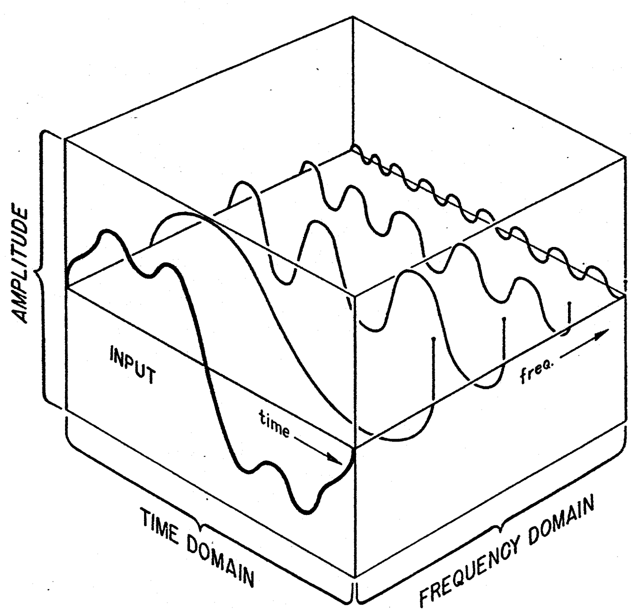
ME 22.403
Mechanical Lab II

Basics of Spectrum Analysis/Measurements and the FFT Analyzer

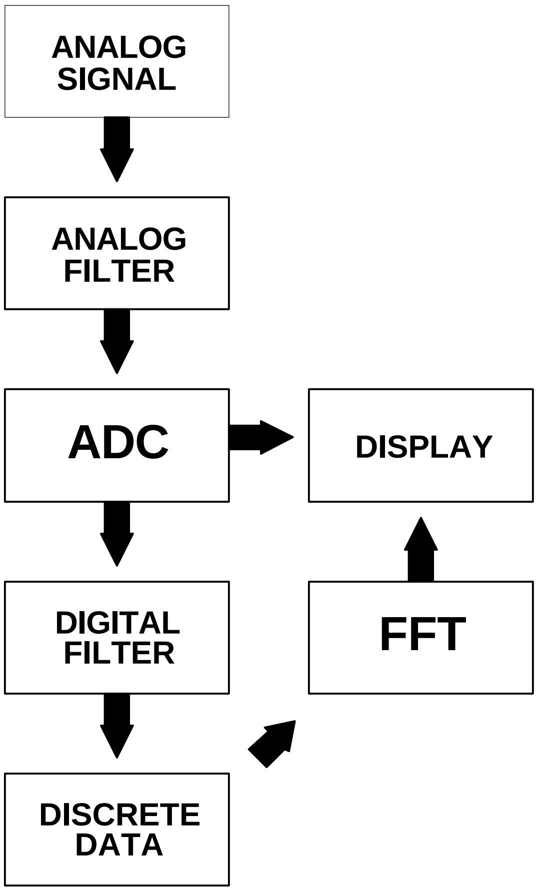
Transformation of Time to Frequency

Many times a transformation is performed to provide a better or clearer understanding of a phenomena. The time representation of a sine wave may be difficult to interpret. By using a Fourier series representation, the original time signal can be easily transformed and much better understood.



Transformations are also performed to respresent the same data with significantly less information. Notice that the original time signal was defined by many discrete time points (ie, 1024, 2048, 4096 …) whereas the equivalent Fourier representation only requires 4 amplitudes and 4 frequencies.

