This problem set covers material from Week 2, dates 2/20- 2/23. Unless otherwise noted, all problems are taken from the textbook. Problems can be found at the end of the corresponding subsection.

Instructions: Write or type complete solutions to the following problems and submit answers to the corresponding Canvas assignment. Your solutions should be neatly-written, show all work and computations, include figures or graphs where appropriate, and include some written explanation of your method or process (enough that I can understand your reasoning without having to guess or make assumptions). A general rubric for homework problems appears on the final page of this assignment.

Tuesday 2/20

1. Suppose $X_1, \ldots, X_n | \theta \stackrel{\text{iid}}{\sim} \text{Bern}(\theta)$ where $\theta \in [0,1]$ is the unknown parameter. Let us use the following prior for θ: $\theta \sim \text{Beta}(a, b)$ where a, b are some fixed values chosen to reflect your prior beliefs. Derive the posterior distribution of θ given the data $\mathbf{x}$. You should provide the name of the well-known distribution as well as its exact parameters.
2. `R` problem 1 (Please find an `.Rmd` template on the previous page under this assignment.)
3. 7.10: Problem 7(b). *Note: the problem description gives the* mean *of the Exponentials, not the* rate. *You can assume the lightbulbs are conditionally independent given θ.*

Thursday 2/22

4. Deriving Normal-Normal conjugacy.
   (a) Useful identity: Suppose $x_1, x_2, \ldots, x_n$ be any list of n numbers with mean $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$. Let $a \in \mathbb{R}$. Prove the following identity:
   $$\sum_{i=1}^n (x_i - a)^2 = \sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - a)^2$$
   Hint: use the classic trick of adding 0 into the squared term in order to introduce $\bar{x}$
   (b) Suppose $X_1, \ldots, X_n | \theta \stackrel{\text{iid}}{\sim} N(\theta, \sigma^2)$ where σ^2 is known and $\theta \in \mathbb{R}$ is unknown. We use the prior $\theta \sim N(\mu_0, \tau^2)$. Provide a derivation demonstrating that $\theta | \mathbf{x} \sim N(\mu_n, \tau_n^2)$, where $\mu_n = \frac{n\bar{x}\tau^2 + \mu_0\sigma^2}{n\tau^2 + \sigma^2}$ and $\tau_n^2 = \frac{\sigma^2\tau^2}{n\tau^2 + \sigma^2}$. *Proportionality is fine, but your solution should clearly show how you go from prior and likelihood to the posterior.*
5. 7.3: Problem 19
6. Let $X_1, X_2, \ldots, X_n | \theta \stackrel{\text{iid}}{\sim} \text{Unif}[0, \theta]$, where $\theta > 0$ is unknown. Let $\theta \sim \text{Pareto}(r, \alpha)$, where $r > 0$ and $\alpha > 0$ are fixed and known. The PDF of this Pareto is given by:
   $$p(\theta) = \begin{cases} \frac{r}{\alpha}\left(\frac{\alpha}{\theta}\right)^{r+1} & \theta \geq \alpha \\ 0 & \theta < \alpha \end{cases}$$

(a) Obtain the posterior distribution of θ given the data.

(b) Is the Pareto distribution a conjugate prior for data from a Unif$[0, \theta]$? Why or why not?

Friday 2/23

7. 7.4: Problem 15

8. Suppose that a random sample of four observations is drawn from a Unif$[0, \theta]$ distribution, where $\theta > 0$ is unknown. The prior we choose for θ is

$$p(\theta) = \begin{cases} \frac{1}{\theta^2} & \theta \geq 1 \\ 0 & o.w. \end{cases}$$

   Suppose that the values of the observations are $0.6, 0.4, 0.8$, and 0.9. Determine the Bayes estimate of θ under absolute loss. *Hint: no need to re-derive the posterior. Where have we seen this Uniform sampling model? How does the prior we chose then compare to the prior in this problem?*

9. `R` Problem 2

General rubric

Points	Criteria
5	The solution is correct *and* well-written. The author leaves no doubt as to why the solution is valid.
4.5	The solution is well-written, and is correct except for some minor arithmetic or calculation mistake.
4	The solution is technically correct, but author has omitted some key justification for why the solution is valid. Alternatively, the solution is well-written, but is missing a small, but essential component.
3	The solution is well-written, but either overlooks a significant component of the problem or makes a significant mistake. Alternatively, in a multi-part problem, a majority of the solutions are correct and well-written, but one part is missing or is significantly incorrect.
2	The solution is either correct but not adequately written, or it is adequately written but overlooks a significant component of the problem or makes a significant mistake.
1	The solution is rudimentary, but contains some relevant ideas. Alternatively, the solution briefly indicates the correct answer, but provides no further justification.
0	Either the solution is missing entirely, or the author makes no non-trivial progress toward a solution (i.e. just writes the statement of the problem and/or restates given information).
Notes:	For problems with multiple parts, the score represents a holistic review of the entire problem. Additionally, half-points may be used if the solution falls between two point values above.
Notes:	For problems with code, well-written means only having lines of code that are necessary to solving the problem, as well as presenting the solution for the reader to easily see. It might also be worth adding comments to your code.