SLR coefficient estimates

11/7/24

Housekeeping

- No TA hours tonight
- Will discuss details of Midterm 2 next week!
- Revisions for proposals due Saturday 11:59pm

Recap

- **Linear regression**: statistical method where the relationship between variable x and variable y is modeled as a **line + error:**

$$y = \underbrace{\beta_0 + \beta_1 x}_{\text{line}} + \underbrace{\epsilon}_{\text{error}}$$

- β_0 and β_1 are population parameters and their corresponding point estimates b_0 and b_1 are estimated from the data
- **Fitted** model: $\hat{y} = b_0 + b_1 x$
- Residual: $e_i = \hat{y}_i - y_i$
- LINE conditions: **L**inearity, **I**ndependence, **N**ormal residuals, **E**qual variance

Fitting the least-squares line

Parameter estimates

- Like in previous topics, we have to estimate the parameters using data
- We want to estimate β_0 and β_1 using the (x_i, y_i)
  - In practice, we let software do this for us
- However, we *can* derive the least-squares estimates using properties of the least-squares line

Estimating slope and intercept

First obtain b_1:

$$b_1 = \frac{s_y}{s_x} R$$

where:

- s_x and s_y are the sample standard deviations of the explanatory and response variables
- R is the correlation between x and y

Then obtain b_0:

$$b_0 = \bar{y} - b_1 \bar{x}$$

where

- $\bar{y}$ is the sample mean of the response variable
- x is the sample mean of the explanatory variable

- Take STAT 0211 or 0311 to see where these formulas come from!

Fitting `cherry` model (by hand)

Verify estimates $b_0 = -36.94$ and $b_1 = 5.07$ from our model for the cherry data:

```r
cherry |>
  pivot_longer(cols = c(diam, volume),
               names_to = "variable",
               values_to = "val") |>
  select(-height) |>
  group_by(variable) |>
  summarise(mean = mean(val), s = sd(val))
```

variable	mean	s
diam	13.248	3.138
volume	30.171	16.438

```r
R <- cor(cherry$diam, cherry$volume)
R
```

```
[1] 0.9671194
```

What does this value of R tell us?

- Set-up the calculations:
  - $b_1 = \frac{s_y}{s_x} R$
  - $b_0 = \bar{y} - b_1 \bar{x}$

- $b_1 = \frac{16.438}{3.138} \times 0.967 = 5.07$
- $b_0 = 30.171 - 5.07 \times 13.248 = -36.94$
- What do these numbers really mean?

Interpreting parameters

Interpreting the parameters (i.e. **coefficients**) in a regression model is one of *the most* important steps in an analysis!

Intercept interpretation

Our fitted model is $\hat{y} = b_0 + b_1 x$.

- To interpret the estimate of the intercept coefficient b_0, simply plug in $x = 0$:

$$\hat{y} = b_0 + b_1 x = b_0 + b_1(0) = b_0$$

- So, the intercept describes the **average/expected** value of the response variable y if $x = 0$

Intercept in `cherry` model

$$\widehat{\text{volume}} = -36.94 + 5.07 \times \text{diameter}$$

- Interpretation of intercept in context: for a tree with a diameter of 0 inches, the expected volume would be -36.94 cubic feet
  - This interpretation is mathematically correct, but practically speaking is useless
- The intercept's interpretation only makes sense when a value of $x = 0$ for the explanatory variable is plausible!
  - This is typically not the case/relevant in many applications
  - Trees with 0 diameter are not able to sampled

Slope interpretation

- Let $\hat{y}_1$ be the estimated response for a given value of x, so $\hat{y}_1 = b_0 + b_1 x$
- What happens when we increase x by 1?
- Let $\hat{y}_2$ be the estimated response for $x + 1$:

$$
\begin{aligned}
\hat{y}_2 &= b_0 + b_1(x+1) \\
&= b_0 + b_1 x + b_1 \\
&= \hat{y}_1 + b_1 \Rightarrow \\
b_1 &= \hat{y}_2 - \hat{y}_1
\end{aligned}
$$

- Interpretation: for a 1 unit increase in the explanatory variable x, we expect the response variable to change by b_1 units

Slope in `cherry` model

$$\widehat{\text{volume}} = -36.94 + 5.07 \times \text{diameter}$$

- Interpretation in context: for every 1 inch increase in diameter, we expect that volume of cherry trees to increase by 5.07 cubic feet

Example: `elmhurst`

The `elmhurst` dataset from `openintro` provides a random sample of 50 students gift aid for students at Elmhurst College.

