Flutter Phenomenon in Aeroelasticity and Its Mathematical Analysis

Article *in* Journal of Aerospace Engineering · January 2006

DOI: 10.1061/(ASCE)0893-1321(2006)19:1(1)

Marianna Shubov
University of New Hampshire

Flutter Phenomenon in Aeroelasticity and Its Mathematical Analysis

Marianna A. Shubov[1]

Abstract: The present paper is the last part of a three-part survey paper, in which I give a review of several research directions in the area of mathematical analysis of flutter phenomenon. Flutter is known as a structural dynamical instability, which occurs in a solid elastic structure interacting with a flow of gas or fluid and consists of violent vibrations of the structure with rapidly increasing amplitudes. The focus of this paper is a collection of models of fluid-structure interaction, for which precise mathematical formulations are available. My main interest is in the analytical results on such models: the results that can be used to explain flutter and its qualitative and even quantitative treatments. This study does not pretend to be a comprehensive review of an enormous engineering literature on analytical, computational, and experimental aspects of the flutter problem. I present a brief exposition of the results obtained in several selected papers or groups of papers. In this paper, I concentrate on the most well-known cases of flutter, i.e., flutter in aeroelasticity. Namely, I discuss aircraft flutter in historical retrospective and outline some future directions of flutter analysis. The last two sections of the paper are devoted to the precise analytical results obtained in my several recent works on a specific aircraft wing model in a subsonic, inviscid, incompressible airflow. I also mention that in the previous papers (Parts I and II of the survey), I discuss such topics as: (1) bending–torsion vibrations of coupled beams; (2) flutter in transmission lines; (3) flutter in rotating blades; (4) flutter in hard disk drives; (5) flutter in suspension bridges; and (6) flutter of blood vessel walls.

DOI: 10.1061/(ASCE)0893-1321(2006)19:1(1)

CE Database subject headings: Flutter; Aeroelasticity; Mathematical models.

Introduction

The objective of this study is to present a review of several selected research directions in the area of mathematical and engineering analysis of flutter. The present work represents Part III of a three-part survey paper. The Introduction to Part I, "Mathematical modelling and analysis of flutter in bending-torsion coupled beams, rotating blades, and hard disk drives" (Shubov 2004a), contains a very detailed description of the motivation and the choice of models discussed in all three parts. Therefore, in the Introduction to the present paper, Part III, I briefly outline the importance of the flutter analysis and control, and then describe the organization of the paper. In addition to the Introduction, this paper contains the following sections: "Flutter Phenomenon in Aeroelasticity," "Theoretical Results on Euler–Bernoulli and Timoshenko Beam Model," and "Mathematical Analysis of Aircraft Wing Model in Subsonic Air Flow." To create the whole picture, I recall that in the aforementioned first paper (Shubov 2004a), I have discussed such topics as: (1) bending–torsion coupled beams under deterministic and random loads; (2) flutter of transmission lines; (3) flutter of helicopter, propeller, and turbine blades; and (4) flutter in hard disk drives. In Part II of the survey, "Mathematical modelling and analysis of flutter in long-span suspension bridges and in blood vessel walls" (Shubov 2004b), I have considered: (1) flutter development in a long-span suspension and partially supported bridges, and active and passive mechanisms for flutter control; and (2) flutter in blood vessel walls, with blood vessels being considered as channels with flexible walls. Even though the present paper is a part of an entire survey, it is self-contained and can be read independently (as can Parts I and II).

I recall that flutter is a physical phenomenon that occurs in a solid elastic structure interacting with a flow of gas or fluid. Flutter is a structural dynamical instability, which consists of violent vibrations of the solid structure with rapidly increasing amplitude. It usually results either in serious damage of the structure or in its complete destruction. Flutter occurs when the parameters characterizing fluid-structure interaction reach certain critical values. The physical reason for this phenomenon is that under special conditions, the energy of the flow is rapidly absorbed by the structure and transformed into the energy of mechanical vibrations.

In engineering practice, flutter must be avoided either by design of the structure or by introducing a control mechanism capable of suppressing harmful vibrations. Flutter is known as an inherent feature of fluid-structure interaction and, thus, it cannot be eliminated completely. However, the critical conditions for flutter onset can be shifted to the safe range of the operating parameters. This is the ultimate goal for the design of flutter control mechanisms.

The most well-known cases of flutter are related to flutter in aircraft wings, tails, and control surfaces. Flutter is an in-flight event that happens beyond some speed-altitude combinations. High-speed aircrafts are most susceptible for flutter although flutter can occur at a speed as low as about 90 km/h. In fact, no speed regime is truly immune from flutter.

[1]Professor, Dept. of Mathematics and Statistics, Univ. of New Hampshire, Durham, NH 03824. E-mail: marianna.shubov@euclid.unh.edu

Note. Discussion open until June 1, 2006. Separate discussions must be submitted for individual papers. To extend the closing date by one month, a written request must be filed with the ASCE Managing Editor. The manuscript for this paper was submitted for review and possible publication on December 13, 2002; approved on December 8, 2004. This paper is part of the ***Journal of Aerospace Engineering***, Vol. 19, No. 1, January 1, 2006. ©ASCE, ISSN 0893-1321/2006/1-1–12/$25.00.

However, flutter vibrations occur in a variety of different engineering and even biomedical situations. Namely, in aeronautic engineering, flutter of helicopter, propeller, and turbine blades is a serious problem. It also affects electric transmission lines, high-speed hard disk drives, and long-spanned suspension bridges. Flutter of cardiac tissue and blood vessel walls is of special concern to medical practitioners. Flutter is an extremely complex physical phenomenon whose complete theoretical explanation is an open problem. I recall that the focus of this review is a collection of models of fluid-structure interaction for which precise mathematical formulations are available. My main interest is in the analytical results of such models, which can be used for flutter explanation. I believe that the analytical treatment of flutter problems is an important component of this area of research. Such treatment can provide insights not available from purely computational or experimental results and is certainly important for designing flutter control mechanisms.

Ideally, a complete picture of a fluid-structure interaction should be described by a system of partial differential equations, a system that contains both the equations governing the vibrations of an elastic structure and the hydrodynamic equations governing the motion of gas or fluid flow. The system of equations of motion should be supplied with appropriate boundary and initial conditions. The structural and hydrodynamic parts of the system must be coupled in the following sense. The hydrodynamic equations define a pressure distribution on the elastic structure. This pressure distribution in turn defines the so-called aerodynamic loads, which appear as forcing terms in structural equations. On the other hand, the parameters of the elastic structure enter the boundary conditions for the hydrodynamic equations.

The above picture is mathematically very complicated, and to make a particular problem tractable, it is necessary to introduce simplifying assumptions. In my selection of papers for this review, I have chosen the ones containing precisely formulated mathematical models of structures in gas–fluid flows and partial analytical results on solutions of these models. Even though most of the results are not completely rigorous mathematically, they contain important qualitative information.

In the second section of the paper, "Flutter Phenomenon in Aeroelasticity," I present an overview of different directions in theoretical and practical aeroelasticity.

In the last two sections of this paper, I present a series of my recent analytical results on two aircraft wing models. The first one, "Theoretical Results on Euler–Bernoulli and Timoshenko Beam Model," describes ground vibrations of a long slender wing, which is modeled as a coupled Euler–Bernoulli–Timoshenko beam with boundary conditions representing control action of self-straining actuators. My results are rigorous and include the following information. The system is treated as a single linear evolution equation in a Hilbert state space of the system. The metric in this Hilbert space is defined by the energy of the system. The dynamics generator of the system is an unbounded non-self-adjoint operator in the state space. It is represented as a 4×4 matrix differential operator, whose domain is defined by the boundary conditions. I give explicit asymptotic formulas for the complex eigenvalues of this operator, which represent the natural frequencies and the rates of energy dissipation for the ground vibrations of the wing. In particular, it follows from my asymptotic formulas that the spectrum consists of two branches: the bending and torsion eigenvalues. Asymptotic formulas for the eigenvectors have been obtained in our papers (Shubov 2000b; Shubov 2001a,b,c; Balakrishnan et al. 2004). However, those formulas are very complicated and are not given in this paper. I present my important result on the generalized eigenvectors of the dynamics generator. Since the dynamics generator is a non-self-adjoint operator, it might have the associate vectors in addition to the eigenvectors. However, the number of associate vectors is always finite and the entire set of the generalized eigenvectors (eigenvectors and associate vectors together) forms a nonorthogonal basis of the state space (the so-called Riesz basis). Using the Riesz basis property, I present the solution of the original initial boundary-value problem in the form of expansion with respect to the Riesz basis of the generalized eigenvectors. I believe that such expansion may be very efficient in computational analysis.

The second of the aforementioned models (see the section "Mathematical Analysis of Aircraft Wing Model in Subsonic Air Flow") describes a wing of a high-aspect ratio in a subsonic, inviscid, incompressible air flow. In this model, the hydrodynamic equations have been solved explicitly and aerodynamic loads are represented as forcing terms in the structural equations in the forms of time-convolution type integrals with very complicated kernels. Thus, the model is described by a system of integro-differential equations. The very notion of spectral analysis for such systems is a new mathematical challenge. My results on this model include the following. I treat the system of equations of motion of the model as a single evolution–convolution equation in the Hilbert state space of the model. (The integral convolution part of this equation vanishes if the speed of an air stream is equal to zero, and we obtain the equation of motion for the previous ground vibration model.)

1. I introduce the notion of the generalized resolvent for the above evolution–convolution equation and represent the solution of the original initial boundary-value problem in the frequency domain in terms of this generalized resolvent. The generalized resolvent is an analytic finite-meromorphic operator-valued function of the spectral parameter. I rigorously define the aeroelastic modes as the poles of the generalized resolvent. The corresponding mode shapes are defined in terms of the residues at the poles.
2. I give explicit asymptotic formulas for the aeroelastic modes. (To the best of my knowledge, these are the first such formulas in the literature on aeroelasticity.) The entire set of aeroelastic modes splits asymptotically into two branches, which are asymptotically close to the eigenvalues of the structural part of the original system.
3. The asymptotic formulas for the aeroelastic mode shapes are also derived in our works (Shubov 2001a,b,c; Shubov and Balakrishnan 2004a,b), but I do not present them in this paper.
4. The set of all mode shapes forms a nonorthogonal basis (Riesz basis) of the state space of the system. (As was already mentioned, the set of the generalized eigenvectors of the structural part of the system has a similar property.)
5. Using the Riesz basis property of the mode shapes, I present the solution of the original initial boundary-value problem in the form of expansion with respect to the mode shapes and the integral along the negative real semiaxis, which is formally associated with the continuous spectrum.

