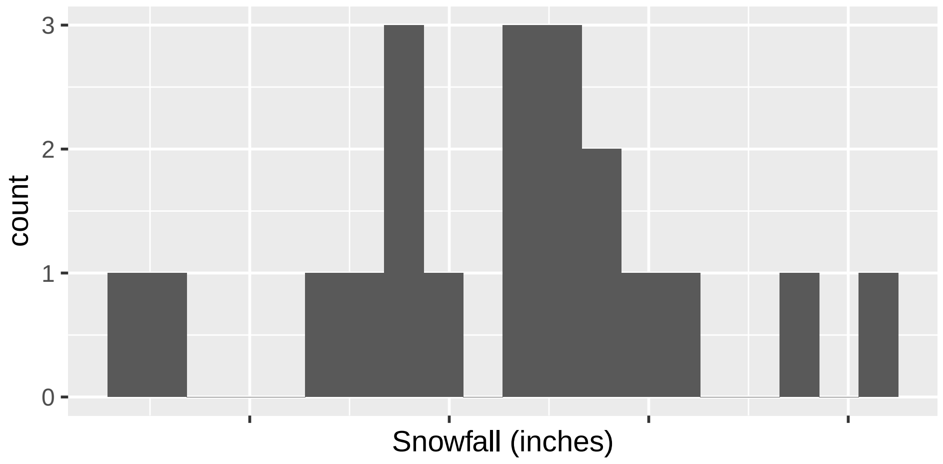
Metropolis example

Data

Maybe I'm thinking about moving to Burlington and I'd like to know the average amount of snowfall θ I should expect in a given year. Luckily, there is some historic data about the observed total snowfall in Burlington from each winter season since 2004, courtesy of NOAA. Before taking a look at the data, we should formulate a prior. Let me first graph the data for you (hiding the values themselves):



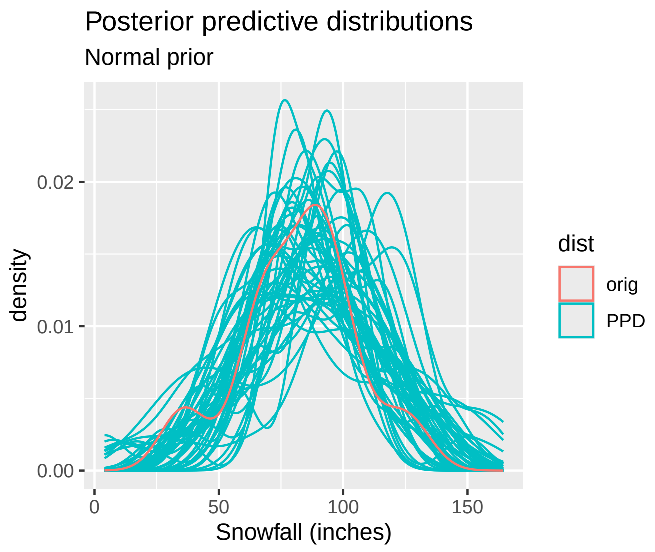
Model 1

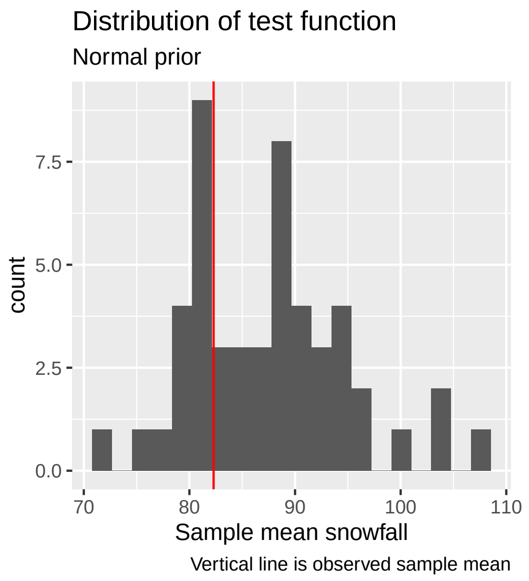
This looks like maybe we could model the yearly snowfall as Normal with unknown mean θ. For now, let's assume that σ^2 is equal to the observed s^2 for simplicity (i.e. $\sigma^2 = 554.885$).

