subject matter of aeroelasticity is more than half a century old. The main difficulty would arise from the fact that the algebra becomes unwieldy when deriving the expressions for generalised mass, generalised stiffness and generalised aerodynamic coefficients of an aircraft wing. With the advent of, and advancement in symbolic computing, it appears that the proposition before us to solve the flutter problem analytically has become reasonable and feasible. This difficulty can now be overcome and the seemingly impossible task is within our grasp. It can only be fruitfully accomplished by rigorous application of symbolic computation for which commercially available software such us REDUCE[8,9] is currently in widespread use. The purpose of this paper is to undertake such an investigation and present an analytical method of flutter analysis by using normal modes, generalised coordinates and two dimensional Theodorsen aerodynamics[7].

The free vibration analysis of beams coupled in bending and torsion (of which an aircraft wing is an important example) has been carried out by many investigators either by using the direct solution of the governing differential equation[10] and substitution of appropriate end-conditions for displacements and forces in the dynamic stiffness method[11-13], or by using the traditional finite element and other approximate methods[14,15]. As expected the solution technique used relies on numerical methods to manipulate mass, stiffness or dynamic stiffness matrices when obtaining natural frequencies and mode shapes of the coupled beam for a given set of end conditions. Based on these earlier works, Banerjee[16] has recently developed exact analytical expressions for the frequency equation and mode shapes of a bending-torsion coupled beam with cantilever end condition by making extensive use of the symbolic algebraic package REDUCE[8,9]. The formulae for frequency equation and mode shapes derived by Banerjee[16] are suitably applicable to cantilever aircraft wings and thus can be further extended to cover flutter analysis by linking the modal analysis to unsteady aerodynamic analysis of the wing forces. Firstly, the analytically derived normal modes (which are usually coupled in both bending and torsional deformation) are implemented in the expressions for the generalised mass and generalised stiffness in a particular mode. The symbolic algebraic package REDUCE[8,9] is then used to obtain the integral expressions in explicit form. Using normal modes, the derivation of the generalised aerodynamic forces in analytical form is probably the most difficult part undertaken during symbolic computation. The difficulty arises because the unsteady aerodynamic forces are usually expressed in terms of real and imaginary parts[7]. Due to the tremendous development and advancement in symbolic algebra in recent years, this became possible and the work has been greatly assisted by REDUCE. Once the analytical expressions for generalised mass, generalised stiffness and generalised aerodynamic force in each mode are obtained individually in explicit form, they are summed algebraically to formulate the complex flutter function which is primarily a function of two unknown variables, namely the air-speed and the frequency. The zeros of this function which give the flutter speed and flutter frequency are obtained by a standard root finding procedure for the real and imaginary parts of the function. Only in this final stage does the problem become numerical in the sense that results are obtained from the roots of a polynomial rather than from rigorous numerical matrix manipulation.

The above theory has been applied to predict the flutter speed of a cantilever wing for which some comparative results are available in the literature. The anticipated accuracy of the proposed theory is confirmed by numerical results. The proposed theory is likely to be well received by those research workers who are working in the field of aeroelastic optimisation.

Theory

A uniform aircraft wing of length L, bending rigidity EI, torsional rigidity GJ, mass per unit length m, and mass moment of inertia per unit length I_α, is shown in Fig. 1. The mass and elastic axes, which are respectively the loci of the mass centre and the shear centre of the wing cross-sections, are shown in the figure, with x_α being the distance of separation between them. In the right handed co-ordinate system shown, the elastic axis which is coincident with the Y-axis, is allowed to deflect out of the plane by h(y,t), whilst the cross-section is allowed to rotate (or twist) about OY by $\psi(y,t)$, where y and t denote distance from the origin and time respectively. The wing is assumed to undergo simple harmonic oscillation with circular (or angular) frequency ω and with a cantilever end-condition, the built-in end being at the origin.

Free vibration analysis: Following the recent work of Banerjee[16] the frequency equation of the aircraft wing shown in Fig. 1 with a cantilever end condition is given by Eq. (1) if appropriate substitutions are made from Eqs. (2)-(15) in the given order of sequence and working out from the basic beam parameters EI, GJ, m, I_α , x_α and L as defined at the beginning of the above paragraph. It should be noted that the left hand side of the frequency equation given by Eq. (1) is primarily a function of the frequency ω, the zeros of which gives the natural frequencies ω_n in free vibration.

$$f(\omega)=\nu_2(\lambda_1+\eta_1+\xi_1)+\nu_3(\lambda_2+\eta_2-\xi_2)-\nu_1(\lambda_3+\eta_3-\xi_3)-\mu_1\varepsilon_1-\mu_2\varepsilon_2-\mu_3\varepsilon_3+\delta_1+\delta_2+\delta_3=0 \quad (1)$$

where

$$a=I_\alpha\omega^2L^2/GJ;\ b=m\omega^2L^4/EI;\ c=1-mx_\alpha^2/I_\alpha \quad (2)$$

$$q=b+a^2/3 \quad (3)$$

$$\phi=\cos^{-1}[(27abc-9ab-2a^3)/\{2(a^2+3b)^{3/2}\}] \quad (4)$$

$$\alpha=[2(q/3)^{1/2}\cos(\phi/3)-a/3]^{1/2}$$
$$\beta=[2(q/3)^{1/2}\cos\{(\pi-\phi)/3)\}+a/3]^{1/2}$$
$$\gamma=[2(q/3)^{1/2}\cos\{(\pi+\phi)/3)\}+a/3]^{1/2} \quad (5)$$

