Final Presentation

Reducing Maximization Bias and Risk in Hyperparameter Optimization for Reinforcement Learning and Learning-Based Control

Master Thesis

Tsung Yuan Tseng [1,2]

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October 31, 2023

Introduction and Motivation

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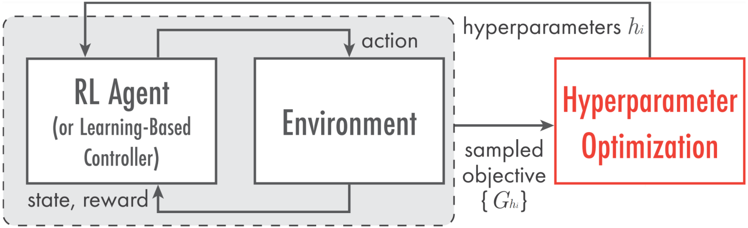

Decision-Making Algorithms

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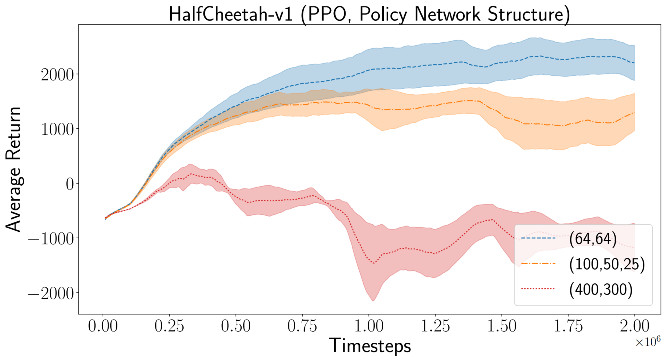

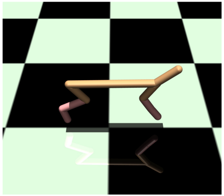

[a] (Bischl et al. 2021; Henderson et al. 2018)

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- Robust hyperparameters are essential.
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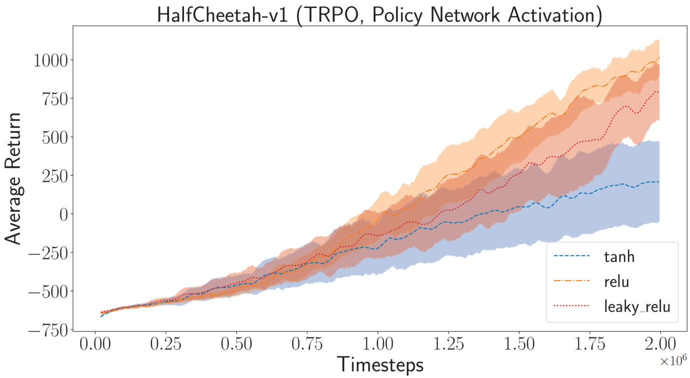

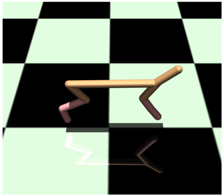

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Related Works

- Hyperparameter optimization (HPO)

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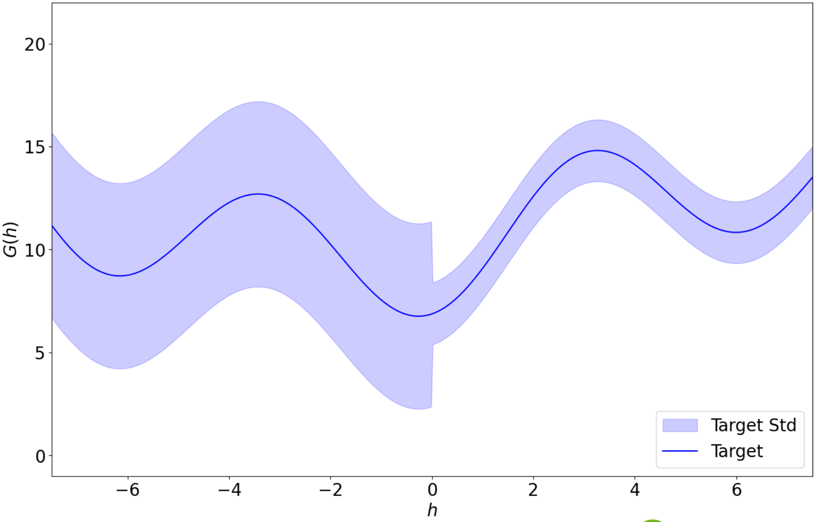
Random Search



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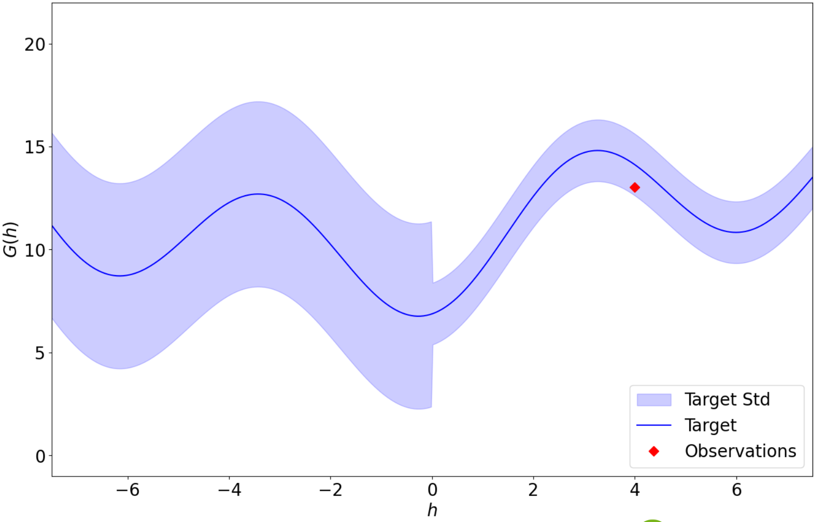

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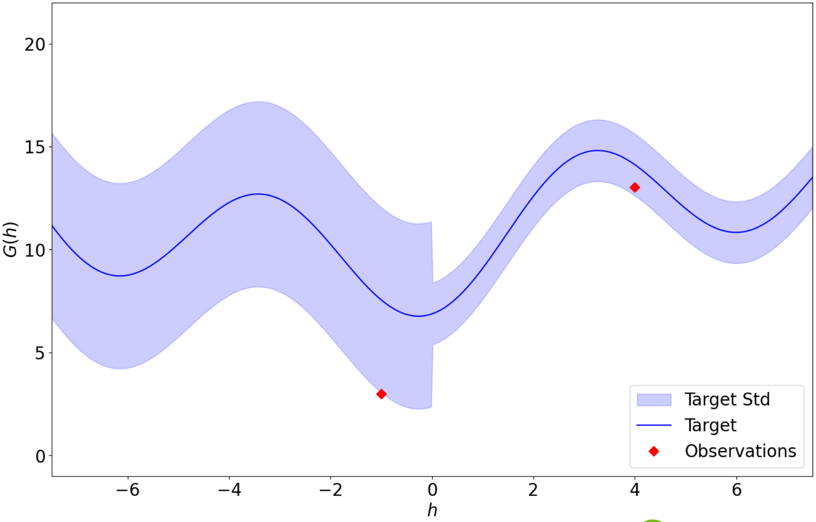

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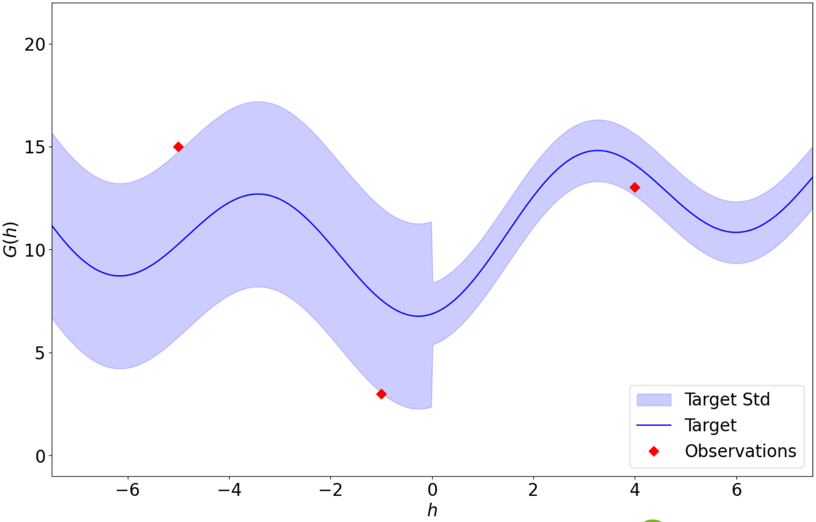

