Reduced order methods for parametric Fluid-Structure Interaction problems: applications to haemodynamics

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Roscoff, Britanny,
YM60
May 2-4, 2017

ERC-CoG-AROMA-CFD GA 681447

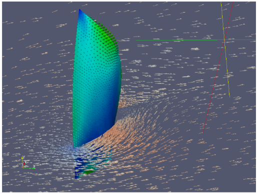
European and American activities: Happy Birthday Yvon!

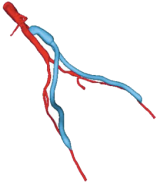
2002	HaeModel RTN, Paris VI
2003	MIT RB "band"
2004	EIM "band"
2006–2008	MIT–Brown U.
2010–2012	LJLL Paris
2012–2017	SISSA & friends

Leading Motivation: Computational Sciences challenges

- **Simulation-based sciences** is a quickly emerging field for mathematics and computational modelling.
- Present and future efforts: towards **multiphysics** problems, as well as systems characterized by **multiple spatial and temporal scales**.
- Growing demand of
  - **efficient computational tools** for
  - **many query** and **real time** computations,
  - **parametrized formulations**,
  - simulations of increasingly **complex systems** with uncertain scenarios,

  by **industrial and clinical** research partners.
- The need of a computational collaboration rather than a competition between **High Performance Computing** (HPC) and **Reduced Order Methods** (ROM), as well as Full/High Order and Reduced Order Methods.





Overview: our current efforts, aims and perspectives

A team developing **Advanced Reduced Order Modelling** techniques with special focus on **Computational Fluid Dynamics**

Overview: our current efforts, aims and perspectives

- Towards **Real-Time** Computing and Visualization, through an **Offline–Online** computational paradigm that combines

  High Performance Computing to **Advanced Reduced Order Modelling techniques**.

Offline:

HPC facilities, time demanding

"Science" driven

Online:

In situ, tablets or smartphones, real time

"Industrial needs" driven

- **Export numerical simulations and scientific computing** in fields and places where at the state of the art there is still little exploitation.
- Development of new open-source tools based on reduced order methods:
  - **ITHACA**, **I**n real **T**ime **H**ighly **A**dvanced **C**omputational **A**pplications, as an add-on to integrate already well established CSE/CFD open-source software libraries with ROMs, and
  - **RBniCS** as educational initiative for newcomer ROM users (training).

ITHACA

RBniCS

http://mathlab.sissa.it/rbnics

Reduced Order Methods: a brief historical background

An old idea...facing difficult problems

- **Reduced Basis Method**: continuation method in non-linear structural mechanics...
- **Propert Orthogonal Decomposition**: transient and turbulent flows...
- **Extensive Scientific Computing** was still a dream...

Pioneers 1980's: Noor, Peters, Brogan, Stern, Almroth, Fink, Rheinboldt, ...

New mathematical and methodological developments 2000's: Maday, Patera, Willcox, Huerta, Ito, Ravindran, Peterson, Farhat, Quarteroni, Hesthaven, Benner, Sorensen, Volkwein, Kunish, Urban, ...

Applications 2010's: several groups around the world focusing on many different aspects and applications. [Last SIAM CSE15: 28 minisymposia on ROM]

First mini on RB methods....ICOSAHOM Upsala 2001 and Providence 2004...organized by Maday and Patera.

Reduced Order Methods in a nutshell

- $()^{\mathcal{N}}$: "truth" high order method (FEM, FV, FD, SEM) – to be accelerated
- $()_N$: reduced order method (ROM) – *the accelerator*

∗ Input parameters:

$$\boldsymbol{\mu} \text{ (geometry, physical properties, etc.)}$$

∗ Parametrized PDE:

$$\mathcal{A}(u(\boldsymbol{\mu});\boldsymbol{\mu}) = 0 \quad \rightsquigarrow \quad \underbrace{\boldsymbol{A}^{\mathcal{N}}(\boldsymbol{\mu})\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}) = 0}_{\text{high order}} \quad \rightsquigarrow \quad \underbrace{\boldsymbol{A}_N(\boldsymbol{\mu})\boldsymbol{u}_N(\boldsymbol{\mu}) = 0}_{\text{reduced order}}$$

∗ Output:

$$s(\boldsymbol{\mu}) \quad \approx \quad \underbrace{s^{\mathcal{N}}(\boldsymbol{\mu})}_{\text{high order}} \quad \approx \quad \underbrace{s_N(\boldsymbol{\mu})}_{\text{reduced order}}$$

∗ Input-Output evaluation:

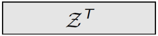
$$\boldsymbol{\mu} \quad \to \quad s^{\mathcal{N}}(\boldsymbol{\mu}) \quad \to \quad s_N(\boldsymbol{\mu})$$

J. S. Hesthaven, G. Rozza, B. Stamm. *Certified Reduced Basis Methods for Parametrized Partial Differential Equations*. SpringerBriefs in Mathematics. Springer, 2015

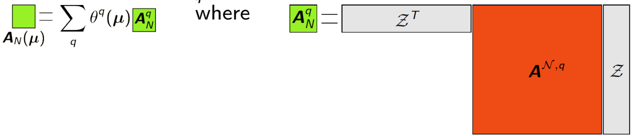
Reduced Order Methods in a nutshell

- $()^{\mathcal{N}}$: "truth" high order method (FEM, FV, FD, SEM) – to be accelerated
- $()_N$: reduced order method (ROM) – *the accelerator*

- **Offline**: very expensive preprocessing (high order): basis calculation (done *once*) after suitable parameters sampling (greedy, POD, ...)



- **Online**: extremely fast (reduced order): real-time input-output evaluation
$$\boldsymbol{\mu} \quad \rightarrow \quad s_N(\boldsymbol{\mu})$$
thanks to an efficient assembly of problem operators
$$\boldsymbol{A}_N(\boldsymbol{\mu}) = \sum_q \theta^q(\boldsymbol{\mu})\boldsymbol{A}_N^q, \text{ where } \boldsymbol{A}_N^q = \mathcal{Z}^T \boldsymbol{A}^{\mathcal{N},q}\mathcal{Z}$$



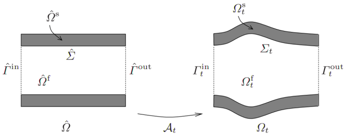
- Numerical issues: stability, error bounds, efficient parametrization, sampling, ...