$$k_\alpha=(b-\alpha^4)/bx_\alpha;\ k_\beta=(b-\beta^4)/bx_\alpha;\ k_\gamma=(b-\gamma^4)/bx_\alpha \quad (6)$$

$$C_{h\alpha}=\cosh\alpha;\ C_\beta=\cos\beta;\ C_\gamma=\cos\gamma \quad (7)$$
$$S_{h\alpha}=\sinh\alpha;\ S_\beta=\sin\beta;\ S_\gamma=\sin\gamma \quad (8)$$

$$\mu_1=k_\alpha+k_\beta;\ \mu_2=k_\beta+k_\gamma;\ \mu_3=k_\gamma+k_\alpha \quad (9)$$

$$\nu_1=k_\alpha-k_\beta;\ \nu_2=k_\beta-k_\gamma;\ \nu_3=k_\gamma-k_\alpha \quad (10)$$

$$\lambda_1=\alpha^4(k_\beta C_\beta-k_\gamma C_\gamma)$$
$$\lambda_2=\beta^4(k_\gamma C_\gamma-k_\alpha C_{h\alpha})$$
$$\lambda_3=\gamma^4(k_\beta C_\beta-k_\alpha C_{h\alpha}) \quad (11)$$

$$\xi_1=\alpha^2k_\alpha(\beta^2C_\beta-\gamma^2C_\gamma)$$
$$\xi_2=\beta^2k_\beta(\alpha^2C_{h\alpha}+\gamma^2C_\gamma)$$
$$\xi_3=\gamma^2k_\gamma(\alpha^2C_{h\alpha}+\beta^2C_\beta) \quad (12)$$

$$\eta_1=\alpha^3S_{h\alpha}(\beta k_\gamma S_\beta C_\gamma-\gamma k_\beta C_\beta S_\gamma)$$
$$\eta_2=\beta^3S_\beta(\gamma k_\alpha C_{h\alpha}S_\gamma+\alpha k_\gamma S_{h\alpha}C_\gamma)$$
$$\eta_3=\gamma^3S_\gamma(\alpha k_\beta S_{h\alpha}C_\beta+\beta k_\alpha C_{h\alpha}S_\beta) \quad (13)$$

$$\varepsilon_1=\alpha\beta k_\gamma C_\gamma(\alpha\beta C_{h\alpha}C_\beta+\gamma^2S_{h\alpha}S_\beta)$$
$$\varepsilon_2=\beta\gamma k_\alpha C_{h\alpha}(\alpha^2S_\beta S_\gamma-\beta\gamma C_\beta C_\gamma)$$
$$\varepsilon_3=\gamma\alpha k_\beta C_\beta(\alpha\gamma C_{h\alpha}C_\gamma+\beta^2S_{h\alpha}S_\gamma) \quad (14)$$

$$\delta_1=2\alpha^2C_{h\alpha}C_\beta C_\gamma(\gamma^2k_\beta^2+\beta^2k_\gamma^2)$$
$$\delta_2=2\alpha\beta\gamma k_\gamma S_\gamma(\alpha k_\beta C_{h\alpha}S_\beta+\beta k_\alpha S_{h\alpha}C_\beta)$$
$$\delta_3=2\beta\gamma^2k_\alpha C_\gamma(\alpha k_\beta S_{h\alpha}-\beta k_\alpha C_{h\alpha}C_\beta) \quad (15)$$

Note that the value of $f(\omega)$ in Eq.(1) is zero when $\omega=0$, which corresponds to a wing at rest so that there is no inertial loading[16]. This known value of $f(0)=0$ can be used to avoid any numerical problem of overflow at zero frequency when computing $f(\omega)$. For any other (non-trivial) values of ω, the expression for $f(\omega)$ given by Eq.(1) can be used in locating the natural frequencies by successively tracking the changes of its sign.

Once the natural frequencies ω_n are found from Eq.(1), the mode shapes for the wing consisting of bending displacement (H_n) and torsional rotation (Ψ_n) can be expressed as[12,16]

$$H_n(\xi)=A_n\cosh\alpha\xi+B_n\sinh\alpha\xi+C_n\cos\beta\xi+D_n\sin\beta\xi++E_n\cos\gamma\xi+F_n\sin\gamma\xi \quad (16)$$

and

$$\Psi_n(\xi)=P_n\cosh\alpha\xi+Q_n\sinh\alpha\xi+R_n\cos\beta\xi+S_n\sin\beta\xi++T_n\cos\gamma\xi+U_n\sin\gamma\xi \quad (17)$$

where $\xi=y/L$ is the non-dimensional spanwise distance from the wing root.

It has been shown earlier by Banerjee[12,16] that the coefficients A_n, B_n, C_n, D_n, E_n, F_n are related to P_n, Q_n, R_n, S_n, T_n, U_n as follows

$$P_n=k_\alpha A_n;\quad R_n=k_\beta C_n;\quad T_n=k_\gamma E_n$$
$$Q_n=k_\alpha B_n;\quad S_n=k_\beta D_n;\quad U_n=k_\gamma F_n \quad (18)$$

where k_α, k_β and k_γ have already been defined in Eqs. (6).

