Introduction to Multiple Linear Regression

4/28/25

Housekeeping

- Study for midterm!
- Today’s content will not be assessed on midterm, but might be useful for your final project and future coursework!

Voice shimmer and jitter data

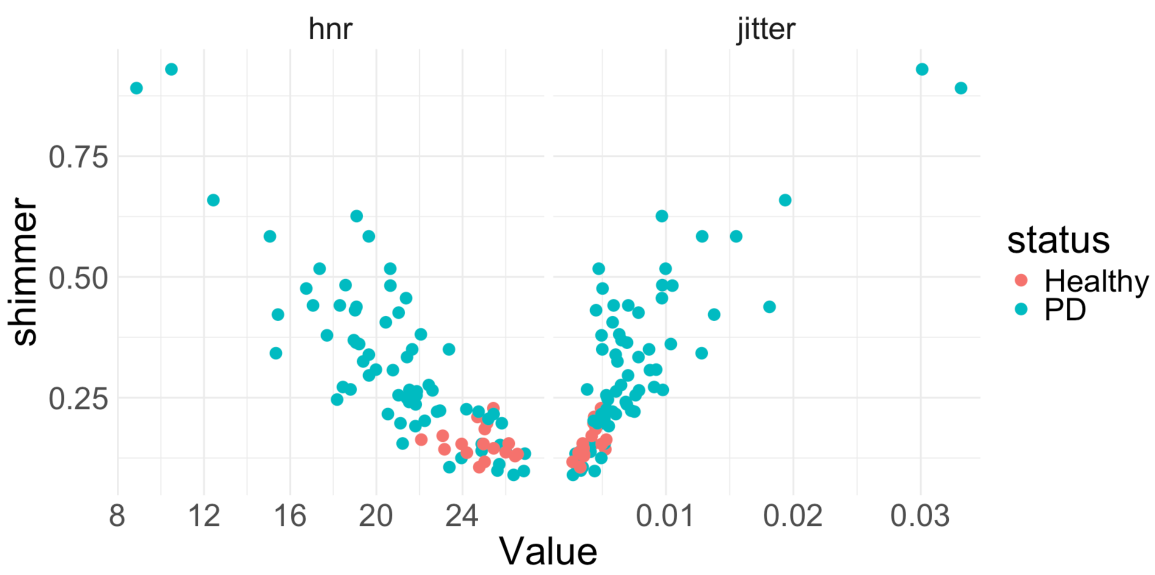
Recall the data from a previous problem set about voice jitter and shimmer among patients with and without Parkinson's Disease (PD).

The variables in the dataset are as follows:

- `clip`: ID of the recording
- `jitter`: a measure of variation in fundamental frequency
- `shimmer`: a measure of variation in amplitude
- `hnr`: a ratio of total components vs. noise in the voice recording
- `status`: PD vs. Healthy
- `avg.f.q`: 1, 2, or 3, corresponding to average vocal fundamental frequency (1 = low, 2 = mid, 3 = high)

Analysis goal

Want to understand what might help explain the voice `shimmer` of a patient with low vocal fundamental frequency.



- What do you notice about how `shimmer` relates to `hnr`, `jitter`, and `status`?
- Can we somehow incorporate all the predictors into the same model for `shimmer`? Do you think we need to?

Multiple linear regression

Multiple linear regression

- We have seen *simple* linear regression, where we had one explanatory variable
- Extend to include multiple explanatory variables
  - Seems natural: usually several factors affect behavior of phenomena
- **Multiple linear regression** takes the form:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p + \epsilon$$

  - Now there are p different explanatory variables $x_1, \ldots, x_p$ per observation
  - Still one response y and error ϵ per observation
- Represents a holistic approach for modeling all of the variables simultaneously

PD data (cont.)

Let's start off by building a model that does not include `status`, as the EDA didn't seem to show a strong relationship between `status` and `shimmer`.