Then a model that we've used before could be the Normal-Normal model:

$$Y_i|\theta \overset{iid}{\sim} N(\theta, 554.885) \qquad i = 1, \dots, n = 20$$
$$\theta \sim N(\mu_0, \sigma_0^2)$$

For my prior probability, I might think that on average, Burlington gets 10 feet = 120 inches of snow in a winter season, with a standard deviation of 1 foot = 12 inches. So $\mu_0 = 120$ and $\sigma_0^2 = 144$. We could obtain Monte Carlo amples for θ from its posterior directly, but let's jump ahead to obtain some PPDs to see if this model is a good fit. I'll simulate 50 PPDs, and graph them alongside the observed data. Additionally, I'll calculate the test function $T(\mathbf{y}^*) = \frac{1}{n}\sum y_i^*$ (i.e. the sample mean) for each PPD and compare them to the observed $\bar{y}$.





What do we think about this model?

Model 2

Let's consider the same sampling distribution model, but a different prior for θ. Maybe I'd like a prior for θ that has thicker/heavier tails than a Normal distribution (i.e. has more prior probability for extreme values than the Normal). This could be useful since I actually don't know what the average snowfall is, and my original prior could be way off.

Let's consider the Cauchy distribution as a prior for θ (a truly terrible distribution; it doesn't have mean). The Cauchy distribution is parameterized by a location parameter $\theta_0 \in \mathbb{R}$ and a scale parameter $\kappa > 0$. If $X \sim \text{Cauchy}(\mu_0, \kappa)$ then its pdf is

$$f(x|\mu_0, \kappa) = \frac{1}{\pi\kappa\left(1 + (\frac{x-\mu_0}{\kappa})^2\right)}, \qquad x \in \mathbb{R}$$

For Model 2, I will use the following prior for θ:

$$\theta \sim \text{Cauchy}(120, 6)$$

Then this has similar prior beliefs as under the Normal above: this Cauchy is roughly centered at 10 feet of snow and has 70% prior probability of being between 9 and 11 feet:

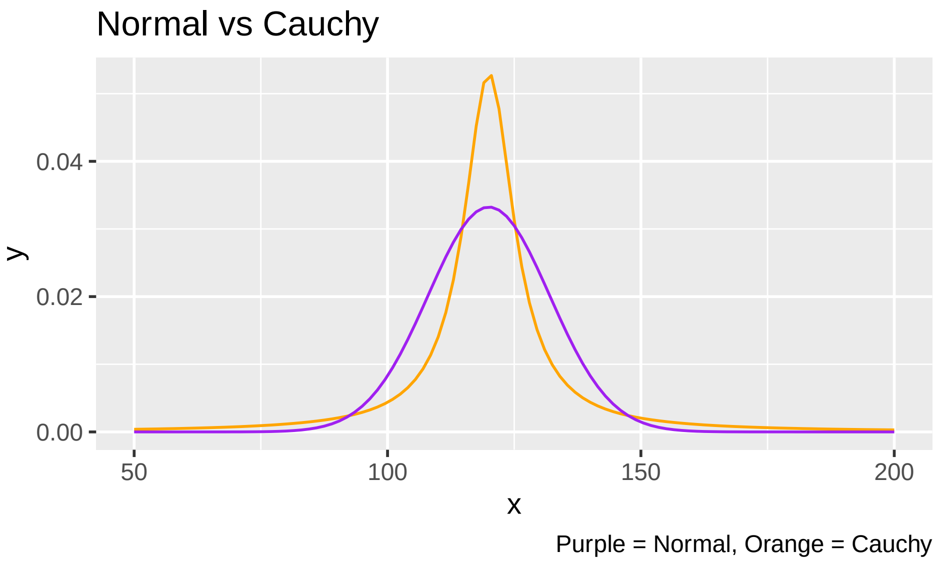
```r
qcauchy(0.15, 120, 6)/12 # convert to feet
```

```
[1] 9.018695
```

```r
qcauchy(0.85, 120, 6)/12
```

```
[1] 10.98131
```

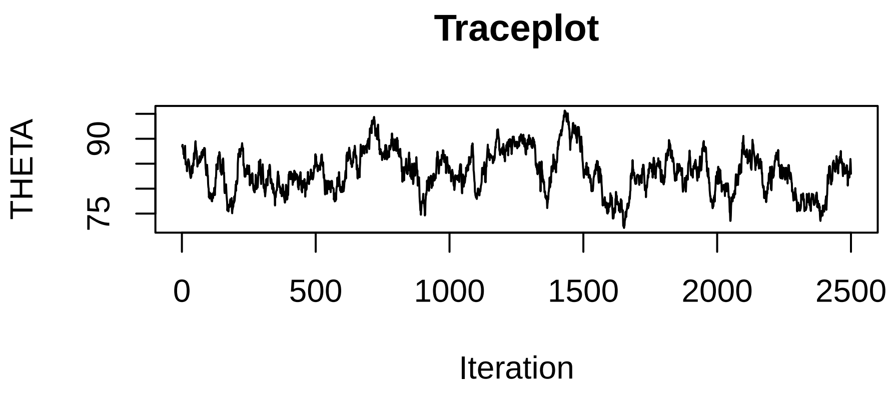
Here, we can see the difference between the Normal prior from Model 1 and the new Cauchy prior from this new Model 2:



