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## Cramer-Rao Lower Bound

1. Suppose we have a single observation  $X|\theta \sim \text{Unif}[0, \theta]$ :

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0 \leq x \leq \theta \\ 0 & \text{o.w.} \end{cases}$$

Note that  $\mathbb{E}[X] = \frac{\theta}{2}$  and  $\text{Var}(X) = \frac{\theta^2}{12}$ . We know that an unbiased estimator for  $\theta$  is  $\delta(X) = 2X$ .

- (a) Using the definition of Fisher Information (and not one of the alternate expressions), what is the Fisher Information  $I(\theta)$  in  $X$ ?
- (b) For this data, what is the CRLB for any unbiased estimator of  $\theta$ ?
- (c) What is the variance of our estimator  $\delta(X) = 2X$ ?
- (d) Compare the CRLB in (b) and your variance in (c). Is there a contradiction? If so, what might be the explanation?
- (e) Now calculate the Fisher Information  $I(\theta)$  in  $X$  using the two alternate expressions for Fisher Information. What do you notice?