- Our multiple linear regression model is:

$$\text{shimmer} = \beta_0 + \beta_1 \text{hnr} + \beta_2 \text{jitter} + \epsilon$$

- Just as in the case of SLR, the estimates of $\beta_0, \beta_1, \beta_2$ parameters are chosen via the least squares criterion

Multiple regression in R

Very easy to code:

```r
shimmer_lm <- lm(shimmer ~ hnr + jitter, data = pd)
tidy(shimmer_lm)
```

term	estimate	std.error	statistic	p.value
(Intercept)	0.732	0.091	8.022	0.0e+00
hnr	-0.025	0.004	-7.066	0.0e+00
jitter	13.467	2.574	5.232	1.2e-06

- Simply identify the estimated coefficients from the output to obtain fitted model
- Try writing down the fitted model

$$\widehat{\text{shimmer}} = 0.732 - 0.025\text{hnr} + 13.467\text{jitter}$$

Interpretation of intercept

- Interpretation of the estimated intercept b_0 in MLR is very similar to SLR!

$$\widehat{\text{shimmer}} = 0.732 - 0.025\text{hnr} + 13.467\text{jitter}$$

- Try interpreting the intercept!
- We simply plug in 0 for all the explanatory variables
  - The estimated voice shimmer of a patient with 0 hnr and 0 voice jitter is 0.732.

Interpretation of non-intercept

- When we have more than one predictor variable, interpretation of the coefficients requires a bit of care
  - Multiple moving parts
- Interpretation of a particular non-intercept coefficient b_k relies on "holding the other variables fixed/constant" (assuming the model is appropriate)

$$\widehat{\text{shimmer}} = 0.732 - 0.025\text{hnr} + 13.467\text{jitter}$$

- For every one unit increase in a person's HNR, their voice shimmer is expected/estimated to $\text{decrease by } 0.025$, **holding their voice jitter value constant**
- Interpret the coefficient associated with jitter

Interpretation on non-intercept (cont.)

$$\widehat{\text{shimmer}} = 0.732 - 0.025\text{hnr}13.467\text{jitter}$$

- For every one unit increase in a patient's voice jitter, their voice shimmer is expected to $\text{increase by } 13.467$ units, holding their HNR value constant

More isn't always better

- You might be tempted to throw in all available predictors into your model! Don't fall into temptation!
- Quality over quantity
- For SLR, we used the coefficient of determination R^2 to assess how good the model was
  - R^2 is less helpful when there are many variables
  - Why? The R^2 will never decrease (and will *almost always* increase) when we include an additional predictor

Adjusted R^2

- For multiple linear regression, we use the **adjusted** R^2 to assess the quality of model fit
  - "Adjusted" for the presence of additional predictors
  - Take STAT 211 to learn the formula and intuition behind it!
- Adjusted R^2 is always less than R^2, and doesn't have a nice interpretation
- When choosing between models, one method is to choose the one with highest adjusted R^2

Adjusted R^2 (cont.)

```r
summary(shimmer_lm)
```

```

Call:
lm(formula = shimmer ~ hnr + jitter, data = pd)

Residuals:
      Min        1Q    Median        3Q       Max 
-0.182276 -0.047886 -0.007739  0.029861  0.236647 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.732203   0.091279   8.022 5.89e-12 ***
hnr         -0.024795   0.003509  -7.066 4.54e-10 ***
jitter      13.466902   2.573728   5.232 1.23e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 
1

Residual standard error: 0.07437 on 83 degrees of freedom
Multiple R-squared:  0.807,	Adjusted R-squared:  0.8024 
F-statistic: 173.5 on 2 and 83 DF,  p-value: < 2.2e-16
```

