Likelihood ratio test

Solutions

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LRT set-up

Recall the coin-flipping scenario again! We have $X_1, \ldots, X_n$ are a random sample from a Bern(θ) distribution, where θ is the probability of Heads. We have the hypotheses

$$H_0 : \theta = \theta_0 \qquad \text{vs.} \qquad H_1 : \theta \neq \theta_0$$

We can conduct a LRT of these hypotheses. We found the likelihood ratio statistic to be:

$$\Lambda(\mathbf{X}) = \left(\frac{n\theta_0}{\sum X_i}\right)^{\sum X_i} \left(\frac{n(1-\theta_0)}{n - \sum X_i}\right)^{n - \sum X_i}$$

The LRT says reject when $\Lambda(\mathbf{x}) \leq k$ for some $k \in [0, 1]$. Notice that $\Lambda(\mathbf{x})$ depends on how many heads we see!

Specific case

Suppose we have $n = 6$ coin flips and $\theta_0 = 0.6$. That is,

$$H_0 : \theta = 0.6 \qquad \text{vs.} \qquad H_1 : \theta \neq 0.6$$

Also suppose we want a level 0.08 LRT of these hypotheses, which means we need to find the value of k such that

$$\Pr(\Lambda(\mathbf{x}) \leq k | \theta = 0.6) \leq 0.08$$

We can evaluate the test statistic for these specific values of n, θ_0, and each possible value of $\sum_{i=1}^{6} X_i = y \in \{0, 1, \ldots, 6\}$:

```
n <- 6
theta0 <- 0.6
y <- 0:n
Lambda <- ((n*theta0/y)^y) * (n*(1-theta0)/(n - y))^(n - y)
Lambda
```

```
## [1] 0.00409600 0.09172943 0.41990400 0.88473600 0.94478400 0.46438023 0.04665600
```

Since $\Lambda(\mathbf{x})$ is a function of $\sum X_i$, $\Pr(\Lambda(\mathbf{x}) = k|\theta = 0.6)$ can be obtained from $\Pr(\sum X_i = y|\theta = 0.6)$. For example, $\Pr(\Lambda(\mathbf{x}) = 0.004096) = \Pr(\sum X_i = 0|\theta = 0.6)$.

We display the possible values of $\Lambda(\mathbf{x})$ below, along with their associated probabilities (which are obtained from $\Pr(\sum X_i = y|\theta = 0.6)$):

	y = 0	y = 1	y = 2	y = 3	y = 4	y = 5	y =
\(Pr(\sum X_i = y \mid \theta_0)\)	0.004096	0.0368640	0.138240	0.276480	0.311040	0.1866240	0.0466
\(\Lambda(\mathbf{x})\)	0.004096	0.0917294	0.419904	0.884736	0.944784	0.4643802	0.0466

Now we can find the value k such that $\Pr(\Lambda(\mathbf{x}) \le k|\theta = 0.6) \le 0.08$!

- $\Pr(\Lambda(\mathbf{x}) \le 0.0041|\theta = 0.6) = \Pr(\sum X_i = 0|\theta = 0.6) = 0.004 \le 0.08$
- $\Pr(\Lambda(\mathbf{x}) \le 0.0467|\theta = 0.6) = \Pr(\sum X_i = \{0, 6\}|\theta = 0.6) = 0.0041 + 0.0467 = 0.051 \le 0.08$
- $\Pr(\Lambda(\mathbf{x}) \le 0.0917|\theta = 0.6) = \Pr(\sum X_i = \{0, 6, 1\}|\theta = 0.6) = 0.0041 + 0.0467 + 0.0369 = 0.088 > 0.08$

So rejecting H_0 when $\Lambda(\mathbf{x}) \le k$ for any $k \in [0, 0.0917)$ yields a level-0.08 test.

In particular, the test that rejects H_0 when $k = 0.0467$ has size 0.051.

The test that rejects H_0 when $k = 0.0041$ has size 0.004.