Also for a cantilever end condition of the wing the ratios of the mode shape coefficients in terms of A_n are given by (see Ref. 16)

$$B_n/A_n=\beta\gamma(\phi_1+\gamma^2\nu_1\sigma_3-\beta^2\nu_3\sigma_2)/\chi$$
$$C_n/A_n=\nu_3/\nu_2$$
$$D_n/A_n=\gamma\alpha(-\phi_2-\gamma^2\nu_1\kappa_3+\alpha^2\nu_2\kappa_1)/\chi$$
$$E_n/A_n=\nu_1/\nu_2$$
$$F_n/A_n=\alpha\beta(\phi_3+\beta^2\nu_3\kappa_2-\alpha^2\nu_2\sigma_1)/\chi \quad (19)$$

where ν_1, ν_2 and ν_3 have already been defined in Eqs. (10), and the following further variables are introduced to compute the expressions in Eqs. (19).

$$\zeta_1=\alpha S_{h\alpha}+\beta C_{h\alpha}\ ;\ \zeta_2=\beta S_\beta-\gamma S_\gamma\ ;\ \zeta_3=\gamma S_\gamma+\alpha S_{h\alpha} \quad (20)$$

$$\tau_1=\alpha^2 C_{h\alpha}+\beta^2 C_\beta;\tau_2=\beta^2 C_\beta-\gamma^2 C_\gamma;\tau_3=\gamma^2 C_\gamma+\alpha^2 C_{h\alpha} \quad (21)$$

$$\sigma_1=\alpha^2-\alpha\beta S_{h\alpha}S_\beta+\beta^2 C_{h\alpha}C_\beta$$
$$\sigma_2=\beta^2-\beta\gamma S_\beta S_\gamma-\gamma^2 C_\beta C_\gamma$$
$$\sigma_3=\gamma^2-\beta\gamma S_\beta S_\gamma-\beta^2 C_\beta C_\gamma \quad (22)$$

$$\kappa_1=\alpha^2-\alpha\gamma S_{h\alpha}S_\gamma+\gamma^2 C_{h\alpha}C_\gamma$$
$$\kappa_2=\beta^2+\alpha\beta S_{h\alpha}S_\beta+\alpha^2 C_{h\alpha}C_\beta$$
$$\kappa_3=\gamma^2+\alpha\gamma S_{h\alpha}S_\gamma+\alpha^2 C_{h\alpha}C_\gamma \quad (23)$$

$$\phi_1=\alpha^2\nu_2(\tau_2 C_{h\alpha}-\alpha\zeta_2 S_{h\alpha})$$
$$\phi_2=\beta^2\nu_3(\tau_3 C_\beta+\beta\zeta_1 S_\beta)$$
$$\phi_3=\gamma^2\nu_1(\tau_1 C_\gamma+\gamma\zeta_1 S_\gamma) \quad (24)$$

$$\chi=\alpha\beta\gamma\nu_2(\alpha^2\zeta_2 C_{h\alpha}-\beta^2\zeta_3 C_\beta+\gamma^2\zeta_1 C_\gamma) \quad (25)$$

Note that α, β, γ, k_α, k_β, k_γ, ν_1, ν_2, ν_3, $C_{h\alpha}$, $S_{h\alpha}$, C_β, S_β, C_γ, and S_γ appearing in Eqs. (19)-(25) must be calculated for the particular natural frequency ω_n at which the mode shape is required.

The coefficients P_n, Q_n, R_n, S_n, T_n, U_n which give the torsional displacements in the n-th mode (see Eq. (17)) can also be expressed in terms of A_n using Eqs. (18) and (19). Thus the mode shape is completely defined in terms of A_n which can be arbitrarily chosen (e.g. A_n=1).

Generalised Mass and Generalised Stiffness: The generalised mass M_n and generalised stiffness K_n in the n-th mode of the cantilever wing can be derived by following the procedure put forward by Bishop and Price[17]. These are respectively given by

$$M_n=\int_0^1 [(mH_n^2+I_\alpha\Psi_n^2)-2mx_\alpha H_n\Psi_n]d\xi \quad (26)$$

and

$$K_n=\int_0^1 [(EI(H_n'')^2+GJ(\Psi')^2]d\xi \quad (27)$$

where H_n and Ψ_n are given by Eqs. (16) and (17) and the terms EI, GJ, m, I_α and x_α have been defined before.

However, a simpler way to calculate the generalised stiffness K_n would be to use the following equation

$$K_n=\omega_n^2 M_n \quad (28)$$

where ω_n, the n-th natural frequency has already been calculated from the frequency equation (see Eq.(1)) prior to the calculation of mode shapes H_n and Ψ_n.

Using Eqs. (16) and (17), the integrals $\int_0^1 H_n^2 d\xi$, $\int_0^1 \Psi_n^2 d\xi$ and $\int_0^1 H_n\Psi_m d\xi$ have been evaluated in explicit analytical form by performing symbolic computation with REDUCE[8,9]. These integrals in the most general forms are given in the Appendix.

