This problem set covers material from Week 5, dates 3/12- 3/14. Unless otherwise noted, all problems are taken from the textbook. Problems can be found at the end of the corresponding subsection. "AP" stands for additional problems not found in the book.

**Instructions**: Write or type complete solutions to the following problems and submit answers to the corresponding Canvas assignment. Your solutions should be neatly-written, show all work and computations, include figures or graphs where appropriate, and include some written explanation of your method or process (enough that I can understand your reasoning without having to guess or make assumptions). A general rubric for homework problems appears on the final page of this assignment.

**This content will be fair game for Midterm 1.**

## Tuesday 3/12

1. Under the conditions set out in class, prove that an alternate expression for the Fisher Information $I(\theta)$ in a sample $X$ for the unknown parameter $\theta$ is:

$$I(\theta) = -\mathbb{E}[l''(\theta|x)],$$

where $l''(\theta(|x) = \frac{d^2}{d\theta^2} \log f(x|\theta)$. I will get you started, but refer back to your notes from class for some very helpful facts!

Start by noting that:

$$l''(\theta(|x) = \frac{d^2}{d\theta^2} \log f(x|\theta) = \frac{d}{d\theta}\left[l'(\theta|x)\right] = \frac{d}{d\theta}\left[\frac{d}{d\theta} \log f(x|\theta)\right] = \frac{d}{d\theta}\left[\frac{\frac{d}{d\theta} f(x|\theta)}{f(x|\theta)}\right]$$

From here, continue by evaluating the right-most term (remember derivatives for fractions!). More hints:

- LoTUS will be helpful.
- Remember where you're trying to go/what you're trying to prove. You'll need to get $I(\theta)$ somehow, so be vigilant about spotting it!

2. Let $X_1, \ldots, X_n|\sigma^2 \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$ where $\mu$ is known but the variance $\sigma^2$ is unknown. What is $I_n(\sigma^2)$? *Note that we are interested in the parameter $\sigma^2$ and not $\sigma$.*

## Thursday 3/14

3. Let $X_1, \ldots, X_n|\theta \overset{\text{iid}}{\sim} \text{Bern}(\theta)$. We have previously shown that $\hat{\theta}_{MLE} = \bar{X}$. Is the MLE a MVUE (minimum variance unbiased estimator) for $p$? Justify why or why not.

4. Let $X_1, \ldots, X_n | \theta \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$. Consider the following two estimators of $\theta$:

$$\hat{\theta}_{MLE} = \bar{X} \qquad \text{and} \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

   (a) Which of the two estimators for $\theta$ is better? Justify/defend your choice.

   (b) What is the asymptotic distribution of the MLE?

5. Recall that we typically use MSE to evaluate an estimator $\delta$ for $\theta$ (dropping the $\mathbf{X}$ in $\delta(\mathbf{X})$ for simplicity):

$$MSE_\theta(\delta) = \text{Var}(\delta) + (\mathbb{E}[\delta] - \theta)^2$$

   If we have two estimators $\delta_1$ and $\delta_2$ that are both unbiased for $\theta$, then comparison of their MSEs reduces to comparison of their variances. Thus, given two unbiased estimators $\delta_1$ and $\delta_2$ for $\theta$, the **relative efficiency** of $\delta_2$ relative to $\delta_1$ is

$$\text{eff}_\theta(\delta_1, \delta_2) = \frac{\text{Var}(\delta_2)}{\text{Var}(\delta_1)}. \tag{1}$$

   (a) What is the interpretation of $\text{eff}_\theta(\delta_1, \delta_2) < 1$?

   (b) Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Unif}[0, \theta]$ (we can never escape this distribution!). Consider two estimators for $\theta$:

$$\delta_1 = 2\bar{X} \qquad \text{vs.} \qquad \delta_2 = \frac{n+1}{n}Y,$$

   where $Y = \max\{X_i\}$. On the previous homework, you showed that both $\delta_1$ and $\delta_2$ are unbiased for $\theta$. So, let's compare the two estimators! Find the relative efficiency of $\delta_2$ relative to $\delta_1$. Feel free to use results from the past.

   (c) Which estimator of $\theta$ would you recommend and why? *Consider various sample sizes...*

## General rubric

| Points | Criteria |
|---|---|
| 5 | The solution is correct *and* well-written. The author leaves no doubt as to why the solution is valid. |
| 4.5 | The solution is well-written, and is correct except for some minor arithmetic or calculation mistake. |
| 4 | The solution is technically correct, but author has omitted some key justification for why the solution is valid. Alternatively, the solution is well-written, but is missing a small, but essential component. |
| 3 | The solution is well-written, but either overlooks a significant component of the problem or makes a significant mistake. Alternatively, in a multi-part problem, a majority of the solutions are correct and well-written, but one part is missing or is significantly incorrect. |
| 2 | The solution is either correct but not adequately written, or it is adequately written but overlooks a significant component of the problem or makes a significant mistake. |
| 1 | The solution is rudimentary, but contains some relevant ideas. Alternatively, the solution briefly indicates the correct answer, but provides no further justification. |
| 0 | Either the solution is missing entirely, or the author makes no non-trivial progress toward a solution (i.e. just writes the statement of the problem and/or restates given information). |
| | |
| Notes: | For problems with multiple parts, the score represents a holistic review of the entire problem. Additionally, half-points may be used if the solution falls between two point values above. |
| Notes: | For problems with code, well-written means only having lines of code that are necessary to solving the problem, as well as presenting the solution for the reader to easily see. It might also be worth adding comments to your code. |