Central Limit Theorem

10/23/24

Recap

- Normal distribution: symmetric, bell-shaped curve that is described by mean μ and standard deviation σ
  - Common model used to describe behavior of continuous variables
- Use area under the Normal curve to obtain probabilities
- 68-95-99.7 rule
- z-score standardizes observations to allow for easier comparison: $z = \frac{x-\mu}{\sigma}$

Warm-up

- Let $Z \sim N(0, 1)$. If the 10th percentile of Z is -1.28, what is the 90th percentile?
- Let $X \sim N(0, 2)$. If the 10th percentile of X is -2.56, what is the 90th percentile? Or can you not say without code?
- Let $Y \sim N(2, 1)$. If the 10th percentile of Y is 0.72, what is the 90th percentile? Or can you not say without code?

Where we’re going

- We are going to learn one of the BIGGEST theorems in Statistics
- Uses the Normal distribution, and will be immensely helpful for inference tasks of confidence intervals and hypothesis testing

Central Limit Theorem

Central Limit Theorem (CLT)

- Assume that you have a **sufficiently large** sample of n **independent** values from a population with mean μ and standard deviation σ.
- Then the distribution of sample means is approximately Normal:

$$\bar{X} \dot{\sim} N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

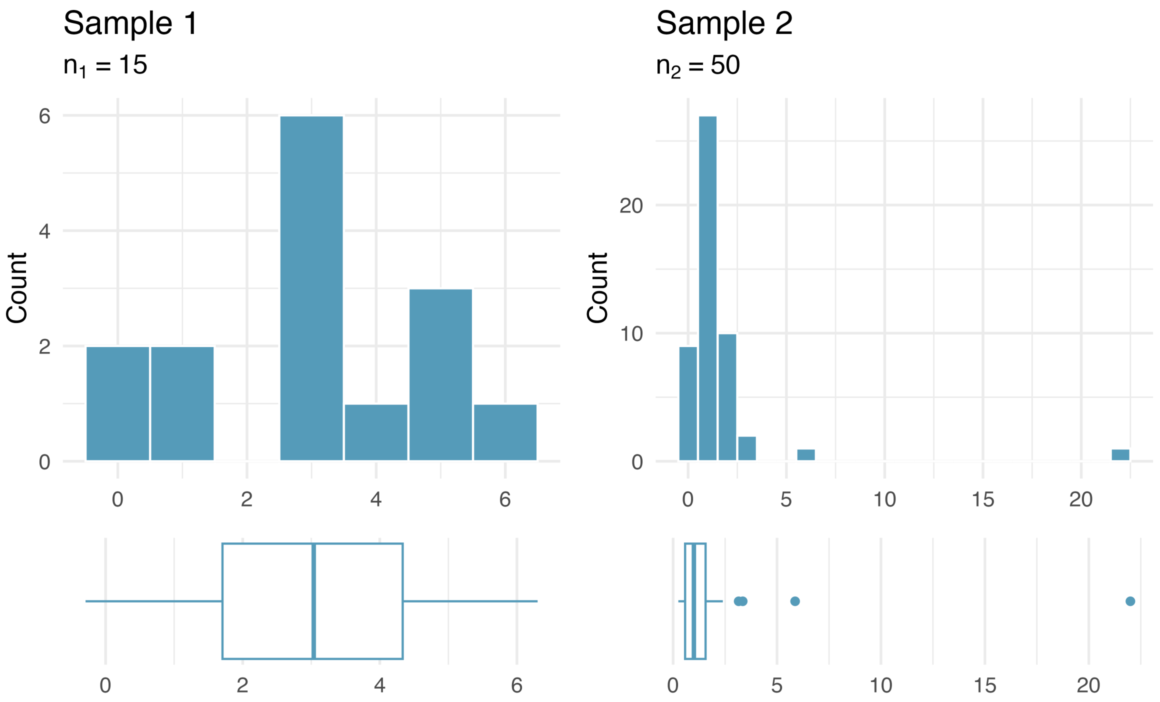
  - That is, the sampling distribution of the sample mean is approximately normal with mean μ and standard error $\sigma/\sqrt{n}$

CLT assumptions

- **Remark #1**: does not require any assumption about how the data $x_1, \ldots, x_n$ behave so long as the following assumptions hold:
  1. Independent samples: usually achieved by random sampling
  2. Sufficiently large sample size n, where large is in relation to total size of population
- **Remark #2**: if the data $x_1, \ldots, x_n$ are known to be Normal and independent, then the distribution of sample means is *exactly* Normal, even for small n
  - For this reason, if n is small we require the data to be Normal
  - How to know? We replace (2) above with the **normality condition**:
    - If n is small $(n < 30)$: we assume data are approximately normal if there are no clear outliers
    - If n is larger $(30 \leq n <?)$: we assume data are approximately normal if there are no particularly extreme outliers

Normality condition

Do you believe the large sample size/normality condition is satisified in the following two samples?