The Anatomy of the FFT Analyzer

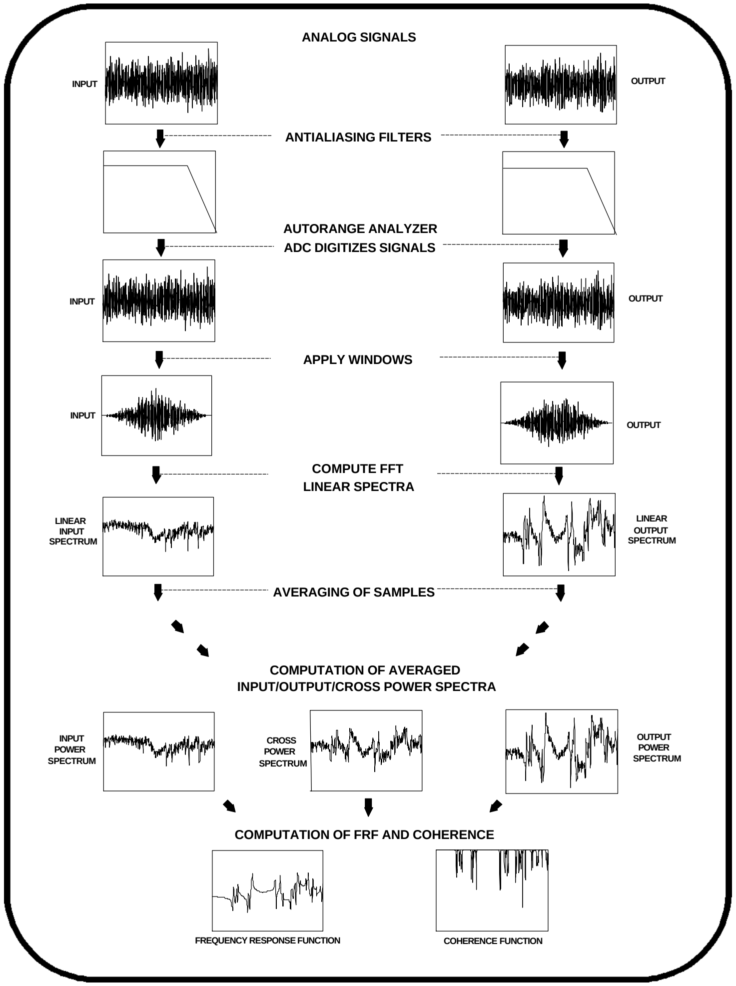


The FFT Analyzer can be broken down into several pieces which involve the digitization, filtering, transformation and processing of a signal.

Several items are important here:

- Digitization and Sampling
- Quantization of Signal
- Aliasing Effects
- Leakage Distortion
- Windows Weighting Functions
- The Fourier Transform
- Measurement Formulation

The Anatomy of the FFT Process



Actual time signals

Analog anti-alias filter

Digitized time signals

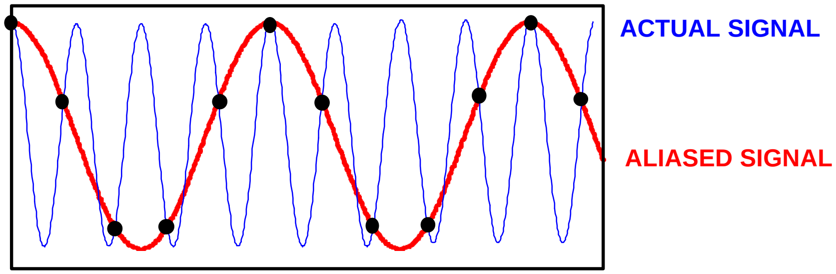
Windowed time signals

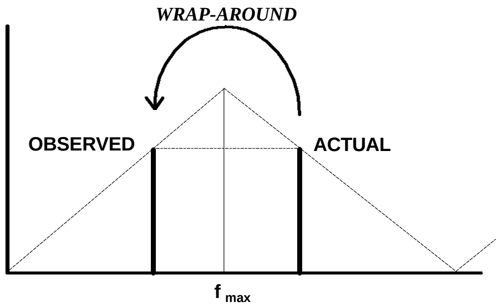
Compute FFT of signal

Average auto/cross spectra

Compute FRF and Coherence

Aliasing (Wrap-Around Error)





Aliasing results when the sampling does not occur fast enough.

Sampling must occur faster than twice the highest frequency to be measured in the data - sampling of 10 to 20 times the signal is sufficient for most time representations of varying signals

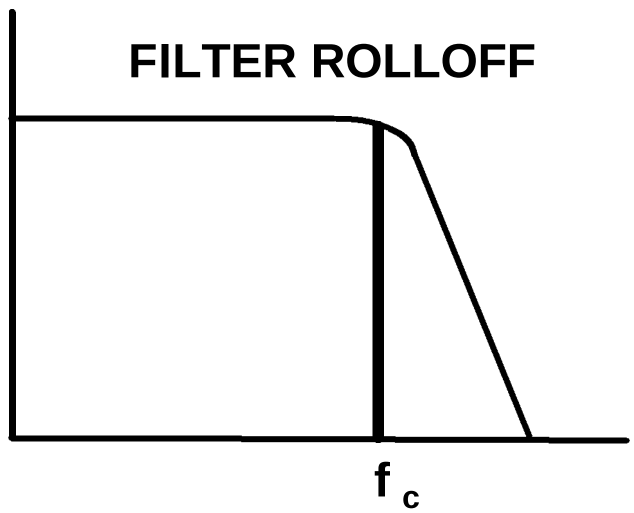
However, in order to accurately represent a signal in the frequency domain, sampling need only occur at greater than twice the frequency of interest