Generalised Aerodynamic Coefficients: The generalised aerodynamic coefficients are derived by applying the principle of virtual work. The aerodynamic strip theory based on Theodorsen expressions for unsteady lift and moment[7] and the normal modes obtained from the analytical formulae of free vibration theory explained above, are used when applying the principle of virtual work. Thus if the bending displacement and torsional rotation in the i-th mode are $H_i(\xi)$ and $\Psi_i(\xi)$ and the corresponding generalised coordinate is $q_i(t)$, then for n number of modes, the time dependent bending displacement $h(\xi,t)$ and torsional rotation $\psi(\xi,t)$ can be expressed respectively as

$$h(\xi,t) = \sum_{i=1}^{n} H_i(\xi)q_i(t) \tag{29}$$

and

$$\psi(\xi,t) = \sum_{i=1}^{n} \Psi_i(\xi)q_i(t) \tag{30}$$

Equations (29) and (30) can be written in matrix form as

$$\begin{bmatrix} h(\xi,t) \\ \psi(\xi,t) \end{bmatrix} = \begin{bmatrix} H_1(\xi) & H_2(\xi) & . & H_n(\xi) \\ \Psi_1(\xi) & \Psi_2(\xi) & . & \Psi_n(\xi) \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ : \\ : \\ q_n(t) \end{bmatrix} \tag{31}$$

If $L(\xi)$ and $M(\xi)$ are respectively the unsteady lift and moment at a spanwise distance ξ=y/L from the root, the virtual work (δW) done by the aerodynamic forces is given by

$$\delta W = \sum_{i=1}^{n} \delta q_i \int_0^1 [L(\xi)H_i(\xi)+M(\xi)\Psi_i(\xi)]d\xi \tag{32}$$

where n is the number of normal modes considered in the analysis.

Equation (32) can be written as

$$\begin{bmatrix} \frac{\delta W_1}{\delta q_1} \\ \frac{\delta W_2}{\delta q_2} \\ : \\ : \\ \frac{\delta W_n}{\delta q_n} \end{bmatrix} = \int_0^1 \begin{bmatrix} H_1 & \Psi_1 \\ H_2 & \Psi_2 \\ : & : \\ : & : \\ H_n & \Psi_n \end{bmatrix} \begin{bmatrix} L(\xi) \\ M(\xi) \end{bmatrix} d\xi \tag{33}$$

The unsteady lift $L(\xi)$ and moment $M(\xi)$ in two dimensional flow given by Theodorsen[7] can be expressed as

$$\begin{bmatrix} L(\xi) \\ M(\xi) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} h(\xi,t) \\ \psi(\xi,t) \end{bmatrix} \tag{34}$$

where

$$\begin{aligned} A_{11} &= \pi\rho U^2\{-k^2+2C(k)ik\} \\ A_{12} &= \pi\rho U^2 b[(a_h k^2+ik)+2C(k)\{1+ik(0.5-a_h)\}] \\ A_{21} &= \pi\rho U^2 b\{2C(k)ik(0.5+a_h)-k^2 a_h\} \\ A_{22} &= \pi\rho U^2 b^2[2(0.5+a_h)C(k)\{1+ik(0.5-a_h)\}+ \\ &\quad k^2/8 + k^2 a_h^2+(a_h-0.5)ik] \end{aligned} \tag{35}$$

In Eqs. (35), U, b, ρ, k, C(k) and a_h are in the usual notation : the airspeed, semi-chord, density of air, reduced frequency parameter (defined as $k=\omega b/U$), Theodorsen function and elastic axis location from mid-chord respectively[1-3].

Substituting Eqs. (34) and (31) into Eq. (33) gives

$$\begin{bmatrix} \frac{\delta W_1}{\delta q_1} \\ \frac{\delta W_2}{\delta q_2} \\ : \\ : \\ \frac{\delta W_n}{\delta q_n} \end{bmatrix} = \int_0^1 \begin{bmatrix} H_1 & \Psi_1 \\ H_2 & \Psi_2 \\ : & : \\ : & : \\ H_n & \Psi_n \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times$$

$$\begin{bmatrix} H_1 & H_2 & . & . & H_n \\ \Psi_1 & \Psi_2 & . & . & \Psi_n \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ : \\ : \\ q_n \end{bmatrix} d\xi$$

$$= \begin{bmatrix} QA_{11} & QA_{12} & & QA_{1n} \\ QA_{21} & QA_{22} & & QA_{2n} \\ : & : & & : \\ : & : & & : \\ QA_{n1} & QA_{n2} & & QA_{nn} \end{bmatrix} \tag{36}$$

where [QA] is the generalised aerodynamic matrix with

$$QA_{ij} = \int_0^1 (A_{11}H_iH_j+A_{12}H_j\Psi_i+ A_{21}H_i\Psi_j +A_{22}\Psi_i\Psi_j)d\xi \tag{37}$$

The elements of generalised aerodynamic matrix [QA] are complex with each element having a real part and an imaginary part. This is as a consequence of the terms A_{11}, A_{12}, ...etc in Eq.(37) being complex (see Eqs. (35)). By contrast, the generalised mass and stiffness terms (see Eqs. (26)-(28)) are both real. Analytical expressions for each of the integrals in Eq. (37) are obtained using REDUCE[8,9] (see Appendix for the most general cases of these integrals).

