# Introduction to Multiple Linear Regression

11/18/24

# Housekeeping

- Study for midterm!

# Birth weight data

Baystate Medical Center, Springfield, MA during 1986 on the birth weights of 189 babies, along with descriptive information about the mother

- Want to understand risk factors for a baby's birth weight (`bwt`)
- Homework 8 explores the effect of mother's `smoke` status on birth weight of baby
- Let's look at a different variable: `race` of mother
  - Variable `race` numerical where 1 = white, 2 = black, 3 = other

```
'data.frame':	189 obs. of  10 variables:
 $ low  : int  0 0 0 0 0 0 0 0 0 0 ...
 $ age  : int  19 33 20 21 18 21 22 17 29 26 ...
 $ lwt  : int  182 155 105 108 107 124 118 103 123 113 ...
 $ race : int  2 3 1 1 1 3 1 3 1 1 ...
 $ smoke: chr  "no" "no" "yes" "yes" ...
 $ ptl  : int  0 0 0 0 0 0 0 0 0 0 ...
 $ ht   : int  0 0 0 0 0 0 0 0 0 0 ...
 $ ui   : int  1 0 0 1 1 0 0 0 0 0 ...
 $ ftv  : int  0 3 1 2 0 0 1 1 1 0 ...
 $ bwt  : int  2523 2551 2557 2594 2600 2622 2637 2637 2663 2665 ...
```

# Converting to factor

We need to convert variable `race` to categorical! Does not make sense to do "math" on variable:

```r
birthwt2 <- birthwt |>
  mutate(race = case_when(race == 1 ~ "white",
                          race == 2 ~ "black",
                          race == 3 ~ "other")) |>
  mutate(race = factor(race, levels = c("white", "black", "other")))

str(birthwt2)
```

```
'data.frame':	189 obs. of  10 variables:
 $ low  : int  0 0 0 0 0 0 0 0 0 0 ...
 $ age  : int  19 33 20 21 18 21 22 17 29 26 ...
 $ lwt  : int  182 155 105 108 107 124 118 103 123 113 ...
 $ race : Factor w/ 3 levels "white","black",..: 2 3 1 1 1 3 1 3 1 1 ...
 $ smoke: chr  "no" "no" "yes" "yes" ...
 $ ptl  : int  0 0 0 0 0 0 0 0 0 0 ...
 $ ht   : int  0 0 0 0 0 0 0 0 0 0 ...
 $ ui   : int  1 0 0 1 1 0 0 0 0 0 ...
 $ ftv  : int  0 3 1 2 0 0 1 1 1 0 ...
 $ bwt  : int  2523 2551 2557 2594 2600 2622 2637 2637 2663 2665 ...
```

# Fit model

```r
bwt_lm <- lm(bwt ~ race, data = birthwt2)
tidy(bwt_lm)
```

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 3102.7188 | 72.92298 | 42.547890 | 0.0000000 |
| raceblack | -383.0264 | 157.96382 | -2.424773 | 0.0162741 |
| raceother | -297.4352 | 113.74198 | -2.614999 | 0.0096546 |

Fitted model:

$$\widehat{\text{birth\_wt}} = 3102.72 - 383.03\text{raceBlack} - 297.44\text{raceOther}$$

$$\text{raceBlack} = \begin{cases} 1 & \text{if race} = \text{Black} \\ 0 & \text{otherwise} \end{cases} \qquad \text{raceOther} = \begin{cases} 1 & \text{if race} = \text{Other} \\ 0 & \text{otherwise} \end{cases}$$

Estimate the birth weight for babies whose mothers are White

# Interpreting coefficients

$$\widehat{\text{birth\_wt}} = 3102.72 + -383.03\text{raceBlack} + -297.44\text{raceOther}$$

- $\widehat{\text{birth\_wt}} = 3102.72 - 383.03 \times 0 - 297.44 \times 0$
- The estimated birth weight of babies whose mothers are White is 3102.72 grams
- More generally: $b_0$ is the estimated value of the response variable for the base level
- What is the interpretation of $b_1$ = -383.03? Of $b_2$ = -297.44?
  - Babies whose mothers are Black have an estimated birth weight about 383.03 grams less than babies whose mothers are White
  - Babies whose mothers are race "Other" (i.e. not Black or White) have an estimated birth weight about 297.44 grams less than babies whose mothers are White

