(*): Assigned to weekly problem set.

Joint distributions (discrete)

1. Suppose we have two random variables X and Y, both of which can only take the values 0 or 1. Define their joint PMF as

$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{6} & x=0, y=0 \\ \frac{1}{6} & x=0, y=1 \\ \frac{1}{3} & x=1, y=0 \\ \frac{1}{3} & x=1, y=1 \\ 0 & o.w. \end{cases}$$

   (a) What is $P(X=Y)$?

   (b) What are the marginal PMFs of X and Y?

   (c) Now suppose we have a different joint PMF for X and Y, given as follows:

$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{12} & x=0, y=0 \\ \frac{5}{12} & x=1, y=0 \\ \frac{3}{12} & x=1, y=1 \\ \frac{3}{12} & x=0, y=1 \end{cases}$$

   Under this joint PMF, what are the marginal PMFs of X and Y?

   (d) Based on what you've found, determine if the following statement is true or false: Knowing the marginal distributions of a set of random variables is sufficient to determine their joint distribution.

2. A fair coin is flipped twice. Let X be the number of Heads in the two tosses, and Y be the indicator r.v for the tosses landing the same way.

   (a) Find the joint PMF of X and Y. *Hint: it may be helpful to think about one of the variables marginally first.*

   (b) Find the marginal PMFs of X and Y.

   (c) Are X and Y independent?

   (d) Find the conditional PMFs of Y given $X=x$ and of X given $Y=y$.

3. (*) A fair die is rolled, and then a coin with probability p of Heads is flipped as many times as the die roll says, e.g., if the result of the die roll is a 3, then the coin is flipped 3 times. Let X be the result of the die roll and Y be the number of times the coin lands Heads. *Throughout this problem and others, don't forget to define supports where appropriate!*

(a) Find the joint PMF of X and Y. Are they independent?

(b) Find the marginal PMFs of X and Y. They may or may not look beautiful!

(c) Find the conditional PMF of Y given $X = x$. Also find the form of the conditional PMF of X given $Y = y$ (you do not necessarily need to evaluate it).