J. S. Hesthaven, G. Rozza, B. Stamm. *Certified Reduced Basis Methods for Parametrized Partial Differential Equations*. SpringerBriefs in Mathematics. Springer, 2015

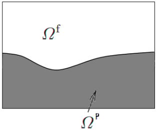
Current efforts

- Efficient management of (physical and numerical) interfaces and subdomains in a ROM setting:
  - physical interfaces:

FSI

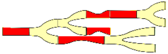


Stokes-Darcy [Martini, Haasdonk, R., 2015]

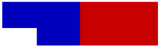


  - non-physical (computational) interfaces:

networks



domain decomposition

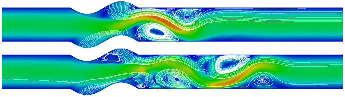


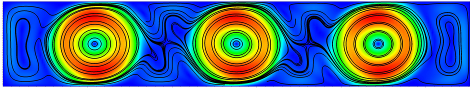
(including same physics with different mathematical models, e.g. viscous-potential coupling).

- Aim:
  - accurate coupling of physics,
  - keep low number of parameters,
  - dealing with moving boundaries, interfaces and domains.

#CFD

ROM and geometrical parametrization for fluid mechanics problems on moving domains Joint work with Francesco Ballarin





Reduced-order numerical simulations by POD-Galerkin ROM

- parametrized formulation of Navier-Stokes equations (modelling Newtonian fluid);
- offline stage:
  - intensive phase, on HPC architectures, to be done once;
  - Finite Element approximation of the problem for few values of the parameters (snapshots):

for $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{\mathbf{u}}^{\mathcal{N}}(\boldsymbol{\mu}), \underline{\mathbf{p}}^{\mathcal{N}}(\boldsymbol{\mu})) \in \mathbb{R}^{\mathcal{N}_\mathbf{u}} \times \mathbb{R}^{\mathcal{N}_p}$, $\boxed{\textbf{large } \mathcal{N}}$

$$\begin{bmatrix} A^{\mathcal{N}}(\boldsymbol{\mu}) + C^{\mathcal{N}}(\underline{\mathbf{u}}^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) & B^{\mathcal{N}}(\boldsymbol{\mu})^T \\ B^{\mathcal{N}}(\boldsymbol{\mu}) & O \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}}^{\mathcal{N}}(\boldsymbol{\mu}) \\ \underline{\mathbf{p}}^{\mathcal{N}}(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \underline{f}^{\mathcal{N}}(\boldsymbol{\mu}) \\ \underline{\mathbf{0}} \end{bmatrix}$$

  - [POD] Proper Orthogonal Decomposition (based on singular value decomposition) to extract optimal basis functions from the set of numerical simulations (snapshots) of the system to build $\mathcal{Z}$. [RB] Greedy as an alternative.

    Aubry et al. J. Fluid Mech. 1988; Ravindran, Int. J. Numer. Meth. Fluids, 2000
- online stage

Reduced-order numerical simulations by POD-Galerkin ROM

- parametrized formulation of Navier-Stokes equations (modelling Newtonian fluid);
- offline stage
- online stage:
  - inexpensive and very fast, on a laptop, to be done multiple times (for each new value of the parameters);
  - Galerkin projection over a reduced basis space:

for $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{\mathbf{u}}_{\mathbf{N}}(\boldsymbol{\mu}), \underline{\mathbf{p}}_{\mathbf{N}}(\boldsymbol{\mu})) \in \mathbb{R}^{N_{\mathbf{u}}} \times \mathbb{R}^{N_p}$, $\boxed{N = N_{\mathbf{u}} + N_p \ll \mathcal{N}}$

$$\begin{bmatrix} A_{\mathbf{N}}(\boldsymbol{\mu}) + C_{\mathbf{N}}(\underline{\mathbf{u}}_{\mathbf{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) & B_{\mathbf{N}}(\boldsymbol{\mu})^T \\ B_{\mathbf{N}}(\boldsymbol{\mu}) & O \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}}_{\mathbf{N}}(\boldsymbol{\mu}) \\ \underline{\mathbf{p}}_{\mathbf{N}}(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \underline{f}_{\mathbf{N}}(\boldsymbol{\mu}) \\ \underline{\mathbf{0}} \end{bmatrix}$$

Inf-sup stabilization and pressure recovery

- inf-sup condition is **not** necessarily preserved by Galerkin projection in the online phase.
- reduced velocity space **enrichment** by supremizer solutions,

$$V_N = POD(\{\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}^i)\}_{i=1}^{N_{train}}; N_{\mathbf{u}}) \quad \oplus \quad POD(\{S^{\boldsymbol{\mu}^i} p^{\mathcal{N}}(\boldsymbol{\mu}^i)\}_{i=1}^{N_{train}}; N_{\mathrm{s}}),$$
$$Q_N = POD(\{p^{\mathcal{N}}(\boldsymbol{\mu}^i)\}_{i=1}^{N_{train}}; N_p),$$

where $S^{\boldsymbol{\mu}} : Q^{\mathcal{N}} \to V^{\mathcal{N}}$ is the **supremizer operator** given by

$$(S^{\boldsymbol{\mu}} q^{\mathcal{N}}, \boldsymbol{w}^{\mathcal{N}})_V = b(q^{\mathcal{N}}, \boldsymbol{w}^{\mathcal{N}}; \boldsymbol{\mu}), \quad \forall \boldsymbol{w} \in V^{\mathcal{N}}.$$

in order to fullfil an ***inf-sup condition at the reduced-order level*** too:

$$\beta_N(\boldsymbol{\mu}) = \inf_{\underline{\mathbf{q}}_N \neq \underline{\mathbf{0}}} \sup_{\underline{\mathbf{v}}_N \neq \underline{\mathbf{0}}} \frac{\underline{\mathbf{q}}_N^T B_N(\boldsymbol{\mu}) \underline{\mathbf{v}}_N}{\|\underline{\mathbf{v}}_N\|_{V_N} \|\underline{\mathbf{q}}_N\|_{Q_N}} \geq \tilde{\beta}_N > 0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

where $B_N(\boldsymbol{\mu})$ is the reduced-order matrix associated to the divergence term. *(Rozza, Veroy. CMAME, 2007, Rozza et al, Numerische Mathematik, 2013. Ballarin et al. IJNME, 2015)*. Other options: residual-based stabilization procedures for POD-Galerkin *(Caiazzo, Iliescu et al. JCP, 2014)*, Petrov-Galerkin *(Dahmen; Carlberg; Abdulle, Budac)*, div-free approach *(Lovgren et al.)*.

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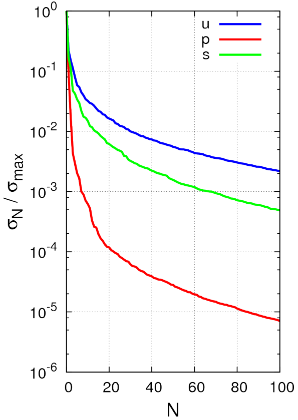
Typical reduced space dimensions and computational speedup for viscous flows



- **offline phase** in parallel on modern HPC clusters ($\mathcal{N} \approx 10^6$ dofs);
- **online phase** in serial on standard personal computer ($N \approx 10^2$ dofs);
- **considerable computational speedup**, from high-fidelity simulations that take O(day) to reduced-order ones that take O(min).

