CIs and HTs for a single mean

CLT-based

4/14/25

Housekeeping

- Dessert social today! 3-4:30pm in WNS 105!
- Modified office hours today: 1:30-2:30pm instead of 2-3pm
- Homework 7 due tonight
- Project proposals due Wednesday night

Recap

- CLT: if we have a sufficiently large sample of n independent observations from a population with mean μ and standard deviation σ, then $\bar{X} \dot{\sim} N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$
- To obtain a $\gamma \times 100\%$ CI via CLT, we use

$$\text{point estimate} \pm \text{critical value} \times \text{SE}$$

  - We may need to replace the standard error with an estimate $\widehat{SE}$

Checking normality

- Remember, CLT requires a sufficiently large sample size n or assumption of Normality of the underlying data.
- No perfect way to check Normality, but rule of thumb:
  - If $n < 30$ small: check that there are no clear outliers
  - If $n \geq 30$ large: check that there are no particularly extreme outliers

CI for a single mean

CI for a single mean (known variance)

Suppose we want a $\gamma \times 100\%$ CI for population mean μ.

- *If* CLT holds, then we know

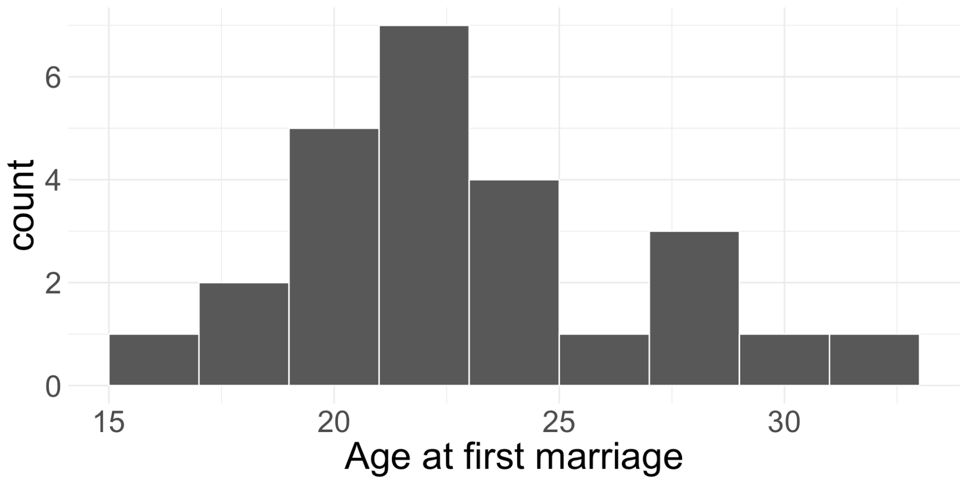
$$\bar{X} \dot{\sim} N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

- So our $\gamma \times 100\%$ CI for μ is:

$$\text{point estimate} \pm \underbrace{\text{critical value} \times \text{SE}}_{\text{Margin of Error}} = \bar{x}_{obs} \pm z^*_{(1+\gamma)/2} \times \frac{\sigma}{\sqrt{n}}$$

Example: age at marriage

In 2006-2010, the CDC conducted a thorough survey asking US women their age at first marriage. Suppose it is known that the standard deviation of the ages at first marriage is 5 years. Suppose we randomly sample 25 US women and ask them their age at first marriage (plotted below). Their average age at marriage was 23.32.



We will obtain an 80% confidence interval for the mean age of US women at first marriage.

- Are conditions of CLT met?
- If so, what does CLT tell us?

What is/are the population parameter(s)? What is the statistic?

Example: age at marriage (cont.)

Obtain an 80% confidence interval for the mean age of US women at first marriage.

- Because we have a random sample (independence) and there are no outliers in the data (normality condition), we can proceed with CLT!

$$\bar{X} \dot{\sim} N\left(\mu, \frac{5}{\sqrt{25}}\right) = N(\mu, 1)$$

- Construct your confidence interval and interpret!

