CS 2050 Discrete Mathematics	10/11/2024

Homework 8: Relations

Michael Chan — *Due: 10/18/2024 11:59 PM*

Submissions submitted at least 24 hours prior to the due date will receive 2.5 points of extra credit. On-time submissions receive no penalty. You may turn it in one day late for a 10-point penalty or two days late for a 25-point penalty. Assignments more than two days late will NOT be accepted. We will prioritize on-time submissions when grading before an exam.

You should submit a typeset or *neatly* written PDF on Gradescope. The grading TA should not have to struggle to read what you've written; if your handwriting is hard to decipher, you will be required to typeset your future assignments. Illegible solutions will be given 0 credit. A 5-point penalty will occur if pages are incorrectly assigned to enumerate or not properly oriented when submitting (this includes submitting all answers on one page). You may collaborate with other students, but any written work should be your own. Write the names of the students you work with on the top of your assignment. **Only Submit cleanly written paragraph proofs. Do not turn in scratch work. You will be graded on your writing.**

1. (20 points each) For any sets A, B, let

$$B^A = \{f \mid f : A \to B \text{ is a function}\}$$

Let $\mathbb{N}^{\mathbb{N}}$ be the set of all functions with domain and co-domain $\mathbb{N}$ and let R be a relation on $\mathbb{N}^{\mathbb{N}}$ such that fRg if $\exists n \in \mathbb{N}(f(n) = g(n))$.

(a) Prove or disprove: R is reflexive.

We show that $\exists n \in \mathbb{N}$ such that $(f(n) = f(n))$. Well surely $n = n$, so R is reflexive.

(b) Prove or disprove: R is symmetric.

We show that if $\exists n \in \mathbb{N}$ such that $f(n) = g(n)$, then $\exists m \in \mathbb{N}$ such that $g(m) = f(m)$. If $f(n) = g(n)$, because equality is symmetric, $g(n) = f(n)$. So there exists an $m = n \in \mathbb{N}$ such that $g(m) = f(m)$. So R is symmetric.

(c) Prove or disprove: R is transitive.

Consider functions $f(n) = 1$, $g(n) = n$, where $n \in \mathbb{N}$. Let $b = 1$. Then $f(b) = 1$ and $g(b) = 1$. So there exists an $n \in \mathbb{N}$ such that $f(n) = g(n)$. So fRg.

Now consider function $h(n) = 2$, where $n \in \mathbb{N}$. Let $c = 2$. Then $g(c) = 2$ and $h(c) = 2$. So there exists an $m \in \mathbb{N}$ such that $g(m) = h(m)$. So gRh.

Suppose R is transitive. Then $(fRg \wedge gRh) \implies fRh$. We have shown that fRg and gRh. But for all $a \in \mathbb{N}$, $f(a) = 1$ and $h(a) = 2$. So $f(a)$ does not equal $h(a)$ for all a. So f does not relate to h. Contradiction. $\square$

For the following problems, please use the formal definitions as follows

- $\forall a, b \in \mathbb{Z}$ we say $a \mid b \iff \exists c \in \mathbb{Z}[ac = b]$
- $a \equiv b \pmod{n} \iff n \mid (a - b)$

2. (20 points) Let $a, b, c \in \mathbb{Z}$. Prove $a \mid b$ implies $ac \mid bc$.

   If $a \mid b$, then there is some $d \in \mathbb{Z}$ such that $ad = b$. We multiply both sides by c to get $adc = bc$, or
   $$ac(d) = bc$$
   Since d is some constant, we may say $ac \mid bc$. So $a \mid b$ implies $ac \mid bc$. $\square$

3. (20 points) Let $s + t \equiv 0 \pmod{n}$ and let $a \equiv b \pmod{n}$. Then $as + bt \pmod{n}$ is equivalent to what element of $\mathbb{Z}_n$? Prove it.

   We are given that $a \pmod{n} = b \pmod{n}$. So there exists some integer k such that $nk = a - b$ or $nk + b = a$. So
   $$\begin{aligned} as + bt &= (nk + b)s + bt \\ &= nks + bs + bt \\ &= b(s + t) + nks \end{aligned}$$
   Since $(s + t) \pmod{n} = 0 \pmod{n}$, and $nks \pmod{n}$ surely is 0,
   $$as + bt \equiv 0 \pmod{n}$$
   So $as + bt$ is equal to the equivalence class of $[0] \in \mathbb{Z}_n$. $\square$