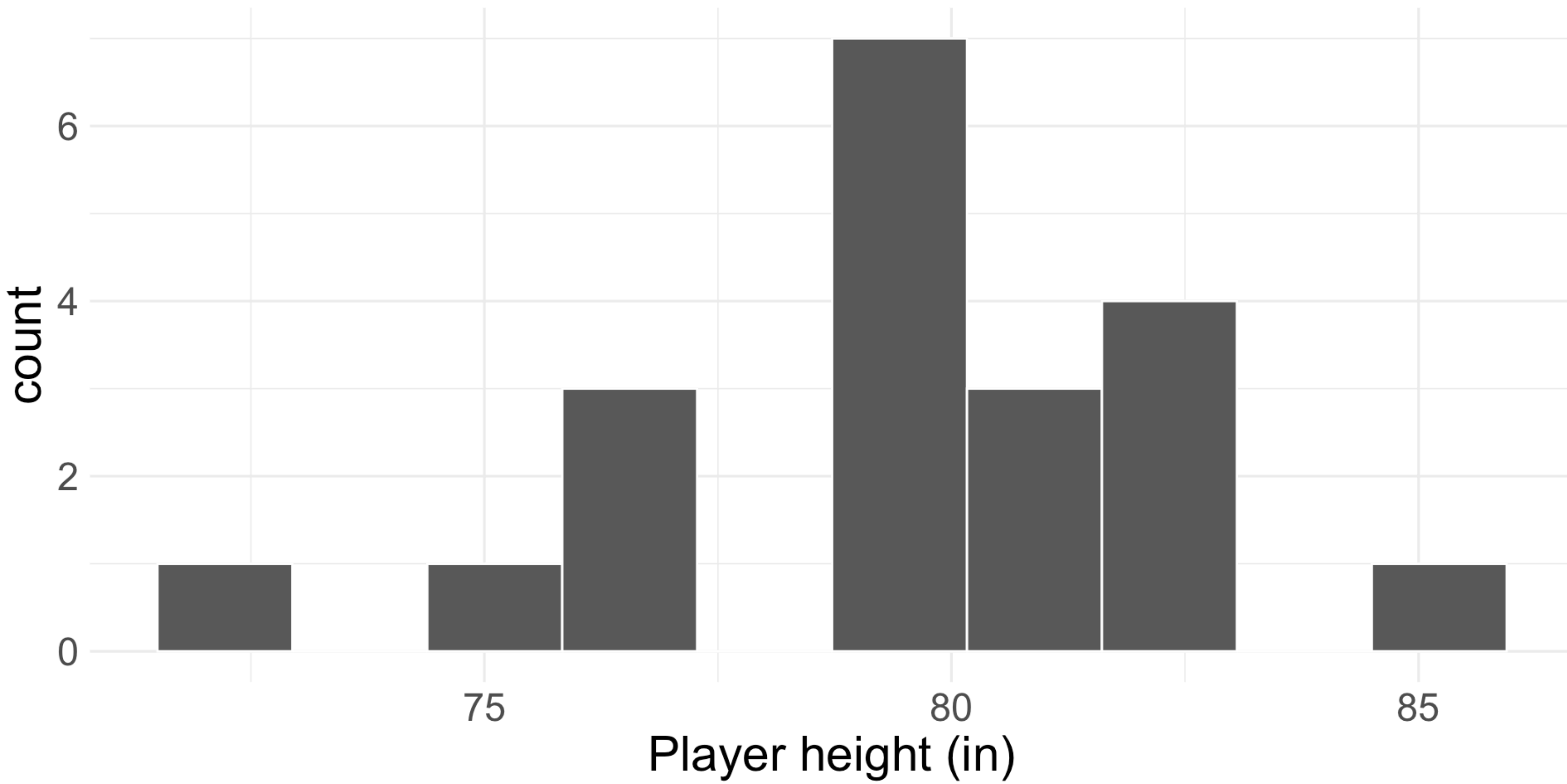


- Sample 1: small $n < 30$. But histogram and boxplot reveals no clear outliers, so I would say normality condition is met.
- Sample 2: larger $n \geq 30$. Even though n is larger, there is a particularly extreme outlier, so I would say normality condition is not met.