F. Ballarin, A. Manzoni, A. Quarteroni, G. Rozza. Int. J. Num. Meth. Eng., 2015.

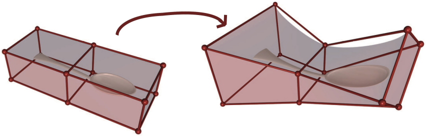
Shape parametrization for ROM

- need to combine solutions defined on different domains because (i) domain is moving and also (ii) initial configuration is parametrized,
- definition of a map

$$\Omega_o(\boldsymbol{\mu}) = \boldsymbol{T}(\Omega; \boldsymbol{\mu})$$

Shape parameterization for ROM

- Free-Form Deformations (FFD) [Lassila, Rozza, CMAME, 2010], [Salmoiraghi *et al.*, AMSES, 2016].
- Radial Basis Functions (RBF) [Manzoni *et al.*, IJNMBE, 2011].
- Transfinite Mapping (TM) [Løvgren, Maday, Rønquist, 2006], [Iapichino *et al.*, CMAME, 2012].
- Vascular shape parametrization [Ballarin *et al.*, JCP, 2016].
- Reduced inverse Distance Weighting [ongoing 2016].



ROM for problems on moving domains

- fluid problem in Arbitrary Lagrangian Eulerian (ALE) formulation;
- **parametrized displacement**;
- the parameter $\boldsymbol{\mu} \in \mathcal{D}$ controls (e.g.) the maximum amplitude of the deformation; $d(t)$ is a prescribed temporal profile:

$$\widetilde{\boldsymbol{\mu}}(t;\boldsymbol{\mu}) = \boldsymbol{\mu}\, d(t)$$

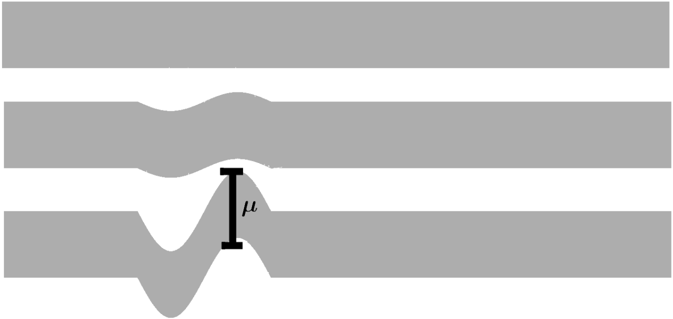
- the moving shape at time t depends both on $\boldsymbol{\mu}$ and t through $\widetilde{\boldsymbol{\mu}}(t) = \widetilde{\boldsymbol{\mu}}(t;\boldsymbol{\mu})$;
- **offline-online** decomposition of the reduced-order model is preserved/recovered **[EIM, Barrault et al., 2004]**, e.g.

$$A(t;\boldsymbol{\mu}) \overset{\text{EIM}}{\approx} \sum_{q=1}^{Q_A} \Theta_q^A(\boldsymbol{\mu}) \theta_q^A(t) A^q$$

  - the stored data structures A^q do not depend explicitly on time because the temporal dependence is stored in the multiplicative factors $\theta_q^A(t)$.

ROM for problems on moving domains: an example

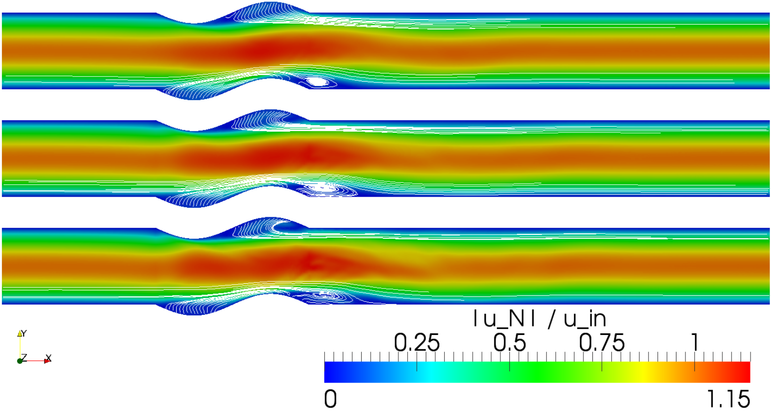
Domain: Rectangular channel of height equal to 1.
Physical parametrization: Reynolds number.
Geometrical parametrization: sinusoidal in space and in time ($T = 2\ s$) displacement of a deformable part of the domain (affine mapping). The amplitude of the peak displacement is the geometrical parameter.



Computational reduction: $N = 25$ modes capture 99.9% of the energy of NS system. Computational savings of $\approx 95\%$ (online POD–Galerkin vs FEM). Offline stage for 60 random snapshots $\times$ 100 timesteps: 1.5 days.

ROM for problems on moving domains: results

$\mu = 0.5, t = 0.75\ s$. From top to bottom: $Re = 400, 600, 800$:

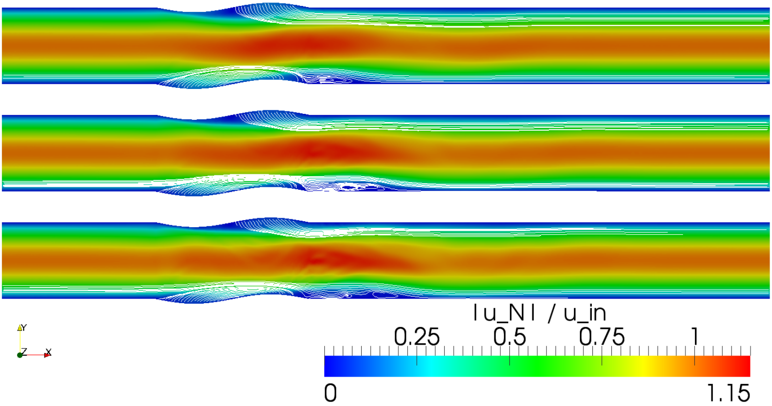


Small displacement: one small vortex propagates downstream...