In Part I of the survey (Shubov 2004a), I have provided a possible explanation of the flutter phenomenon in the framework of a specific solid structure–gas–fluid interaction model. Due to its importance, I reproduce the reasoning here. Flutter should be viewed as a sharp increase of the amplitude of the solution of the corresponding initial boundary-value problem. Such an increase may be due to several reasons.

The first most obvious reason consists of the following. In a

J. Aerosp. Eng., 2006, 19(1): 1-12

specific model, the aeroelastic modes (or their analogs in a nonaeroelastic setting) are either eigenvalues of a certain operator (the dynamics generator) or poles of the generalized resolvent (as has been explained above). These modes are the functions of the parameters of the problem, i.e., the functions of the air stream speed, the control gain parameters in the boundary conditions, etc. If all of the aeroelastic modes are complex points located in the left half plane, then the solution of the problem is expected to be stable. However, as I point out below, even in this case flutter may occur. For some critical combination of the parameters (e.g., for sufficiently high speed of the air stream), one or several modes may cross the imaginary axis and move into the right half plane. This shift will cause an instability of the solution of the corresponding initial boundary-value problem and should be considered as one of the possible sources of flutter. The specific value of the air stream speed, at which at least one of the aeroelastic modes crosses the imaginary axis, should be defined as the flutter speed.

The aforementioned reason for the flutter instability is not the only one. It is accepted in extensive engineering literature that flutter in a wing (or any beamlike structure) may occur when two eigenmodes corresponding to bending and torsional vibrations become coalescent. More precisely, classic coupled bending–torsion flutter in wings occurs when with an increase of air speed, the bending frequency increases and the torsional frequency falls until both frequencies become coincident. Such a flutter is well documented based on experimentations. The natural question is: "Why does the coincidence cause instability?" A possible explanation may be as follows. Coalescence of two eigenmodes can lead to an appearance of the eigenmode, which has a nontrivial Jordan block in the spectral decomposition of the dynamics generator. In other words, the corresponding eigenvector has a chain of associate vectors. In that case, equations of motion have a solution, whose amplitude as a function of time can be represented in the form $f(t)=Ct^n \exp(-\alpha t)$, where $(n+1)$=size of the Jordan block (n is the number of the associate vectors) and $(-\alpha)<0$=real part of the corresponding eigenmode. Clearly, the function $f(t)$ has a peak of the amplitude $A=C(n/\alpha e)^n$ before it goes to zero as $t\to\infty$. This peak may be large enough to cause flutter-type instability. (The amplitude A may be large even if $n=1$ but $\alpha>0$ is small, i.e., the aeroelastic mode is stable but is close to the imaginary axis.) The dynamics of the elastic structure is also subject to nonlinear effects, which are neglected for small amplitude vibrations. When the amplitude suddenly increases, the vibrations are likely to interact with nonlinearities and destroy the stability of the system.

In the conclusion of this Introduction, I point out that in the section "Theoretical Results on Euler–Bernoulli and Timoshenko Beam Model" I discuss the problem of natural frequencies and mode shapes of the wing when an aircraft is not in flight; i.e., I study the so-called ground vibrations. I allow smart-material inclusions into the wing structure, which provide a stabilizing effect. I emphasize that the problem considered in this section is a modification of the bending–torsion vibration model studied in numerous papers discussed in the section "Bending–Torsion Coupled Beams under Deterministic and Random Loads" of Part I (Shubov 2004a). The differences in the statements of the problems in the aforementioned section of (Shubov 2004a) and in the section "Theoretical Results on Euler–Bernoulli and Timoshenko Beam Model" of the present paper are, first, that I consider the case when the damping coefficients c_1 and c_2 are zeros, while in Shubov (2004c) those coefficients are positive constants, and second, in this paper the boundary conditions take into account the action of self-straining actuators, i.e., energy exchange with the environment is allowed, while the boundary conditions in Shubov (2004a) are conservative. Moreover, in all papers of (Shubov 2004a), the writers suggest either an experimental or a numerical approach, while I obtain the results using methods and ideas of rigorous mathematical analysis.

Finally, in the section "Mathematical Analysis of Aircraft Wing Model in Subsonic Air Flow," I use the results of the section "Theoretical Results on Euler–Bernoulli and Timoshenko Beam Model" to analyze the response of an aircraft wing to a turbulent air flow and clarify the notion of a flutter frequency.

Flutter Phenomenon in Aeroelasticity

In the well-known paper "Renaissance of aeroelasticity and it's future" (Friedmann 1999), the writer writes, "The primary objective of this paper is to demonstrate that the field of aeroelasticity continues to play a critical role in the design of modern aerospace vehicles, and several important problems are still far from being well understood. Furthermore, the emergence of new technologies, such as the use of adaptive materials (sometimes denoted as smart structures technology), providing new actuator and sensor capabilities, has invigorated aeroelasticity, and generated a host of new and challenging research topics that can have a major impact on the design of a new generation of aerospace vehicles."

Accounting to the writer, aeroelasticity deals with the behavior of an elastic body or vehicle in an airstream, wherein there is significant reciprocal interaction, or feedback, between deformation and flow. While dramatic instabilities are often a cause for concern, it is important to emphasize that the subcritical aeroelastic response problem is equally if not even more important for classes of vehicles like helicopters and tilt rotors. In a modern aerospace vehicle, there is also a strong interaction between aeroelasticity and high-gain control systems leading to aeroservoelasticity. Furthermore, in high-speed supersonic or hypersonic vehicles, thermal effects become important, producing an even more complex class of aerothermo-servoelastic problems. The writer of (Friedmann 1999) focuses on the state of the art in selected important topics, such as: (1) experimental aeroelasticity in wind tunnels; (2) aeroservoplasticity; (3) computational and nonlinear aeroelasticity; (4) rotary-wing aeroelasticity; (5) impact of new technologies on aeroelasticity; and (6) experimental verification of aeroelastic behavior, aeroelastic problems in new configurations, and aeroelasticity and design. Aeroservoelasticity is a multidisciplinary technology dealing with the interaction of the aircraft flexible structure, the steady and unsteady aerodynamic forces resulting from the motion of the aircraft, and the flight control system. Its role and importance are increasing in modern aircraft with high-gain digital control systems.

In another survey-type paper (Garrick and Reed 1981), the writers give an overview of an aircraft flutter in historical retrospective. When an aircraft is in flight, it necessarily deforms appreciably under load. Such deformations change the distribution of the aerodynamic load, which in turn changes the deformations: the interacting feedback process may lead to flutter, a self-excited oscillation, often destructive, wherein energy is absorbed from the airstream. The initiation of flutter depends directly on the stiffness and only indirectly on the strength of an airplane, analogous to depending on the slope of the lift curve rather than on the maximum lift. This implies that the airplane must be treated not as a rigid body but as an elastic structure. Despite the fact that this subject is an old one, it requires for a modern airplane a large

J. Aerosp. Eng., 2006, 19(1): 1-12

effort in many areas, including ground vibration testing, use of dynamically scaled wind-tunnel models, theoretical analysis, and flight flutter testing.

From the present perspective, flutter is included in the broader term aeroelasticity, the study of the static and dynamic responses of an elastic airplane. Since flutter involves the problems of interaction of aerodynamics and structural deformation, including inertial effects at subcritical as well as at critical speeds, it really involves all aspects of aeroelasticity. In a broad sense, aeroelasticity is at work in natural phenomena such as in the motion of insects, fish, and birds (biofluid dynamics).

Aeroelastic model tests in wind tunnels, supported by mathematical analysis, gave designers a much needed feeling of confidence that neither theory nor experiment alone could provide. These wind-tunnel model investigations ranged from measurements of the oscillating air loads to flutter-proof tests using complete aeroelastic models of a prototype aircraft. In addition to providing designers with solutions that might not be obtainable by theory in a reasonable length of time, such experiments also are extremely useful tools for evaluating and guiding the development of theory.

The usefulness of wind-tunnel flutter model tests to validate theory, study flutter trends, and determine margins of safety for full-scale prototypes had already been well established for low-speed aircraft. Decades later, with aircraft of all-metal construction and flight speeds approaching that of sound, new requirements arose for the design and fabrication of aeroelastically scaled flutter models.

The replica model concept was replaced by a much simpler model design approach wherein only those modes of vibration that were expected to be significant from the standpoint of flutter were represented. With this approach, beamlike wings could be simulated by a single metal spar having the proper stiffness distribution.

With the advent of flight at transonic speeds brought about mainly by the jet engine came a host of new and challenging aeroelastic problems, many of which remain to this day as the transonic speed range is nearly always the most critical one from the standpoint of flutter.

Supersonic speeds also produce a new type of flutter. The panel flutter can occur involving the skin covering wherein standing or traveling ripples in the skin have persisted, potentially leading to an abrupt fatigue failure. Panels are natural structural elements of both aircraft and spacecraft so avoidance of panel flutter is important. Panel flutter depends on many parameters, including the Mach number and the boundary layer, but especially on compressive or thermal effects that tend to create low buckles in the skin.

The carriage of external stores affects the aeroelastic stability of an aircraft. The store carriage problem is still significant today (Kehoe 1995), particularly given the many store configurations an aircraft can carry. Certain combinations of external stores (carried, e.g., by F-16, F-18, and F-111 aircraft) produce an aeroelastic instability known as limit cycle oscillations. Although these oscillations are mostly characterized by sinusoidal oscillations of limited amplitude, flight testing has shown that the amplitudes may either decrease or increase as a function of load factor (angle of attack) and airspeed. Much has been learned about the prevention of flutter through proper aircraft design and testing. Flutter testing, however, is still a hazardous test for a number of reasons (Kehoe 1995). First, it is necessary to fly close to actual flight speeds before imminent instabilities can be detected. Second, subcritical damping trends cannot be extrapolated to predict stability at higher speeds. Third, the aeroelastic stability may change abruptly from a stable to an unstable one with only a few knots' change in airspeed (Kehoe 1995).