# General interpretation

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 3102.7188 | 72.92298 | 42.547890 | 0.0000000 |
| raceblack | -383.0264 | 157.96382 | -2.424773 | 0.0162741 |
| raceother | -297.4352 | 113.74198 | -2.614999 | 0.0096546 |

- When fitting a regression model with a categorical variable with $k > 2$ levels, the software will always provide a coefficient for $k - 1$ of the levels
  - The base level does not receive a coefficient
  - Interpretation of the coefficient associated with a non-base level is the expected change in the response **relative to** the base level
- Note: the fitted model has more than one "slope" coefficient, but the `race` variable is still a single explanatory variable
- What happens if we explicitly want to include more than one explanatory variable?

# Multiple linear regression

# Multiple linear regression

- We have seen *simple* linear regression, where we had one explanatory variable
- Extend to include multiple explanatory variables
  - Seems natural: usually several factors affect behavior of phenomena
- **Multiple linear regression** takes the form:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p + \epsilon$$

  - Now there are $p$ different explanatory variables $x_1, \ldots, x_p$ per observation
  - Still one response $y$ and error $\epsilon$ per observation
- Represents a holistic approach for modeling all of the variables simultaneously

# Birthweight data (cont.)

Suppose we would also like to include the mother's age (`age`) and weight at last period (`lwt`) into the model:

$$\text{birth\_wt} = \beta_0 + \beta_1 \text{raceBlack} + \beta_2 \text{raceOther} + \beta_3 \text{age} + \beta_4 \text{lwt} + \epsilon$$

- Just as in the case of SLR, the estimates of $\beta_0, \ldots, \beta_4$ parameters are chosen via the squared deviation criterion

# Multiple regression in R

Very easy to code:

```r
bwt_mlr <- lm(bwt ~ race + age + lwt, data = birthwt2)
```

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 2461.147482 | 314.722327 | 7.8200600 | 0.0000000 |
| raceblack | -447.614691 | 161.369310 | -2.7738527 | 0.0061108 |
| raceother | -239.356515 | 115.188920 | -2.0779474 | 0.0391022 |
| age | 1.298831 | 10.107701 | 0.1284991 | 0.8978943 |
| lwt | 4.619545 | 1.787729 | 2.5840294 | 0.0105407 |

- Simply identify the estimated coefficients from the output to obtain fitted model

$$\begin{aligned}\widehat{\text{birth\_wt}} = 2461.15 &- 447.61\text{raceBlack} - 239.36\text{raceOther} + 1.3\text{age} \\ &+ 4.62\text{lwt}\end{aligned}$$

- Note that the number of explanatory variables need not equal the number of parameters in the model!

# Interpretation

- When we have more than one predictor variable, interpretation of the coefficients requires a bit of care
  - Multiple moving parts
- Interpretation of a particular coefficient $b_m$ relies on "holding the other variables fixed/constant" (assuming the model is appropriate)

$$\begin{aligned}\widehat{\text{birth\_wt}} = 2461.15 &- 447.61\text{raceBlack} - 239.36\text{raceOther} + 1.3\text{age} \\ &+ 4.62\text{lwt}\end{aligned}$$

- For every one year older the mother is, the baby's birth weight is expected to increase by $1.3$ grams, **holding all other variables constant**
- Interpret the coefficient associated with the mother's weight (`lwt`)

# Interpretation (cont.)

$$\begin{aligned}\widehat{\text{birth\_wt}} = 2461.15 - 447.61\text{raceBlack} - 239.36\text{raceOther} + 1.3\text{age}\\ + 4.62\text{lwt}\end{aligned}$$

- For every one pound heavier the mother's weight at last period was, the baby's birth weight is expected to increase by $4.62$ grams, holding all other variables constant

# More isn’t always better

- You might be tempted to throw in all available predictors into your model! Don’t fall into temptation!
- Quality over quantity
- For SLR, we used the coefficient of determination $R^2$ to assess how good the model was
  - $R^2$ is less helpful when there are many variables
  - Why? The $R^2$ will never decrease (and will *almost always* increase) when we include an additional predictor

# Adjusted $R^2$

- For multiple linear regression, we use the **adjusted** $R^2$ to assess the quality of model fit
  - "Adjusted" for the presence of additional predictors
  - Take STAT 211 to learn the formula and intuition behind it!
- Adjusted $R^2$ is always less than $R^2$

