(*): Assigned to weekly problem set.

Gamma and Beta Distributions

1. (*) Let $X \sim \text{Gamma}(a, \lambda)$ and $Y \sim \text{Gamma}(b, \lambda)$ be independent. Show that $X + Y \sim \text{Gamma}(a + b, \lambda)$ in three ways:

   (a) Using MGFs.

   (b) Using a story about a Poisson process. (For this part only, assume a and b are positive integers).

   (c) By going through the following process:

      i. Define $Z = X + Y$. Find the conditional CDF of Z given $X = x$. Then, differentiate this CDF to obtain the conditional PDF of Z given $X = x$, Don't forget to specify the support of the PDF and define the CDF everywhere.

      ii. Use continuous LoTP (Theorem 7.1.18) to express the marginal PDF of Z in terms of the conditional PDF of Z given $X = x$ and the marginal PDF of X.

      iii. Evaluate the integral from above by recognizing it as a Beta integral (you may need to make an appropriate u-substitution first).

      iv. Verify that the marginal PDF of Z is indeed the PDF for $\text{Gamma}(a + b, \lambda)$.

2. Let $B \sim \text{Beta}(a, b)$. Find the distribution of $W = 1 - B$ using change of variables. Can you also find the distribution of W using a story proof?

Poisson Processes

A process of arrivals in continuous time is called a *Poisson process* with rate λ if the following two conditions hold:

- The number of arrivals that occur in an interval of length t is a $\text{Pois}(\lambda t)$ random variable
- The number of arrivals that occur in disjoint intervals are independent of each other. For example, the number of arrivals in the interval $(0, 10]$, $(10, 12]$ and $(15, 100]$ are independent.

We'll now define a few variables related to this process:

- For each real number $t \geq 0$, let N_t be the number of arrivals in the interval $(0, t]$; then $N_t \sim \text{Pois}(\lambda t)$.
- For $s < t$, the variable $N_t - N_s$ counts the number of arrivals in the interval $(s, t]$; then $N_t - N_s \sim \text{Pois}(\lambda(t - s))$.
- For each integer $k \geq 1$, let T_j the time of the jth arrival; we will show $T_j \sim \text{Gamma}(j, \lambda)$.
- For each integer $k \geq 1$, let R_j be the time between the kth and $(k-1)$th arrival; we will show $R_j \sim \text{Expo}(\lambda)$.

3. The following exercises outline a proof that $T_j \sim \text{Gamma}(j, \lambda)$

   (a) Find the CDF for T_1 using the fact that $P(T_1 \leq t) = P(N_t \geq 1)$. Then calculate the PDF of T_1.

   (b) Express the probability $P(T_j > t)$ as a finite sum, using the fact that $P(T_j > t) = P(N_t < j)$.

   (c) Use your previous answer to write the CDF of T_j as a **finite** sum.

   (d) Differentiate and use the product rule to write the PDF of T_j as a finite sum. Be careful about the derivative of the constant term.

   (e) Explicitly write out the terms of the sum for the PDF of T_2 and T_3. What pattern do you notice?

   (f) Make a conjecture for how to simplify the sum for the PDF of T_j.

   (g) Based on your formula in the previous part, what is the name of the distribution of T_j?