Inclusion-Exclusion

1. Alice attends a small college in which each class meets only once a week. She is deciding between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. This also implies that on a given day, Alice can take at most 6 classes.

   Alice decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that she will have classes every day, Monday through Friday? (This problem can be done either directly using the naive definition of probability, or using inclusion-exclusion.)

   Let A_i be the event that Alice has class on day i, $i = 1, \ldots, 5$. We are interested in $P(\cap_{i=1}^{5} A_i)$. We will find the probability of the complement (at least one day off). Define $B_i = A_i^c$ (i.e. no class on day i), so we are finding $P(\cup_{i=1}^{5} B_i)$. Note that it's not possible to have all five days off or four days off, because Alice can only take a maximum of six classes in a given day. $P(B_i) = \binom{24}{7}/\binom{30}{7}$, $P(B_i \cap B_j) = \binom{18}{7}/\binom{30}{7}, i \neq j$, and $P(B_i \cap B_j \cap B_k) = \binom{12}{7}/\binom{30}{7}$ for $i \neq j \neq k$. By principle of Inclusion-Exclusion:

$$P(\cup_{i=1}^{5} B_i) = \sum_{i=1}^{5} P(B_i) - \sum_{i<j} P(B_i \cap B_j) + \sum_{i<j<k} P(B_i \cap B_j \cap B_k)$$

   The problem is symmetric (i.e. Monday and Tuesday are no different), so

$$P(A_1 \cap \cdots \cap A_5) = 1 - P(\cup_{i=1}^{5} B_i) = 1 - \left(5\frac{\binom{24}{7}}{\binom{30}{7}} - \binom{5}{2}\frac{\binom{18}{7}}{\binom{30}{7}} + \binom{5}{3}\frac{\binom{12}{7}}{\binom{30}{7}} \right)$$

   which simplifies to $1 - 263/377 = 114/377 \approx 0.302$.