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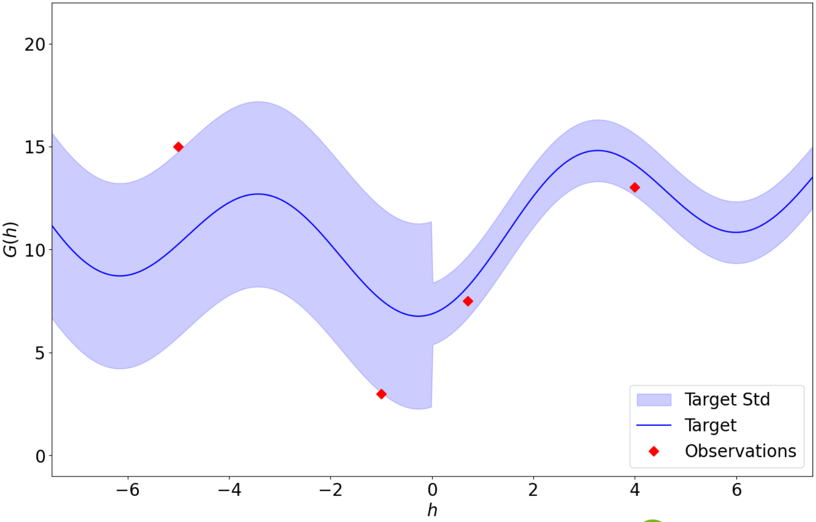

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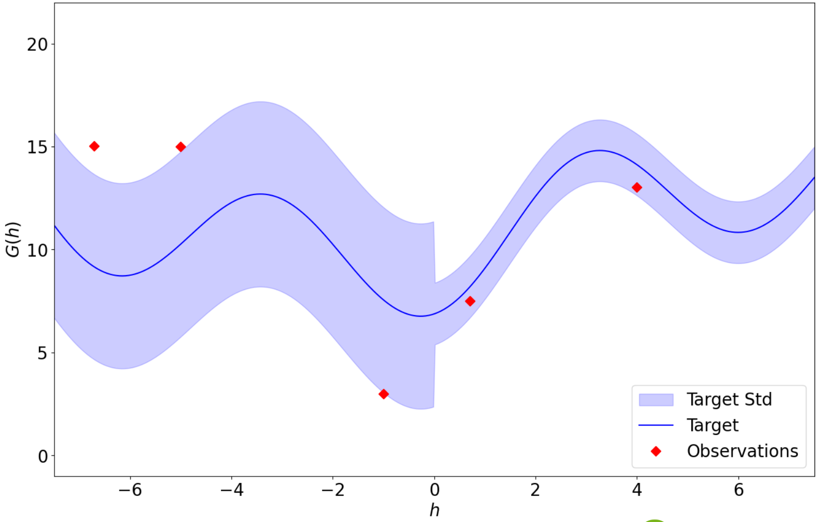

Current Challenges: Maximization bias

- Maximization bias in HPO (synonyms are overconfidence, overtuning, and overfitting). [5]

[5] Hutter et al. 2007; Patterson et al. 2023

Problem Statement

Problem Formulation

- Evaluation metrics G;
- Hyperparameter $\boldsymbol{h} \in \mathcal{H}$;

$$\boldsymbol{h}^* = \arg\max_{\boldsymbol{h} \in \mathcal{H}} \rho\left[G|\boldsymbol{h}\right],$$

where ρ is a risk functional.

Goal of the Thesis

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Goal of the Thesis

Extract robust $\boldsymbol{h}^*$ by observing data sequences $\{(\boldsymbol{h}_i, \hat{\rho}[G|\boldsymbol{h}_i])\}_{i=1}^{M}$.

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Analysis on Maximization Bias

- The value of $\max\{\hat{\mu}_{\boldsymbol{h}_i}\}_{i=1}^{M}$ has a great impact on final selection of $\boldsymbol{h}^*$
- $\max\{\hat{\mu}_{\boldsymbol{h}_i}\}_{i=1}^{M} \stackrel{?}{=} \max\{\mathbb{E}[G_{\boldsymbol{h}_i}]\}_{i=1}^{M}$

Analysis on Maximization Bias

Lemma: Lower Bound of Maximization Bias

Consider

- Sampled set of hyperparameters $\mathbb{H} = \{\boldsymbol{h}_i\}_{i=1}^{M}$

Use

Then

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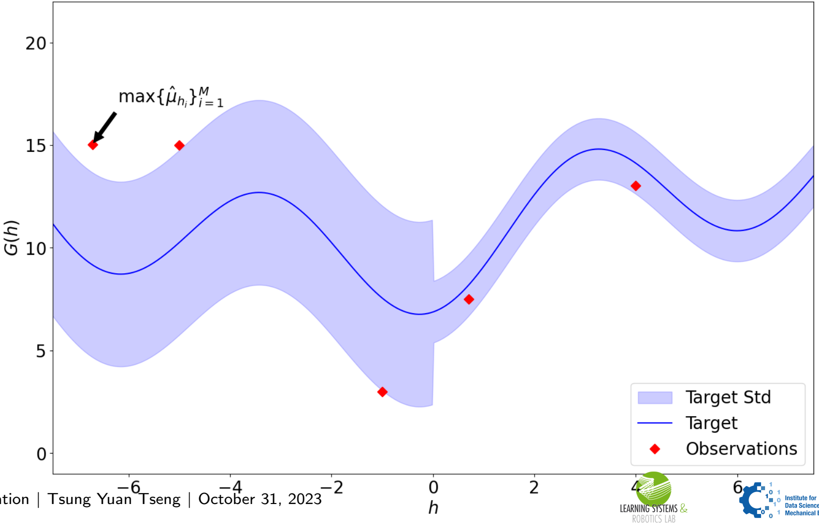
- $\mathbb{E}[\max_{\boldsymbol{h}_i \in \mathbb{H}} \hat{\mathcal{G}}] - \max_{\boldsymbol{h}_i \in \mathbb{H}} \mathbb{E}[\mathcal{G}] \geq 0$

Analysis on Maximization Bias

- Uncertainties in $\hat{\mathcal{G}} = \{\hat{\mu}_{h_i}\}_{i=1}^{M}$ leads to sub-optimal decisions.



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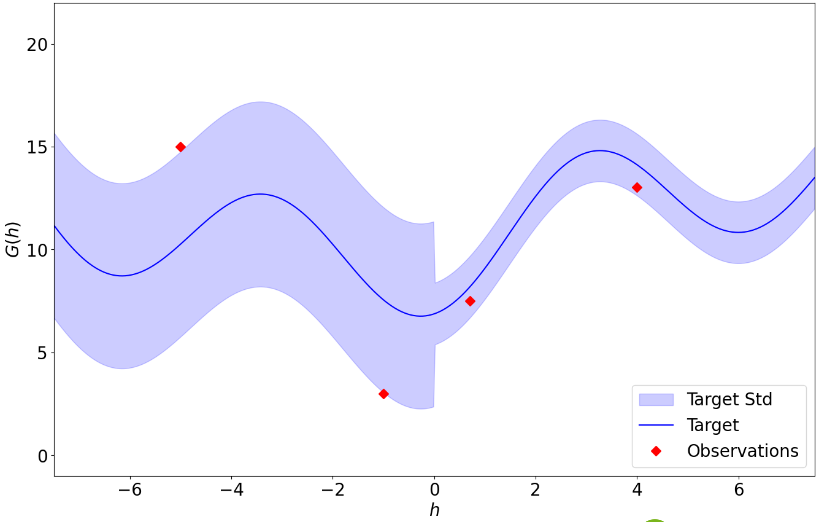
A run means one realization of G

Methods: Adaptive Variance Reduction



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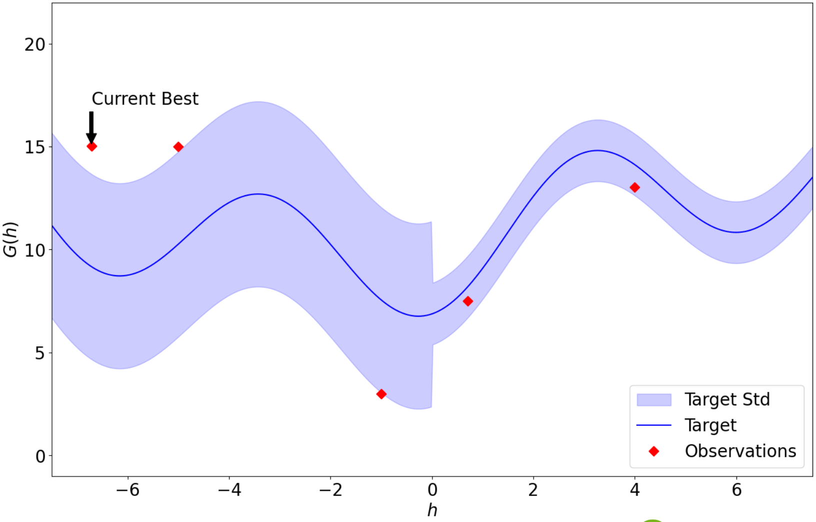