Activity

Height example

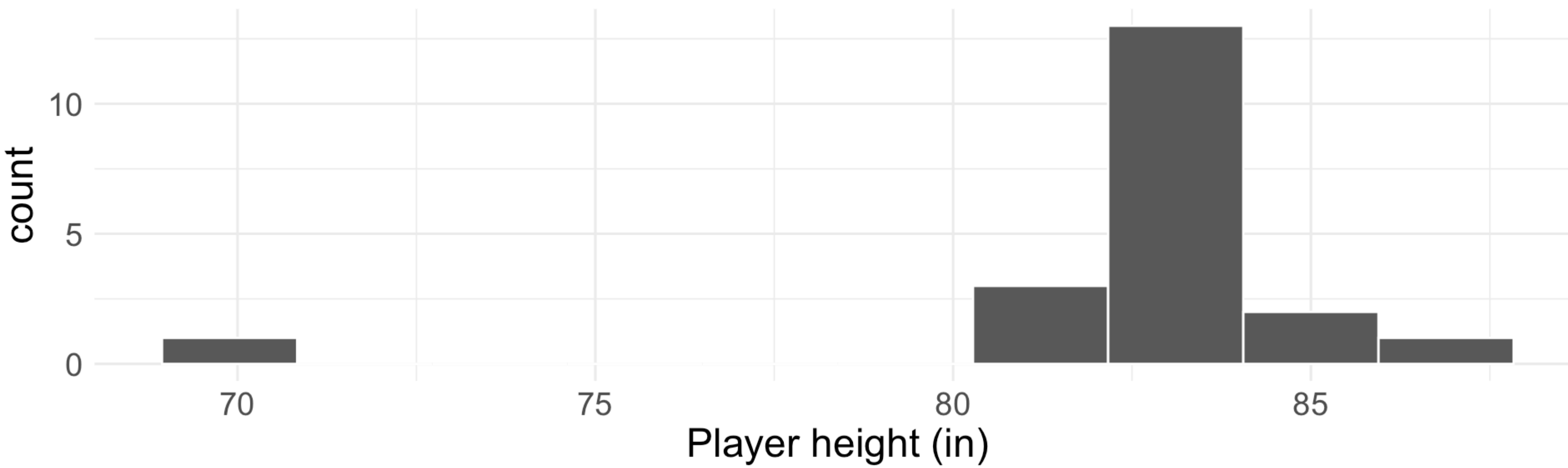
The average height of all NBA players in the 2008-9 season is 79.21 inches, with a standard deviation of 3.57 inches. We randomly sample 20 of these players and record their heights.



What is the sampling distribution of the sample mean heights?

Height example: solution

- We have independent samples, but not a large sample size. However, the histogram of the data looks approximately Normal (no clear outliers).
- CLT applies! By CLT: $\bar{X} \dot{\sim} N\left(79.21, \frac{3.57}{\sqrt{20}}\right)$
- If the data instead looked like the following, I would say normality condition is violated:



Bank example

Customers are standing in line at a bank. The service time for each customer i is represented by X_i. Suppose that the average service time for all customers is 5 minutes, with a standard deviation of 6 minutes.

Assume that a bank currently has 36 customers in it, and all customers are independent of each other. What is the probability that the average service time of all these customers is less than 4 minutes?

Bank example: solution

- We want $\Pr(\bar{X} < 4)$
- Conditions for CLT met: independence (random sample) and sufficiently large sample size ($n = 36$).
  - So by CLT, $\bar{X} \dot{\sim} N(5, \frac{6}{\sqrt{36}}) = N(5, 1)$
- Using 68-95-99.7 rule, probability that the average service time of all these customers is less than 4 minutes is about $1 - (0.34 + 0.5) = 0.16$
  - `pnorm(4, 5, 1)` = 0.159

Proportion as a mean

Remember $\hat{p}$ is a sample mean! So the CLT applies to proportions as well!

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad\qquad x_i = \{0, 1\}$$

- Typically, $x_i = 1$ is read as “success” and $x_i = 0$ as “failure”, so p is the population-level probability of success

CLT for proportions

CLT for sample proportions: if we have n independent binary observations with $np \geq 10$ and $n(1-p) \geq 10$, then:

$$\hat{p} \dot{\sim} N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

- What do the conditions $np \geq 10$ and $n(1-p) \geq 10$ mean?
- For this reason, this is called the **"success-failure" condition** for CLT for proportions

M&M's example

Mars, Inc. is the company that makes M&M's. In 2008, Mars changed their color distribution to have 13% red candies.