Flutter analysis is an extremely important and active area of research in modern aeronautical studies. In recent years, extensive research has been carried out to develop active flutter suppression and gust alleviation systems, in which aerodynamic control surfaces are operated according to a control law, which relates the motion of the controls to some measurements taken on the vehicle. The aerodynamic forces generated by the controls modify the overall forces in such a way as to suppress flutter and alleviate structural turbulence response within the aircraft flight envelope (including safety margins). Some recent analytical developments, wind-tunnel tests, and flight-test demonstrations show the potential and feasibility of active control systems.

Papers by Lu and Huang (1992, 1993) present theoretical analysis of the flutter suppression of oscillating thin airfoils using active acoustic excitations in incompressible flow. The aerodynamic control surfaces driven by the hydraulic power units, such as leading and trailing-edge flaps, ailerons, spoilers, tailerons, and additional vanes, are often employed for flutter suppression and gust alleviation. Known difficulties in active control techniques, such as design of the feedback control laws and the implementation of the hydroservo systems, are usually accompanied by a specific problem, namely, the hydraulic actuator is usually sluggish in response and sometimes cannot cope with the high-frequency oscillations.

Acoustic devices have the advantage of being simple in hardware, having wide operational frequency range, and having the response, which is quicker in comparison with conventional aerodynamic control surfaces. As shown in the paper by Lu and Huang (1992), an appropriately operated loudspeaker system can stabilize a fluttering airfoil. It is the conversion and amplification of the incident acoustic wave into shedding vortices from the trailing edge, which is known as the trailing-edge receptivity, that holds the key to the acoustic flutter suppression technique discussed in Lu and Huang (1992).

As demonstrated in the unsteady flow analysis, the location of the sound source can critically affect the aerodynamic flow field, especially the circulatory part. The flutter boundary improvements in relation to the sound source location are calculated. As expected, the trailing edge is an important region for the acoustic flutter control design. No matter what gain constant is considered, a rapid flutter boundary improvement (or deterioration) occurs as the sound source is placed in the vicinity of the trailing edge. The importance of the phases of gain constants is emphasized. A wrong choice of the phase may lead to an unexpectedly low flutter boundary. The writers carried out analytic derivation of the unsteady, acoustically induced aerodynamic loads for the two-dimensional incompressible flow. It is found that the lift and pitching moment, induced by the acoustic excitation, are attributed mostly to the circulatory part of the acoustically induced unsteady flow field. The emitted sound field actually serves as a catalyst rather than a direct control force converting the incident pressure wave into the shedding vorticity wave that in turn results in a change of the corresponding condition, therefore playing a combined critical role in this acoustic control force generation process.

Open-loop flutter analysis and closed-loop flutter suppression control studies are also conducted. A parametric study of the closed-loop dynamics indicates that the phase of the feedback gain constant is a critical factor for determining the stability of the acoustically interfered closed-loop system. Moreover, the enlarge-

J. Aerosp. Eng., 2006, 19(1): 1-12

ment of the flutter boundary can be extraordinary when the locations of the excitations are placed close to the trailing edge. However, this trailing-edge excitation strategy must be used with correct phase angles; otherwise, a dramatic reduction in flutter boundary could occur (Lu and Huang 1993).

In papers by Friedmann and Resente (2001), Fujimori et al. (1984), Leger et al. (1999), and Lee et al. (1999), the writers study analytically, numerically, and experimentally the effect of the trailing-edge flap on flutter control of incompressible flow, and in addition, an important paper (Friedmann and Resente 2001) addresses the question of aeroelastic scaling.

Recent advances in the area of smart structures have led to the use of such materials as actuators for aeroservoelastic applications in order to introduce continuous structural deformations of the lifting surface that can be exploited to manipulate the unsteady aerodynamic loads and prevent flutter. Many papers exist that deal with analysis and applications of piezoelectric materials to the control of static aeroelastic problems in a composite wing and also demonstrating flutter suppression by using piezoelectric actuation on small-scale wind-tunnel models in incompressible flow. This research was extended to flutter suppression using piezoelectric actuation, culminating in a wind-tunnel test of a swept wing that is controlled by piezoelectric patches. As reported in Friedmann and Resente (2001), an increase in flutter dynamic pressure of approximately 12% was demonstrated in those tests. Although the potential of piezoelectric actuators in aeroservoelasticity is substantial, currently such materials have major limitations on their stroke- and force-producing capabilities. Furthermore, most of the successful demonstration tests have been conducted on small models in incompressible flow. Aeroelastic scaling has been disregarded and the question of how one would scale such actuators for differently sized models, or for an actual full-scale vehicle, has not been carefully addressed.

According to the writers Friedmann and Resente (2001), aeroelastically scaled wind-tunnel models have been widely used in testing during the last 40 years, and aeroelastic scaling laws that enable one to relate wind-tunnel test results to the behavior of the full-scale system have played an important role in aeroelasticity. Such scaling laws have relied on dimensional analysis to establish a set of scaling parameters used for aeroelastically scaled models suitable for wind-tunnel testing. More refined laws can be obtained using similarity solutions, which represent closed-form solutions to the equations of motion. However, for complex aeroelastic problems such solutions are impractical. It is important to emphasize that since the 1960s practically no research has been done on aeroelastic or aeroservoelastic scaling.

The general thrust of the paper by Friedmann and Resente (2001) is the development of innovative scaling laws for aeroelastic problems in compressible flow, where control is implemented either by conventional trailing-edge surfaces, that is flaps, or by piezoelectrically induced actuation, so as to expand the flutter margin. The writers have developed a new, two-pronged approach to aeroelastic and aeroservoelastic scaling, an approach which combines the classical approach with computer simulation of the specific model.

The aforementioned specific model consists of a typical cross section of a wing with plunge and pitch degrees of freedom, and it also contains a trailing-edge control surface representing an actively controlled flap. It is one of the goals of the paper to obtain equivalence relations between a conventional airfoil–trailing-edge flap combination and a continuously deforming wing section that is piezoelectrically actuated. The method is capable of providing useful scaling information on hinge moments and power requirements for flutter suppression. It is completely general in scope and applicable to any aeroelastic system. Solutions to the nondimensional aeroelastic or aeroservoelastic problems provide similarity solutions. Only such solutions correctly predict the behavior of a full-scale configuration. Saturation of flap deflection and rate pose realistic limitations on the extent of flutter margin expansion. Saturation introduces a strong nonlinearity that must be addressed at a technical level. Power requirements for flutter suppression were calculated for two models of a typical cross section, i.e., one with an actively controlled flap and the other with a piezoelectrically actuated continuously distributed twist. The writers discovered that the average power required for piezoelectric actuation is significantly larger than that required by a typical wing section employing an actively controlled flap. Using time-domain unsteady aerodynamics with full-state feedback requires reconstruction of the unsteady aerodynamic states that cannot be observed. This introduces considerable complexity in the aeroservoelastic problem. The reconstruction of such nonobservable states is essential in practical applications.

In the paper by Lee et al. (1999), the writers suggest mathematical analysis and then numerical verification of a specific structural nonlinearity. Namely, the effect of a cubic structural restoring force on the flutter characteristics of a two–dimensional airfoil placed in an incompressible flow is investigated.

The method developed in Wong et al. (1995) and Gong et al. (1998) for coupled nonlinear mechanical systems is extended to study the dynamic response of an airfoil with soft and hard cubic structural nonlinearities. The aeroelastic equations are formulated as a set of eight first-order ordinary differential equations. Given the initial values of plunge and pitch displacements and their velocities, the system of equations is integrated numerically using a fourth-order Runge–Kutta scheme. Investigation of pre- and post-Hopf bifurcation is carried out using methods developed for studying stability near equilibrium points of nonlinear oscillating systems. The amplitudes of pitch and plunge motions of limit cycle oscillations for post–Hopf bifurcation are computed for a hard spring and compared with numerical simulations. Small values of the ratio of cubic to linear structural stiffness are chosen in such a way that chaos is not encountered in this study. The presented numerical results are for a single nonlinearity in the pitch degree of freedom. Examples of coupled cubic nonlinearities in the plunge and pitch motions are not studied since, even in the absence of aerodynamics, the equations of motion describing the coupled Duffing's oscillators show an extremely complex jump condition as demonstrated in the paper by Lee et al. (1997). There are many possibilities, and perhaps unknown complexities, of airfoil motion that may occur when different parameters of the nonlinearities are varied. This fact, combined with the fact that the number of variables needed to define the airfoil is indeed very large, makes a complete parametric study of the 2-degree-of-freedom system practically impossible. In addition, four initial conditions specifying the plunge and pitch displacements and their time derivatives are required to uniquely define the problem. A systematic study of initial conditions has not been attempted; in determining the flutter boundary, only one pair of initial conditions (pitch displacement and velocity or plunge displacement and velocity) is varied at a time, while the other pair is kept equal to zero. The examples given in this paper on cubic structural nonlinearities are, therefore, incomplete but leave opportunities for other researchers to explore the more complex airfoil motions that can be encountered with various coupled nonlinearities, airfoil parameters, and initial conditions. Numerical simulations related to the effect of a cubic structural restoring force in the pitch

J. Aerosp. Eng., 2006, 19(1): 1-12

degree of freedom on the flutter characteristics of a two-dimensional airfoil placed in an incompressible flow have been carried out and led to the following new results: (1) the dependence of the flutter boundary on initial conditions for a soft spring has been discovered. For the spring constants and airfoil parameters investigated, the flutter speed decreases with increasing values of the initial pitch and plunge displacements and their velocities; (2) a hard spring, on the other hand, has a flutter boundary that is practically independent from initial conditions; (3) the nonlinear flutter speed practically coincides with the linear flutter speed; and (4) divergent flutter is not encountered, but instead limit-cycle oscillation is observed for velocities greater than the flutter speed. At the flutter speed, a supercritical Hopf bifurcation occurs.

In the post-Hopf bifurcation range for a hard spring, the frequency of the limit-cycle oscillations is evaluated using an approximate method. For the airfoil parameters chosen in this study, agreement with numerical simulations is affected only by the plunge to pitch natural frequency ratio. The amplitude of the limit-cycle oscillations as a function of the frequency is predicted using an asymptotic theory. The results compare favorably with numerical solutions for small values of the limit-cycle frequency because in such cases the amplitude is weakly dependent on the frequency. For larger values of the frequency, the agreement deteriorates. The method used to determine the frequency of the limit-cycle oscillation is valid only in the region close to the Hopf point (i.e., $U^*/U_L^*=1$). This deficiency in the present analysis can be eliminated by using more rigorous mathematical study that employs the center manifold theory and the principle of normal forms.