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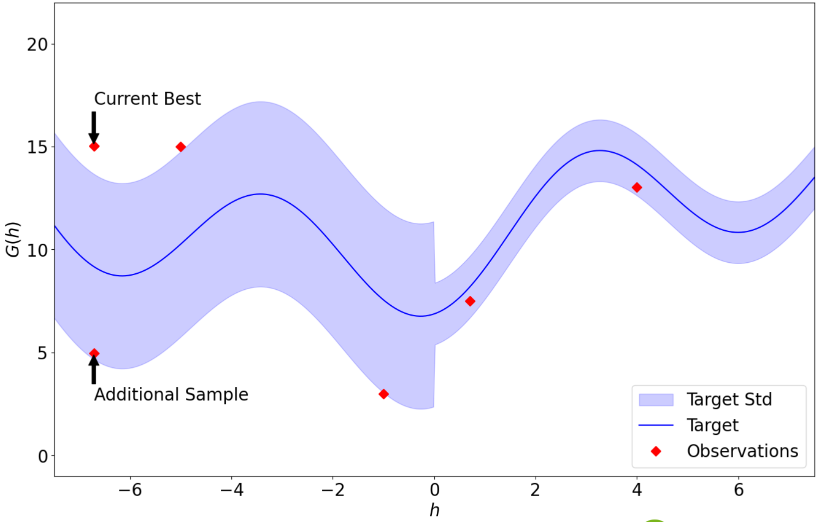

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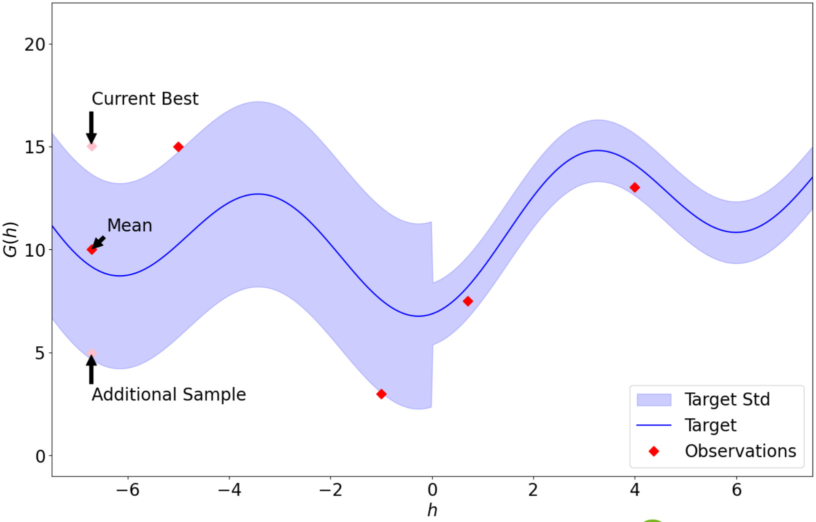

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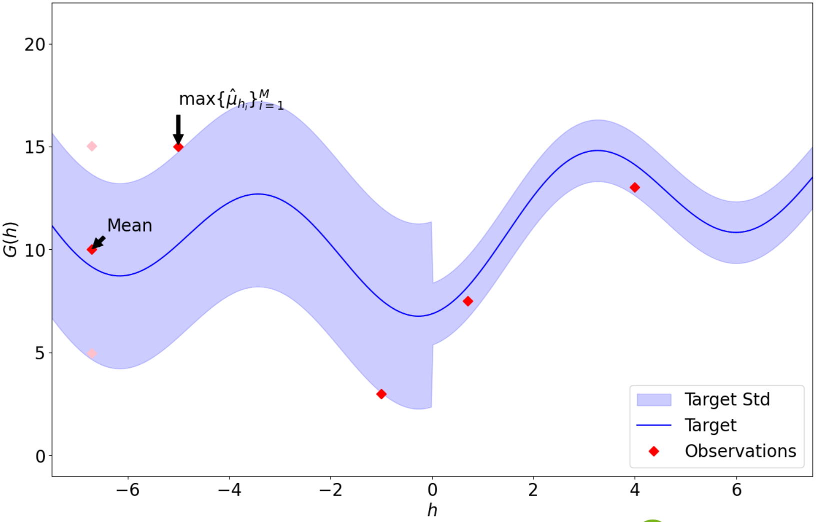

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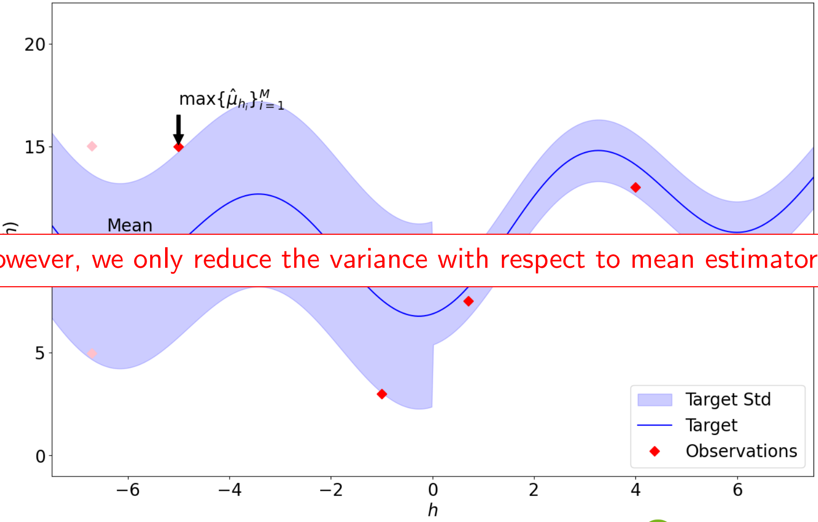
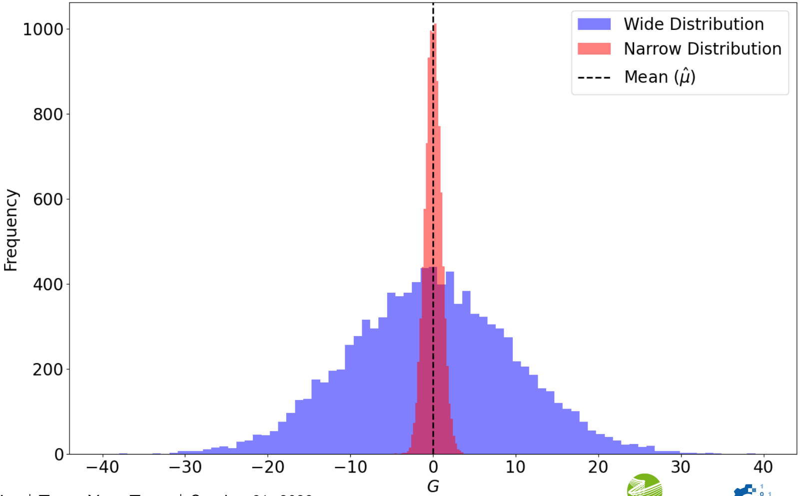


However, we only reduce the variance with respect to mean estimators...

Methods: Risk in Tail Distributions

- Is the expectation a good risk measure (risk functional)?



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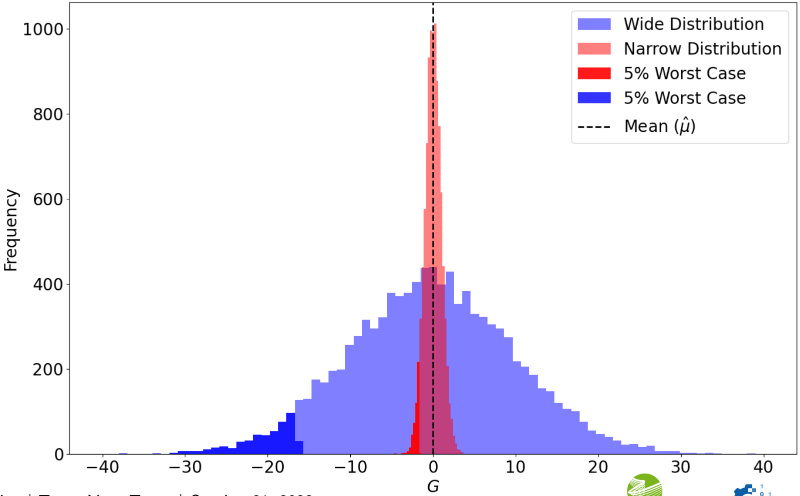
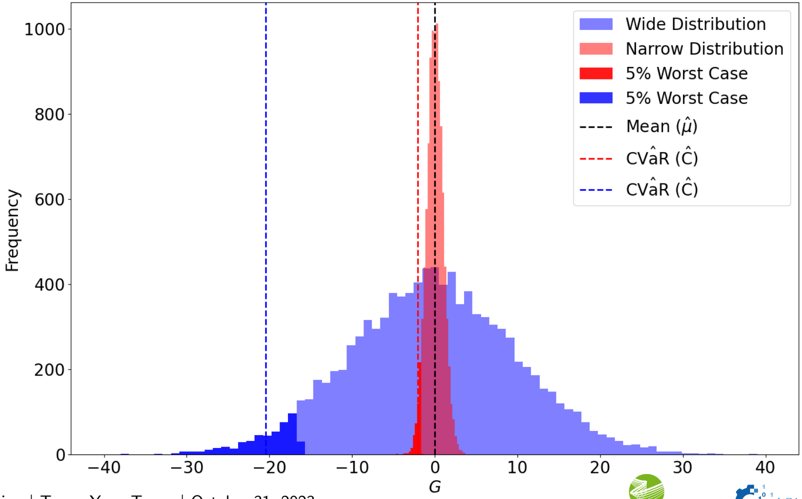