1. Point estimate: $\bar{x}_{obs} = 23.32$
2. Standard error: $\mathrm{SE} = 1$
3. Critical value: $z^*_{0.9} =$ `qnorm(0.9, 0, 1)` $= 1.28$

So our 80% confidence interval is $23.32 \pm 1.28 \times 1 = (22.04, 24.6)$

Utility of this model

- The previous formula for the confidence interval for μ relies on knowing σ
- But wait…
  - Want to construct a CI for μ because we don't know its value
  - If we don't know μ, it seems highly unlikely that we would know σ!
- So in practice, we will have to estimate standard error for $\bar{X}$:

$$\widehat{\mathrm{SE}} = \frac{s}{\sqrt{n}}$$

where s is the observed sample standard deviation

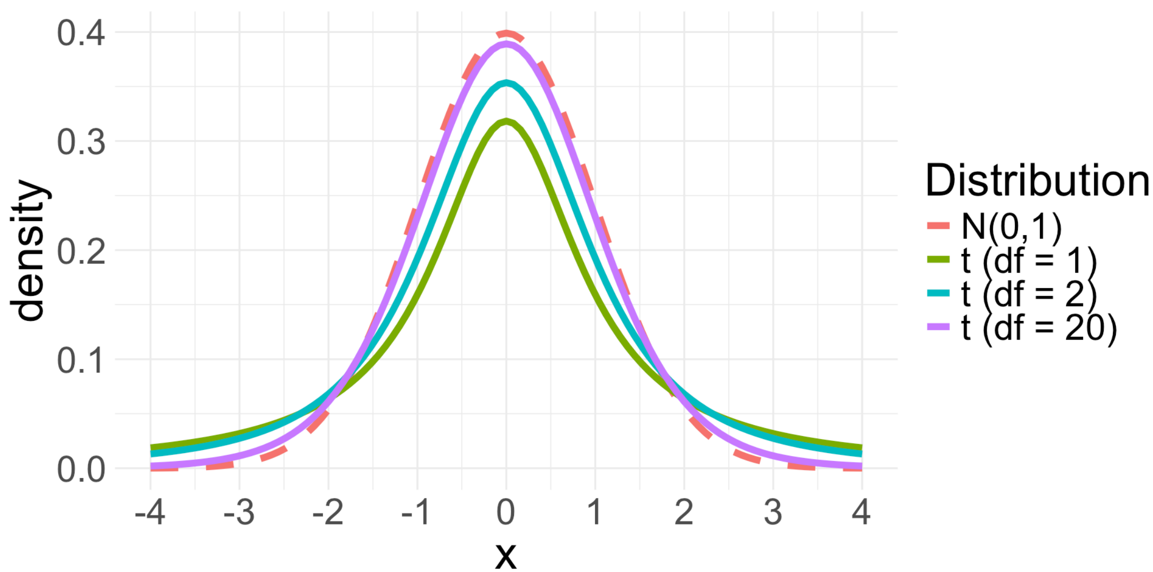
- Recall we did something similar for CI for p, where we replaced p with $\hat{p}_{obs}$

Variance issue

- Estimating variance is extremely difficult when n is small, and still not great for large n
  - Thus, replacing σ with s invalidates CLT
- So if σ is unknown, we ***cannot*** use the Normal approximation to model $\bar{X}$ for inferential tasks
- Instead, we will use a new distribution for inference calculations, called the t-distribution

t-distribution

- The t**-distribution** is symmetric and bell-curved (like the Normal distribution)
- Has "thicker tails" than the Normal distribution (the tails decay more slowly)