Formulation of the Flutter Problem: Using the standard classical approach, the flutter determinant is formed from the flutter matrix, and this is formed by summing algebraically the generalised mass, generalised stiffness and the generalised aerodynamic matrices. Thus for a system without structural damping the flutter matrix [QF] can be formed as given by Eq. (38) below. (Structural damping has generally a small effect on the oscillatory motion and is not included here.)

$$[QF]\{q\}=[\,-\omega^2[M]+[K]-[QA]\,]\{q\} \tag{38}$$

where [QA] is the complex nxn generalised aerodynamic matrix defined in Eqs. (36)-(37), [M] and [K] are nxn diagonal matrices of generalised mass and generalised stiffness respectively (with the i-th diagonal representing the generalised mass M_i and generalised stiffness K_i), q_i is the column vector of n generalised coordinates and ω is the circular frequency in rad/s.

For flutter to occur, the determinant of the complex flutter matrix must be zero so that from Eq. (38)

$$|QF| = |\,-\omega^2[M]+[K]-[QA]\,| = 0 \tag{39}$$

The solution of the flutter determinant can now be sought by expanding the above determinant in algebraic form because each of the terms of [M], [K] and [QA] and hence each of the elements of [QF] is now available in an analytical form.

Application to Binary Flutter Problem : In the case of binary flutter of a cantilever wing, two modes are chosen, usually one of them is bending dominated and the other torsion dominated. Let these two modes be 1 and 2 so that (H_1,Ψ_1) and (H_2,Ψ_2) are the chosen two (not necessarily the first two) of the normal modes of vibration.

Defining the following integrals based on the general formulae given in the Appendix

$$I_1=\int_0^1 H_1H_2\,d\xi \;;\quad I_2=\int_0^1 H_2\Psi_1\,d\xi$$
$$I_3=\int_0^1 H_1\Psi_2\,d\xi \;;\quad I_4=\int_0^1 \Psi_1\Psi_2\,d\xi$$
$$I_5=\int_0^1 H_1^2\,d\xi \;;\quad I_6=\int_0^1 H_1\Psi_1\,d\xi$$
$$I_7=\int_0^1 \Psi_1^2\,d\xi \;;\quad I_8=\int_0^1 H_2^2\,d\xi$$
$$I_9=\int_0^1 H_2\Psi_2\,d\xi \;;\quad I_{10}=\int_0^1 \Psi_2^2\,d\xi \tag{40}$$

where I_1-I_{10} can be easily calculated using the analytical expressions given for the general integral forms (see Appendix).

Thus the expression for generalised masses and stiffnesses in modes 1 and 2 respectively become (see Eqs. (26) and (28))

$$M_1=mI_5+I_\alpha I_7-2mx_\alpha I_6$$
$$M_2=mI_8+I_\alpha I_{10}-2mx_\alpha I_9 \tag{41}$$

and

$$K_1=\omega_1^2M_1\;;\quad K_2=\omega_2^2M_2 \tag{42}$$

In a similar manner, the results given in the Appendix can be used to generate the elements of the unsteady aerodynamic matrix [QA] in explicit analytical form as follows (see Eq. (37))

$$QA_{11}=A_{11}I_5+A_{12}I_6+A_{21}I_6+A_{22}I_7$$
$$QA_{12}=A_{11}I_1+A_{12}I_2+A_{21}I_3+A_{22}I_4$$
$$QA_{21}=A_{11}I_1+A_{12}I_3+A_{21}I_2+A_{22}I_4$$
$$QA_{22}=A_{11}I_8+A_{12}I_9+A_{21}I_9+A_{22}I_{10} \tag{43}$$

Finally with the help of Eqs. (41)-(43), the elements of the 2x2 flutter matrix [QF] can be written using Eq. (38) as

$$QF_{11}=K_1-\omega^2M_1-QA_{11}$$
$$QF_{12}=-QA_{12}$$
$$QF_{21}=-QA_{21}$$
$$QF_{22}=K_2-\omega^2M_2-QA_{22} \quad (44)$$

For flutter to occur, the determinant of [QF] must be zero yielding the flutter speed and flutter frequency. Thus for the present binary case, Eq. (44) gives the condition for flutter as

$$QF_{11}QF_{22}-QF_{12}QF_{21}=0 \quad (45)$$

Having known analytical expressions for QF_{11}, QF_{12}, QF_{21} and QF_{22}, the flutter problem is thus solved analytically without performing any numerical matrix manipulation.

An Illustrative Example

An illustrative example is chosen which examines the cantilever wing of Goland[4]. Firstly the two natural frequencies are located using the explicit frequency equation, see Eq. (1). These are shown in Fig.2 in a graphical plot of $f(\omega)$ against ω. Next the bending-torsion coupled mode shapes corresponding to these natural frequencies are computed using the explicit formulation given in Eqs. (16)-(25). These modes are shown in Fig.3. The frequencies and modes obtained from the above expressions agreed completely with the results given in Ref. 18 which uses a dynamic stiffness approach. Finally the flutter speed and flutter frequency of the wing were calculated using these two modes and by applying Eq. (45). These were respectively 137 m/s and 70 rad/s which are in accord with the results obtained using CALFUN[19] which solves the complex flutter determinant using a completely numerical approach.

Conclusions

An analytical method of flutter analysis is presented by deriving in explicit form each term needed for the flutter analysis. This involved extensive symbolic computation to obtain expressions for generalised mass, generalised stiffness and generalised unsteady aerodynamic terms. This was greatly assisted by the symbolic computation package REDUCE. The correctness and accuracy of the method is validated by numerical results. The proposed method offers prospects for aeroelastic developments in an optimisation environment.