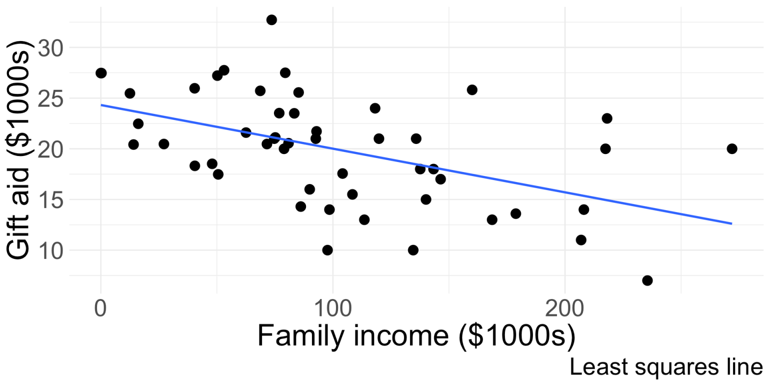
- We will examine the relationship between the family income of the student and the gift aid that student received (in $1000s)

Are the first two conditions of LINE satisfied?

Example: `elmhurst` (cont.)

We run the model in R, and the output looks something like this:

term	estimate	std.error	statistic	p.value
(Intercept)	24.319	1.291	18.831	0
family_income	-0.043	0.011	-3.985	0

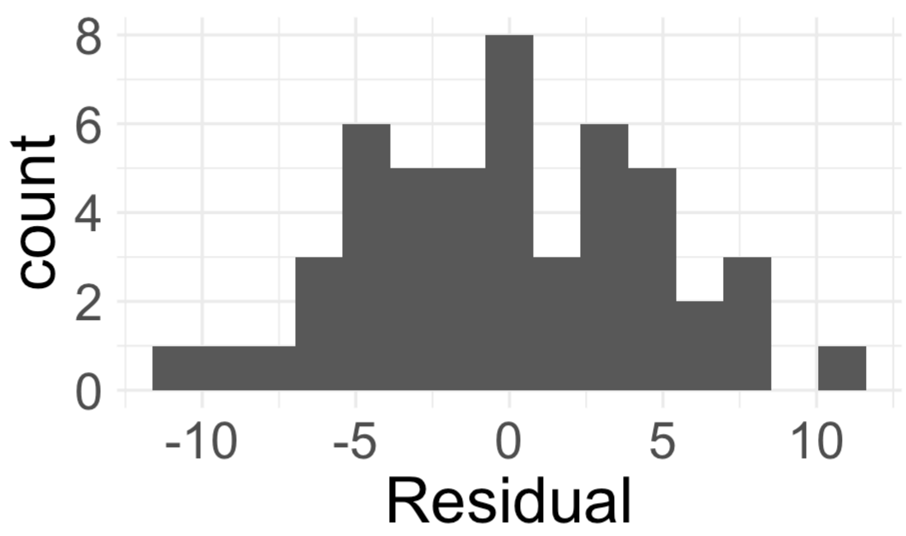


- The values in the estimate column are our b_0 and b_1:
  - $b_0 = ?$ and $b_1 = ?$
  - What do you think the second column is?
- Write out our fitted model in context

Example: `elmhurst` model

$$\widehat{\text{aid}} = 24.319 + -0.043 \times \text{family\_income}$$

- Before we interpret the coefficients, we should verify that the linear model is appropriate for the data!



Do you believe the last two conditions of LINE are satisfied?

Example: `elmhurst` interpretation

$$\widehat{\text{aid}} = 24.319 + -0.043 \times \text{family\_income}$$

- Interpret the slope in context
- Interpret the intercept in context
- Is the meaning of the intercept relevant?