# Adjusted $R^2$ (cont.)

```r
summary(bwt_mlr)
```

```
Call:
lm(formula = bwt ~ race + age + lwt, data = birthwt2)

Residuals:
     Min       1Q   Median       3Q      Max 
-2103.50  -429.68    41.74   486.10  1902.20 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 2461.147    314.722   7.820 3.97e-13 ***
raceblack   -447.615    161.369  -2.774  0.00611 ** 
raceother   -239.357    115.189  -2.078  0.03910 *  
age            1.299     10.108   0.128  0.89789    
lwt            4.620      1.788   2.584  0.01054 *  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 704.9 on 184 degrees of freedom
Multiple R-squared:  0.08536,	Adjusted R-squared:  0.06548 
F-statistic: 4.293 on 4 and 184 DF,  p-value: 0.00241
```

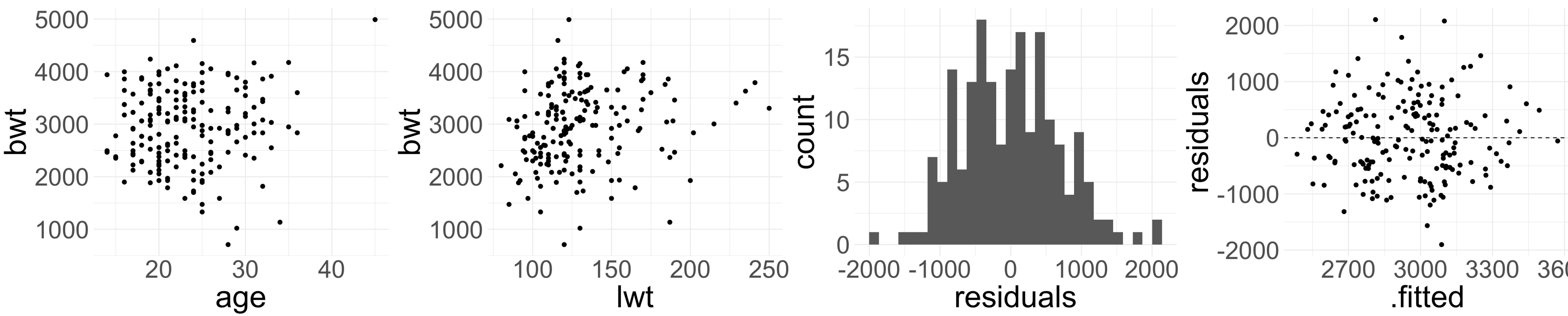
```r
glance(bwt_mlr)
```

| r.squared | adj.r.squared | sigma | st |
|---|---|---|---|
| 0.0854 | 0.0655 | 704.9368 | |

# Conditions for inference

We still need LINE to hold

- **Linearity**: harder to assess now that multiple predictors are involved. Good idea to make several scatter plots
- **Independence**: same as before
- **Nearly normal residuals**: same as before
- **Equal variance**: residual plot has *fitted* values on the x-axis, instead of an explanatory variable



# Inference in MLR

# Hypothesis testing in MLR

- In MLR, we are interested in the effect of a variable $m$ on the response $y$.
  - Need to account for presence of other predictors in the model
- $H_0 \,:\, \beta_m = 0$, given other predictors in the model
- $H_A \,:\, \beta_m \neq 0$, given other predictors in the model (or $>$, $<$)
- We can write down one null hypothesis for each coefficient in the model

# Hypothesis tests from `lm()`

Returning to the larger model:

$$\text{birth\_wt} = \beta_0 + \beta_1 \text{raceBlack} + \beta_2 \text{raceOther} + \beta_3 \text{age} + \beta_4 \text{lwt} + \epsilon$$

- We can test the following null hypotheses (no need to write down):
  - $H_0 \ : \beta_1 \ = 0$, given `age` and `lwt` are included in the model
  - $H_0 \ : \beta_2 \ = 0$, given `age` and `lwt` are included in the model
  - $H_0 \ : \beta_3 \ = 0$, given `race` and `lwt` are included in the model
  - $H_0 \ : \beta_4 \ = 0$, given `race` and `age` are included in the model

# Hypothesis tests from `lm()`

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 2461.1475 | 314.7223 | 7.8201 | 0.0000 |
| raceblack | -447.6147 | 161.3693 | -2.7739 | 0.0061 |
| raceother | -239.3565 | 115.1889 | -2.0779 | 0.0391 |
| age | 1.2988 | 10.1077 | 0.1285 | 0.8979 |
| lwt | 4.6195 | 1.7877 | 2.5840 | 0.0105 |

- Output from `lm()` provides:
  - Test statistic, which follows $t_{n-p}$ where $p$ = total number of unknown parameters (i.e. $\beta$ terms)
  - p-values for testing two-sided $H_A$ provided

Based on the model fit, which variables seem to be important predictors of birth weight of a baby? Why?