F. Ballarin, G. Rozza, IJNMF, 2016.

ROM for problems on moving domains: results

$\mu = 0.5, t = 0.9\ s$. From top to bottom: $Re = 400, 600, 800$:

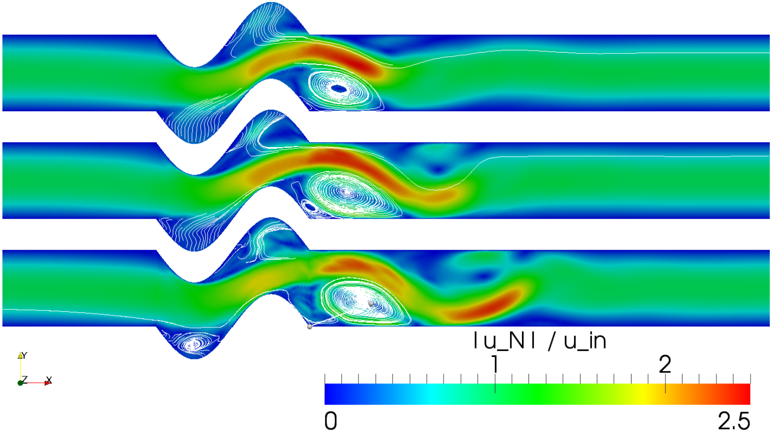


Small displacement: ... and is rapidly dissipated.

F. Ballarin, G. Rozza, IJNMF, 2016.

ROM for problems on moving domains: results

$\mu = 1.5, t = 0.75\ s$. From top to bottom: $Re = 400, 600, 800$:

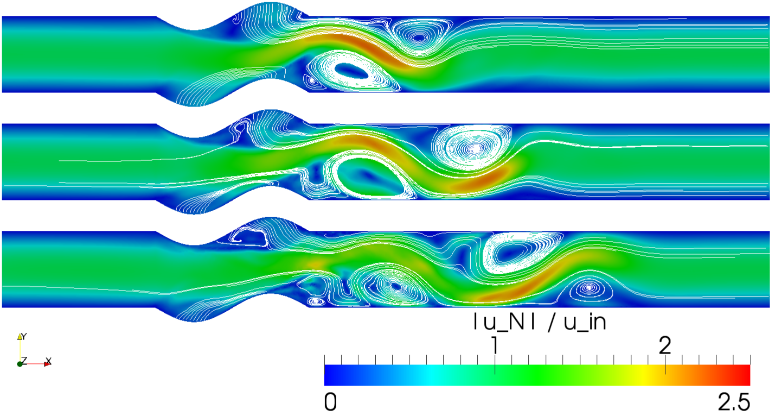


Large displacement: a bigger vortex propagates downstream, together with a jet at higher velocity...

F. Ballarin, G. Rozza, IJNMF, 2016.

ROM for problems on moving domains: results

$\mu = 1.5, t = 0.9\ s$. From top to bottom: $Re = 400, 600, 800$:

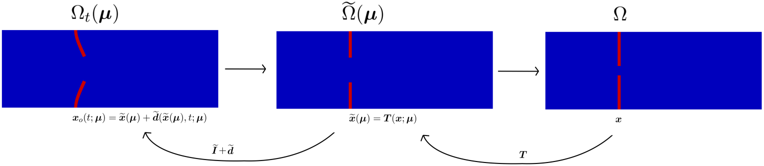


Large displacement: ... and other vortices may appear, depending on the Reynolds number.

F. Ballarin, G. Rozza, IJNMF, 2016.

#FSI

Monolithic ROMs for FSI problems
Joint work with Francesco Ballarin.



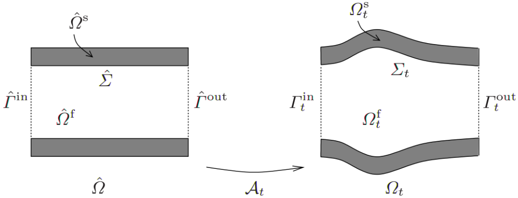
Formulation of FSI problems

- Fluid variables: $(\boldsymbol{u}_f, p, \boldsymbol{d}_f)$,
- Structure variables: $(\boldsymbol{u}_s, \boldsymbol{d}_s)$,
- Fluid-structure interaction problem three-fields formulation:

$$\begin{cases} F(\boldsymbol{u}_f, p, \boldsymbol{d}_f; \boldsymbol{d}_s) = 0, & \text{Fluid} \\ S(\boldsymbol{u}_s, \boldsymbol{d}_s) = 0, & \text{Structure} \\ I(\boldsymbol{d}_f, \boldsymbol{d}_s) = 0, & \text{Interface} \end{cases}$$

subject to interface (coupling) conditions

$$\begin{cases} \boldsymbol{d}_s - \boldsymbol{d}_f = 0 & \text{on } \Gamma, & \text{geometric continuity} \\ \boldsymbol{u}_s - \boldsymbol{u}_f = 0 & \text{on } \Gamma, & \text{velocity continuity} \\ \sigma_f \cdot n_f + \sigma_s \cdot n_s = 0 & \text{on } \Gamma, & \text{balance of normal forces.} \end{cases}$$



Previous reduced order approaches to FSI problems

- [Lassila *et al.*, 2012] and [Lassila *et al.*, 2013] (1D structural model, parametric interface coupling to RB fluid problem Stokes/Navier-Stokes, respectively, axialsymmetry),
- [Colciago, Ph.D. thesis, 2014] (fixed domain and thin-walled structure, RB for fluid problem with generalized Robin boundary conditions),
- [Bertagna, Veneziani, 2014] (1D structural model, POD–Galerkin),
- [Forti, Rozza, 2014] (efficient geometrical parametrization of interfaces, modal greedy),
- [Lieu *et al.*, 2006], ..., [Amsallem *et al.*, 2013], [Amsallem *et al.*, 2015] (aeroelasticity),

Our approach:

- no simplifications for structural model,
- POD–Galerkin method for **global** variables $\boldsymbol{u}, p, \boldsymbol{d}$ (monolithic approach), time dependent,
- capability to parametrize the initial configuration (geometry).

Reduced order monolithic formulation of FSI problems

Truth Finite Element discretization (P2-P1 Taylor-Hood)

For $\boldsymbol{\mu} \in \mathcal{D}$, solve large $\mathcal{N}$

$$
\begin{aligned}
F^{\mathcal{N}}(\boldsymbol{u}_f^{\mathcal{N}}(\mu), p^{\mathcal{N}}(\mu), \boldsymbol{d}_f^{\mathcal{N}}(\mu); \boldsymbol{d}_s^{\mathcal{N}}(\mu); \mu) &= 0 \quad \text{Fluid}\\
S^{\mathcal{N}}(\boldsymbol{u}_s^{\mathcal{N}}(\mu), \boldsymbol{d}_s^{\mathcal{N}}(\mu); \mu) &= 0 \quad \text{Structure}\\
I^{\mathcal{N}}(\boldsymbol{d}_f^{\mathcal{N}}(\mu), \boldsymbol{d}_s^{\mathcal{N}}(\mu); \mu) &= 0 \quad \text{Interface}\\
\text{coupling conditions} &
\end{aligned}
$$

OFFLINE – Space construction and matrices assembling

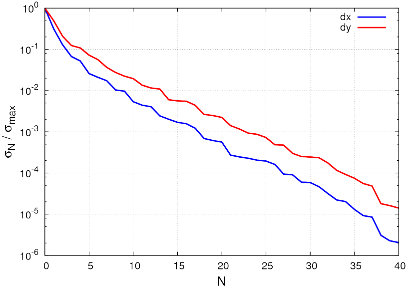
- Space construction by **Proper Orthogonal Decomposition** for **global** variables.
- Additional computations related to inf-sup stabilization procedure by means of supremizer enrichment $\rightarrow$ accurate pressure recovery for balance of normal forces. [Ballarin et al., 2015], [Rozza *et al.*, 2012], [Rozza, Veroy, 2007].