Methods: Risk in Tail Distributions

- Conditional Value at Risk (CVaR)



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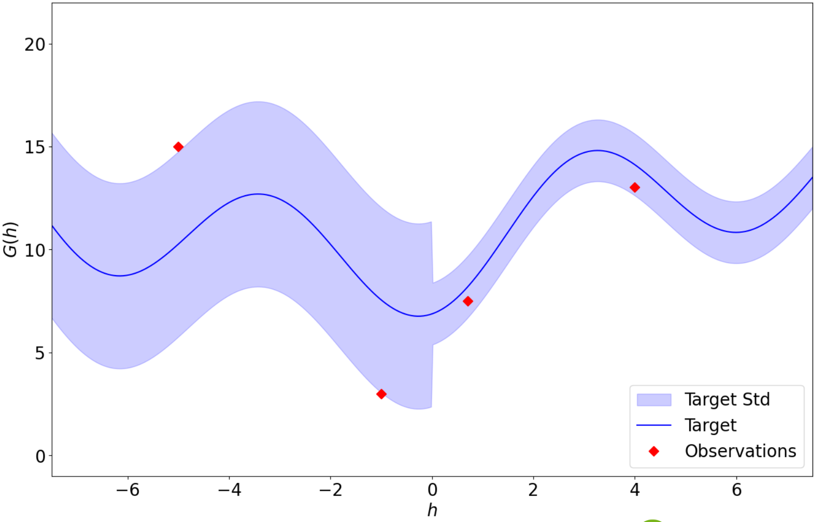

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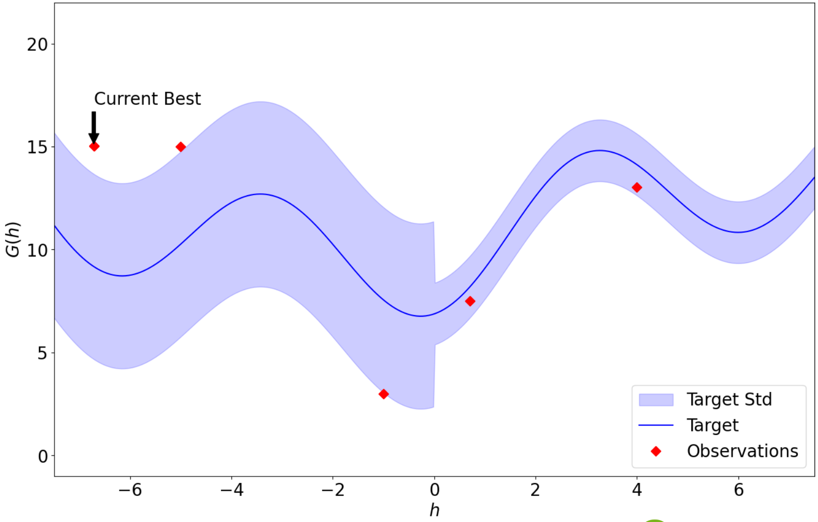

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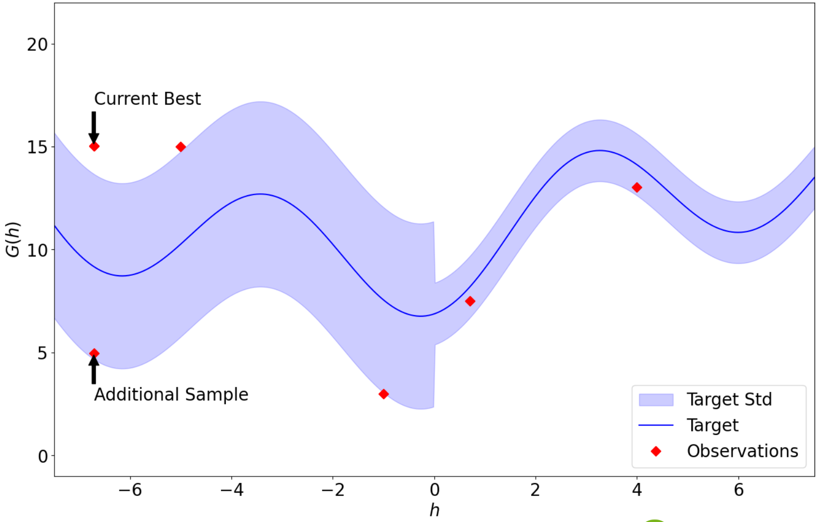

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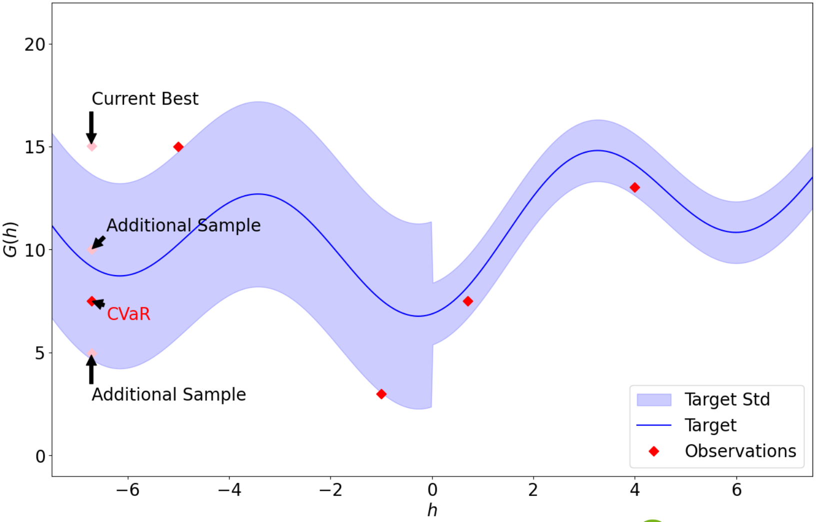

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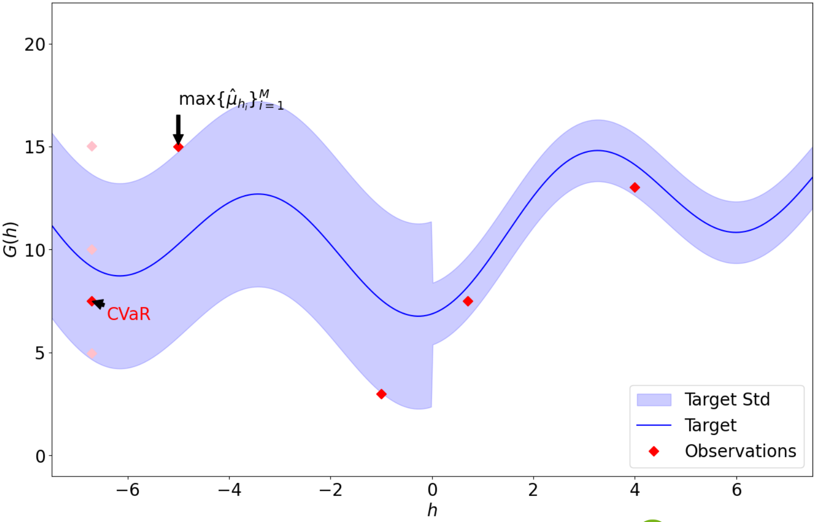

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- Convergence criterion?

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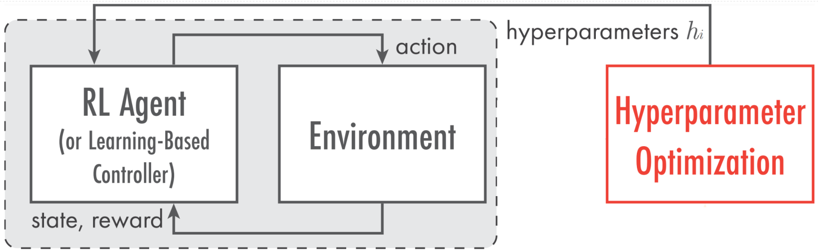
- $|\hat{\text{CVaR}}^{s}[G_{\boldsymbol{h}_i}] - \hat{\text{CVaR}}^{ss}[G_{\boldsymbol{h}_i}]| < \delta$.