# Hypothesis tests from `lm()` (cont.)

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 2461.1475 | 314.7223 | 7.8201 | 0.0000 |
| raceblack | -447.6147 | 161.3693 | -2.7739 | 0.0061 |
| raceother | -239.3565 | 115.1889 | -2.0779 | 0.0391 |
| age | 1.2988 | 10.1077 | 0.1285 | 0.8979 |
| lwt | 4.6195 | 1.7877 | 2.5840 | 0.0105 |

- `lwt` **does** seem to be an important predictor for birth weight, despite inclusion of `race` and `age` in the model
  - Low p-value suggests it would be extremely unlikely to see data that produce $b_4 = 4.62$ if the true relationship between `lwt`and `bwt` was non-existent (i.e., if $\beta_4 = 0$) and the model also included `age` and `race`
- `race` **does** seem to be an important predictor, despite inclusion of `age` and `lwt`
- `age` **does not** seem to be an important predictor after including `race` and `lwt`

# Simpler model

Let's see the model that does not include mother's age in the model:

```r
bwt_mlr_no_age <- lm(bwt ~ race + lwt, data = birthwt2)
tidy(bwt_mlr_no_age)
```

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 2486.9039 | 241.9933 | 10.2767 | 0.0000 |
| raceblack | -451.8381 | 157.5662 | -2.8676 | 0.0046 |
| raceother | -241.3008 | 113.8869 | -2.1188 | 0.0354 |
| lwt | 4.6634 | 1.7501 | 2.6646 | 0.0084 |

Write out the fitted model. Interpret the intercept and the coefficient for `lwt` in context

# Simpler model (cont.)

| term | estimate | std.error | statistic | p.value |
|---|---|---|---|---|
| (Intercept) | 2486.9039 | 241.9933 | 10.2767 | 0.0000 |
| raceblack | -451.8381 | 157.5662 | -2.8676 | 0.0046 |
| raceother | -241.3008 | 113.8869 | -2.1188 | 0.0354 |
| lwt | 4.6634 | 1.7501 | 2.6646 | 0.0084 |

- Intercept: the birth weight of babies whose mothers are White and weigh 0 lbs have an estimated birth weight of 2486.9 grams
- Coefficient for `lwt`: for every one pound increase in the mother's weight at last period, the birth weight of the baby is expected to increase by 4.66 grams, holding all other variables (i.e. `race`) constant

# Comparing models

Let's compare the model that includes age to the model without age:

```r
tidy(bwt_mlr) |>
  select(term, estimate, p.value)
```

| term | estimate | p.value |
|---|---|---|
| (Intercept) | 2461.1475 | 0.0000 |
| raceblack | -447.6147 | 0.0061 |
| raceother | -239.3565 | 0.0391 |
| age | 1.2988 | 0.8979 |
| lwt | 4.6195 | 0.0105 |

```r
glance(bwt_mlr)
```

| r.squared | adj.r.squared | sigma | statistic |
|---|---|---|---|
| 0.0854 | 0.0655 | 704.9368 | 4.293 |

```r
tidy(bwt_mlr_no_age) |>
  select(term, estimate, p.value)
```

| term | estimate | p.value |
|---|---|---|
| (Intercept) | 2486.9039 | 0.0000 |
| raceblack | -451.8381 | 0.0046 |
| raceother | -241.3008 | 0.0354 |
| lwt | 4.6634 | 0.0084 |

```r
glance(bwt_mlr_no_age)
```

| r.squared | adj.r.squared | sigma | statistic |
|---|---|---|---|
| 0.0853 | 0.0704 | 703.0605 | 5.7491 |

What do you notice about the estimated coefficients, $R^2$, and adjusted $R^2$?

# Remarks

- We have only scratched the surface of MLR
- Things to consider:
  - Multicollinearity (when the predictor variables are correlated with each other)
  - Model selection
  - More than one categorical variable
  - Interaction effects