Comparing estimators of Normal variance

Let $X_1, \ldots, X_n | \mu, \sigma^2 \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$ where both μ and σ^2 are unknown. Recall the following estimators of the variance:

$$\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \qquad \text{and} \qquad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

We have shown in a previous lecture that the MLE is a biased estimator for σ^2 whereas s^2 is an unbiased estimator. For this reason, s^2 is used in practice most often. But we know that we shouldn't just look at the bias to compare estimators, but we should also consider their MSEs! **Using properties/moments of the χ^2 distribution**:

1. Find $\mathbb{E}[\hat{\sigma}^2_{MLE}]$.
2. Find $\mathbb{E}[s^2]$.
3. Find $\text{Var}(\hat{\sigma}^2_{MLE})$.
4. Find $\text{Var}(s^2)$.
5. Find $MSE_{\sigma^2}(\hat{\sigma^2}_{MLE})$ and $MSE_{\sigma^2}(s^2)$. Which estimator for σ^2 is better? *Hint: use the fact that* $(a-1) < a$ *for all* $a \in \mathbb{R}$.