(*): Assigned to weekly problem set.

Continuous Random Variables

1. Determine whether the following statements are true or false. Briefly explain.
   (a) If X is a continuous random variable and $a \in \mathbb{R}$, then $P(X \leq a) = P(X < a)$.
   (b) If X is a continuous random variable with density f and $\epsilon > 0$ is small, then $f(x) \cdot \epsilon \approx P(x - \epsilon/2 < X < x + \epsilon/2)$.
   (c) The density function f for any continuous random variable must satisfy $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$.
   (d) The support of a continuous random variable X is the set of all real numbers x so that $P(X = x) > 0$.
   (e) The CDF F for a continuous random variable is an antiderivative for its density function.

2. (*) In calculus, the change-of-variable formula allows you to express the integral of one function in terms of the integral of another. In probability, there is an analogous formula for comparing probabilities of random variables

   (Change-of-Variables Formula) Suppose X is a random variable with support S_X and CDF $F_X(x)$, and let $Y = g(X)$. If g is increasing (and hence, invertible) on S_X, then the CDF $H_Y(y)$ of Y is given by

$$H_Y(y) = F_X(g^{-1}(y)).$$

   Moreover, if X is a continuous random variable with PDF $f_X(x)$, then the PDF $h_Y(y)$ of Y is

$$h_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}.$$

   where $S_Y = \{g(x) : x \in S_X\}$

   (a) Prove the change-of-variables formula above (note, there are two statements to prove: one about the CDF and one about the PDF). *Hint: Use the chain-rule from calculus, along with the formula for the derivative of an inverse function.*
   (b) Use the Change-of-Variable formula from part (a), along with u-substitution from calculus, to prove LOTUS for continuous random variables (at least in the case when g is an increasing function):

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

   where X is a continuous random variable with density $f_X(x)$.