Theoretical Results on Euler–Bernoulli and Timoshenko Beam Model

As already mentioned in the "Introduction," in this section I present a series of results on the spectral and asymptotic properties of non-self-adjoint operators, the operators that are the dynamics generators for hyperbolic systems, which govern the motion of a coupled Euler–Bernoulli and Timoshenko beam model subject to a two–parameter family of nonconservative boundary conditions (Balakrishnan et al. 2004).

There exists extensive mathematical and engineering literature (both theoretical and computational) devoted to different aspects of the Euler–Bernoulli beam model and the Timoshenko beam model. Each model has been studied individually by numerous writers in connection with the spectral and asymptotic properties; many problems on boundary and distributed controllability have been stated and solved (Chattopadhyay et al. 1984; Grandhi and Moradmand 1989; Rao 1992; Balakrishnan 1997, 1998a,c; Geist and McLaughlin 1997, 2001; Shubov 1999; 2000a,b; Shubov and Peterson 2003). However, to the best of my knowledge, the only works containing rigorous mathematical investigation of the coupled beams are papers (Balakrishnan 1998b; Balakrishnan et al. 2004). At the same time, the model is not an artificial combination of two well-known beam models; such a model naturally occurs in aeroelasticity and the derivation of the corresponding boundary-value problem with conservative boundary conditions can be found in the classical textbooks on aeroelasticity, e.g., (Fung 1993; Bisplinghoff et al. 1996). I note that the bending–torsion model is not a specific case of a Timoshenko beam model. It is totally different both from the physical (or engineering) point of view and from the point of view of mathematical investigation. In all of the aforementioned papers and books, one can find the derivation of the model as well as its numerical and experimental verification, but none of these sources suggests complete analytical study of the model (let alone the solution of the corresponding initial boundary–value problem). In my study, I present first in the literature on aeroelasticity rigorous results on the asymptotical distribution of the eigenfrequencies and on the properties of the eigenmodes of the vibrating flexible structure.

I consider a system of two coupled hyperbolic differential equations with a very general family of physically meaningful boundary conditions depending on two complex parameters. The parameters have been introduced to model the action of self-sensing, self-straining materials as is the custom in current mathematical and engineering literature (Tsou and Gadre 1989; Shubov 1995, 1999, 2000a,b; Balakrishnan 1997, 1998b,c, 1999; Shubov and Peterson 2003; Balakrishnan et al. 2004). My main object of interest is the class of non-self-adjoint operators in the energy space of four-component Cauchy data; they are the dynamics generators of the corresponding initial boundary-value problems. One of my main results is the fact that the set of the eigenfunctions of the dynamics generator forms a Riesz basis in the state space of the system, which is called an energy space. I recall that a Riesz basis is not an orthogonal basis, but a mild deformation of an orthonormal basis. More precisely, a Riesz basis can be obtained from an orthonormal basis by modifying the latter with the help of a bounded and boundedly invertible operator (Gohberg and Krein 1996). The most important fact about a Riesz basis is that any vector from the appropriate Hilbert space can be expanded with respect to this Riesz basis and this expansion converges unconditionally to that vector in the metric of the Hilbert space. In other words, a Riesz basis is as efficient as any orthonormal basis. A non-self-adjoint operator with a discrete spectrum is called a Riesz spectral operator (Dunford and Schwartz 1963) if the entire set of its eigenfunctions forms a Riesz basis.

Control—Theoretical Aspect of Problem

The fact that the dynamics generator of the evolution equation is a Riesz spectral operator allows us to solve different boundary and distributed controllability problems via the spectral decomposition method [see, e.g., Shubov (2000a) and references therein]. As is well known, for the case when the dynamics generator is Riesz spectral, one can give precise formulas for the control laws in terms of the spectral characteristics of the model. However, the bending–torsion model is special in its control-theoretical part, as well. Let us briefly explain this. I recall that in the case of a Timoshenko beam model (Geist and McLaughlin 1997, 2001; Balakrishnan 1998b; Shubov 1999, 2002), the spectrum has two branches; i.e., the set of eigenvalues splits into two asymptotically disjoint sets. The eigenvalues in each branch form a set of asymptotically equidistant complex points. In the present problem, though I also have a two-branch spectrum, the distribution of the eigenvalues in each individual branch is different from the one that I have observed in the Timoshenko beam model. Namely, in one of the branches, the eigenvalues are asymptotically equidistant [see formula (15) below]; in the other branch, the distance between two consecutive eigenvalues tends to infinity at the rate of $|n|$, where n is the number of the eigenvalue [see formula (13) below]. The latter fact makes the accompanying moment problem much more challenging than for the case of a damped Timoshenko beam model.

J. Aerosp. Eng., 2006, 19(1): 1-12

Operator Statement of Problem

To describe the model of a dynamical coupled Euler–Bernoulli and Timoshenko beam model without an external load (Balakrishnan et al. 2004), I use the following notations: m=mass per unit length; S=mass moment per unit length; EI=bending stiffness; GJ=torsion stiffness; I_α=moment of inertia; $h(t,x)$=bending displacement at location x at time t; and $\alpha(t,x)$=torsion angle (twist) at location x at time t. The coupled bending–torsion equations are

$$mh_{tt}(t,x) + S\alpha_{tt}(t,x) + EIh''''(t,x) = 0, \quad x \in [-L,0], \quad t \geqslant 0 \tag{1}$$

$$Sh_{tt}(t,x) + I_\alpha \alpha_{tt}(t,x) - GJ\alpha''(t,x) = 0, \quad x \in [-L,0], \quad t \geqslant 0 \tag{2}$$

By the superprimes I have denoted the derivatives with respect to x. I assume that this flexible structure is clamped at $x=-L$, i.e.

$$h(t,-L) = h'(t,-L) = \alpha(t,-L) = 0 \tag{3}$$

and is subject to the following boundary conditions at the end $x=0$

$$h'''(t,0) = 0$$

$$EIh''(t,0) + g_h h_t'(t,0) = 0 \tag{4}$$

$$GJ\alpha'(t,0) + g_\alpha \alpha_t(t,0) = 0$$

g_h and g_α=arbitrary complex parameters, i.e., $g_h \in \mathbb{C} \cup \{\infty\}$, $g_\alpha \in \mathbb{C} \cup \{\infty\}$. When $g_h > 0$ and $g_\alpha > 0$, they are called the bending and the torsion control gains. I also assume

$$\Delta \equiv \det M = \det \begin{bmatrix} m & S \\ S & I_\alpha \end{bmatrix} > 0 \tag{5}$$

Energy Considerations

A convenient expression for the energy of the system is

$$\boldsymbol{\mathcal{E}}(t) = \frac{1}{2}\int_{-L}^{0} \{EI|h''(t,x)|^2 + GJ|\alpha'(t,x)|^2 + m|h_t(t,x)|^2 + I_\alpha|\alpha_t(t,x)|^2 + S[\alpha_t(t,x)\bar{h}_t(t,x) + \bar{\alpha}_t(t,x)h_t(t,x)]\}dx \tag{6}$$

Proposition 1. *(1) Under condition (5), the energy of vibrations (6), is nonnegative and is equal to zero if and only if* $h(t,x)=\alpha(t,x)=0$, $x \in [-L,0]$, $t \geqslant 0$.

(2) If $\Re g_h \geqslant 0$ *and* $\Re g_\alpha \geqslant 0$, *then the energy of the system governed by Eqs. (1) and (2), and the boundary conditions (3) and (4) dissipates, i.e.*

$$\boldsymbol{\mathcal{E}}_t(t) = -EI|h_t'(t,0)|^2 \Re g_h - GJ|\alpha_t(t,0)|^2 \Re g_\alpha \leqslant 0 \tag{7}$$

Operator Setting of Problem

Let $\mathcal{H}$ be a set of four-component vector-valued functions $\Psi(t,x)=[h(t,x),h_t(t,x),\alpha(t,x),\alpha_t(t,x)]^T \equiv (\psi_0,\psi_1,\psi_2,\psi_3)^T$ (the superscript T means the transposition), the space obtained as a closure of smooth functions satisfying the conditions $\psi_0(-L)=\psi_0'(-L)=\psi_2(-L)=0$ in the following energy norm:

$$\|\Psi\|_{\mathcal{H}}^2 = \frac{1}{2}\int_{-L}^{0} \{EI|\psi_0''(x)|^2 + GJ|\psi_2'(x)|^2 + m|\psi_1(x)|^2 + I_\alpha|\psi_3(x)|^2 + S[\bar{\psi}_1(x)\psi_3(x) + \psi_1(x)\bar{\psi}_3(x)]\}dx \tag{8}$$

By a direct calculation, I can verify that the system given by Eqs. (1) and (2) and conditions (3) and (4) can be written as the first order in a time-evolution equation in the energy space $\mathcal{H}$

$$\Psi_t(t,x) = (i\mathcal{L}_{g_h g_\alpha}\Psi)(t,x), \quad \Psi(t,x)|_{t=0} = \Psi_0(x) \quad t \geqslant 0 \tag{9}$$

The dynamics generator $\mathcal{L}_{g_h g_\alpha}$ is a matrix differential operator, which is given by the following matrix differential expression [Δ is defined in Eq. (5)]:

$$\mathcal{L}_{g_h g_\alpha} = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{I_\alpha EI}{\Delta}\dfrac{d^4}{dx^4} & 0 & -\dfrac{SGJ}{\Delta}\dfrac{d^2}{dx^2} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{SEI}{\Delta}\dfrac{d^4}{dx^4} & 0 & \dfrac{mGJ}{\Delta}\dfrac{d^2}{dx^2} & 0 \end{bmatrix} \tag{10}$$

defined on the domain

$$\begin{aligned} D(\mathcal{L}_{g_h g_\alpha}) = [\Psi \in \mathcal{H}\colon\ & \psi_0 \in H^4(-L,0), \quad \psi_1 \in H^2(-L,0), \\ & \psi_2 \in H^2(-L,0), \quad \psi_3 \in H^1(-L,0), \\ & \psi_1(-L) = \psi_1'(-L) = \psi_3(-L) = 0, \quad \psi_0'''(0) = 0, \\ & EI\psi_0''(0) + g_h\psi_1'(0) = 0, \quad GJ\psi_2'(0) + g_\alpha\psi_3(0) = 0] \end{aligned} \tag{11}$$

Eq. (9) defines a strongly continuous semigroup of transformations in the energy space $\mathcal{H}$. The generator of this semigroup, i.e., the operator $\mathcal{L}_{g_h g_\alpha}$, is my main object of interest.