Methods: Overview



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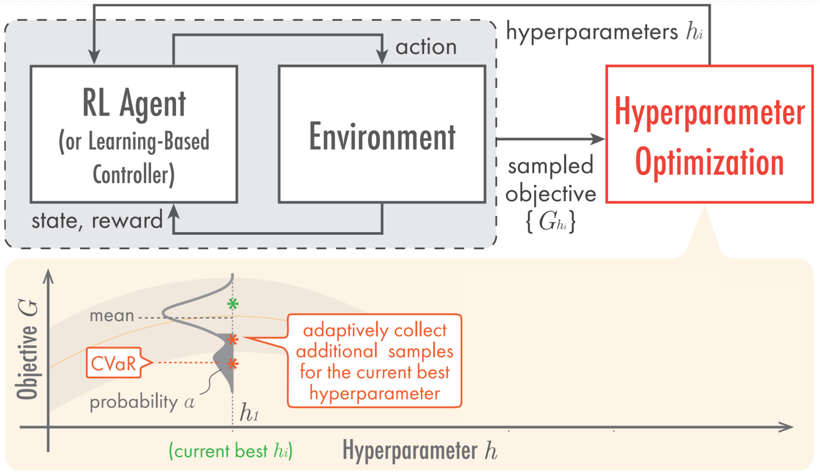

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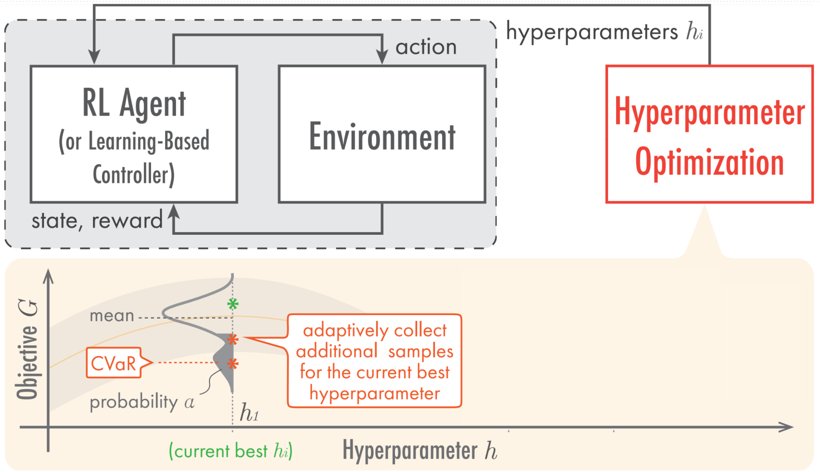

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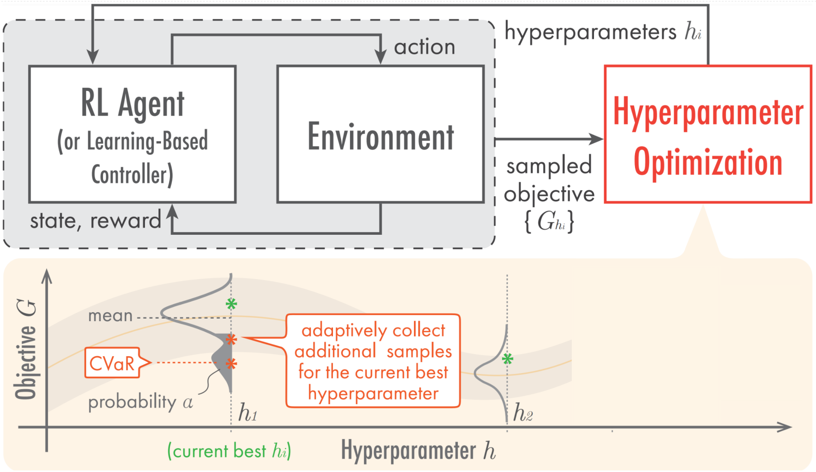

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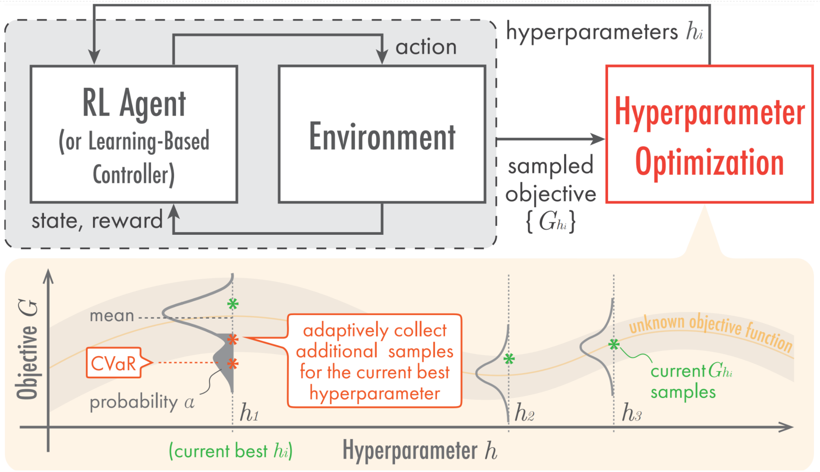

HPO Strategy Comparisons

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- Impact of number of samples per $G(h)$?

HPO Strategy Comparisons

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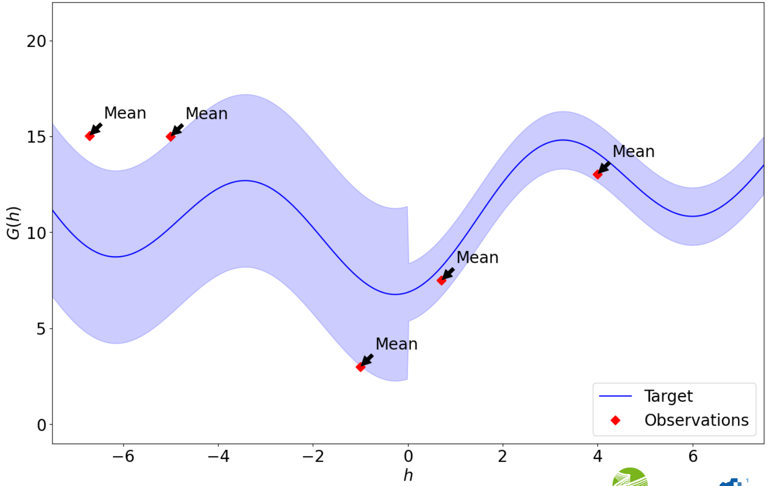
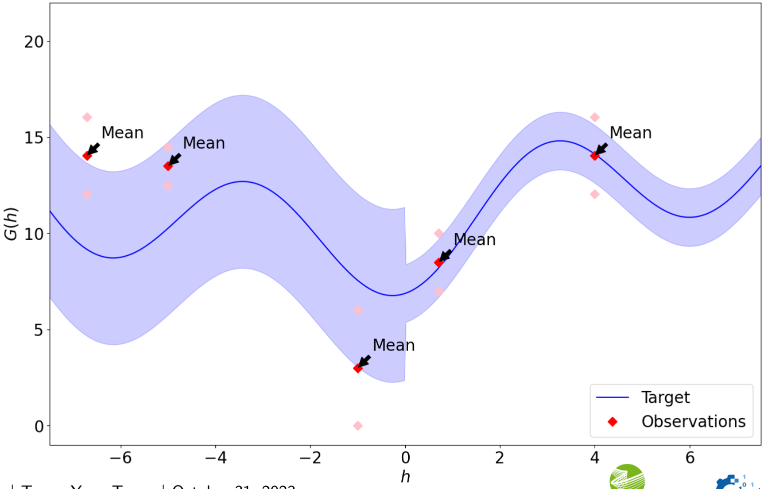


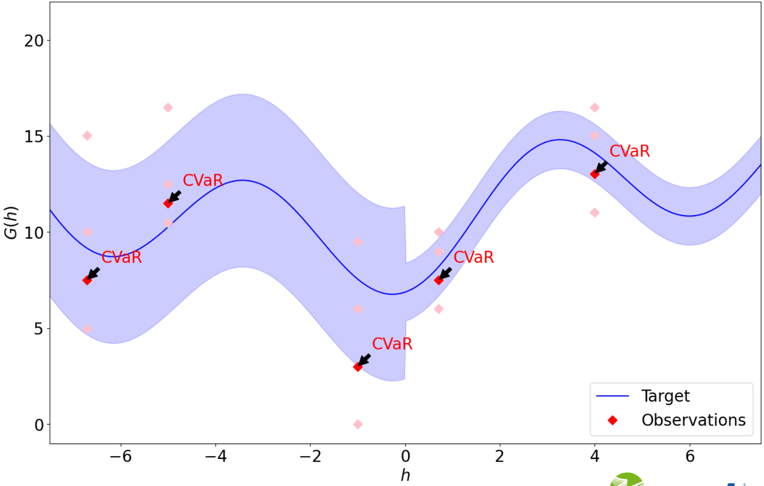
HPO Strategy Comparisons

- Multiple runs: $\mathbb{E}[G_{h_i}] \approx \sum_j S^j_{G_{h_i}}/N$.



