

Probability basics

Part 1

9/30/24

Probability basics

We spend a whole semester on this in MATH/STAT 310!

Key terms

- **Random process:** a situation in which a particular result, called an **outcome**, is random/not known ahead of time
 - Examples: flipping a coin, rolling six-sided die, sports game, if a treatment is effective
- A **sample space** S is the set of all possible outcomes of the random process
 - What are possible sample spaces for the above examples?
- An **event** is a set of outcomes from a random process

Random variable

- A **random variable** is a variable whose value is unknown and depends on random events
 - Often denoted with a capital letter like X or Y
- There are two types: discrete and continuous (just like in numeric variables)
 - **Discrete**: represents random process where sample space is countable (i.e. finite, or distinct counts)
 - **Continuous**: sample space is uncountable (i.e. can take on any value within a specified interval with infinite number of possible values)
- **NOTE**: we will focus on *discrete* random variables for now

Probability

- For us, the **probability** of an outcome is the proportion of times the outcome would occur if we observed the random process an infinite number of times
 - Probability is used to express the likelihood that some outcome or event will or will not occur
 - Think of as a proportion
- Let A denote some outcome or event. We denote the probability of A occurring as $P(A)$ or $\Pr(A)$.
- When the sample space S is discrete with a finite size, then
$$\Pr(A) = \frac{\text{number of outcomes favorable to } A}{\text{number of total outcomes possible}} = \frac{\text{number of outcomes favorable to } A}{\text{size of } S}$$

Example

Let the random process rolling a fair, six-sided die. Let X a random variable representing the value of the die.

- For each of the following, determine the outcome(s) and event under consideration, along with the value of the probability itself:
 - $\Pr(X = 1)$
 - $\Pr(X = 1 \text{ and } 2)$
 - $\Pr(X \text{ is even})$

Operations with events

Let A and B be two possible events.

- The **intersection** of A and B is denoted as $A \cap B$, and is the set of outcomes that belong to *both* events A and B
- The **union** of A and B is denoted as $A \cup B$, and is the set of outcomes that belong to A and/or B

When we have only two or three events, Venn diagrams can be very useful for visualizing probabilities!

Disjoint events

Two events are **disjoint** or **mutually exclusive** if they cannot simultaneously happen.

- That is, if A and B are disjoint, then $\Pr(A \cap B) = ?$
- If our random process is rolling a six-sided die one time, what are some examples of disjoint events?

Rules of probability

Kolmogorov axioms

1. The probability of any event is non-negative real number
2. The probability of the entire sample space 1
3. If A and B are disjoint, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

These axioms imply that all probabilities are between 0 and 1 inclusive, and lead to some important rules!

Addition rule

Let A and B be two possible events. Then the **addition rule** states that the probability that at least one will occur is:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Venn diagram
- Example: in a standard deck of 52 cards, we have four suits (diamond, heart, club, spade) with 13 cards within each suit (1-10, Jack, Queen, King).
 - Suppose we randomly draw one card from the shuffled deck.
 - Let A be the event that the card is a spade.
 - Let B be the event that the card is a face card (Jack, Queen or King).
 - Find $P(A \cup B)$.

Complement

- The **complement** of an event A is the set of all outcomes in S that are not in A
 - Denoted as A^c
- Continuing the dice example, if A is the event that a 1 or 2 is rolled, what is A^c ?
- **Complement rule:** $\Pr(A^c) = 1 - \Pr(A)$
- Let our random process be the sum of two dice. What is the probability that...
 - the sum of the dice is *not* 6?
 - the sum is *at least* 4?

DeMorgan's Laws

Let's use Venn diagrams to try and determine formulas for the following:

1. Complement of union: $(A \cup B)^c = ?$
2. Complement of intersection: $(A \cap B)^c = ?$

Independence

- Qualitatively, two processes are **independent** if knowing the outcome of one does not provide any information about the outcome of the other process
 - Examples and non-examples? How to formalize this?
- If A and B are independent events from two different and independent processes, then $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
- More generally, if $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ and A and B are events from the same process, then A and B are independent.
 - This is known as the **multiplication rule for independent events**

Probability distributions

When a random variable is discrete, it can be useful to discuss its **probability distribution**, which is a table of all (disjoint) outcomes and their associated probabilities.

- Let X be the sum of two fair, six-sided dice. What is the sample space S ?
- Fill out the table below to display the probability distribution of X :

X	2	3	4	5	6	7
Probability						
X	8	9	10	11	12	
Probability						

Probability distributions (cont.)

The probability distribution of a discrete random variable must satisfy the following three rules:

1. The outcomes listed must be disjoint
2. Each probability must be between 0 and 1 (inclusive)
3. The probabilities must sum to 1

Let's confirm that the distribution we found on the previous slide satisfies these rules!

Practice

A Pew Research survey asked 2,373 randomly sampled registered voters their political affiliation (Republican, Democrat, or Independent) and whether or not they identify as swing voters. 35% of respondents identified as Independent, 23% identified as swing voters, and 11% identified as both.

- a. What percent of voters are Independent but not swing voters?
- b. What percent of voters are Independent or swing voters?
- c. What percent of voters are neither Independent nor swing voters?
- d. Is the event that someone is a swing voter independent of the event that someone is a political Independent?