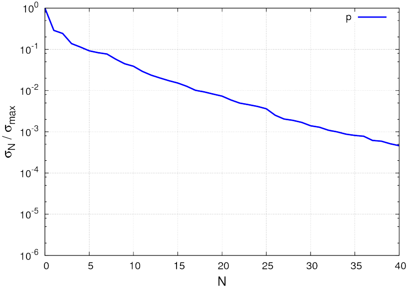
ONLINE – Galerkin projection over the enriched space

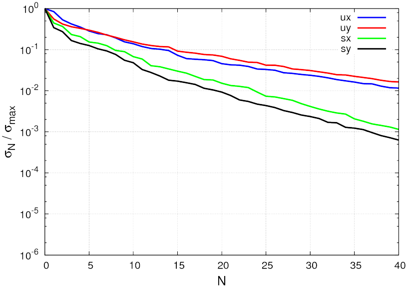
For $\boldsymbol{\mu} \in \mathcal{D}$, solve $\mathbf{N} \ll \mathcal{N}$

$$
\begin{aligned}
F^{\mathbf{N}}(\boldsymbol{u}_f^{\mathbf{N}}(\mu), p^{\mathbf{N}}(\mu), \boldsymbol{d}_f^{\mathbf{N}}(\mu); \boldsymbol{d}_s^{\mathbf{N}}(\mu); \mu) &= 0 \quad \textbf{Reduced}\text{ fluid}\\
S^{\mathbf{N}}(\boldsymbol{u}_s^{\mathbf{N}}(\mu), \boldsymbol{d}_s^{\mathbf{N}}(\mu); \mu) &= 0 \quad \textbf{Reduced}\text{ structure}\\
I^{\mathbf{N}}(\boldsymbol{d}_f^{\mathbf{N}}(\mu), \boldsymbol{d}_s^{\mathbf{N}}(\mu); \mu) &= 0 \quad \textbf{Reduced}\text{ interface}\\
\text{coupling conditions} &
\end{aligned}
$$

Reduced order monolithic formulation of FSI problems: results (1)







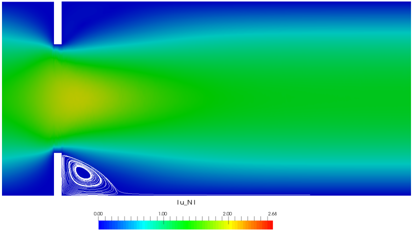
POD singular values for (global) displacement, pressure, velocity and supremizers.

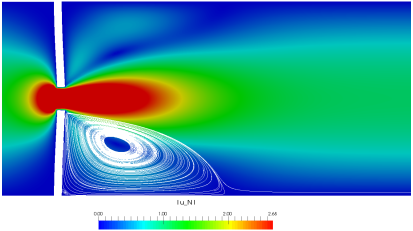
Fastest decay: displacement (top left).

Slower decay for velocity and supremizers modes (bottom left).

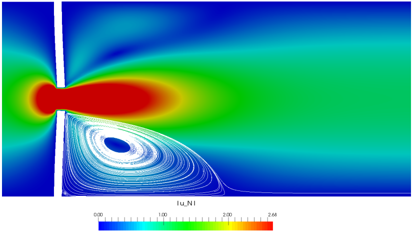
Ongoing applications to cardiovascular modelling

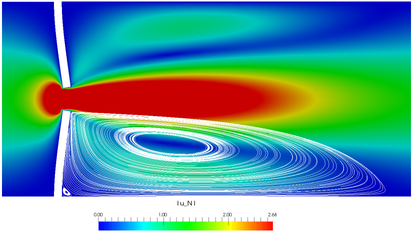
Increase leaflet length: *(same inlet velocity)*





Increase inlet velocity: *(same leaflet length)*



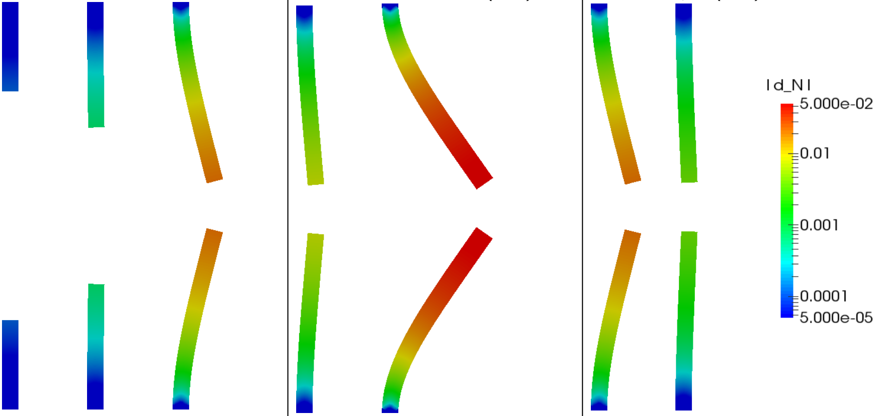


Ongoing applications to cardiovascular modelling

Increase leaflet length:

Increase inlet vel. (5×):

Increase μ_s (8×):



(same inlet velocity, same material properties)

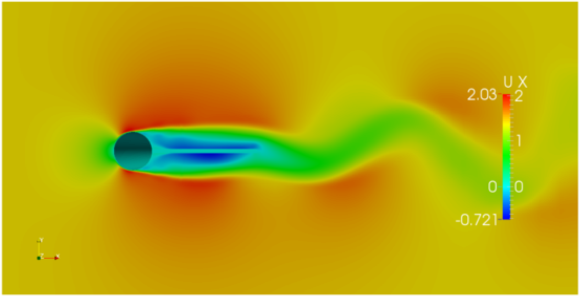
(same leaflet length, same material properties)

(same leaflet length, same inlet velocity)

Ballarin, Rozza. IJNMF, 2016.