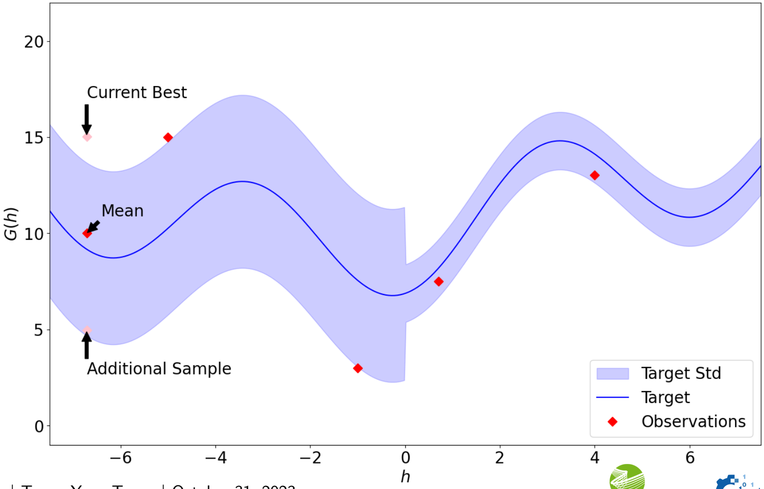
HPO Strategy Comparisons

- Multiple runs with CVaR: CVaR$[G_{h_i}] \approx \hat{\text{CVaR}}_N[G_{h_i}]$.



HPO Strategy Comparisons

- AMRA with expectations: $\mathbb{E}[G_{h_i}] \approx \sum_j S^j_{G_{h_i}} / N_i$.



HPO Strategy Comparisons

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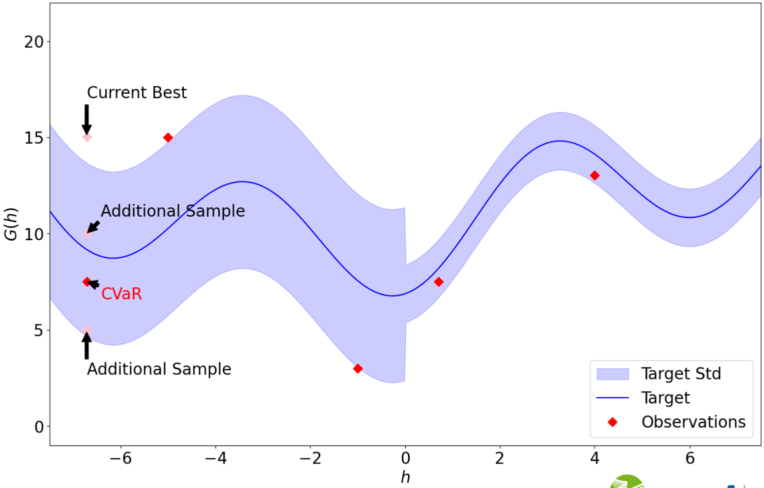
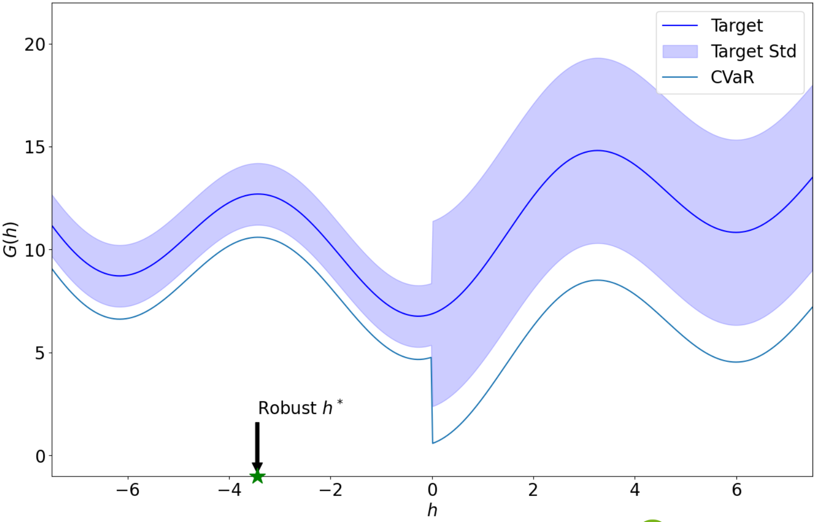


Experiments: Synthetic Examples



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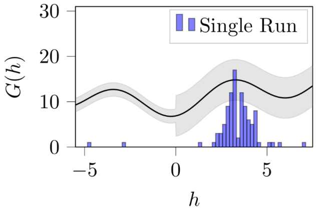

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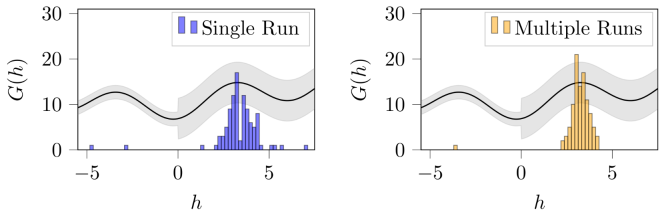

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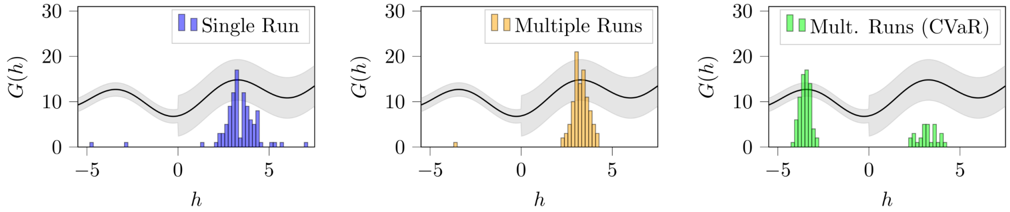

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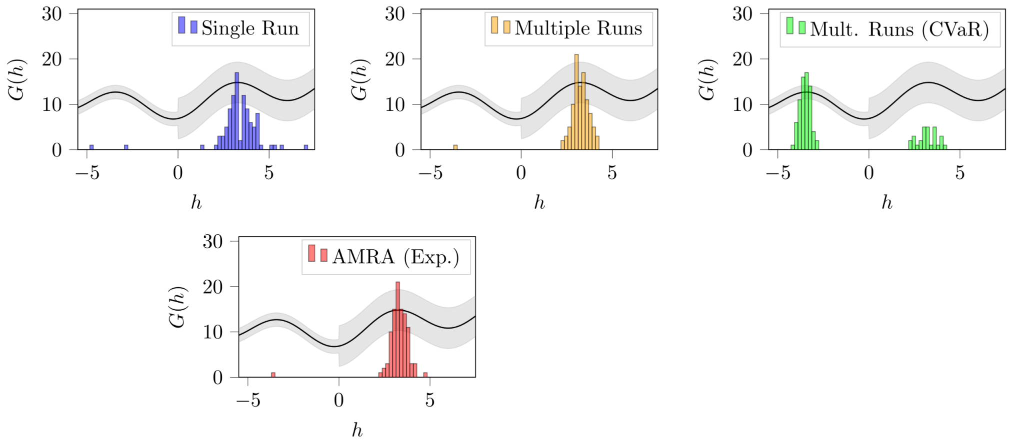

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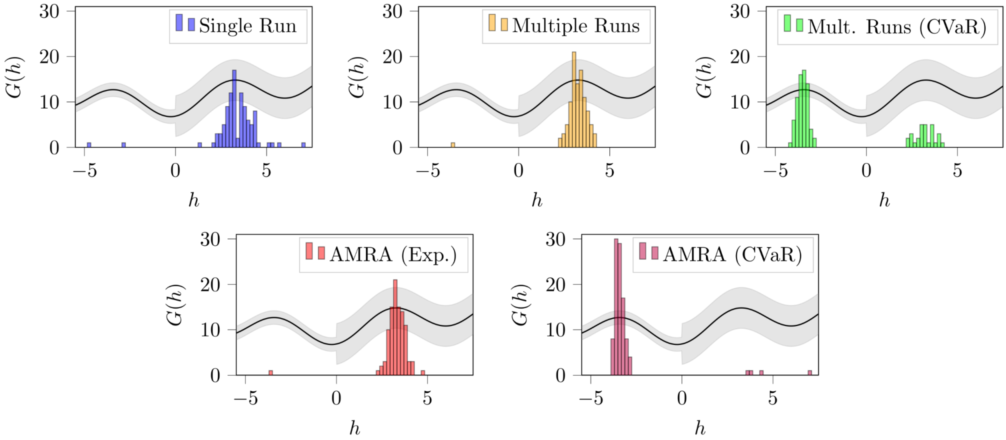
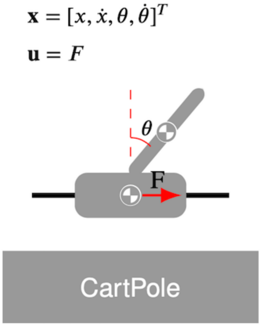


Experiments: Decision-Making Algorithms

- Benchmarking environment: Safe-Control-Gym (SCG).
- Control tasks: Cartpole system.
- DDPG, PPO, SAC, and GP-MPC.