- Slope: for every $1000 increase in family income, we expect that the student's gift aid will decrease by $43.
- Intercept: for a student whose family income is $0, we expect that average amount of aid they will receive is $2.4319^{4}
- Since a family could have an income of $0, the intercept does seem relevant

Words of caution

- The estimates from the fitted model will always be imperfect
  - The linear equation is good at capturing trends, no individual outcome will be perfectly predicted
- **Do not try to use the model for x values beyond the range of the observed x!**
  - The true relationship between x and y is almost always much more complex than our simple line
  - We do not know how the relationship behaves outside our limited window

Extrapolation

Suppose we would like to use our fitted model to estimate the expected gift aid for someone whose family income is $1,000,000:

- Find the estimated gift aid (careful with units)
  - $\widehat{\text{aid}} = 24.319 + -0.043 \times 1000 = -18.681$
  - This is ridiculous!
- This is an example of **extrapolation**: using the model to estimate values outside the scope of the original data
  - We should *never* extrapolate!

Strength of fit

If we fit a model and determine LINE was met, we still need a way to describe how “good” the fit is!

Describing the fit

- Recall sample correlation R describes the linear relationship between variables x and y
- We typically use the **coefficient of determination** or R^2 (**R-squared**) to describe strength of linear fit
  - Describes amount of variation in y that is explained by predictor x in the least squares line
- It turns out that R^2 in SLR is exactly … R squared (i.e. the square of the sample correlation)
  - What are the possible values of R^2? What are desirable values of R^2?

Example: `elmhurst` model fit

- The sample correlation between `family income` and `aid` is $R =$ -0.499
- So the coefficient of determination is $R^2 = (-0.499)^2 = 0.249$
  - **Interpretation**: using a linear model, about 24.9% of the variability in `aid` received by the student is explained by `family income`

Categorical predictor

Thus far, we have assumed that x is numerical. Now let x be categorical.

Categorical predictor with two levels

- Remember that the different groupings/categories of categorical variables are called levels
- Now assume that x is categorical with two levels
- Running example: the `possum` data from `openintro`
  - Response variable: `tail_l` (tail length in cm)
  - Explanatory variable: `pop` (either `Vic` or `other`)
- Maybe we would think to write our regression as

$$\text{tail length} = \beta_0 + \beta_1 \text{pop} + \epsilon$$

- Why doesn't this work?
  - Functions require a numerical input!

Indicator variables

We need a mechanism to convert the categorical levels into numerical form!

- This is achieved through an **indicator variable** which takes the value 1 for one specific level and the value 0 otherwise:

$$\text{pop\_other} = \begin{cases} 0 & \text{if pop} = \text{Vic} \\ 1 & \text{if pop} = \text{other} \end{cases}$$

tail_l	pop	pop_new
38.0	other	0
34.0	Vic	1
36.0	Vic	1
36.5	Vic	1
41.5	other	0

- The level that corresponds to 0 is called the **base level**
  - So `Vic` is the base level
  - Choosing which level is the base level can sometimes be important

Example: `possum` model

This yields the SLR model

$$\text{tail length} = \beta_0 + \beta_1 \text{pop\_other} + \epsilon$$

Our estimates are as follows:

term	estimate	std.error	statistic	p.value
(Intercept)	35.935	0.253	142.065	0
popother	1.927	0.339	5.690	0

- Write out the equation of our fitted model

Intercept for categorical x

Our fitted model is:

$$\widehat{\text{tail length}} = 35.935 + 1.927 \times \text{pop\_other}$$

- Let's interpret the intercept by plugging in 0 for the explanatory variable:

$$\widehat{\text{tail length}} = 35.935 + 1.927 \times 0 = 35.935$$

- But wait, when is $\text{pop\_other} = 0$? When the possum is from Victoria!
- So when x is categorical, the interpretation of b_0 is the expected value of the response variable for the base level of x
- Interpret b_0 in context
  - The expected tail length of possums from Victoria is 35.935 cm

Slope for categorical x

$$\widehat{\text{tail length}} = 35.935 + 1.927 \times \text{pop\_other}$$

$$\text{pop\_other} = \begin{cases} 0 & \text{if pop} = \text{Vic} \\ 1 & \text{if pop} = \text{other} \end{cases}$$

- Remember, the slope coefficient is interpreted as the expected change in y for a one unit increase in x
- What does it mean for the indicator variable to increase by one unit here?
  - pop_other increases by one unit by going from 0 to 1. This corresponds to a `pop` value of “other”
- When x is categorical, the interpretation of b_1 is the expected change in y when moving from the base level to the non-base level
- Try interpreting b_1 in context!