Spectral Properties of Dynamics Generator $\mathcal{L}_{g_h g_\alpha}$

My main result of this section is related to general properties of the dynamics generator.

Theorem 2. *The operator $\mathcal{L}_{g_h g_\alpha}$ has the following properties.*

1. $\mathcal{L}_{g_h g_\alpha}$ is an unbounded closed non-self-adjoint (unless $\Re g_h = \Re g_\alpha = 0$) operator in $\mathcal{H}$.

2. If $\Re g_h \geqslant 0$ and $\Re g_\alpha \geqslant 0$, then $\mathcal{L}_{g_h g_\alpha}$ is a dissipative operator in $\mathcal{H}$ (i.e., if $\Psi \in D(\mathcal{L}_{g_h g_\alpha})$, then $\Im(\mathcal{L}_{g_h g_\alpha}\Psi,\Psi)_{\mathcal{H}} \geqslant 0$ (Dunford and Schwartz 1963; Gohberg and Krein 1996).

3. The inverse operator $\mathcal{L}^{-1}_{g_h g_\alpha}$ exists and it is a compact operator in $\mathcal{H}$. Therefore, $\mathcal{L}^{-1}_{g_h g_\alpha}$ has a purely discrete spectrum consisting of normal eigenvalues. [Recall that λ is a normal eigenvalue of a bounded operator A in the space H if (1) λ is an isolated point of the spectrum of A, (2) the algebraic multiplicity of λ is finite, (3) the range $(A-\lambda I)H$ of the operator $(A-\lambda I)$ is closed (Dunford and Schwartz 1963; Gohberg and Krein 1996)].

As mentioned above, I would not provide any proofs. As a commentary to Theorem 2, I would like to note that the mathematical fact (1), i.e., that $\mathcal{L}_{g_h g_\alpha}$ is a non-self-adjoint operator, means that the mechanical system participates in the energy exchange with the environment. Indeed, when the boundary parameters are such that $\Re g_h \geqslant 0$ and $\Re g_\alpha \geqslant 0$, then $\boldsymbol{\mathcal{E}}_t(t) \leqslant 0$, which indicates some energy losses through the boundary.

The fact (2) provides important information about the properties of the operator $\mathcal{L}_{g_h g_\alpha}$. Because this operator is non-self-adjoint, no general theory is available. Examples exist in mathematical literature when a non-self-adjoint operator possesses the

J. Aerosp. Eng., 2006, 19(1): 1-12

spectrum, which fills out the entire half plane of the complex plane. Part (2) of Theorem 2 guarantees that the operator $\mathcal{L}_{g_h g_\alpha}$ has only a discrete spectrum and that the eigenvalues cannot accumulate on any finite point on the complex plane. From this theorem, it follows that $\mathcal{L}_{g_h g_\alpha}$ may have either a finite number of the eigenvalues or an infinite number of eigenvalues, and in the latter case, the eigenvalues may accumulate only at infinity. The fact that $\mathcal{L}_{g_h g_\alpha}$ has an infinite amount of eigenvalues (though countably many) requires an independent proof. In my case, this proof follows from asymptotic analysis; in addition, since the dynamics generator is a non-self-adjoint operator, I might have multiple eigenvalues with finite multiplicities. Each such multiple eigenvalue may have an eigenvector or several eigenvectors and each eigenvector may have a chain of the associate vectors (similar to the nontrivial Jordan cells in the case of matrices). A formal definition says that a vector $\Phi \neq 0$ is an associate vector of a linear operator A of order m corresponding to an eigenvalue λ if $(A-\lambda I)^m \Phi \neq 0$ and $(A-\lambda I)^{m+1}\Phi = 0$. If $m=0$, then Φ is an eigenvector. Eigenvectors and associate vectors form a set of the root vectors. By Theorem 3 below, in my case the spectrum asymptotically splits into two branches and any distant eigenvalue in the appropriate branch is simple, i.e., for such an eigenvalue, one has only one eigenvector. Therefore, for the operator $\mathcal{L}_{g_h g_\alpha}$, there can be only a finite number of the associate vectors.

Theorem 3. *(1) The operator $\mathcal{L}_{g_h g_\alpha}$ has a countable set of complex eigenvalues. If*

$$g_\alpha \neq \sqrt{I_\alpha G I} \tag{12}$$

then all eigenvalues are located in a strip in the upper half plane parallel to the real axis.

(2) The entire set of the eigenvalues asymptotically splits into two disjoint subsets. I call them the h and the α branches and denote those branches by $\{\lambda_n^h\}_{n\in\mathbb{Z}}$ and $\{\lambda_n^\alpha\}_{n\in\mathbb{Z}}$, respectively. If $\Re g_\alpha \geqslant 0$ and $\Re g_h > 0$, then the α branch is asymptotically close to some horizontal line in the upper half plane. If $\Re g_h = \Re g_\alpha = 0$, then the operator $\mathcal{L}_{g_h g_\alpha}$ is self-adjoint and thus its spectrum is real. The entire set of eigenvalues may have only two points of accumulation: $+\infty$ and $-\infty$ in the sense that $\Re\lambda_n^{h(\alpha)} \to \pm\infty$ and $\Im\lambda_n^{h(\alpha)} < \text{const}$ as $n \to \pm\infty$ [see formulas (13)–(15) below].

(3) The following asymptotic formula is valid for the h branch of the spectrum:

$$\lambda_n^h = \pm \pi^2/L^2 \sqrt{I_\alpha E I/\Delta}(|n| - 1/4)^2 + \kappa_n(\omega), \quad \omega = |g_h|^{-1} + |g_\alpha|^{-1}, \quad |n| \to \infty \tag{13}$$

A complex-valued sequence $\{\kappa_n\}$ is bounded above in the following sense:

$$\sup_{n\in\mathbb{Z}}\{|\kappa_n(\omega)|\} = C(\omega), \quad C(\omega) \to 0 \text{ as } \omega \to 0 \tag{14}$$

In formula (13), *the + should be taken for $n>0$ and the − for $n<0$.*

(4) The following asymptotic formula is valid for the α branch of the spectrum:

$$\lambda_n^\alpha = \frac{\pi n}{L\sqrt{I_\alpha/GJ}} + \frac{i}{2L\sqrt{I_\alpha/GJ}}\ln\frac{g_\alpha + \sqrt{I_\alpha GJ}}{g_\alpha - \sqrt{I_\alpha GJ}} + O(\omega|n|^{-1/2}), \quad |n| \to \infty \tag{15}$$

where ω is defined in Eq. (13). In Eq. (15), the principal value of the logarithm is understood. There may be only a finite number of multiple eigenvalues of a finite multiplicity each. For such an eigenvalue, the geometric multiplicity may be less than the algebraic multiplicity, i.e., in addition to the eigenvector or eigenvectors, there may be associate vectors (see Fig. 1).

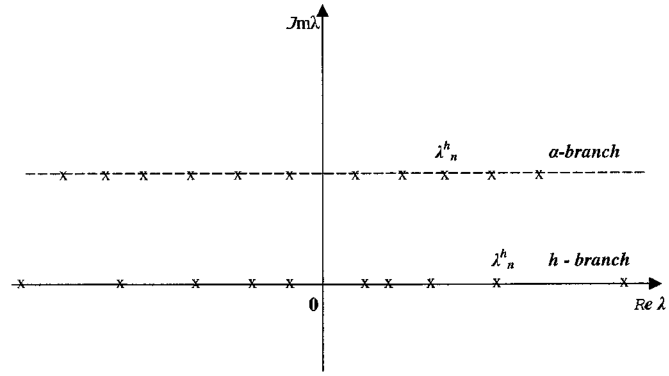

Fig. 1. Asymptotic distribution of eigenvalues; λ_n^α-α-branch eigenvalues; λ_n^h-h-branch eigenvalues

As follows from formulas (13) and (15), the leading term of the asymptotics for the h branch eigenvalues does not depend on the control parameter g_h, while the leading term for the α branch contains the control parameter g_α explicitly. This mathematical result suggests that piezoelectric inclusions are not strong enough to efficiently control the bending component of vibrations, while the torsional component can be affected by such inclusions.

My next results are related to the properties of the eigenfunctions of the dynamics generator. Namely, I address the questions of completeness and minimality. I recall that an infinite set of vectors in a Hilbert space H is complete if finite linear combinations of elements of this set form a dense set in H.

Now I recall that minimality is the generalization to the case of an infinite number of vectors of the notion of linear independence, which is used in the case of a finite number of vectors. I say that an infinite set of vectors is minimal if any vector from the set does not belong to a closed linear span of the remaining vectors. The main result is the following statement.

Theorem 4. *For any boundary controls, g_h and g_α, the root vectors (eigenvectors and associate vectors together) form complete and minimal sets in the energy space.*

Finally, I address an extremely important question whether the set of the root vectors forms an unconditional basis. First, I notice that because $\mathcal{L}_{g_h g_\alpha}$ is a non-self-adjoint (though not normal) operator, there is no chance to have an orthogonal basis of the root vectors. A mild modification of an orthonormal basis is the so-called Riesz basis, which is an unconditional basis obtained from an orthonormal basis by means of bounded and boundedly invertible linear operators. To formulate my next result, let us introduce the notation that $\{\Phi_n^h\}_{n\in\mathbb{Z}}$ be a set of the root vectors corresponding to the h branch of the spectrum and $\{\Phi_n^\alpha\}_{n\in\mathbb{Z}}$ be a set of the root vectors corresponding to the α branch of the spectrum. Note, I do not make any difference in labeling eigenvectors and associate vectors. The latter fact should not cause any problem because there could be only a finite number of the associate vectors. To continue the exposition, I need some information on the operator $\mathcal{L}^*_{g_h g_\alpha}$, which is adjoint to the operator $\mathcal{L}_{g_h g_\alpha}$. Namely, it can be shown that the adjoint operator $\mathcal{L}^*_{g_h g_\alpha}$ is given by the same matrix differential expression (10) defined on the domain, which can be obtained from $D(\mathcal{L}_{g_h g_\alpha})$ [see Eq. (11)], in which g_h and g_α have been replaced with $(-\bar{g}_h)$ and $(-\bar{g}_\alpha)$. Thus, $\mathcal{L}^*_{g_h g_\alpha}$ also has a two-branch spectrum $\{\bar{\lambda}_n^h\}_{n\in\mathbb{Z}} \cup \{\bar{\lambda}_n^\alpha\}_{n\in\mathbb{Z}}$ and, moreover, the numeration in the set of the root vectors $\{\Phi_n^{h*}\}_{n\in\mathbb{Z}} \cup \{\Phi_n^{\alpha*}\}_{n\in\mathbb{Z}}$ of the operator

$\mathcal{L}^*_{g_h g_\alpha}$ can be introduced in such a way, that the sets of the root vectors of the operators $\mathcal{L}_{g_h g_\alpha}$ and $\mathcal{L}^*_{g_h g_\alpha}$ are mutually biorthogonal, i.e.

$$(\Phi_n^h, \Phi_m^{h*})_{\mathcal{H}} = \delta_{nm}, \quad (\Phi_n^\alpha, \Phi_m^{\alpha*})_{\mathcal{H}} = \delta_{nm},$$
$$(\Phi_n^h, \Phi_m^{\alpha*})_{\mathcal{H}} = (\Phi_n^\alpha, \Phi_m^{h*})_{\mathcal{H}} = 0 \tag{16}$$

The following important result is valid.