Let p be the proportion of red M&M's in a random sample of n M&M's. What is the distribution of $\hat{p}$ if we take random sample of size:

- $n = 100$
- $n = 10$

M&M's example: solution

We have independence due to the random sample. Need to check success-failure condition:

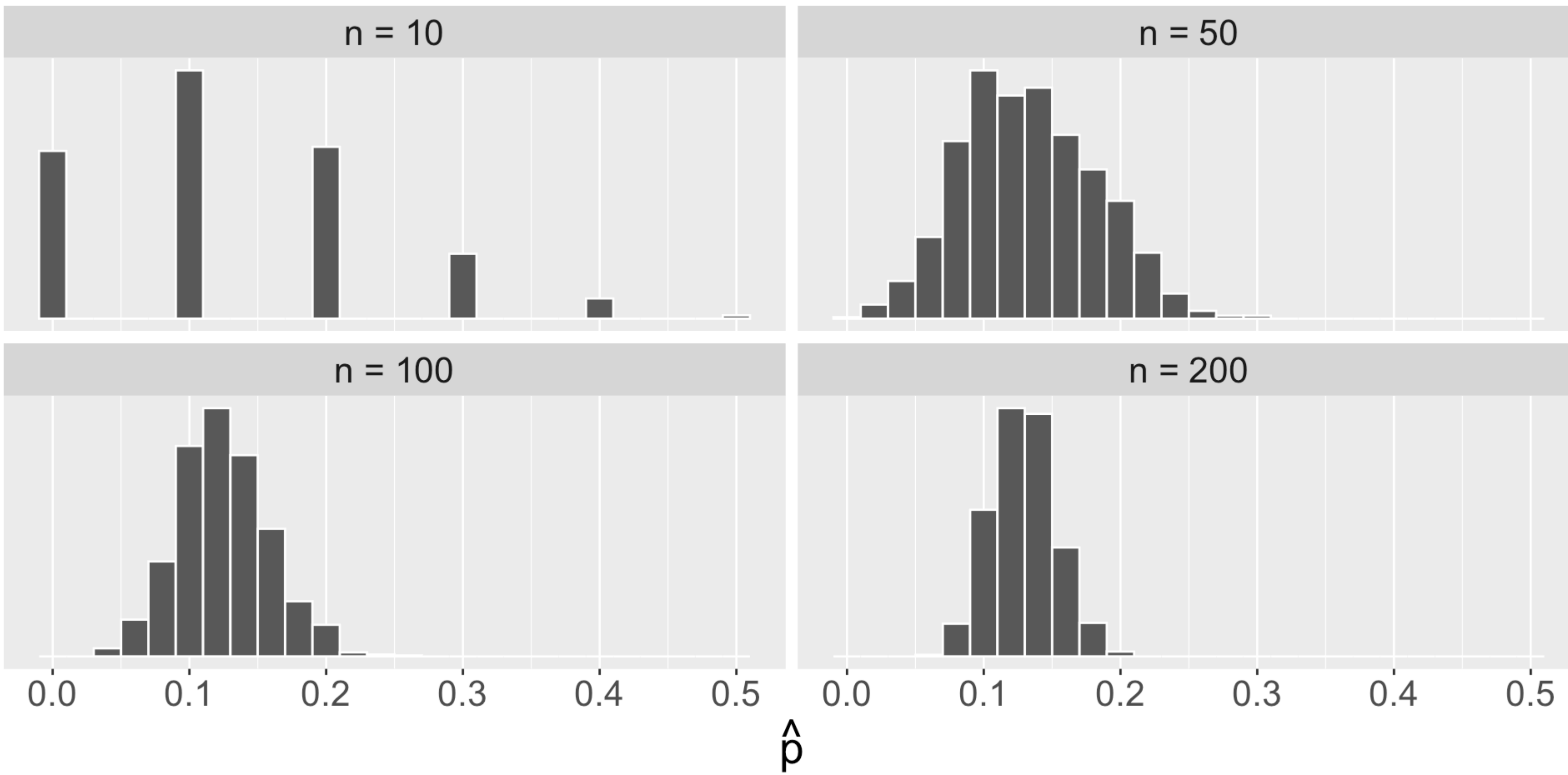
- **If $n = 100$:**
  - $np = 100(0.13) = 13 \geq 10$
  - $n(1-p) = 100(0.87) = 87 \geq 10$
- So CLT applies:

$$\hat{p} \dot{\sim} N\left(0.13, \sqrt{\frac{0.13(1-0.13)}{100}}\right)$$
$$= N(0.13, 0.034)$$

- **If $n = 10$:**
  - $np = 10(0.13) = 1.3 < 10$
- Success-failure condition not met. Cannot use CLT.

