Duality Theorem 1

Theorem: Suppose for every value θ_0 of $\theta \in \Omega_0$, there exists a level α_0 test δ_{θ_0} of the simple null $H_0 : \theta = \theta_0$. Denote the acceptance region for $r(\mathbf{X})$ of the test as R^c. Then the set $c(\mathbf{X}) = \{\theta : r(\mathbf{x}) \in R^c\}$ is a $\gamma = 1 - \alpha_0$ coefficient confidence set for θ.

Proof: Let the level α_0 test δ_{θ_0} be as defined above. Then

$$\begin{aligned}
\alpha_0 &\geq \Pr(\text{reject} H_0 | \theta = \theta_0) \\
&= \Pr(r(\mathbf{X}) \in R | \theta = \theta_0) \\
&= 1 - \Pr(r(\mathbf{X}) \in R^c | \theta = \theta_0) \Rightarrow \\
\Pr(r(\mathbf{X}) \in R^c | \theta = \theta_0) &\geq 1 - \alpha_0
\end{aligned}$$

Define a confidence set $c(\mathbf{X})$ as $c(\mathbf{X}) = \{\theta : r(\mathbf{x}) \in R^c\}$, where R^c is from the above test. Then

$$\Pr(\theta \in c(\mathbf{X}) | \theta = \theta_0) = \Pr(r(\mathbf{X}) \in R^c | \theta = \theta_0) \geq 1 - \alpha_0 = \gamma$$

where the first equality follows by how $c(\mathbf{X})$ is defined. Thus by definition, $c(\mathbf{X})$ is a γ coefficient confidence set for θ.