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Appendix

Explicit Integral Expressions Using REDUCE

The integrals used in the theoretical derivations of generalised mass, stiffness and aerodynamic terms can be classified under two general forms which are

$$\int_0^1 H_n^2(\xi)d\xi \qquad (A1)$$

and

$$\int_0^1 H_n(\xi)\,\Psi_m(\xi)d\xi \qquad (A2)$$

where $H_n(\xi)$ and $\Psi_m(\xi)$ are respectively bending and torsional modes corresponding to the n-th and m-th natural frequencies as given below.

$$H_n(\xi)=A_n\cosh\alpha_n\xi+B_n\sinh\alpha_n\xi+C_n\cos\beta_n\xi+D_n\sin\beta_n+ \\ +E_n\cos\gamma_n\xi+F_n\sin\gamma_n\xi \qquad (A3)$$

$$\Psi_m(\xi)=P_m\cosh\alpha_m\xi+Q_m\sinh\alpha_m\xi+ \\ R_m\cos\beta_m\xi+S_m\sin\beta_m+T_m\cos\gamma_m\xi+U_m\sin\gamma_m\xi \qquad (A4)$$

The analytical expressions for the integrals were obtained using REDUCE and manipulating the algebra very considerably.

The integral of Eq. (A1) in explicit form is given by

$$\int_0^1 H_n^2(\xi)d\xi = \mu_n+\nu_n+\rho_n+\tau_n+\sigma_n+\lambda_n+\varepsilon_n+\zeta_n+\theta_n \tag{A4}$$

where

$$\begin{aligned}
&\mu_n=(A_n^2-B_n^2+C_n^2+D_n^2+E_n^2+F_n^2)/2\\
&\nu_n=\{\alpha_n\gamma_n(C_n^2-D_n^2)\sin 2\beta_n+\alpha_n\beta_n(E_n^2-F_n^2)\sin 2\gamma_n+\\
&\quad \beta_n\gamma_n(A_n^2+B_n^2)\sinh 2\alpha_n\}/(4\alpha_n\beta_n\gamma_n)\\
&\rho_n=-2C_n(\eta_{\gamma n}\gamma_n\cos\beta_n-\xi_{\gamma n}\beta_n\sin\beta_n+F_n\gamma_n)/(\beta_n^2-\gamma_n^2)\\
&\tau_n=2C_n(\eta_{\alpha n}\alpha_n\cos\beta_n+\xi_{\alpha n}\beta_n\sin\beta_n-B_n\alpha_n)/(\alpha_n^2+\beta_n^2)\\
&\sigma_n=2A_n(\eta_{\gamma n}\gamma_n\cosh\alpha_n+\xi_{\gamma n}\alpha_n\sinh\alpha_n+F_n\gamma_n)/(\alpha_n^2+\gamma_n^2)\\
&\lambda_n=-2D_n(\eta_{\gamma n}\gamma_n\sin\beta_n+\xi_{\gamma n}\beta_n\cos\beta_n-E_n\beta_n)/(\beta_n^2-\gamma_n^2)\\
&\varepsilon_n=2B_n(\eta_{\gamma n}\gamma_n\sinh\alpha_n+\xi_{\gamma n}\alpha_n\cosh\alpha_n-E_n\alpha_n)/(\alpha_n^2+\gamma_n^2)\\
&\zeta_n=2D_n(\eta_{\alpha n}\alpha_n\sin\beta_n-\xi_{\alpha n}\beta_n\cos\beta_n+A_n\beta_n)/(\alpha_n^2+\beta_n^2)\\
&\theta_n=(C_nD_n\alpha_n\gamma_n\sin^2\beta_n+E_nF_n\alpha_n\beta_n\sin^2\gamma_n+\\
&\quad A_nB_n\beta_n\gamma_n\sinh^2\alpha_n)/(\alpha_n\beta_n\gamma_n)
\end{aligned} \tag{A5}$$

with

$$\eta_{\alpha n}=A_n\sinh\alpha_n+B_n\cosh\alpha_n\ ;\ \eta_{\gamma n}=E_n\sin\gamma_n-F_n\cos\gamma_n \tag{A6}$$

$$\xi_{\alpha n}=A_n\cosh\alpha_n+B_n\sinh\alpha_n\ ;\ \xi_{\gamma n}=E_n\cos\gamma_n+F_n\sin\gamma_n \tag{A7}$$

The integral of Eq. (A2) in explicit form is given as

$$\begin{aligned}\int_0^1 H_n(\xi)\Psi_m(\xi)=&\mu_{mn}\cos\beta_m+\nu_{mn}\sin\beta_m+\rho_{mn}\cos\gamma_m+\\
&\tau_{mn}\sin\gamma_m+\sigma_{mn}\cosh\alpha_m+\lambda_{mn}\sinh\alpha_m+\\
&\varepsilon_{mn}+\zeta_{mn}+\theta_{mn}\end{aligned} \tag{A8}$$

where

$$\begin{aligned}\mu_{mn}=&-(\eta_{\beta n}\beta_nR_m+\xi_{\beta n}\beta_mS_m)/\varphi_{mn}-\\
&(\eta_{\gamma n}\gamma_nR_m+\xi_{\gamma n}\beta_mS_m)/\kappa_{mn}+\\
&(\eta_{\alpha n}\alpha_nR_m-\xi_{\alpha n}\beta_mS_m)/\psi_{nm}\end{aligned} \tag{A9}$$

