In-class Exam

1. Surprise eggs are type of collectible popular among children; each egg contains a randomized toy that is revealed once the egg is purchased and opened. Suppose there are a total of n unique toys types, and that each egg has equal chance of containing each type. If you purchase t eggs, what is the expected number of distinct toys you will collect?

2. Let $Z \sim N(0,1)$ and $Y = |Z|$. We say that Y has the *folded Normal* distribution. Find **two** expressions for the MGF of Y as unsimplified integrals, one integral based on the PDF of Y and one based on the PDF of Z.

3. Let X have PDF $f(x) = 3x^2$ for $x \in (0,1)$. Find the median of X.

4. A variable X is said to have the *arcsine* distribution if its CDF F is given by

$$F(x) = \frac{2}{\pi}\sin^{-1}(\sqrt{x}) \qquad \text{for } 0 < x < 1$$

   and $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 1$.

   (a) Check that F is indeed a valid CDF.

   (b) Find the corresponding PDF f. (Recall that the derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$)

   (c) Find a formula for the quantile function F^{-1} of X.

   (d) Suppose $U \sim \text{Unif}(0,1)$ and let F^{-1} be the quantile function of X. What is the name of the distribution of $F^{-1}(U)$?

5. Let $X \sim \text{Pois}(\lambda)$. (While proofs of each of the following are in the text, it's good practice with Taylor Series to compute them yourself).

   (a) Show directly that the PMF for X,

$$p(k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad \text{for} \qquad k \ge 0$$

   is a valid PMF.

   (b) Show using the PMF of X that the mean and variance of X are both λ.

   (c) Give an alternative proof of the preceding fact by finding the MGF of X using LOTUS and then computing moments.

6. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Exp}(\lambda)$. This means that each $X_i \sim \text{Exp}(\lambda)$ and they are independent of each other. Is $Y = X_1 + \ldots + X_n$ an Exponential random variable? If so, provide the correct rate parameter. If not, explain why not.

Take-home Exam

1. Let $A_1, A_2, \dots, A_m$ be an arbitrary collection of events (we do not assume that the A_i's are either disjoint or independent). Show that

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq \left(\sum_{i=1}^{m} P(A_i) \right) - m + 1$$

by first considering indicator variables and then using the fundamental bridge.

2. Following the example in the textbook of deriving the Binomial(n, p) variance, derive the variance of $X \sim \text{HyperGeom}(w, b, n)$.

3. An immortal ant wanders along an infinitely long stick that bears striking resemblance to the x-axis. Suppose at the dawn of time, the ant starts at position 0, and that each second thereafter, the ant moves 1cm in either the positive or the negative direction, each with equal probability, independent of the ant's previous movement. Let X_n be the ant's position at time n. Compute $\mathbb{E}[X_n]$.

4. Rigorously show that if X is a continuous random variable with pdf $f_X(x)$ that is symmetric about a, then $\text{Skew}(X) = 0$.

5. It's almost Christmas! On average, one in every 500 Christmas tree light bulb manufactured by a certain company that are sold are defective. The company sells strings of Christmas lights that contain 300 bulbs each, where each light on the string is assumed to act independently of the others.

   (a) Provide the exact form of the probability that there will no more than two defective bulbs in five strings of Christmas lights. Also obtain the answer numerically.

   (b) Provide an approximate probability that there will no more than two defective bulbs in five strings of Christmas lights. Be sure to justify why your approximation is valid. How good is your approximation?

   (c) Use appropriate functions in R of named distributions to verify your answers to (a) and (b).

6. Suppose $X \sim \text{Bin}(20, 0.25)$.

   (a) Use `dbinom` in R to show that the mean X is 5. *Hint: If `v` and `w` are vectors, then `sum(v*w)` is the sum of pairwise products of entries of `v` and `w`.*

   (b) Use `qbinom` in R to show that 5 is a median of X.

   (c) Use `dbinom` in R to show that the mode of X is 5. *Note that the function `max(v)` computes the largest value of the vector `v` and that the function `which.max(v)` finds the position of the maximal value of vector `v`.*

   (d) Verify that X is **not** symmetric by creating a plot of the PMF in R, despite the fact the mean, median and mode of X are all equal.