#FSI 2

Partitioned ROMs for FSI problems
Joint work with Francesco Ballarin and Yvon Maday

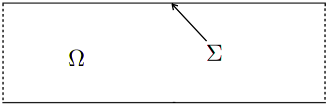


FSI problem with semi-implicit scheme

- **monolithic scheme** requires an **extensive preprocessing** of parametrized tensors, but is capable of handling **geometrical parametrization**, both as variation of the initial configuration and as interface coupling;
- goal: investigate reduced-order **semi-implicit operator-splitting** schemes;
- simplifying assumptions: **Stokes** equations, on a **fixed** fluid domain, under **thin wall** assumption and generalized string model for the **vertical** motion of the structure.

find fluid velocity $\boldsymbol{u}(t) : \Omega \to \mathbb{R}^2$, fluid pressure $p(t) : \Omega \to \mathbb{R}$ and structure (**vertical**) displacement $\eta(t) : \Sigma \to \mathbb{R}$ such that

$$\begin{cases} \rho_f \partial_t \boldsymbol{u} - \operatorname{div}(\boldsymbol{\sigma}_f(\boldsymbol{u}, p)) = \boldsymbol{0} & \text{in } \Omega \times (0, T], \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \times (0, T], \\ \boldsymbol{u} = \partial_t \eta \, \boldsymbol{n} & \text{on } \Sigma \times (0, T], \\ \rho_s h_s \partial_{tt} \eta - c_1 \partial_{xx} \eta + c_0 \eta = -\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{n} & \text{on } \Sigma \times (0, T]. \end{cases}$$



High-fidelity semi-implicit operator-splitting scheme

The high-fidelity discretization is based on the **Chorin-Temam** projection scheme:

1. Explicit step (fluid viscous part): find $\boldsymbol{u}^{k+1} : \Omega \to \mathbb{R}^2$ such that:

$$\begin{cases} \rho_f \frac{\boldsymbol{u}^{k+1} - \boldsymbol{u}^k}{\Delta t} - 2\mu_f \operatorname{div} \boldsymbol{\varepsilon}(\boldsymbol{u}^{k+1}) = -\nabla p^k & \text{in } \Omega, \\ \boldsymbol{u}^{k+1} = D_t \eta^k \, \boldsymbol{n} & \text{on } \Sigma. \end{cases}$$

2. Implicit step:
   2.1. Fluid projection substep: find $p^{k+1} : \Omega \to \mathbb{R}$ such that:

$$\begin{cases} -\operatorname{div}(\nabla p^{k+1}) = -\frac{\rho_f}{\Delta t} \operatorname{div} \boldsymbol{u}^{k+1} & \text{in } \Omega, \\ \frac{\partial}{\partial \boldsymbol{n}} p^{k+1} = -\rho_f D_{tt} \eta^{k+1} & \text{on } \Sigma. \end{cases}$$

   2.2. Structure substep: find $\eta^{k+1} : \Sigma \to \mathbb{R}$ such that:

$$\rho_s h_s D_{tt} \eta^{k+1} - c_1 \partial_{xx} \eta^{k+1} + c_0 \eta^{k+1} = -\boldsymbol{\sigma}(\boldsymbol{u}^{k+1}, p^{k+1}) \boldsymbol{n} \cdot \boldsymbol{n} \quad \text{on } \Sigma.$$

[Fernández, Gerbeau, Grandmont, IJNME, 2007]
[Astorino, Chouly, Fernandez, SISC, 2010]

High-fidelity semi-implicit operator-splitting scheme (II)

We further discretize in space by the **finite element method**:

1_h. Explicit step (fluid viscous part): find $\boldsymbol{u}_h^{k+1} \in V_h$ such that:

$$\int_\Omega \frac{\rho_f}{\Delta t} \boldsymbol{u}_h^{k+1} \cdot \boldsymbol{v}_h \, d\boldsymbol{x} + \int_\Omega 2\mu_f \boldsymbol{\varepsilon}(\boldsymbol{u}_h^{k+1}) : \nabla \boldsymbol{v}_h \, d\boldsymbol{x} = \int_\Omega \frac{\rho_f}{\Delta t} \boldsymbol{u}_h^{k} \cdot \boldsymbol{v}_h \, d\boldsymbol{x} - \int_\Omega \nabla p_h^k \cdot \boldsymbol{v}_h \, d\boldsymbol{x}$$

for all $\boldsymbol{v}_h \in V_h$, subject to the coupling condition

$$\boldsymbol{u}_h^{k+1} = D_t \eta_h^k \, \boldsymbol{n} \quad \text{on } \Sigma \times [0, T],$$

2_h. Implicit step: for any $j = 0, \ldots,$ until convergence:

2.1_h. Fluid projection substep: find $p_h^{k+1} \in Q_h$ such that:

$$\int_\Omega \nabla p_h^{k+1} \cdot \nabla q_h \, d\boldsymbol{x} = -\int_\Omega \frac{\rho_f}{\Delta t} \operatorname{div} \boldsymbol{u}_h^{k+1} \, q_h \, d\boldsymbol{x} - \int_\Sigma \rho_f D_{tt} \eta_h^{k+1,j} \, q_h \, ds$$

for all $q_h \in Q_h$.

2.2_h. Structure substep: find $\eta_h^{k+1} \in E_h$ such that:

$$\int_\Sigma \frac{\rho_s h_s}{\Delta t^2} \eta_h^{k+1} \zeta_h \, ds + \int_\Sigma c_1 \partial_x \eta_h^{k+1} \, \partial_x \zeta_h \, ds + \int_\Sigma c_0 \eta_h^{k+1} \zeta_h \, ds =$$

$$\int_\Sigma \frac{\rho_s h_s}{\Delta t^2} \eta_h^{k} \zeta_h \, ds + \int_\Sigma \frac{\rho_s h_s}{\Delta t} D_t \eta_h^{k} \zeta_h \, ds - \int_\Sigma \boldsymbol{\sigma}(\boldsymbol{u}^{k+1}, p^{k+1}) \boldsymbol{n} \cdot \zeta_h \boldsymbol{n} \, ds$$

for all $\zeta_h \in E_h$.

An operator-splitting ROM scheme: advantages and questions

- ☺ as for the high-fidelity method, online systems (for explicit and implicit steps) have smaller dimensions than a monolithic approach;

- ☺ thanks to the Chorin-Temam projection scheme, no supremizer enrichment (see *[Ballarin et al., IJNME, 2015]*, *[Rozza, Veroy, CMAME, 2007]*) is required, resulting in a smaller online dimension for the explicit step;

- 😐 in order to enhance the convergence of the implicit step, a Robin-Neumann scheme must be adopted (*[Astorino, Chouly, Fernandez, SISC, 2010]*). However, since it requires on the evaluation of a mass matrix on the interface, it is straightforward to adapt it to a reduced order setting;

- 😐 how will the number of iterations of the implicit-step at the reduced-order level compare to the one at the high-fidelity? Will it increase?

