(*): Assigned to weekly problem set.

Axioms of Probability and Inclusion-Exclusion

1. **Boole's Inequality** states that for a countable set of events $A_1, A_2, A_3, \ldots$, we have

$$P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i).$$

   If you've heard of the Bonferroni correction in multiple testing, then know that it comes from Boole's inequality!

   (a) State in plain English what the inequality means.

   (b) Using the axioms of probability and derived properties, prove that the inequality is true. The following will be useful:

   - Define a new sequence of events $B_1, B_2, B_3, \ldots$ as follows: $B_1 = A_1$ and $B_k = A_k \setminus (\bigcup_{i=1}^{k} A_i)$ for $k > i$, where $\setminus$ represents "set minus". For example, B_2 is the set of elements in A_2 that are not in A_1.
   - It is true that $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$. (You do not need to prove it, though it could be good to think about why.)

2. Alice attends a small college in which each class meets only once a week. She is deciding between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. This also implies that on a given day, Alice can take at most 6 classes. Trusting in the benevolence of randomness, Alice decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that she will have classes every day, Monday through Friday? (This problem can be done either directly using the naive definition of probability, or using inclusion-exclusion.)

3. (*) A fair die is rolled n times. What is the probability that at least one of the six values never appears?