Unfortunately, the Cauchy is not a conjugate prior for the Normal sampling model. Rather than try to derive the posterior exactly, let's instead use the Metropolis algorithm to approximate the posterior for θ!

```r
set.seed(412)
snow <- read_csv("../handouts/burlington_snow.csv")
y <- snow$snowfall
n <- length(y)
s2 <- var(y)
ybar <- mean(y)
loc0 <- 10*12
scale0 <- 12*(0.5)
S <- 5000
THETA <- rep(NA, S)

# keep track of how many times we accept
accept_vec <- rep(0, S)
delta <- 1 # a proposal of 1 inch
# initialize
theta <- loc0
for(s in 1:S){
  # step 1: propose
  theta_prop <- rnorm(1, theta, delta)

  # step 2: calculate acceptance ratio (demonstrate why we log instead of prod)
  log_r <- sum(dnorm(y, theta_prop, sqrt(s2), log = T)) -
    sum(dnorm(y, theta, sqrt(s2), log = T)) +
    dcauchy(theta_prop, loc0, scale0, log = T) -
    dcauchy(theta, loc0, scale0, log = T)

  # step 3: decide
  u <- runif(1)
  if(log(u) < log_r){ # remember: r is on log scale
    theta <- theta_prop
    accept_vec[s] <- 1
  }
  # else: keep theta where it currently is

  ## STORE like usual
  THETA[s] <- theta
}

# burn
THETA <- THETA[-c(1:(S/2))]
```

Let’s examine the traceplot for the chain after burn-in:

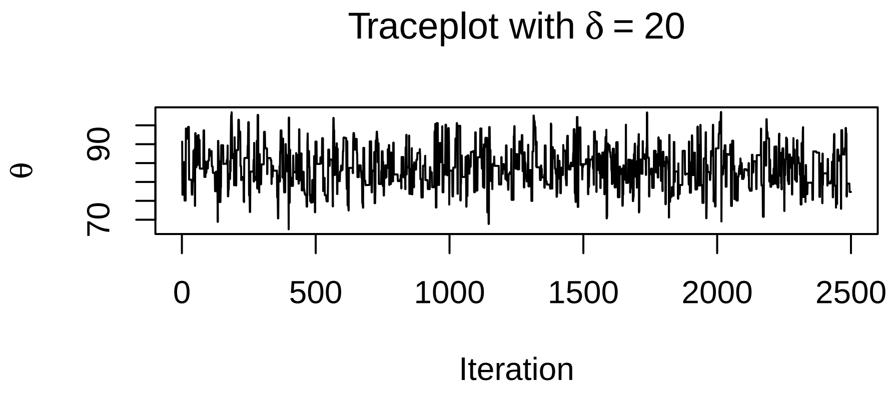


This looks bad! Why is that? Let's also examine the proportion of times we accepted the proposal in our sampler:

```r
mean(accept_vec)
```