- 😐 for matching meshes the interface condition

$$\boldsymbol{u}^{k+1} = D_t \eta^k \, \boldsymbol{n} \quad \text{on } \Sigma.$$

is easy to impose at the high-fidelity level. How to efficiently impose it at the reduced-order level since $\boldsymbol{u}$ and η belong to different reduced spaces?

ROM FSI-1: velocity continuity by Lagrange multipliers

Offline stage:

- collect snapshots of the high-fidelity approximation of the FSI problem, and build reduced basis spaces $V_N^{(1)}, Q_N^{(1)}, E_N^{(1)}$ carrying out a Proper Orthogonal Decomposition for each unknown.
- moreover, also collect snapshots of the residual of the fluid viscous part (explicit step) for test functions that do *not* vanish on the interface, denoted by λ_k. Carry out a Proper Orthogonal Decomposition, that will serve as reduced basis space $L_N^{(1)}$ for Lagrange multipliers to enforce velocity continuity.

Online stage:

$1_N^{(1)}$. Explicit step (fluid viscous part): find $(\boldsymbol{u}_N^{k+1}, \lambda_N^{k+1}) \in V_N^{(1)} \times L_N^{(1)}$ such that:

$$\begin{cases} \int_\Omega \frac{\rho_f}{\Delta t} \boldsymbol{u}_N^{k+1} \cdot \boldsymbol{v}_N \, d\boldsymbol{x} & + \int_\Omega 2\mu_f \varepsilon(\boldsymbol{u}_N^{k+1}) : \nabla \boldsymbol{v}_N \, d\boldsymbol{x} \\ & + \int_\Sigma \lambda_N^{k+1} \boldsymbol{n} \cdot \boldsymbol{v}_N \, ds = \int_\Omega \frac{\rho_f}{\Delta t} \boldsymbol{u}_N^k \cdot \boldsymbol{v}_N \, d\boldsymbol{x} \\ & - \int_\Omega \nabla p_N^k \cdot \boldsymbol{v}_N \, d\boldsymbol{x}, \\ \int_\Sigma \boldsymbol{u}_N^{k+1} \cdot \Upsilon_N \boldsymbol{n} \, ds & = \int_\Sigma D_t \eta_h^k \, \Upsilon_N \, ds, \end{cases}$$

for all $(\boldsymbol{v}_N, \Upsilon_N) \in V_N^{(1)} \times L_N^{(1)}$.

$2_N^{(1)}$. Implicit step: standard Galerkin projection on $Q_N^{(1)} \times E_N^{(1)}$.

ROM FSI-1: velocity continuity by Lagrange multipliers

☺ offline-online decomposition is straightforward, thanks to the simplifying assumptions of this model problem. In a more general setting, one can resort to EIM, as done in *[Ballarin, Rozza, IJNMF, 2016]* for monolithic FSI problems.

☹ even though it is not necessary to enrich the velocity space by (LBB) supremizers, online we still end up solving a saddle point problem due to the imposition of the interface condition by Lagrange multipliers:

- the advantage (in terms of online system dimension) of using a reduced Chorin-Temam approach is squandered;
- we may still need to add supremizers for the velocity-Lagrange multipliers formulation! (although not needed in practice).

ROM FSI-2: velocity continuity by change of variable for fluid velocity

The idea: online, aim at approximating the following *auxiliary* fluid velocity

$$\boldsymbol{z}^{k+1} = \boldsymbol{u}^{k+1} - D_t \widehat{\eta}^k\, \boldsymbol{n}, \qquad \Rightarrow \qquad \boldsymbol{z}^{k+1} = \boldsymbol{0} \quad \text{on } \Sigma.$$

Here $\widehat{\eta}^k$ is an **harmonic extension** of the displacement η^k.

Offline stage:

- load all velocity and displacement snapshots to compute the auxiliary fluid velocity $\boldsymbol{z}^{k+1}$, compute a POD and store the first modes in the (auxiliary) velocity space $V_N^{(2)}$.
- reduced pressure and displacement spaces are unchanged, $Q_N^{(2)} := Q_N^{(1)}$ and $E_N^{(2)} := E_N^{(1)}$.

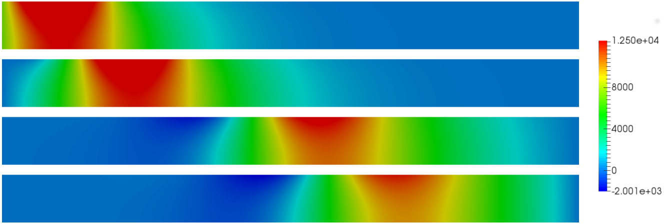
Online stage:
formally rewrite the weak formulation problem in terms of the unknowns $(\boldsymbol{z}, p, \eta)$ and carry out a standard Galerkin projection over the reduced space $V_N^{(2)} \times Q_N^{(2)} \times E_N^{(2)}$.

ROM FSI-2: velocity continuity by change of variable for fluid velocity

- ☺ offline-online decomposition is still straightforward.
- ☺ there is no need to enlarge the system size for the reduced explicit step, neither for supremizer enrichment nor for Lagrange multipliers
- ☺ the interface velocity continuity condition is imposed strongly also at the reduced-order level.
- ☺ the harmonic extension does not require any additional online problem. Indeed, each displacement basis function can be harmonically extended once and for all during the offline stage, and then linearly combined once the solution of the structure problem has been computed without any Galerkin projection for the extension problem.

Numerical test case

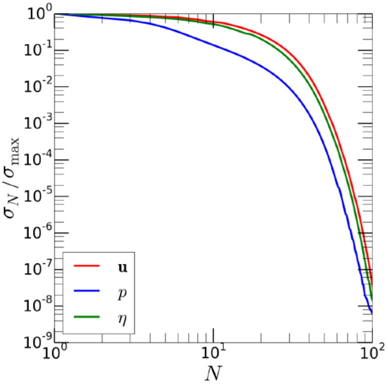
We compare the accuracy and efficiency of the proposed ROMs in a test case characterized by the propagation of a pressure wave inside the fluid domain.



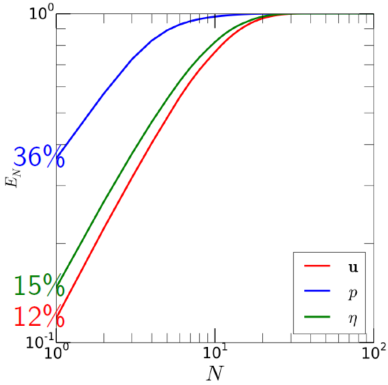
RBniCS

[Formaggia, Gerbeau, Nobile, Quarteroni, CMAME, 2001]

POD singular values for ROM FSI-1

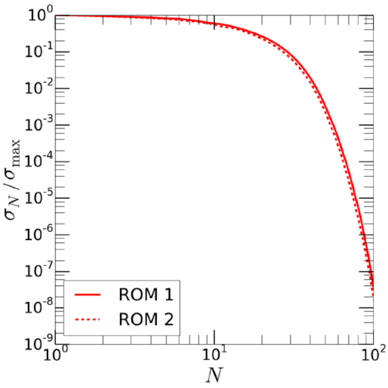


(a) POD singular values.