Theorem 5. *If condition (12) is satisfied, then the set of the root vectors* $\{\Phi_n^h\}_{n\in\mathbb{Z}} \cup \{\Phi_n^\alpha\}_{n\in\mathbb{Z}}$ *forms a Riesz basis in* $\mathcal{H}$. *The same fact is valid for the biorthogonal set* $\{\Phi_n^{h*}\}_{n\in\mathbb{Z}} \cup \{\Phi_n^{\alpha*}\}_{n\in\mathbb{Z}}$. *If condition (12) fails, then neither of the sets forms an unconditional basis in* $\mathcal{H}$.

Solution of Initial Problem

Now I am in a position to give a solution of the problem given by Eq. (9). Let the initial state Φ_0 be expanded with respect to the Riesz basis as follows:

$$\Phi_0(x) = \sum_{n\in\mathbb{Z}} (\Phi_0, \Phi_n^{h*})_{\mathcal{H}} \Phi_n^h(x) + \sum_{n\in\mathbb{Z}} (\Phi_0, \Phi_n^{\alpha*})_{\mathcal{H}} \Phi_n^\alpha(x) \tag{17}$$

With Eq. (17), I immediately conclude that the solution of the initial problem Eq. (9) can be given in the following form:

$$\Phi(x) = \sum_{n\in\mathbb{Z}} (\Phi_0, \Phi_n^{h*})_{\mathcal{H}} e^{i\lambda_n^h t} \Phi_n^h(x) + \sum_{n\in\mathbb{Z}} (\Phi_0, \Phi_n^{\alpha*})_{\mathcal{H}} e^{i\lambda_n^\alpha t} \Phi_n^\alpha(x) \tag{18}$$

The norm of the solution Eq. (18) can be estimated as follows:

$$\begin{aligned}\|\Phi(t,\cdot)\|^2_{\mathcal{H}} &\leqslant C_1 \sum_{n\in\mathbb{Z}} [|e^{i\lambda_n^h t}|^2 |(\Phi_0, \Phi_n^{h*})_{\mathcal{H}}|^2 + |e^{i\lambda_n^\alpha t}|^2 |(\Phi_0, \Phi_n^{\alpha*})_{\mathcal{H}}|^2] \\ &\leqslant C_1 \sup_{n,m\in\mathbb{Z}} \{|e^{i\lambda_n^h t}|^2, |e^{i\lambda_m^\alpha t}|^2\} \\ &\quad \times \sum_{n\in\mathbb{Z}} [|(\Phi_0, \Phi_n^{h*})_{\mathcal{H}}|^2 + |(\Phi_0, \Phi_n^{\alpha*})_{\mathcal{H}}|^2]\end{aligned} \tag{19}$$

Taking into account that the spectrum of $\mathcal{L}_{g_h g_\alpha}$ is located in the closed upper half plane, I have

$$\sup_{n,m\in\mathbb{Z}} \{|e^{i\lambda_n^h t}|^2, |e^{i\lambda_m^\alpha t}|^2\} \leqslant 1$$

and, therefore, estimate (19) becomes

$$\|\Phi(t,\cdot)\|^2_{\mathcal{H}} \leqslant C_1 \sum_{n\in\mathbb{Z}} [|(\Phi_0, \Phi_n^{h*})_{\mathcal{H}}|^2 + |(\Phi_0, \Phi_n^{\alpha*})_{\mathcal{H}}|^2] \leqslant C_2 \|\Phi_0\|^2_{\mathcal{H}} \tag{20}$$

where C_1 and C_2=absolute constants, the precise values of which are immaterial for us. Estimate (20) shows that the solution Eq. (18) indeed belongs to the energy space $\mathcal{H}$.

Mathematical Analysis of Aircraft Wing Model in Subsonic Air Flow

In this section, I extend my results to a boundary-value problem arising in modeling of the flutter phenomenon in an aircraft wing in a subsonic airflow. I use the two-dimensional (2D) strip model, which applies to bare wings of high-aspect ratio (Balakrishnan 1998a, 2001; Shubov 2000b, 2001b,c). The structure is modeled by a uniform cantilever beam which bends and twists. The aerodynamics is assumed to be subsonic, incompressible, and inviscid. The action of the self-straining actuators is also incorporated in this model.

Statement of Problem in Energy Space

Let us introduce the following dynamical variables:

$$X(x,t) = [h(x,t), \alpha(x,t)]^T \quad -L \leqslant x \leqslant 0, t \geqslant 0 \tag{21}$$

where h=bending and α=torsion angle. The model can be described by the following linear system:

$$M\ddot{X}(x,t) + D\dot{X}(x,t) + KX = [f_1(x,t), f_2(x,t)]^T \tag{22}$$

where the dot means differentiation with respect to t. All 2×2 matrices in Eq. (22) are given by the formulas

$$M = \begin{bmatrix} \tilde{m} & \tilde{S} \\ \tilde{S} & \tilde{I}_\alpha \end{bmatrix}, \quad D = (\pi\rho u)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$K = \begin{bmatrix} EI\dfrac{\partial^4}{\partial x^4} & 0 \\ 0 & -GJ\dfrac{\partial^2}{\partial x^2} - \pi\rho u^2 \end{bmatrix} \tag{23}$$

$$\tilde{m} = m + \pi\rho, \quad \tilde{S} = S - a\pi\rho, \quad \tilde{I}_\alpha = I_\alpha + \pi\rho(a^2 + 1/8) \tag{24}$$

In Eqs. (23) and (24), m=density of structure; S=mass moment; I_α=moment of inertia; ρ=density of air; a=linear parameter of the structure $(-1 \leqslant a \leqslant 1)$; E=bending stiffness; G=torsion stiffness; and u=stream velocity. The right-hand side of Eq. (22) is given as a system of two convolution-type integral operations

$$f_1(x,t) = -2\pi\rho \int_0^t [uC_2(t-\sigma) - \dot{C}_3(t-\sigma)] g(x,\sigma) d\sigma$$
$$\begin{aligned} f_2(x,t) = -2\pi\rho \int_0^t [[0.5C_1(t-\sigma) - auC_2(t-\sigma) \\ + a\dot{C}_3(t-\sigma) + uC_4(t-\sigma) + 0.5\dot{C}_5(t-\sigma)] g(x,\sigma)] d\sigma \end{aligned} \tag{25}$$

$$g(x,t) = u\dot{\alpha}(x,t) + \ddot{h}(x,t) + (0.5-a)\ddot{\alpha}(x,t) \tag{26}$$

The aerodynamical functions C_i, $i=1\cdots 5$, are defined in the following ways (Balakrishnan 1998a, 2001; Shubov 2000b, 2001a,b,c):

$$\hat{C}_1(\lambda) \equiv \int_0^\infty e^{-\lambda t} C_1(t) dt = \frac{u}{\lambda} \frac{e^{-\lambda/u}}{K_0(\lambda/u) + K_1(\lambda/u)} \quad \text{Re } \lambda > 0$$
$$C_3(t) = \int_0^t C_1(t-\sigma)(u\sigma - \sqrt{u^2\sigma^2 + 2u\sigma}) d\sigma$$
$$C_4(t) = C_2(t) + C_3(t), \quad C_2(t) = \int_0^t C_1(\sigma) d\sigma \tag{27}$$

J. Aerosp. Eng., 2006, 19(1): 1-12

$$C_5(t) = \int_0^t C_1(t-\sigma)[(1+u\sigma)\sqrt{u^2\sigma^2+2u\sigma} - (1+u\sigma)^2]d\sigma$$

where K_0 and K_1=modified Bessel functions of the zero and first orders. The self-straining control actuator action is modeled by the boundary conditions as

$$EIh''(0,t) + \beta\dot{h}'(0,t) = 0, \quad h'''(0,t) = 0 \tag{28}$$

$$GJ\alpha'(0,t) + \delta\dot{\alpha}(0,t) = 0, \quad \beta,\delta \in \mathbb{C}^+ \cup \{\infty\} \tag{29}$$

where $\mathbb{C}^+$=closed right half plane. I consider the boundary conditions at $x=-L$

$$h(-L,t) = h'(-L,t) = \alpha(-L,t) = 0 \tag{30}$$

Here and below, I use the prime for derivative with respect to x. Let the initial state be

$$\begin{aligned} h(x,0) &= h_0(x), \quad \dot{h}(x,0) = h_1(x) \\ \alpha(x,0) &= \alpha_0(x), \quad \dot{\alpha}(x,0) = \alpha_1(x) \end{aligned} \tag{31}$$

I will consider the problem given by Eqs. (22) and conditions (28)–(31) in the energy space $\mathcal{H}$. To introduce the metric of $\mathcal{H}$, I assume that

$$\det\begin{bmatrix} m & S \\ S & I_\alpha \end{bmatrix} > 0, \quad 0 < u \leqslant \frac{\sqrt{2GJ}}{L\sqrt{\pi\rho}} \tag{32}$$