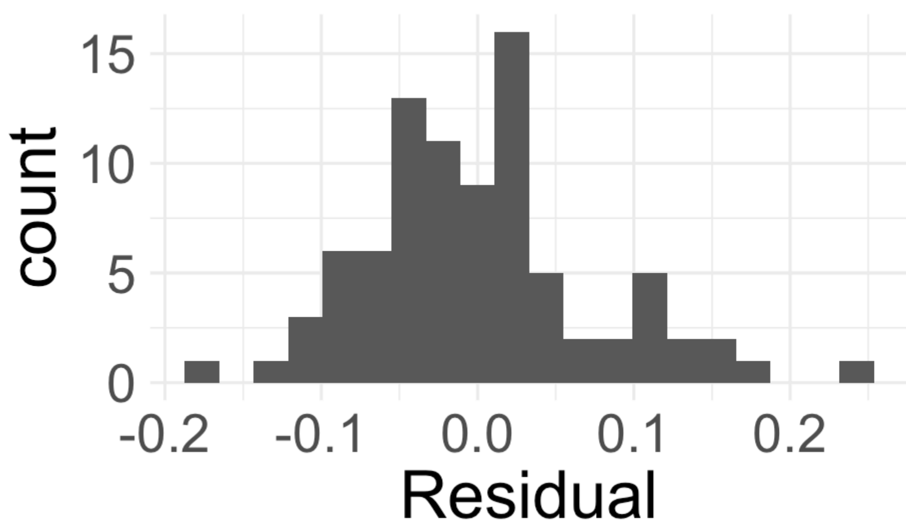
```r
glance(shimmer_lm)
```

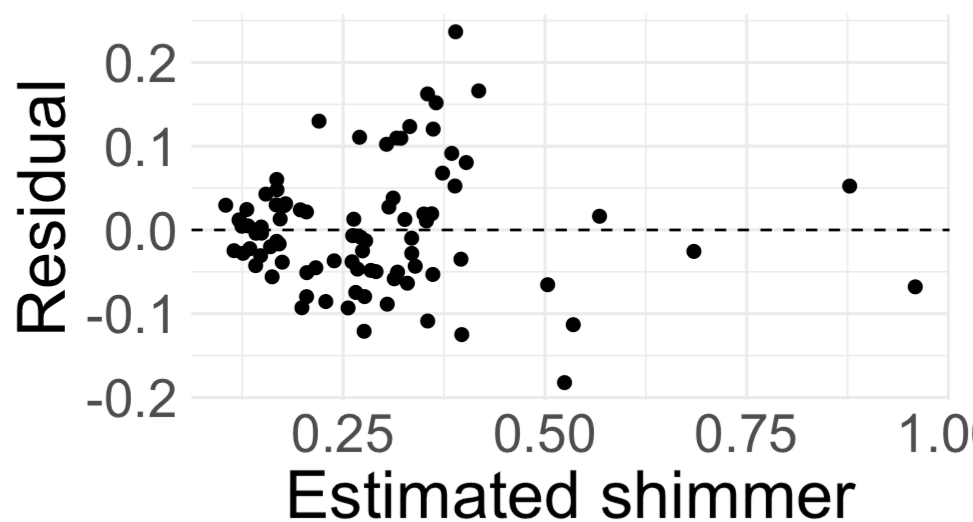
r.squared	adj.r.squared	sigma	statistic
0.807	0.8024	0.0744	173.5385

Conditions for inference

We still need LINE to hold

- **Linearity**: harder to assess now that multiple predictors are involved. Good idea to make several scatter plots
- **Independence**: same as before
- **Nearly normal residuals**: same as before
- **Equal variance**: residual plot has *fitted* values $\hat{y}$ on the x-axis





Inference in MLR

Hypothesis testing in MLR

- In MLR, we are interested in the effect of each predictor variable on the response y.
  - Need to account for presence of other predictors in the model
- $H_0 : \beta_k = 0$, given other predictors in the model
- $H_A : \beta_k \neq 0$, given other predictors in the model (or $>$, $<$)
- We can write down one null hypothesis for each coefficient in the model

Hypothesis tests from `lm()`

$$\text{shimmer} = \beta_0 + \beta_1 \text{hnr} + \beta_2 \text{jitter} + \epsilon$$

We can test the following null hypotheses (no need to write down):

- $H_0 : \beta_1 = 0$, given jitter is included in the model
  - i.e. HNR has no effect on shimmer once we account for jitter
- $H_0 : \beta_2 = 0$, given HNR is included in the model

Hypothesis tests from `lm()`

term	estimate	std.error	statistic	p.value
(Intercept)	0.73	0.091	8.022	0.0e+00
hnr	-0.02	0.004	-7.066	0.0e+00
jitter	13.47	2.574	5.232	1.2e-06

- Output from `lm()` provides:
  - Test statistic, which follows t_{n-p} where $p =$ total number of unknown parameters (i.e. β terms)
  - **p-values for testing two-sided H_A provided**

Based on the model fit, which variables seem to be important predictors of voice `shimmer`? Why?

