(*): Assigned to weekly problem set.

Expected Value and LoTUS

1. (Good practice for midterm 1!) Two teams are going to play a best-of-7 match (the match will end as soon as either team has won 4 games). Each game ends in a win for one team and a loss for the other team. Assume that each team is equally likely to win each game, and that the games played are independent. Find the mean number of games played.

2. You might see the expression $\mathbb{E}[X^2]$ abbreviated as $\mathbb{E}X^2$. However, we should take care to distinguish this from $(\mathbb{E}[X])^2$. Show that $\mathbb{E}[X^2]$ and $(\mathbb{E}[X])^2$ are not in general equal by computing both values for $X \sim \text{DUnif}\{-1, 1\}$ (i.e. $S_X = \{-1, 1\}$).

3. (*) Suppose we have two random variables X and Y. Let $V = \min(X, Y)$ and $W = \max(X, Y)$. So if we observe $X = x$ and $Y = y$, then V is observed as $\min(x, y)$ and similarly for W.

   (a) Show the following (interesting but not necessarily useful) addition law for expectations:
   $$\mathbb{E}[W] = \mathbb{E}[X] + \mathbb{E}[Y] - \mathbb{E}[V]$$

   (b) Let $X \sim \text{Binomial}(n, \frac{1}{2})$ and $Y \sim \text{Binomial}(n+1, \frac{1}{2})$. Find $\mathbb{E}[V] + \mathbb{E}[W]$.

4. Suppose that a person types n letters, writes the corresponding addresses on n envelopes, and then randomly places each letter in an envelope. Let X be the number of letters that are placed in the correct envelopes. Find the expectation of X.