Midterm Practice Problems

Note: Some of these problems are lengthier/more difficult than what you might expect on the midterm given the time constraint. I've indicated these problems with (*).

1. Suppose X and Y are two independent Poisson random variables. It is known that $\text{Var}(X) + \text{Var}(Y) = 5$. Find an expression for $P(X + Y < 2)$.

2. The probability that each specific child in a given family will inherit a certain disease is p. If it is known that at least two children in a family of n children have inherited the disease, what is the probability that exactly k children have inherited the disease?

3. (*) If a random variable X has a discrete distribution with PMF $p(x)$, then the value of $x = x^* \in S_X$ that maximizes $p(x)$ is called the *mode* of the distribution. If this same maximum $f(x^*)$ is attained at more than one value of x, then all such values of x are called the modes of the distribution. If $X \sim \text{Poisson}(\lambda)$, find the mode or modes of this distribution. . *Hint 1: consider the ratio* $\frac{p_X(k)}{p_X(k-1)}$, *and try to understand why this ratio is useful for solving this problem. Hint 2: consider the cases when* λ *is an integer versus when it is not.*

4. On a certain statistics test (not necessarily this one!), 10 out of 100 key terms will be randomly selected to appear on an exam. A student then must choose 7 of these 10 to define. Since the student knows the format of the exam in advance, the student is trying to decide how many of these terms to memorize.

   (a) Suppose the student studies s key terms, where s is an integer between 0 and 100. What is the distribution of X? Give the name and parameters of the distribution, in terms of s.

   (b) Write an expression involving the explicit formula for the PMF or CDF of the named distribution for the probability that the student knows at least 7 of the 10 key terms, assuming the student studies $s = 75$ key terms.

5. Surprise eggs are type of collectible popular among children; each egg contains a randomized toy that is revealed once the egg is purchased and opened. Suppose there are a total of n unique toys types, and that each egg has equal chance of containing each type. If you purchase t eggs, what is the expected number of distinct toys you will collect?

6. Suppose $X \sim \text{Binomial}(8, 0.25)$ and let $Y = 2X + 1$. Find and simplify a formula for the PMF of Y. Then compute the expected value of Y.

7. (*) Derive the variance of $X \sim \text{HyperGeom}(w, b, n)$. *Hint, use the technique we learned for deriving variance of a Binomial.*

8. (*) Let $A_1, A_2, \ldots, A_m$ be an arbitrary collection of m events (we do not assume that the A_j's are either disjoint or independent). Show that

$$P(A_1 \cap A_2 \cap \cdots \cap A_m) \geq \left(\sum_{j=1}^{m} P(A_j) \right) - m + 1$$

   by first considering indicator variables and then using the fundamental bridge.

9. Let $X \sim \text{Poisson}(\lambda_1)$ be independent of $Y \sim \text{Poisson}(\lambda_2)$. Show that the distribution of $X|X+Y=n$ is $\text{Binom}(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$

10. On average, one in every 500 Christmas tree light bulb manufactured by a certain company that are sold are defective. The company sells strings of Christmas lights that contain 300 bulbs each, where each light on the string is assumed to act independently of the others.

    (a) Provide the exact form of the probability that there will no more than two defective bulbs in five strings of Christmas lights. Also obtain the answer numerically.

    (b) Provide an approximate probability that there will no more than two defective bulbs in five strings of Christmas lights. Be sure to justify why your approximation is valid. How good is your approximation?

Previous midterm problems

1. For each of the following statements, determine if the stated distribution is correct or incorrect for the random variable X. In the case of correct, briefly explain/justify your answer. If incorrect, explain why and either provide the correct distribution or the information necessary to make the distribution correct for the random variable.

    (a) We are flipping 10 coins. Let X represent the number of Heads flipped. $X \sim \text{Binomial}(10, \frac{1}{2})$.

    (b) A county contains 350 eligible voters, of which 250 are Democrats and 100 are Republicans. We choose to poll 30 different voters. Let X be the number of Republicans chosen. $X \sim \text{Binomial}(350, \frac{100}{350})$.

    (c) A drawer of socks contains three red socks, two white socks, and two gold socks. You randomly take two socks from the drawer. X is the number of red socks drawn. $X \sim \text{HyperGeometric}(3, 2, 2)$.

2. Suppose that two four-sided die are rolled, where each dice is fair. Let X denote the absolute value of the difference between the two numbers that appear.

    (a) Determine the PMF of X.

    (b) Sketch the CDF of X.

    (c) Now, find the expected value of $(X-1)(X+1)$. Simplify as much as possible.

3. Determine whether the following statement is true or false. If true, justify why it is true. If false, provide a counterexample or proof showing it is false:

    If $X \sim \text{Binomial}(n,p)$ *and* $Y \sim \text{Binomial}(m,q)$ *for* $0 \le p, q \le 1$ *and* $n, m \ge 1$, *then* $X + Y \sim \text{Binomial}\left(m+n, \frac{p+q}{2}\right)$.

4. Let $X \sim \text{Poisson}(\lambda)$. Find $\mathbb{E}[X!]$ if it is finite.