M&M's example (cont.)

The following histograms display sampling distributions for $\hat{p}$ = proportion of red candies in random samples of size $n = \{10, 50, 100, 200\}$:



Why is CLT so important?

1. Allows statisticians safely assume that the mean's sampling distribution is approximately Normal. The Normal distribution has nice properties and is easy to work with.
2. Can be applied to both continuous and discrete numeric data!
3. Does not depend on the underlying distribution of the data.

- For many of these reasons, we can use the CLT for inference!

Confidence Intervals via CLT

Mathematical CIs

- Rather than using simulation techniques (i.e. bootstrap) to obtain the sampling distribution, the CLT gives us the sampling distribution of a mean “for free”
  - (assuming conditions are met)
- Formula for a (symmetric) $\gamma \times 100\%$ confidence interval:

$$\text{point estimate} \pm \underbrace{\text{critical value} \times \text{SE}}_{\text{Margin of Error}}$$

1. **point estimate**: the “best guess” statistic from our observed data (e.g. $\hat{p}$ and $\bar{x}$)
2. **SE**: standard error of the statistic
3. **critical value**: percentile that guarantees the $\gamma \times 100$. This will vary depending on your data/assumptions

Towards a CI for a single proportion

Suppose that I have a sample of n binary (0/1) values. I want a $\gamma \times 100\%$ confidence interval for the probability of success p using the sample.

<u>**If**</u> assumptions of CLT for sample proportions hold, then we know

$$\hat{p} \dot{\sim} N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

- We can use/manipulate this result to obtain a confidence interval for the unknown p!
- How do we know if success-failure condition holds without knowing p?
  - Let's use our best guess: $\hat{p}$
  - Success-failure condition *for inference*: $n\hat{p}$ and $n(1-\hat{p})$ both ≥ 10

Towards a CI for a single proportion (cont.)

1. Point estimate: observed $\hat{p}$ from our sample
2. Standard error: $\sqrt{p(1-p)/n}$
    - But we still don't have p!
    - Instead, use the following approximation for CI:

$$\mathrm{SE}(\hat{p}) \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

3. Critical value: to obtain the middle $\gamma \times 100\%$ part, we use the $(1-\gamma)/2$ and $(1+\gamma)/2$ percentiles of the $N(0,1)$ distribution
    - $z^*_{(1-\gamma)/2}$ (lower bound) and $z^*_{(1+\gamma)/2}$ (upper bound)
    - Note: $z^*_{(1+\gamma)/2} = -z^*_{(1-\gamma)/2}$

CI for single proportion

So the formula for a (symmetric) $\gamma \times 100\%$ CI for $\hat{p}$ is:

$$\hat{p} \pm z^*_{(1+\gamma)/2} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where the critical value is obtained from $N(0, 1)$ distribution

Come take STAT 311 to see why this is our CI!

Example

A poll of 100 randomly sampled registered voters in a town was conducted, asking voters if they support legalized marijuana. It was found that 60% of respondents were in support.

What is the population parameter? What is the point estimate/statistic?

Find a (symmetric) 90% confidence interval for the true proportion of town residents in favor of legalized marijuana.

- Conditions for CLT met?
  - Independence: random sample
  - Success-failure condition: $n\hat{p} = 100(0.6) = 60 \geq 10$ and $n(1-\hat{p}) = 100(0.4) = 40 \geq 10$

Example (cont.)

Find a (symmetric) 90% confidence interval for the true proportion of town residents in favor of legalized marijuana.

Gathering components for CI:

1. Point estimate: $\hat{p} = 0.6$
2. Standard error: $\mathrm{SE}(\hat{p}) \approx \sqrt{\frac{0.6(0.4)}{100}} \approx 0.049$
3. Critical value: what percentiles do we want?
   - $z^*_{0.95} =$ `qnorm(0.95, mean = 0, sd = 1)` ≈ 1.645

So our 90% confidence interval for p is:

$$0.6 \pm 1.645(0.049) = (0.519, 0.681)$$

Interpret the confidence interval in context!

Comprehension questions

- What is the main takeaway of the CLT?
- What are the assumptions of the CLT?
- How do we construct a $\gamma \times 100\%$ confidence interval using a mathematical model?