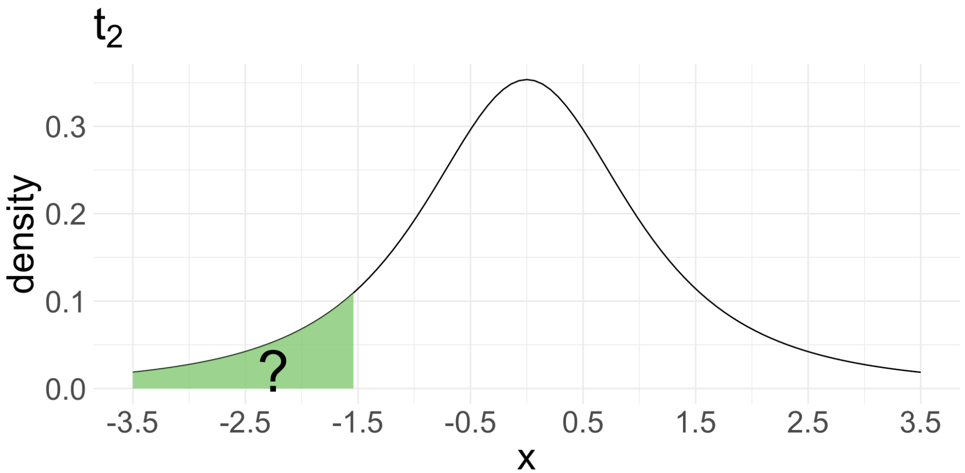


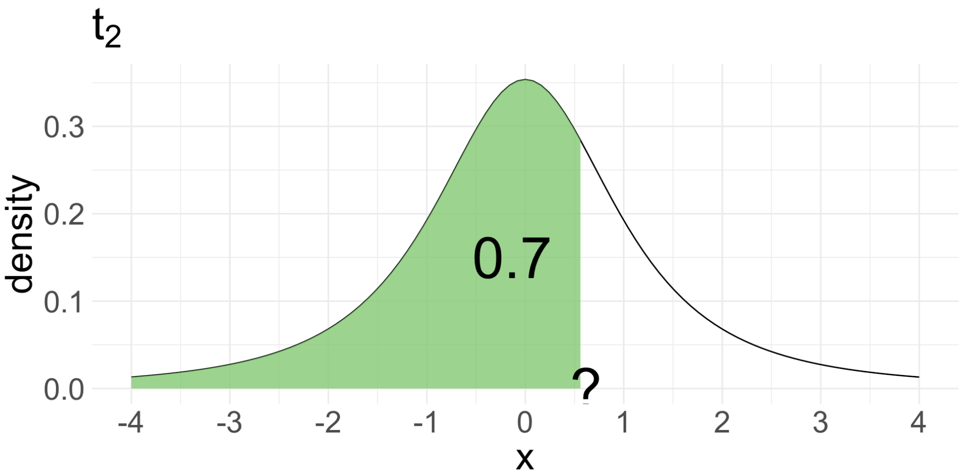
- t-distribution is always centered at 0
- One parameter: **degrees of freedom (df)** defines exact shape of the t
  - Denoted t_{df} (e.g. t_1 or t_{20})

- As df increases, t resembles the $N(0, 1)$. When $df \geq 30$, the t_{df} is nearly identical to $N(0, 1)$

t distribution in R

- `pnorm(x, mean, sd)` and `qnorm(%, mean, sd)` used to find probabilities and percentiles for the Normal distribution
- Analogous functions for t-distribution: `pt(x, df)` and `qt(%, df)`





`pt(-1.5,df =2)` = 0.1361966

`qt(0.7, df =2)` = 0.6172134

CI for a single mean (unknown variance)

- Still require independent observations and the Normality condition for CLT
- General formula for $\gamma \times 100\%$ CI is the same, but we simply change what goes into the margin of error.

$$\text{point estimate} \pm t^*_{df,(1+\gamma)/2} \times \widehat{\text{SE}} = \bar{x}_{obs} \pm t^*_{df,(1+\gamma)/2} \times \frac{s}{\sqrt{n}}$$

- $df = n - 1$ (always for this CI)
- critical value $t^*_{df,(1+\gamma)/2}$ = $(1+\gamma)/2$ percentile of the t_{df} distribution

Example: age at marriage (cont.)

Let's return to the age at marriage example. Once again, obtain an 80% CI for the average age of first marriage for US women, but now suppose we **don't know** σ.

In our sample of $n = 25$ women, we observed a sample mean of 23.32 years and a sample standard deviation of $s = 4.03$ years.

1. Point estimate: $\bar{x}_{obs} = 23.32$
2. Standard error: $\widehat{\text{SE}} = \frac{s}{\sqrt{n}} = \frac{4.03}{\sqrt{25}} = 0.806$
3. Critical value:
   - $df = n - 1 = 24$
   - $t^*_{24,0.9}$ = `qt(0.9, df =24)` = 1.32

So our 80% confidence interval for μ is:

$$23.32 \pm 1.32 \times 0.806 = (22.26, 24.38)$$

Remarks

- Interpretation of CI does not change even if we use a different model!
- If you have access to both σ and s, would should you use?
  - You should use σ!

Test for a single mean

Hypothesis test recap

1. Set hypotheses
2. Collect and summarise data, set α
3. Obtain null distribution and p-value
   - For CLT-based method, obtain *test statistic*
4. Decision and conclusion

Hypotheses and null distribution

Want to conduct a hypothesis test for the mean μ of a population.