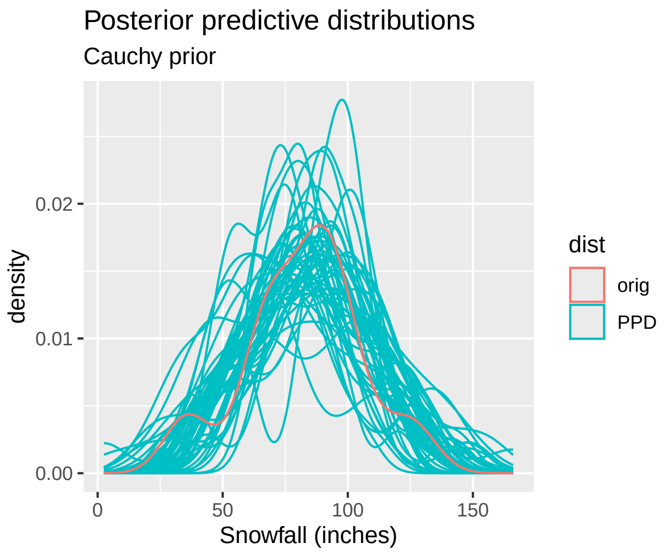
```
[1] 0.9446
```

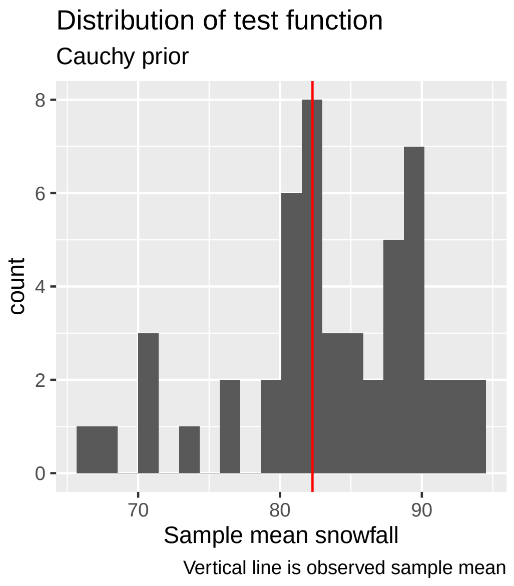
This is a very high acceptance probability! This means that about 95% of the time, we are accepting the proposed $\theta^{(prop)}$. This means that the δ we chose is too small; all proposed values are very similar to the "current" θ and are thus getting accepted. This leads to high autocorrelation and stickiness in our chain, and thus the bad-looking traceplots. So we should increase δ! Let's try increasing δ to see what happens. There is literature that suggests an ideal acceptance ratio for Metropolis is 0.234. Let's try to get close to that:



This traceplot with $\delta = 20$ looks much better. The acceptance ratio now is 0.3166, which is much closer to "ideal".

Let's now do the same PPCs to compare Model 2 to Model 1: Obtain 50 PPDs and calculate the test function of interest.





We can do posterior inference with θ as we usually would! For example, I can approximate the posterior mean of θ under this model by taking the mean of `THETA` vector.

Under Model 2, the posterior mean for average snowfall in a season is about 6.991 feet of snow. Compare this to the posterior mean under Model 1: 7.365 feet. Note that $\bar{y}_{obs} = 6.858$ feet. So what was the effect of changing the prior?

Remark on coding acceptance ratio

Note that we might that we are "lucky" that R provides the density functions `dnorm()` and `dcauchy()` for us. But we actually don't need that! As long as we have closed-form expressions for the likelihood and prior (up to proportionality), we can calculate the acceptance ratio:

```r
# set for demonstration
theta <- 100
theta_prop <- 102

# using R's functions
sum(dnorm(y, theta_prop, sqrt(s2), log = T)) -
    sum(dnorm(y, theta, sqrt(s2), log = T)) +
    dcauchy(theta_prop, loc0, scale0, log = T) -
    dcauchy(theta, loc0, scale0, log = T)
```

```
[1] -1.157209
```

In the following, we leverage that when we take log densities, things cancel out:

$$\log f(\mathbf{y}|\theta) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \theta)^2 \Rightarrow$$

$$\log f(\mathbf{y}|\theta^{(prop)}) - \log f(\mathbf{y}|\theta^{(s)}) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \theta^{(prop)})^2 - \left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \theta^{(s)})^2\right)$$

Similarly

$$\log f(\theta) = \log\left(\frac{1}{\pi\kappa\left(1 + \left(\frac{\theta-\mu_0}{\kappa}\right)^2\right)}\right) = -\left(\log(\pi k) + \log\left(1 + \left(\frac{\theta - \mu_0}{\kappa}\right)^2\right)\right) \Rightarrow$$

$$\log f(\theta^{(prop)}) - \log f(\theta^{(s)}) = -\log\left(1 + \left(\frac{\theta^{(prop)} - \mu_0}{\kappa}\right)^2\right) - \left(\log\left(1 + \left(\frac{\theta^{(s)} - \mu_0}{\kappa}\right)^2\right)\right)$$

```r
# typing out the densities ourselves (up to proportionality)
(-0.5)/s2 * sum((y - theta_prop)^2) - # log likelihood under theta_prop
  (-0.5)/s2 * sum((y - theta)^2) + # log likelihood under current theta
  - log(1 + ((theta_prop - loc0)/scale0)^2) - # log prior at theta_prop
  - log(1 + ((theta- loc0)/scale0)^2) # log prior at current theta
```

```
[1] -1.157209
```

Notice that we get the same acceptance ratio!

This is especially useful when the likelihood is not one already provided by R.