$$\begin{aligned}\nu_{mn}==&-(\eta_{\beta n}\beta_nS_m-\xi_{\beta n}\beta_mR_m)/\varphi_{mn}-\\
&(\eta_{\gamma n}\gamma_nS_m-\xi_{\gamma n}\beta_mR_m)/\kappa_{mn}+\\
&(\eta_{\alpha n}\alpha_nS_m+\xi_{\alpha n}\beta_mR_m)/\psi_{nm}\end{aligned} \tag{A10}$$

$$\begin{aligned}\rho_{mn}==&(\eta_{\beta n}\beta_nT_m+\xi_{\beta n}\gamma_mU_m)/\kappa_{mn}-\\
&(\eta_{\gamma n}\gamma_nT_m+\xi_{\gamma n}\gamma_mU_m)/\Delta_{mn}+\\
&(\eta_{\alpha n}\alpha_nT_m-\xi_{\alpha n}\gamma_mU_m)/\Omega_{nm}\end{aligned} \tag{A11}$$

$$\begin{aligned}\tau_{mn}==&(\eta_{\beta n}\beta_nU_m-\xi_{\beta n}\gamma_mT_m)/\kappa_{mn}-\\
&(\eta_{\gamma n}\gamma_nU_m-\xi_{\gamma n}\gamma_mT_m)/\Delta_{mn}+\\
&(\eta_{\alpha n}\alpha_nU_m+\xi_{\alpha n}\gamma_mT_m)/\Omega_{nm}\end{aligned} \tag{A12}$$

$$\begin{aligned}\sigma_{mn}==&(\eta_{\beta n}\beta_nP_m+\xi_{\beta n}\alpha_mQ_m)/\psi_{mn}+\\
&(\eta_{\gamma n}\gamma_nP_m+\xi_{\gamma n}\alpha_mQ_m)/\Omega_{mn}-\\
&(\eta_{\alpha n}\alpha_nP_m-\xi_{\alpha n}\alpha_mQ_m)/\delta_{mn}\end{aligned} \tag{A13}$$

$$\begin{aligned}\lambda_{mn}==&(\eta_{\beta n}\beta_nQ_m+\xi_{\beta n}\alpha_mP_m)/\psi_{mn}+\\
&(\eta_{\gamma n}\gamma_nQ_m+\xi_{\gamma n}\alpha_mP_m)/\Omega_{mn}-\\
&(\eta_{\alpha n}\alpha_nQ_m-\xi_{\alpha n}\alpha_mP_m)/\delta_{mn}\end{aligned} \tag{A14}$$

$$\begin{aligned}\lambda_{mn}==&(\eta_{\beta n}\beta_nQ_m+\xi_{\beta n}\alpha_mP_m)/\psi_{mn}+\\
&(\eta_{\gamma n}\gamma_nQ_m+\xi_{\gamma n}\alpha_mP_m)/\Omega_{mn}-\\
&(\eta_{\alpha n}\alpha_nQ_m-\xi_{\alpha n}\alpha_mP_m)/\delta_{mn}\end{aligned} \tag{A14}$$

$$\begin{aligned}\lambda_{mn}==&(\eta_{\beta n}\beta_nQ_m+\xi_{\beta n}\alpha_mP_m)/\psi_{mn}+\\
&(\eta_{\gamma n}\gamma_nQ_m+\xi_{\gamma n}\alpha_mP_m)/\Omega_{mn}-\\
&(\eta_{\alpha n}\alpha_nQ_m-\xi_{\alpha n}\alpha_mP_m)/\delta_{mn}\end{aligned} \tag{A14}$$

$$\begin{aligned}\lambda_{mn}==&(\eta_{\beta n}\beta_nQ_m+\xi_{\beta n}\alpha_mP_m)/\psi_{mn}+\\
&(\eta_{\gamma n}\gamma_nQ_m+\xi_{\gamma n}\alpha_mP_m)/\Omega_{mn}-\\
&(\eta_{\alpha n}\alpha_nQ_m-\xi_{\alpha n}\alpha_mP_m)/\delta_{mn}\end{aligned} \tag{A14}$$

with

$$\begin{aligned}&\eta_{\alpha n}=A_n\sinh\alpha_n+B_n\cosh\alpha_n\\
&\eta_{\beta n}=C_n\sin\beta_n-D_n\cos\beta_n\\
&\eta_{\gamma n}=E_n\sin\gamma_n-F_n\cos\gamma_n\end{aligned} \tag{A18}$$

$$\begin{aligned}&\xi_{\alpha n}=A_n\cosh\alpha_n+B_n\sinh\alpha_n\\
&\xi_{\beta n}=C_n\cos\beta_n+D_n\sin\beta_n\\
&\xi_{\gamma n}=E_n\cos\gamma_n+F_n\sin\gamma_n\end{aligned} \tag{A19}$$

and

$$\psi_{mn}=\alpha_m^2+\beta_n^2;\ \kappa_{mn}^2=\beta_m^2-\gamma_n^2;\ \Omega_{mn}=\alpha_m^2+\gamma_n^2 \tag{A20}$$

$$\psi_{nm}=\alpha_n^2+\beta_m^2;\ \kappa_{nm}^2=\beta_n^2-\gamma_m^2;\ \Omega_{nm}=\alpha_n^2+\gamma_m^2 \tag{A21}$$

$$\varphi_{mn}=\beta_m^2-\beta_n^2;\ \Delta_{mn}=\gamma_m^2-\gamma_n^2;\ \delta_{mn}=\alpha_m^2-\alpha_n^2 \tag{A22}$$