- Hypotheses: $H_0 : \mu = \mu_0$ versus $H_A : \mu \neq \mu_0$ (or $\mu > \mu_0$ or $\mu < \mu_0$)
- Verify conditions for CLT
  1. Independence
  2. Approximate normality or large sample size
- Then from population with mean μ and standard deviation σ, we have $\bar{X} \dot{\sim} N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$
- What does the (approximate) **null distribution** for $\bar{X}$ look like?

$$\bar{X} \dot{\sim} N\left(\boldsymbol{\mu_0}, \frac{\sigma}{\sqrt{n}}\right)$$

z-test and *t*-test statistics

Our test statistic is always of the form:

$$\frac{\text{observed} - \text{null}}{\text{SE}} \quad \text{or} \quad \frac{\text{observed} - \text{null}}{\widehat{\text{SE}}}$$

- If σ known and CLT met, we perform a **z-test** where our test-statistic is:

$$z = \frac{\bar{x}_{obs} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$

and we obtain our p-value using `pnorm()`

- If σ unknown and CLT met, we perform a ***t*-test** by estimating σ with s. Our test statistic is:

$$t = \frac{\bar{x}_{obs} - \mu_0}{\frac{s}{\sqrt{n}}} \sim t_{df} \qquad df = n - 1$$

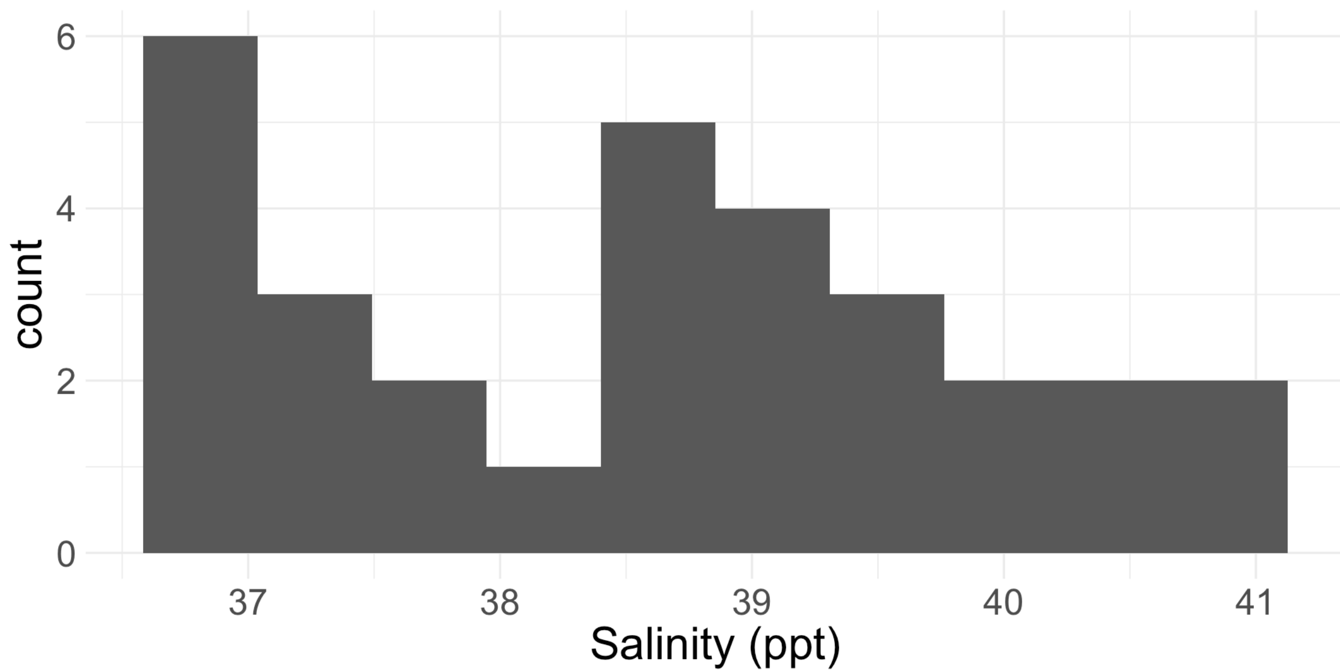
and we obtain our p-value using `pt()`

- Everything else proceeds as usual!

