Joint distributions (discrete)

1. Suppose we have two random variables X and Y, both of which can only take the values 0 or 1. Define their joint PMF as

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{6} & x=0, y=0 \\ \frac{1}{6} & x=0, y=1 \\ \frac{1}{3} & x=1, y=0 \\ \frac{1}{3} & x=1, y=1 \\ 0 & o.w. \end{cases}$$

   (a) What is $P(X=Y)$?

   (b) What are the marginal PMFs of X and Y?

   (c) Now suppose we have a different joint PMF for X and Y, given as follows:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{12} & x=0, y=0 \\ \frac{5}{12} & x=1, y=0 \\ \frac{3}{12} & x=1, y=1 \\ \frac{3}{12} & x=0, y=1 \end{cases}$$

   Under this joint PMF, what are the marginal PMFs of X and Y?

   (d) Based on what you've found, determine if the following statement is true or false: Knowing the marginal distributions of a set of random variables is sufficient to determine their joint distribution.

2. A fair coin is flipped twice. Let X be the number of Heads in the two tosses, and Y be the indicator r.v for the tosses landing the same way.

   (a) Find the joint PMF of X and Y. *Hint: it may be helpful to think about one of the variables marginally first.*

   (b) Find the marginal PMFs of X and Y.

   (c) Are X and Y independent?

   (d) Find the conditional PMFs of Y given $X=x$ and of X given $Y=y$.

Joint distributions (continuous)

1. Let X and Y have a joint PDF

$$f_{X,Y}(x,y) = cxy \quad \text{for } 0 < x < y < 1.$$

   (a) Find the value of c that makes this a valid joint PDF.

   (b) Find the marginal PDFs of X and Y.

   (c) Explain why X and Y are **not** independent (even though superficially, their joint PDF might appear to factor). *Hint: carefully consider the domain of* $f_{X,Y}$

   (d) Find the conditional PDF of Y given $X = x$.

2. Alice and Bob arrange to meet for lunch on a certain day at noon. However, neither is known for punctuality. They both arrive independently at uniformly distributed times between noon and 1pm on that day. Each is willing to wait up to 15 minutes for the other to show up. What is the probability they will meet for lunch that day?

   Hint: Construct a graph of the region in the xy-plane corresponding to the event that Alice and Bob meet for lunch.