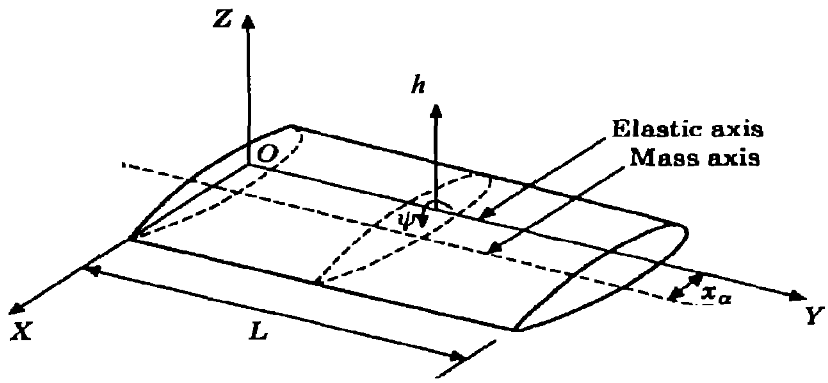

Figure 1. Coordinate system and notation for a bending-torsion coupled beam

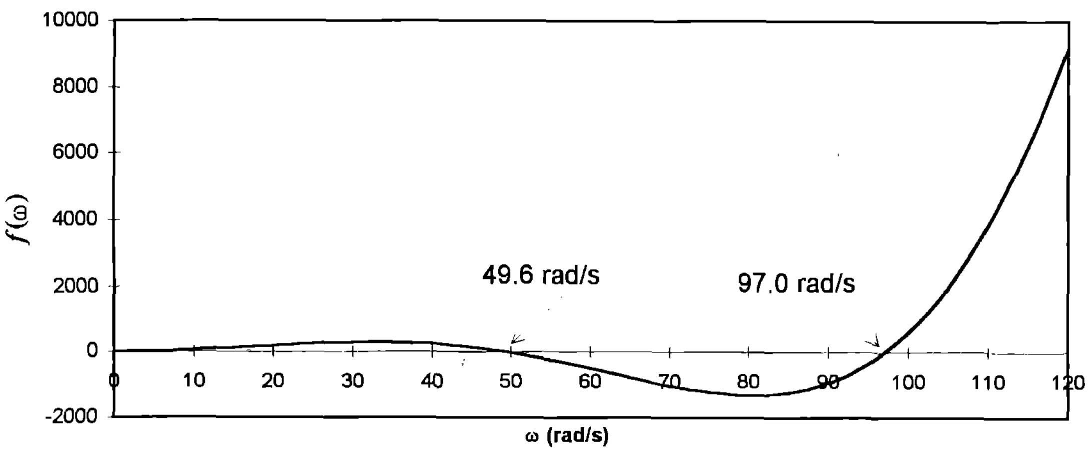

Figure 2. The variation of $f(\omega)$ against frequency (ω)

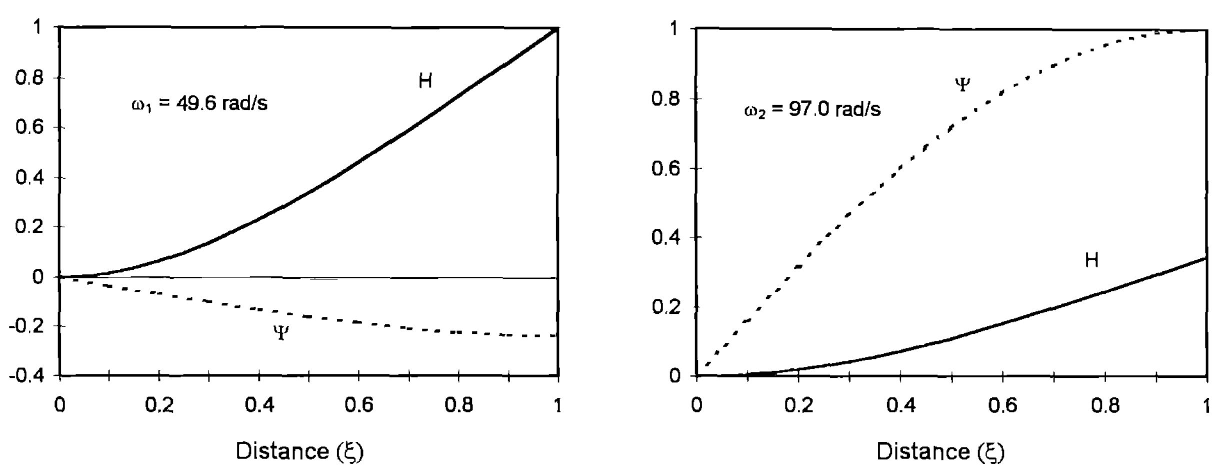

Figure 3. Coupled bending-torsional natural frequencies and mode shapes of an aircraft wing: ______ bending displacement (H), - - - - - torsional rotation (Ψ).