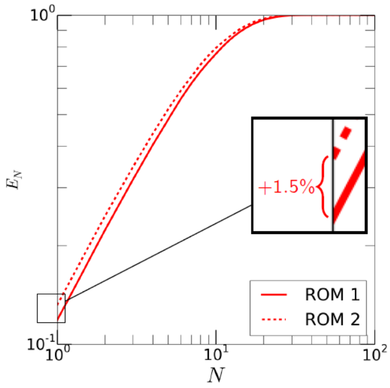


(b) POD retained energy.

POD singular values for ROM FSI-2 vs ROM FSI-1 (velocity only)



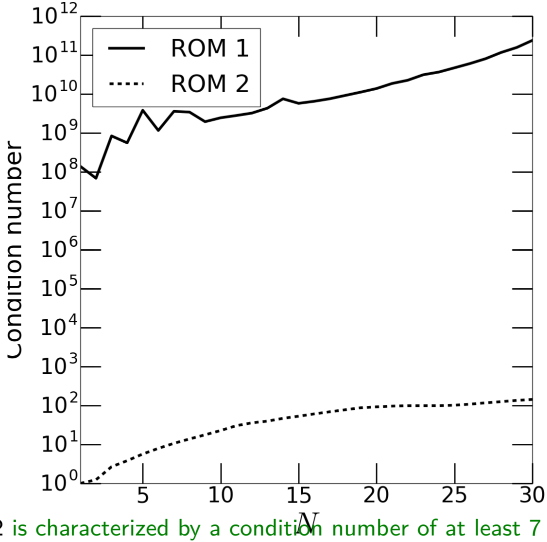
(a) POD singular values.



(b) POD retained energy.

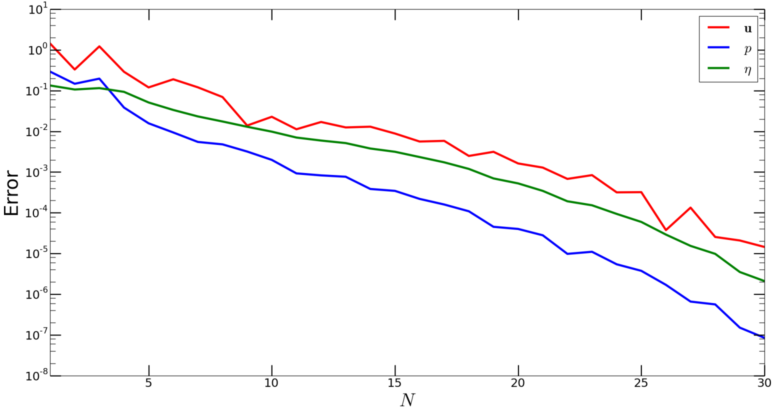
☺ the first (auxiliary) velocity mode of ROM FSI-2 retains more energy than the corresponding mode of ROM FSI-1.

Condition number of the reduced explicit step, ROM FSI-2 vs ROM FSI-1

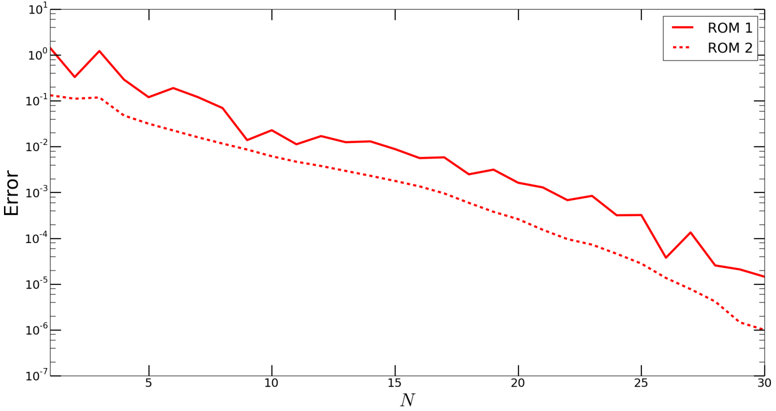


☺ ROM FSI-2 is characterized by a condition number of at least 7 orders of magnitude lower than ROM FSI-1.

Error analysis of ROM FSI-1

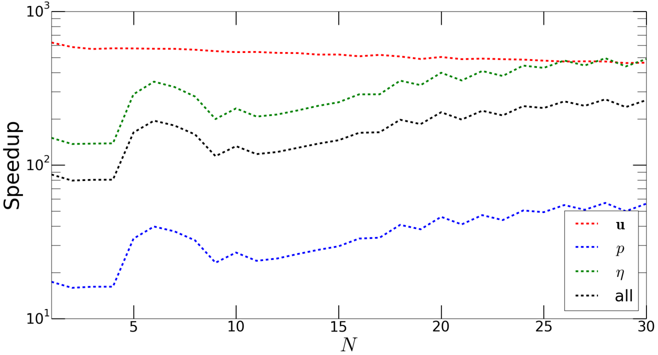


Error analysis of ROM FSI-2 vs ROM FSI-1 (velocity only)



☺ thanks to the lower condition number, strong imposition of interface conditions and higher retained energy, ROM FSI-2 is more accurate (1 order of magnitude) than ROM FSI-1.

Speedup analysis, ROM FSI-2



☺ The overall ROM/HF speedup is of two orders of magnitude. Furthermore, the speedup increases with N because a lower number of iterations is required in the implicit step.

- F. Ballarin, G. Rozza, Y. Maday. *Reduced-order semi-implicit schemes for fluid-structure interaction problems*. Submitted, 2016.

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Sponsors

- **European Research Council Executive Agency**, ERC CoG 2015 AROMA-CFD, GA 681447, 2016-2021.
- MIUR-PRIN project "Mathematical and numerical modelling of the cardiovascular system, and their clinical applications", 2014-2016
- INDAM-GNCS 2015, "Computational Reduction Strategies for CFD and Fluid-Structure Interaction Problems"
- INDAM-GNCS 2016 "Numerical methods for model order reduction of PDEs"
- COST, **European Union Cooperation in Science and Technology**, TD 1307 EU-MORNET Action (`http://www.eu-mor.net`)
- PAR-FSC 2014-2020, Regione Friuli Venezia Giulia
- TRIM, INSEAN-CNR, 2016
- HPC resources: CINECA, INFN, SISSA-ICTP

Thanks for your attention!

European Commission

Horizon 2020
European Union funding
for Research & Innovation