Let $\mathcal{H}$ be the set of four-component vector-valued functions $\Psi=(h,\dot{h},\alpha,\dot{\alpha})^T \equiv (\psi_0,\psi_1,\psi_2,\psi_3)^T$ obtained as a closure of smooth functions, satisfying the boundary conditions $\psi_0(-L)=\psi_0'(-L)=\psi_2(-L)=0$, in the following energy norm:

$$\|\Psi\|_{\mathcal{H}}^2 = 1/2\int_{-L}^0 [EI|\psi_0''(x)|^2 + GJ|\psi_2'(x)|^2 + \tilde{m}|\psi_1(x)|^2 + \tilde{I}_\alpha|\psi_3(x)|^2$$
$$+ \tilde{S}[\psi_3(x)\bar{\psi}_1(x) + \bar{\psi}_3(x)\psi_1(x)] - \pi\rho u^2|\psi_2(x)|^2]dx \tag{33}$$

Let us denote the integral operator from the right-hand side of Eq. (22) by $\mathcal{F}$, i.e., $(\mathcal{F}X)(x,t)=[f_1(x,t),f_2(x,t)]^T$. In terms of the operator $\mathcal{F}$, Eq. (22) can be rewritten in the form

$$M\ddot{X}(x,t) + D\dot{X}(x,t) + KX(x,t) = \mathcal{F}\dot{X}(x,t) \tag{34}$$

My goal is to rewrite Eq. (34) as the first order in the time evolution–convolution equation in the energy space. Due to Eq. (32), M^{-1} exists and Eq. (34) can be represented in the form

$$\ddot{X}(x,t) + M^{-1}D\dot{X}(x,t) + M^{-1}KX(x,t) = M^{-1}\mathcal{F}\dot{X}(x,t) \tag{35}$$

The initial value problem defined by Eq. (35) and conditions (28)–(31) can be written as

$$\dot{\Psi} = i\mathcal{L}_{\beta\delta}\Psi + \tilde{\mathcal{F}}\dot{\Psi}, \quad \Psi = (\psi_0,\psi_1\psi_2,\psi_3)^T, \quad \Psi|_{t=0} = \Psi_0 \tag{36}$$

$\mathcal{L}_{\beta\delta}$ is the following matrix differential operator in $\mathcal{H}$:

$$\mathcal{L}_{\beta\delta} = -i\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{EI\tilde{I}_\alpha}{\Delta}\dfrac{d^4}{dx^4} & -\dfrac{\pi\rho u\tilde{S}}{\Delta} & -\dfrac{\tilde{S}}{\Delta}\left(GJ\dfrac{d^2}{dx^2} + \pi\rho u^2\right) & -\dfrac{\pi\rho u\tilde{I}_\alpha}{\Delta} \\ 0 & 0 & 0 & 1 \\ \dfrac{EI\tilde{S}}{\Delta}\dfrac{d^4}{dx^4} & \dfrac{\pi\rho u\tilde{m}}{\Delta} & \dfrac{\tilde{m}}{\Delta}\left(GJ\dfrac{d^2}{dx^2} + \pi\rho u^2\right) & \dfrac{\pi\rho u\tilde{S}}{\Delta} \end{bmatrix} \tag{37}$$

defined on the domain

$$\mathcal{D}(\mathcal{L}_{\beta\delta}) = [\Psi \in \mathcal{H}\colon \psi_0 \in H^4(-L,0), \psi_1 \in H^2(-L,0) \tag{38}$$

$$\psi_2 \in H^2(-L,0), \psi_3 \in H^1(-L,0)$$

$$\psi_1(-L) = \psi_1'(-L) = \psi_3(-L) = 0$$

$$\psi_0'''(0) = 0; EI\psi_0''(0) + \beta\psi_1'(0) = 0, GJ\psi_2'(0) + \delta\psi_3(0) = 0$$

where H^i, $i=1,2,4,$=Sobolev spaces; and $\tilde{\mathcal{F}}=M^{-1}\mathcal{F}$=linear integral operator. I point out that if $u=0$, then the problem considered in the current section coincides with the problem of the previous section.

Remark 6. The aircraft wing model can be described by an abstract evolution–convolution equation of the form

$$\dot{\Psi}(t) = iA\Psi(t) + \int_0^t F(t-\tau)\dot{\Psi}(\tau)d\tau \tag{39}$$

Here $\Psi(\cdot) \in \mathcal{H}$=energy space of the system; Ψ=four-component vector-valued function; $A(A=\mathcal{L}_{\beta\delta})$=matrix differential operator; and $F(t)$=matrix-valued function. Eq. (39) does not define an evolution semigroup and does not have a dynamics generator. Let us take the Laplace transformation of Eq. (39), then the formal solution is

$$\hat{\Psi}(\lambda) = [\lambda I - iA - \lambda\hat{F}(\lambda)]^{-1}[I - \hat{F}(\lambda)]\Psi_0 \tag{40}$$

where Ψ_0=initial state and by ˆ I denote the Laplace transform. To obtain the space–time-domain solution, I have "to calculate" the inverse Laplace transform of Eq. (40). To this end, I need to investigate the generalized resolvent operator

$$R(\lambda) = [\lambda I - iA - \lambda\hat{F}(\lambda)]^{-1} \tag{41}$$

In my case, $R(\lambda)$=operator-valued meromorphic function on the complex plane with a branch cut along the negative real semiaxis. The poles of $R(\lambda)$ are called the eigenvalues, or the aeroelastic modes. The residues of $R(\lambda)$ at the poles are the projectors on the corresponding generalized eigenspaces. The branch cut corresponds to the continuous spectrum. To find the space–time-domain solution, I have to solve the following problems: (1) to obtain asymptotic formulas for the set of aeroelastic modes; (2) to prove the Riesz basis property of the mode shapes; (3) to obtain an expansion theorem with respect to the eigenfunctions of the continuous spectrum; and (4) to obtain asymptotic formulas for the eigenfunctions of the continuous spectrum. (These eigenfunctions can be expressed in terms of the jump of the kernel of the generalized resolvent $R(\lambda)$ across the branch cut.)

Statement of Main Asymptotical Results

I start with the general properties of the operator $\mathcal{L}_{\beta\delta}$.

Theorem 7. *(1) $\mathcal{L}_{\beta\delta}$ is a closed linear operator in $\mathcal{H}$ whose* *resolvent is compact, and therefore, the spectrum is discrete.*

(2) Operator $\mathcal{L}_{\beta\delta}$ is non-self-adjoint unless β and δ are pure imaginary. If $\Re\beta \geqslant 0$ and $\Re\delta \geqslant 0$, then $\mathcal{L}_{\beta\delta}$ is dissipative, i.e., $\Im(\mathcal{L}_{\beta\delta}\Psi, \Psi) \geqslant 0$ for $\Psi \in \mathcal{D}(\mathcal{L}_{\beta\delta})$. The adjoint operator $\mathcal{L}^*_{\beta\delta}$ is given by expression (37) on the domain obtained from Eq. (38) by replacing β and δ with $(-\bar{\beta})$ and $(-\bar{\delta})$.

In the next theorem, I provide the spectral asymptotics for the operator $\mathcal{L}_{\beta\delta}$.

Theorem 8. *(1) The operator $\mathcal{L}_{\beta\delta}$ has a countable set of complex eigenvalues. If*

$$|\delta| \neq \sqrt{GJ\tilde{I}_\alpha} \tag{42}$$

then the set of eigenvalues is located in a strip parallel to the real axis.

(2) The entire set of eigenvalues asymptotically splits into two disjoint subsets. I call them the β and δ branches and denote them by $\{\lambda_n^\beta\}_{n\in\mathbb{Z}}$ and $\{\lambda_n^\delta\}_{n\in\mathbb{Z}}$, respectively. If $\Re\beta \geqslant 0$ and $\Re\delta > 0$, then the δ branch is asymptotically close to some horizontal line in the upper half plane. If $\Re\beta > 0$ and $\Re\delta = 0$, then the horizontal asymptote coincides with the real axis. If $\Re\beta = \Re\delta = 0$, then the operator $\mathcal{L}_{\beta\delta}$ is self-adjoint and, thus, its spectrum is real. The set of eigenvalues may have only two points of accumulation: $+\infty$ and $-\infty$ in the sense that $\Re\lambda_n^{\beta(\delta)} \to \pm\infty$ and $\Im\lambda_n^{\beta(\delta)} \leqslant \mathrm{const}$ as $n \to \pm\infty$.

(3) The following asymptotics is valid for the β branch of the spectrum:

$$\lambda_n^\beta = (\operatorname{sgn} n)(\pi^2/L^2)\sqrt{EI\tilde{I}_\alpha/(\tilde{m}\tilde{I}_\alpha - \tilde{S}^2)}(n - 1/4)^2 + \kappa_n(\omega) \quad |n| \to \infty \tag{43}$$

where $\omega = |\beta|^{-1} + |\delta|^{-1}$. The complex-valued sequence $\{\kappa_n(\omega)\}_{n\in\mathbb{Z}}$ is bounded in the following sense:

$$\sup_{n\in\mathbb{Z}}\{|\kappa_n(\omega)|\} = C(\omega) \quad C(\omega) \to 0 \text{ as } \omega \to 0$$

(4) The following asymptotics is valid for the δ branch of the spectrum:

$$\lambda_n^\delta = \frac{\pi n}{L\sqrt{\tilde{I}_\alpha/GJ}} + \frac{i}{2L\sqrt{\tilde{I}_\alpha/GJ}}\ln\frac{\delta + \sqrt{GJ\tilde{I}_\alpha}}{\delta - \sqrt{GJ\tilde{I}_\alpha}} + O(|n|^{-1/2}) \quad |n| \to \infty \tag{44}$$

In Eq. (44), ln *means the principal value of the logarithm.*

(5) The entire spectrum may have a finite number of multiple eigenvalues of a finite algebraic multiplicity each. For such an eigenvalue, the geometric multiplicity may be less than the corresponding algebraic multiplicity, i.e., in addition to the eigenvector or eigenvectors, there may be a finite number of the associate vectors.

(6) The asymptotic distribution of the aeroelastic modes coincides with the asymptotic distribution of the points $\{i\lambda_n^\beta\}_{n\in\mathbb{Z}}$ and $\{i\lambda_n^\delta\}_{n\in\mathbb{Z}}$. In particular, the entire set of aeroelastic modes asymptotically splits into two disjoint branches approaching two vertical asymptotes.