Example: salinity

The salinity level in a body of water is important for ecosystem function.

We have 30 salinity level measurements (ppt) collected from a random sample of water masses in the Bimini Lagoon, Bahamas.



- We want to test if the average salinity level in Bimini Lagoon is different from 38 ppm at the $\alpha = 0.05$ level.

Example: salinity (cont.)

1. Set hypotheses (define parameters as necessary).
    - Let μ be the average salinity level in Bimini Lagoon in ppt.
    - $H_0 : \mu = 38$ versus $H_A : \mu \neq 38$
2. Collect summary information, set α.

- $\bar{x}_{obs} = 38.6$
- $s = 1.29$
- $n = 30$
- $\alpha = 0.05$

Example: salinity (cont.)

3. Obtain null distribution, test statistic, and p-value

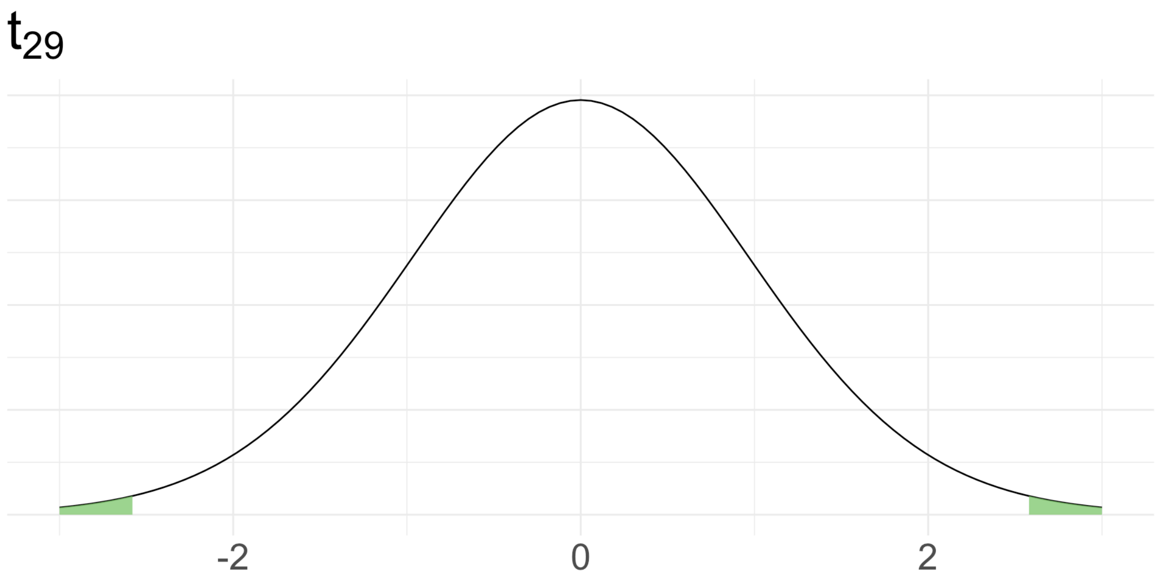
   i. Check conditions for CLT

   ii. If conditions met, obtain null distribution and test-statistic, and determine distribution of test-statistic

- Conditions:
  - Independence: random sample
  - Approximate normality: $n = 30$, but no clear outliers
- So by CLT, null dist. is $\bar{X} \dot{\sim} N\left(38, \frac{\sigma}{\sqrt{30}}\right)$
- Since we don't know σ, we perform a t-test and obtain the following test-statistic:
  - $t = \frac{\bar{x}_{obs} - \mu_0}{\widehat{SE}} = \frac{38.6 - 38}{1.29/\sqrt{30}} = 2.543$
  - This test-statistic follows a t_{29} distribution

Example: salinity (cont.)

iii. Use test-statistic to obtain p-value (draw picture and/or write code using appropriate distribution)



Want

$P(T \geq 2.54) + P(T \leq 2.54)$

because H_A is two-sided!

```r
p_val <- 2 * (1 - pt(t, df = n-1))
p_val
```

```
[1] 0.01658569
```

Example: salinity (cont.)

4. Decision and conclusion

- Since our p-value 0.017 is less than 0.05, we reject H_0.
- The data do provide sufficient evidence to suggest that the average salinity level in Bimini Lagoon is different from 38 ppt.

Let’s code it up together!