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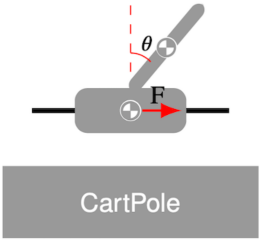
$$\mathbf{x} = [x, \dot{x}, \theta, \dot{\theta}]^T$$

$$\mathbf{u} = F$$



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- Unifying evaluation metric G

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- Reward per step i as $J_i = \exp\left(-\left(\boldsymbol{x}_i^T \boldsymbol{Q} \boldsymbol{x}_i + \boldsymbol{u}_i^T \boldsymbol{R} \boldsymbol{u}_i\right)\right)$

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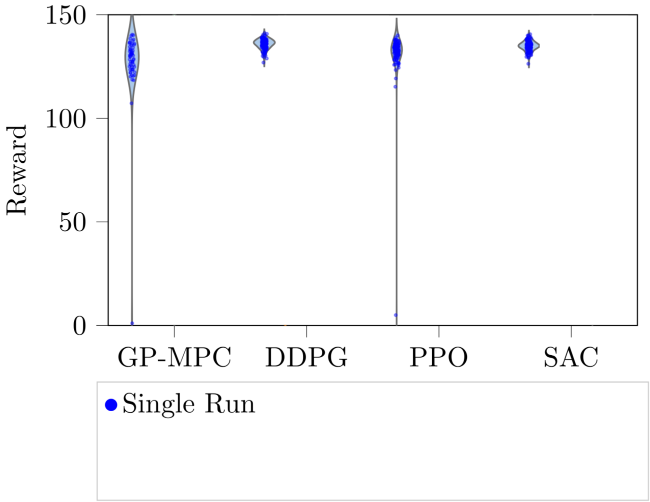
A run means one training-and-evaluation loop of a control algorithm

Experiments: Decision-Making Algorithms



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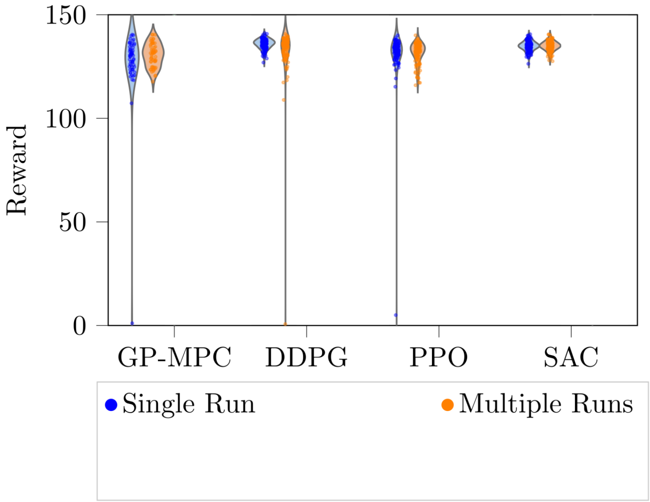

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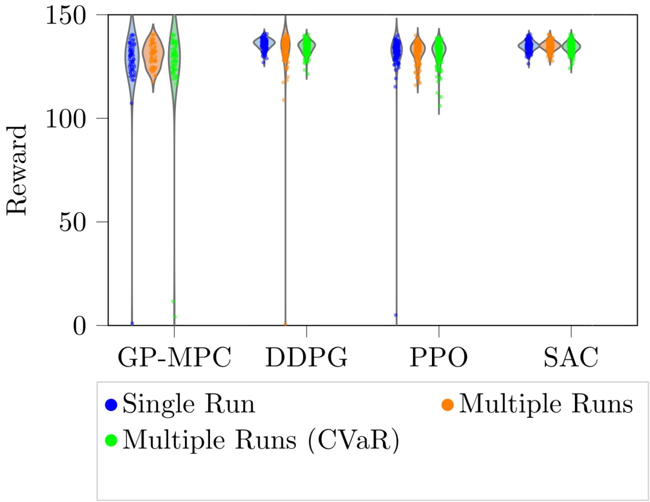

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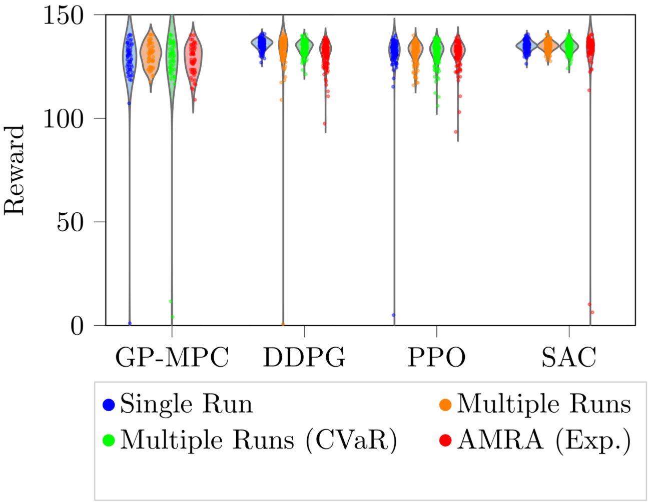

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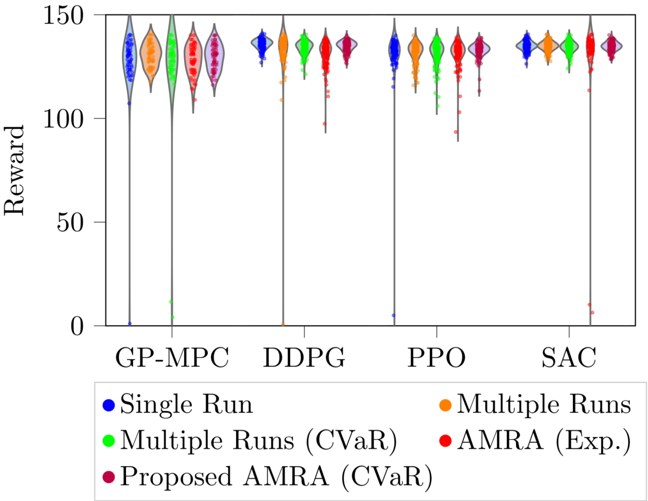

LEARNING SYSTEMS & ROBOTICS LAB

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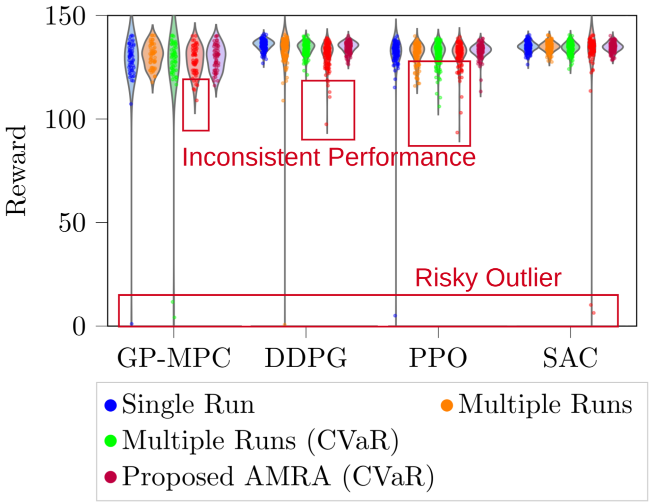

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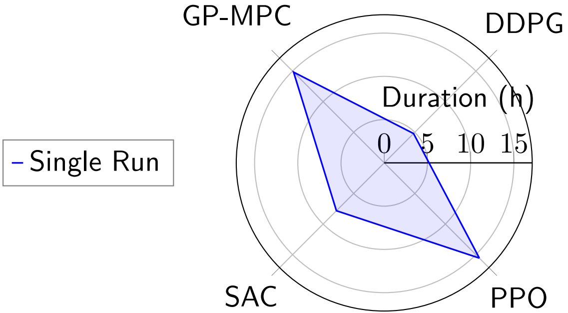