Hypothesis tests from `lm()` (cont.)

term	estimate	std.error	statistic	p.value
(Intercept)	0.732	0.091	8.022	0.0e+00
hnr	-0.025	0.004	-7.066	0.0e+00
jitter	13.467	2.574	5.232	1.2e-06

- HNR **does** seem to be an important predictor for voice shimmer, even when including jitter in the model
  - Low p-value suggests it would be extremely unlikely to see data that produce $b_1 = -0.025$ if the true relationship between shimmer and HNR was non-existent (i.e., if $\beta_1 = 0$) and the model also included jitter
- Jitter **does** seem to be an important predictor, even when including HNR in the model

More complex model

Let’s see a model that now includes the `status` of the patient as a predictor:

```
shimmer_lm2 <- lm(shimmer ~ hnr + jitter + status, data = pd)
tidy(shimmer_lm2)
```

term	estimate	std.error	statistic	p.value
(Intercept)	0.688	0.103	6.668	0.0000000
hnr	-0.024	0.004	-6.273	0.0000000
jitter	13.662	2.585	5.285	0.0000010
statusPD	0.020	0.022	0.915	0.3628131

- Remember, `status` is categorical with two levels. `lm()` converted to indicator variable for us: `statusPD = 1` when `status = "PD"`

Write out the fitted model.

Interpretation with categorical variable

$$\widehat{\text{shimmer}} = 0.688 - 0.024\text{hnr} + 13.662\text{jitter} + 0.02\text{statusPD}$$

- Try interpreting the intercept here!
- What does it mean for the explanatory variables to be 0? It means `hnr = 0`, `jitter = 0`, and the patient does not have `PD`
  - A "healthy" patient with HNR and jitter values of 0 is estimated to have a voice shimmer of 0.688

Interpretation of slope coefficients

$$\widehat{shimmer} = 0.688 - 0.024\text{hnr} + 13.662\text{jitter} + 0.02\text{statusPD}$$

Try interpreting the coefficients of `hnr`, `jitter`, and `statusPD`. Remember the special wording/acknowledgement now that we are in MLR world!

- Coefficient for `hnr`: for every one unit increase in HNR, we expect the patient's shimmer to decrease by 0.024 units, holding the other variables (jitter and status) constant.
- Coefficient for `jitter`: for every one unit increase in jitter, we expect the patient's shimmer to increase by 13.662 units, holding the other variables constant.
- Coefficient for `statusPD`: patients with PD are estimated to have a voice shimmer 0.02 units higher than patients without PD, holding the other variables constant

Effect of `status`

term	estimate	std.error	statistic	p.value
(Intercept)	0.688	0.103	6.668	0.0000000
hnr	-0.024	0.004	-6.273	0.0000000
jitter	13.662	2.585	5.285	0.0000010
statusPD	0.020	0.022	0.915	0.3628131

Based off the model output, does it appear that `status` is an important predictor of a patient’s voice `shimmer`? Why or why not? What about the other two variables `hnr` and `jitter`?

Comparing models

Let's compare the two models:

```
tidy(shimmer_lm) |>
  select(term, estimate, p.value)
```

term	estimate	p.value
(Intercept)	0.732	0e+00
hnr	-0.025	0e+00
jitter	13.467	1e-06

```
glance(shimmer_lm)
```

r.squared	adj.r.squared	sigma	statistic
0.807	0.8024	0.0744	173.5385

```
tidy(shimmer_lm2) |>
  select(term, estimate, p.value)
```

term	estimate	p.value
(Intercept)	0.688	0.000000
hnr	-0.024	0.000000
jitter	13.662	0.000001
statusPD	0.020	0.362813

```
glance(shimmer_lm2)
```

r.squared	adj.r.squared	sigma	statistic
0.809	0.802	0.0744	115.7449

What do you notice about the estimated coefficients, R^2, and adjusted R^2 across the two models?

Remarks

- We have only scratched the surface of MLR
- Things to consider (beyond our course):
  - Multicollinearity (when the predictor variables are correlated with each other)
  - Model selection
  - More than one categorical variable
  - Interaction effects