(*): Assigned to weekly problem set.

Continuous Uniform Distribution

Note: when assessing if an expectation of a (function of a) RV exists, this is essentially asking if the integral converges or not. So you may need to revisit Calc II for tests of convergence/divergence, and revisit the definition of what it means for an integral to converge or diverge!

1. (*) A stick of length 1 is broken at a uniformly random point, yielding two pieces. Let X and Y be the length of the shorter and longer pieces, respectively, and let $R = \frac{X}{Y}$ be the random variable representing the ratio of the lengths X and Y.
   (a) Find the CDF and PDF of R. *Hint: Think about how X and Y relate to each other. Then somehow define a uniform random variable U and try to write X in terms/as some function of U. Then relate R to X.*
   (b) Find the expected value of R (if it exists).
   (c) Find the expected value of $\frac{1}{R}$ (if it exists).
2. (*) The *Pareto distribution* with parameter $a > 0$ has PDF $f(x) = a/x^{a+1}$ for $x \geq 1$ (and 0 otherwise).
   (a) Find the CDF of a Pareto r.v with parameter a.
   (b) Pareto distributions are said to be *heavy-tailed*, which means they have relatively high probability of generating large values. For what values of a does a Pareto variable have a mean? A variance? Compute the mean and variance for those Pareto variables where it makes sense to do so.
   (c) R does not have a formula for generating Pareto random variables (unlike `rbinom` for the binomial distribution). But R does have a function to generating Uniform random variable (`runif`). Explain how to use `runif` to generate 100 samples of a variable with the Pareto(a) distribution.
3. Let $U \sim \text{Uniform}(-1, 1)$.
   (a) Compute $\mathbb{E}[U]$ and $\text{Var}(U)$.
   (b) Find the CDF and PDF of U^2. Is the distribution of U^2 Uniform$(0, 1)$? Why or why not?