Slope for categorical x (cont.)

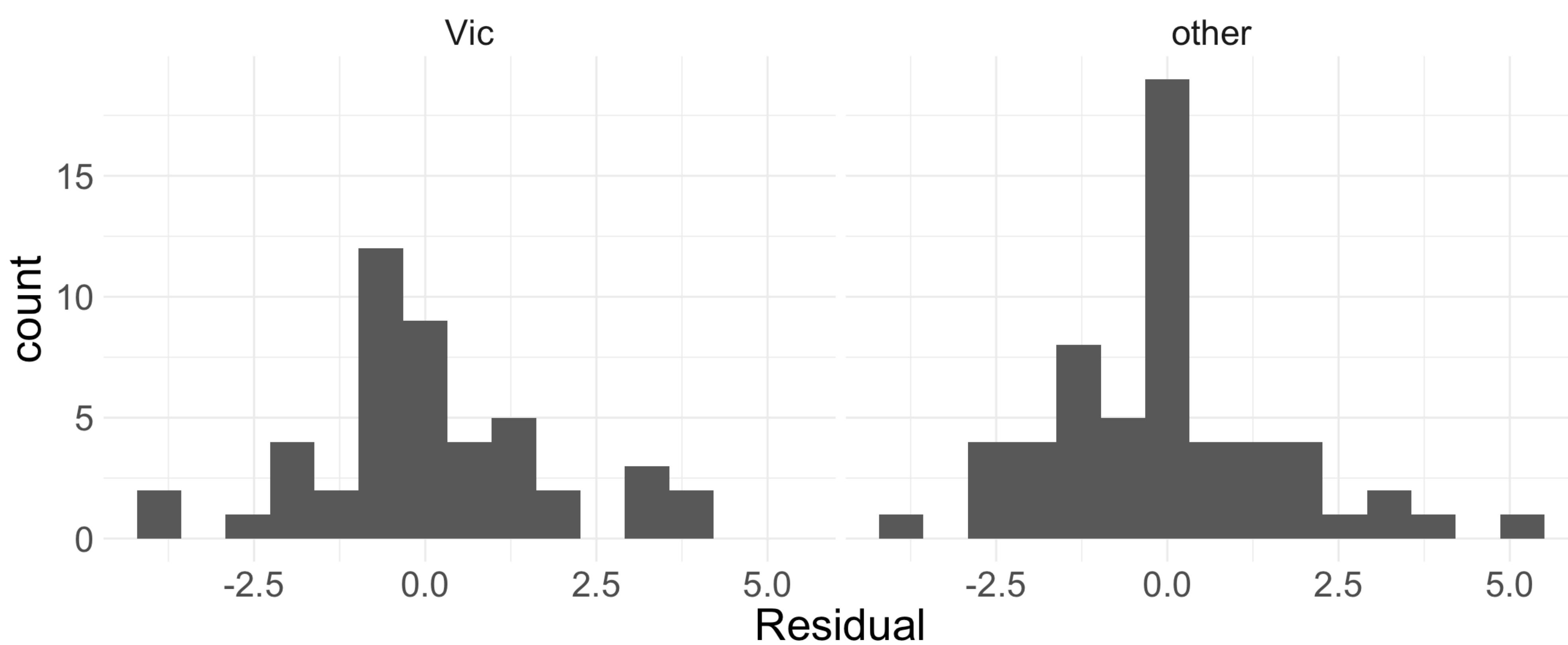
$$\widehat{\text{tail length}} = 35.935 + 1.927 \times \text{pop\_other}$$

$$\text{pop\_other} = \begin{cases} 0 & \text{if pop} = \text{Vic} \\ 1 & \text{if pop} = \text{other} \end{cases}$$

- Interpretation of slope: possums from outside of Victoria are expected to have tail lengths about 1.927 cm longer than possums from Victoria
- Note: interpretations for b_0 and b_1 for categorical x are the same as for numerical x, but they have more specific/nuanced interpretations when placed in context

Assessing linear fit

- When x is categorical, the LINE conditions still need to hold
- When x only has two levels, the **L**inearity assumption will always be satisfied
- We need to evaluate **N**early normal residuals and **E**qual variance for each level:



Live code