LEARNING SYSTEMS & ROBOTICS LAB TUM AND UofT
Institute for Data Science in Mechanical Engineering
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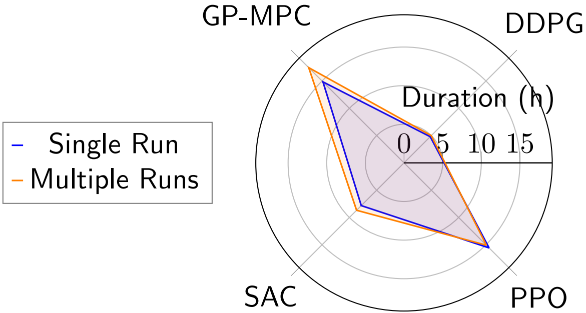
Experiments: Decision-Making Algorithms



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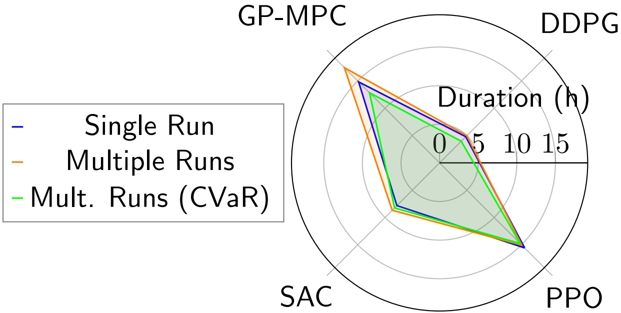

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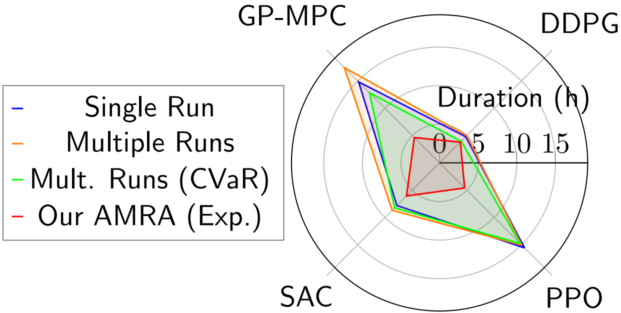

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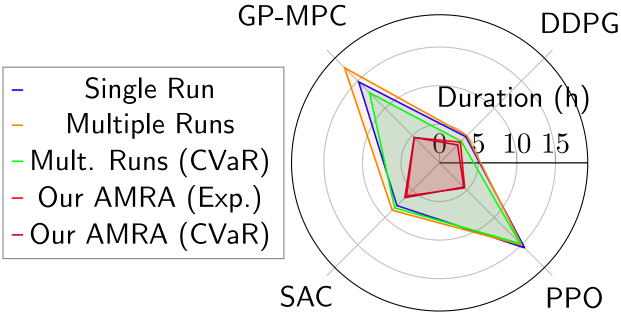

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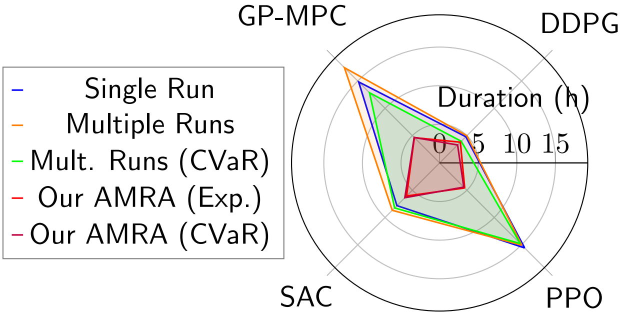

AMRA requires less computational time.

Summary

- We propose a robust and efficient HPO algorithm (AMRA with CVaR).

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- We propose a robust and efficient HPO algorithm (AMRA with CVaR).
  - Adaptive sampling scheme $\rightarrow$ (i) efficiency (ii) bias.
  - CVaR $\rightarrow$ risk in tail distributions.
- Robust performance of selected control algorithms is finally attained.
- Reduce computation effort by 25% to 66% across different control algorithms.

Future Works

- Evaluate our method on well-established and widely-used RL platforms such as OpenAI gym.

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  - Optimize h that yields consistent good performance for selected algorithms for fairer comparisons.

Q & A

Thank You!

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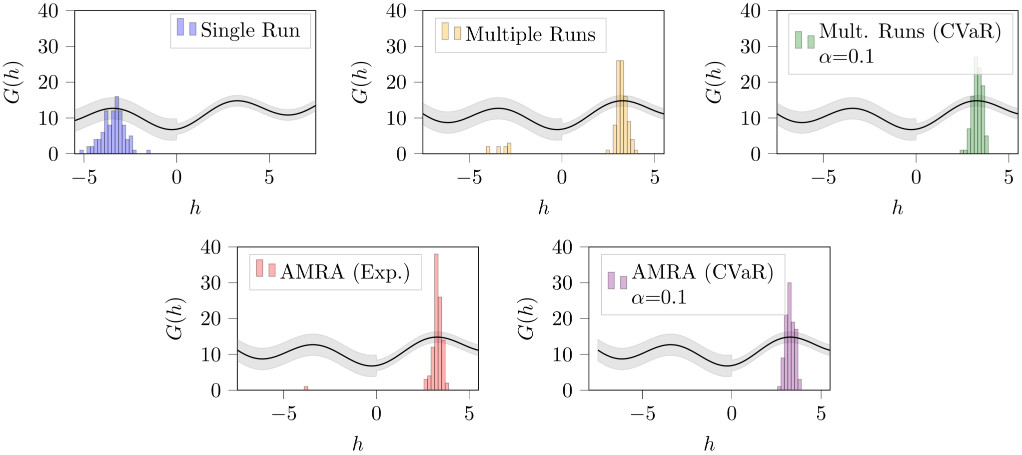
Appendix: Method

Algorithm 2 Adaptive Multiple Runs Algorithm with General Risk Functional

procedure ADAPTIVE MULTIPLE RUNS ALGORITHM WITH GENERAL RISK FUNCTIONAL
    Initialize an HPO algorithm $\mathcal{A}$, estimator $\hat{\rho}_N[G_{\boldsymbol{h}}]$ of an agent given a environment ϵ, number of basic agent runs N, incremented agent runs N^+, number of trials M, dataset $\mathcal{D} = \emptyset$, and $\hat{\rho}^{ss}[G_{\boldsymbol{h}}] = \inf$
    for each trial **do**
        $\boldsymbol{h}_i \leftarrow \mathcal{A}(\mathcal{D})$ ▷ Sample next HPs
        Reset N to the initial value
        $\hat{\rho}^s[G_{\boldsymbol{h}_i}] \leftarrow \hat{\rho}_N[G_{\boldsymbol{h}_i}]$
        $\hat{\rho}[G_{\boldsymbol{h}_i}] \leftarrow \hat{\rho}^s[G_{\boldsymbol{h}_i}]$
        if $\hat{\rho}^s[G_{\boldsymbol{h}_i}]$ is current best trial **then**
            while $|\hat{\rho}^s[G_{\boldsymbol{h}_i}] - \hat{\rho}^{ss}[G_{\boldsymbol{h}_i}]| > \delta$ **do**
                if not the first iteration **then**
                    $\hat{\rho}^s[G_{\boldsymbol{h}_i}] \leftarrow \hat{\rho}^{ss}[G_{\boldsymbol{h}_i}]$
                end if
                $N \leftarrow N + N^+$
                $\hat{\rho}^{ss}[G_{\boldsymbol{h}_i}] \leftarrow \hat{\rho}_N[G_{\boldsymbol{h}_i}]$
            end while
            $\hat{\rho}[G_{\boldsymbol{h}_i}] \leftarrow \hat{\rho}^{ss}[G_{\boldsymbol{h}_i}]$
        end if
        $\mathcal{D} \leftarrow \mathcal{D} \cup (\boldsymbol{h}_i, \hat{\rho}[G_{\boldsymbol{h}_i}])$
    end for
    return $\boldsymbol{h}^* \leftarrow \underset{\boldsymbol{h}_i \in \mathbb{H}}{\arg\max}\ \hat{\rho}[G_{\boldsymbol{h}_i}]$
end procedure

Appendix: Additional Experiments



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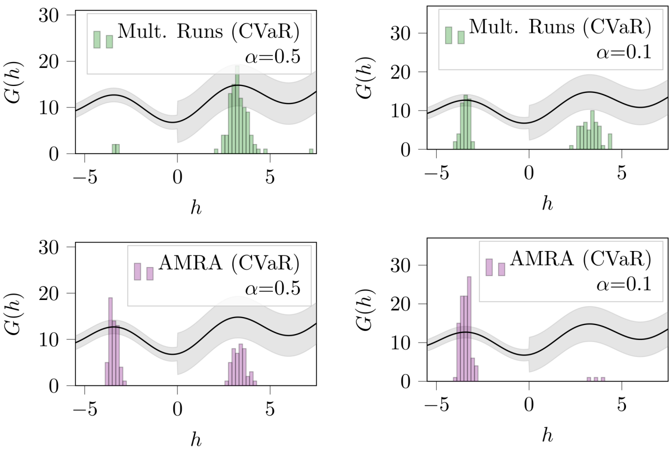

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