Geometric Properties of Root Vectors

Theorem 9. *(1) The entire set of the root vectors (eigenvectors and associate vectors together) of the operator $\mathcal{L}_{\beta\delta}$ is complete in the energy space $\mathcal{H}$. The same fact is valid for the root vectors of the adjoint operator $\mathcal{L}^*_{\beta\delta}$.*

(2) The sets of the root vectors of the operators $\mathcal{L}_{\beta\delta}$ and $\mathcal{L}^*_{\beta\delta}$ are mutually biorthogonal.

(3) If condition (42) is satisfied, then the set of the root vectors of the operator $\mathcal{L}_{\beta\delta}$ forms a Riesz basis in the energy space.

The main result of this section is the following statement.

Theorem 10. *If condition (42) is satisfied, then the set of mode shapes forms a Riesz basis in the state space.*

Flutter Control Problem

The flutter or critical speed u_f and frequency ω_f are defined as the lowest airspeed and corresponding circular frequency at which a given structure flying at given atmospheric density and temperature will exhibit sustained, simple harmonic oscillations. Flight at u_f represents a borderline condition or neutral stability boundary, because all small motions must be stable at speeds below u_f, whereas divergent oscillations can ordinarily occur in a range of speeds (or at all speeds) above u_f. Probably, the most important type of aircraft flutter results from coupling between the bending and torsional motions of a relatively large aspect-ratio wing.

Now I will describe what kind of a control problem will be considered in the future in connection with the flutter suppression. In the wing model, both the matrix differential operator and the matrix integral operator contain entries depending on the speed u of the surrounding air flow. Therefore, the aeroelastic modes are functions of u: $\lambda_k = \lambda_k(u)$ $(k \in \mathbb{Z})$. The wing is stable if $\Re\lambda_k(u) < 0$ for all k. However, if u is increasing, some of the modes move to the right half plane. The flutter speed u_k^f for the kth mode is defined by the relation $\Re\lambda_k(u_k^f) = 0$. Flutter cannot be eliminated completely. To suppress flutter, one should design self-straining actuators (i.e., in the mathematical language, to select parameters in the boundary conditions [control gains g_h and g_α in formulas (28) and (29)], in such a way that flutter does not occur in the desired speed range. This is a highly nontrivial boundary control problem.

Acknowledgments

Partial support by the National Science Foundation Grants Nos. ECS 0080441, DMS 0072247, and the Advanced Research Program-01 of Texas Grant No. 0036-44-045 is highly appreciated by the writer.

References

Balakrishnan, A. V. (1997). "Theoretical limits of damping attainable by smart beams with rate feedback." *Proc. SPIE*, 3039, 204–215.

Balakrishnan, A. V. (1998a). "Aeroelastic control with self-straining actuators: Continuum models." *Proc. 5th Int. Symp. on Smart Structures*, Vol. 3323, SPIE–International Society for Optical Engineers, Washington, D.C., 44–54.

Balakrishnan, A. V. (1998b). "Control of structures with self-straining actuators: Coupled Euler/Timoshenko model." *Nonlinear problems in aviation and aerospace*, Gordon and Breach, Reading, U.K.

Balakrishnan, A. V. (1998c). "Dynamics and control of articulated anisotropic Timoshenko beams." *Dynamics and control of distributed systems*, H. S. Tzou and L. A. Bergman, eds., Cambridge University Press, Cambridge, U.K., 121–201.

Balakrishnan, A. V. (1999). "Damping performance of strain actuated beams." *Comput. Appl. Math.*, 18(1), 31–86.

Balakrishnan, A. V. (2001). "Subsonic flutter suppression using self-straining actuators." *J. Franklin Inst.*, 338(2/3), 149–170.

Balakrishnan, A. V., Shubov, M. A., and Peterson, C. A. (2004). "Spectral analysis of coupled Euler–Bernoulli and Timoshenko beam

J. Aerosp. Eng., 2006, 19(1): 1-12

model." *Z. Angew. Math. Mech.*, 84(2), 291–313.
Bisplinghoff, R. L., Ashley, H., and Halfman, R. L. (1996). *Aeroelasticity*, Dover, New York.
Chattopadhyay, A., Hanaguad, S., and Smith, C. V. (1987). "Optimal design of a vibrating beam with coupled bending and torsion." *AIAA J.*, 25(9), 1231–1240.
Dunford, N., and Schwartz, J. T. (1963). *Linear operators, Part III. Spectral operators*, Interscience, New York.
Friedmann, P. P. (1999). "Renaissance in aeroelasticity and its future." *J. Aircr.*, 36(1), 105–121.
Friedmann, P. P., and Resente, E. (2001). "Active control of flutter in compressible flow and its aeroelastic scaling." *J. Guid. Control Dyn.*, 24(1), 167–175.
Fujimori, A., Gupta, M. M., Ohta, H., and Nikiforuk, P. N. (1984). "Active flutter suppression for two-dimensional airfoils." *J. Guid. Control Dyn.*, 12(2), 188–194.
Fung, Y. C. (1993). *An introduction to the theory of aeroelasticity*, Dover, New York.
Garrick, E. I. and Reed, W. H., III. (1981). "Historical development of aircraft flutter." *J. Aircr.*, 18(11), 897–912.
Geist, B., and McLaughlin, J. R. (1997). "Double eigenvalues for the uniform Timoshenko beam." *Appl. Math. Lett.*, 10, 129–134.
Geist, B., and McLaughlin, J. R. (2001). "Asymptotic formulas for the eigenvalues of the Timoshenko beam." *J. Math. Anal. Appl.*, 253, 341–380.
Gohberg, I. Ts., and Krein, M. G. (1996). *Introduction to the theory of linear non-self-adjoint operators*, Translated in Mathematical Monographs, Vol. 18, American Mathematical Society, Providence, R.I.
Gong, L., Wong, Y. S., and Lee, B. H. K. (1998). "Dynamics of a coupled system of Duffing's equations." *Dynamics of Continuous, Discrete, and ImpulsiveSsystems*, 4, 99–119.
Grandhi, R. V. and Moradmand, J. K. (1989). "Optimum synthesis of thin-walled vibrating beams with coupled bending and torsion." *ASME J. Mech., Transm., Autom. Des.*, 111, 555–560.
Kehoe, M. W. (1995). "A historical overview of flight flutter testing." *NASA Technical Memorandum No. 4720*, Dryden Flight Research Center, Edwards, Calif., 1–20.
Lee, B. H. K., Gong, L., and Wong, Y. S. (1997). "Analysis and computation of a two-degree of freedom system and its application to aeroelasticity." *J. Fluids Struct.*, 11, 225–246.
Lee, B. H. K., Tiang, L. Y. and Wong, Y. S. (1999). "Flutter of airfoil with a cubic restoring force." *J. Fluids Struct.*, 13, 75–101.
Leger, T. J., Wolff, J. M., and Beran, P. S. (1999). "Accurate determination of aeroelastic stability properties using a direct method." *AIAA J.*, 1214, 99–109.
Lu, P.-J., and Huang, L.-J. (1992). "Flutter suppression of thin airfoils using active acoustic excitations." *AIAA J.*, 30(12), 2873–2881.
Lu, P.-J., and Huang, L.-J. (1993). "Optimal control law synthesis for flutter suppression using active acoustic excitations." *J. Guid. Control Dyn.*, 16(1), 124–131.
Rao, J. S. (1992). *Advanced theory of vibrations*, Wiley, New York.
Shubov, M. A. (1995). "Asymptotics of resonances and geometry of resonance states in the problem of scattering of acoustic waves by a spherically symmetric inhomogeneity of the density." *Diff. Integral Eq.*, 5, 1073–1115.
Shubov, M. A. (1999). "Spectral operators generated by Timoshenko beam model." *Syst. Control Lett.*, 38(4–5), 249–258.
Shubov, M. A. (2000a). "Exact controllability of damped Timoshenko beam." *IMA J. Math. Control Inf.*, 17, 375–395.
Shubov, M. A. (2000b). "Riesz basis property of root vectors of non-self-adjoint operators generated by aircraft wing model in subsonic air flow." *Math. Methods Appl. Sci.*, 23, 1585–1615.
Shubov, M. A. (2001a). "Asymptotic representation for the root vectors of non-self-adjoint operators generated by aircraft wing model in subsonic air flow." *J. Math. Anal. Appl.*, 260, 341–366.
Shubov, M. A. (2001b). "Asymptotics of aeroelastic modes and basis property of mode shapes for aircraft wing model." *J. Franklin Inst.*, 338(2/3), 171–185.
Shubov, M. A. (2001c). "Mathematical analysis of aircraft wing model in subsonic airflow." *IMA J. Appl. Math.*, 66, 319–356.
Shubov, M. A. (2002). "Asymptotic and spectral analysis of a spatially inhomogeneous Timoshenko beam model." *Math. Nachr.*, 241, 125–162.
Shubov, M. A. (2004a). "Mathematical modelling and analysis of flutter in bending-torsion coupled beams, rotating blades, and hard disk drives." *J. Aerospace Eng.*, in press.
Shubov, M. A. (2004b). "Mathematical modelling and analysis of flutter in long-span suspension bridges and in blood vessel walls." *J. Aerosp. Eng.*, 17(2), 70–82.
Shubov, M. A., and Balakrishnan, A. V. (2004a). "Asymptotic and spectral properties of the operator-valued functions generated by aircraft wing model." *Math. Methods Appl. Sci.*, 27, 329–362.
Shubov, M. A., and Balakrishnan, A. V. (2004b). "Asymptotic behaviour of aeroelastic modes for aircraft wing model in subsonic air flow." *Proc. R. Soc., Math. Physic. Eng. Sci.*, 460, 1057–1093.
Shubov, M. A., and Peterson, C. A. (2003). "Asymptotic distribution of eigenfrequencies for a coupled Euler–Bernoulli and Timoshenko beam model." *NASA Technical Publication NASA/CR-2003-212022*, NASA Dryden Center, Edwards, Calif., 1–78.
Tzou, H. S., and Gadre, M. (1989). "Theoretical analysis of a multi-layered thin shell coupled with piezoelectric shell actuators for distributed vibration controls." *J. Sound Vib.*, 132, 433–450.
Wong, Y. S., Lee, B. H. K., and Gong, L. (1995). "Dynamic response of a two-degree of freedom system with a cubic nonlinearity." *Proc., 3rd Int. Conf. on Computational Physics*, Chung Li, Taiwan.

J. Aerosp. Eng., 2006, 19(1): 1-12