Anti-aliasing filters are used to prevent aliasing

These are typically Low Pass Analog Filters

Anti-Aliasing Filters

Anti-aliasing filters are typically specified with a cut-off frequency. The roll-off of the filter will determine how quickly the signal will be attenuated and is specified in dB/octave



$$G_{dB} = 20\log_{10} G = 20\log_{10} \frac{V_{out}}{V_{in}}$$

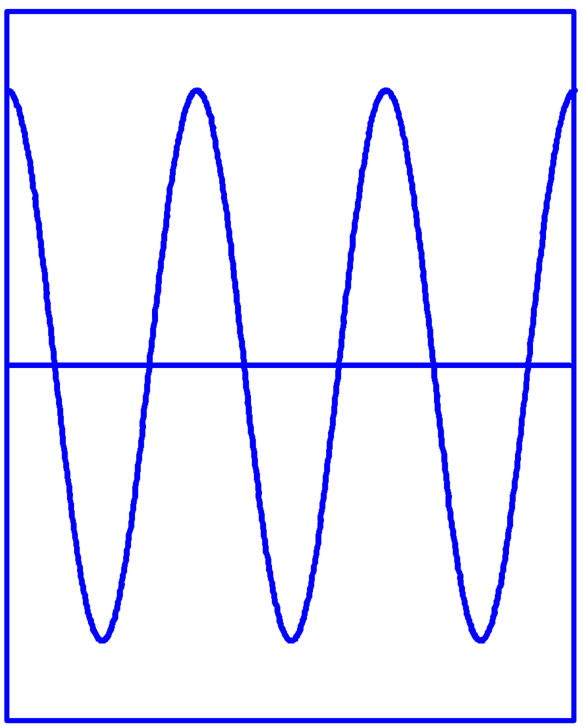
The cut-off frequency is usually specified at the 3 dB down point (which is where the filter attenuates 3 dB of signal).

Butterworth, Chebyshev, elliptic, Bessel are common filters

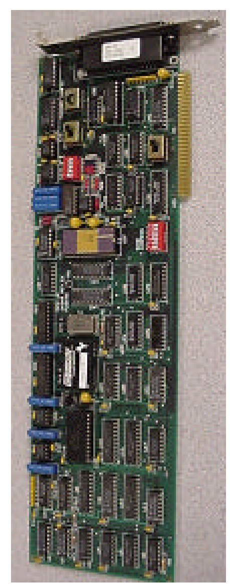
Digitization of a Signal

Sampling rate of the ADC is specified as a maximum that is possible. Basically, the digitizer is taking a series of “snapshots” at a very fast rate as time progresses

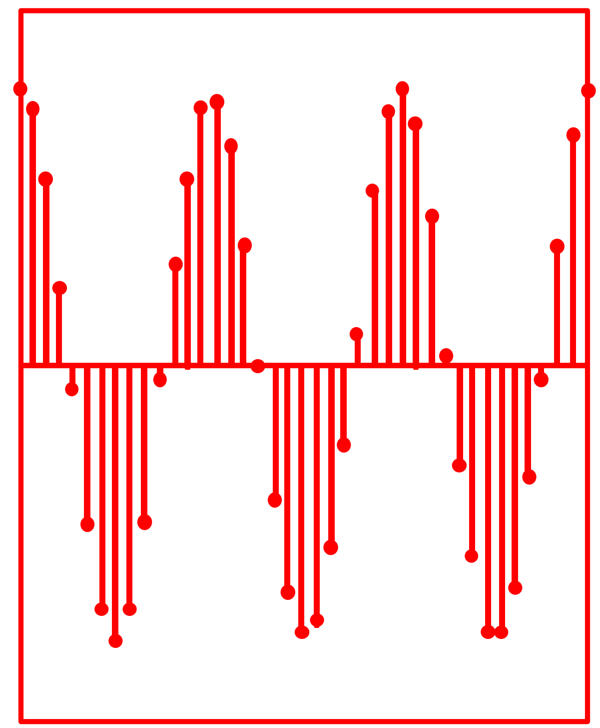
Analog Signal



ADC

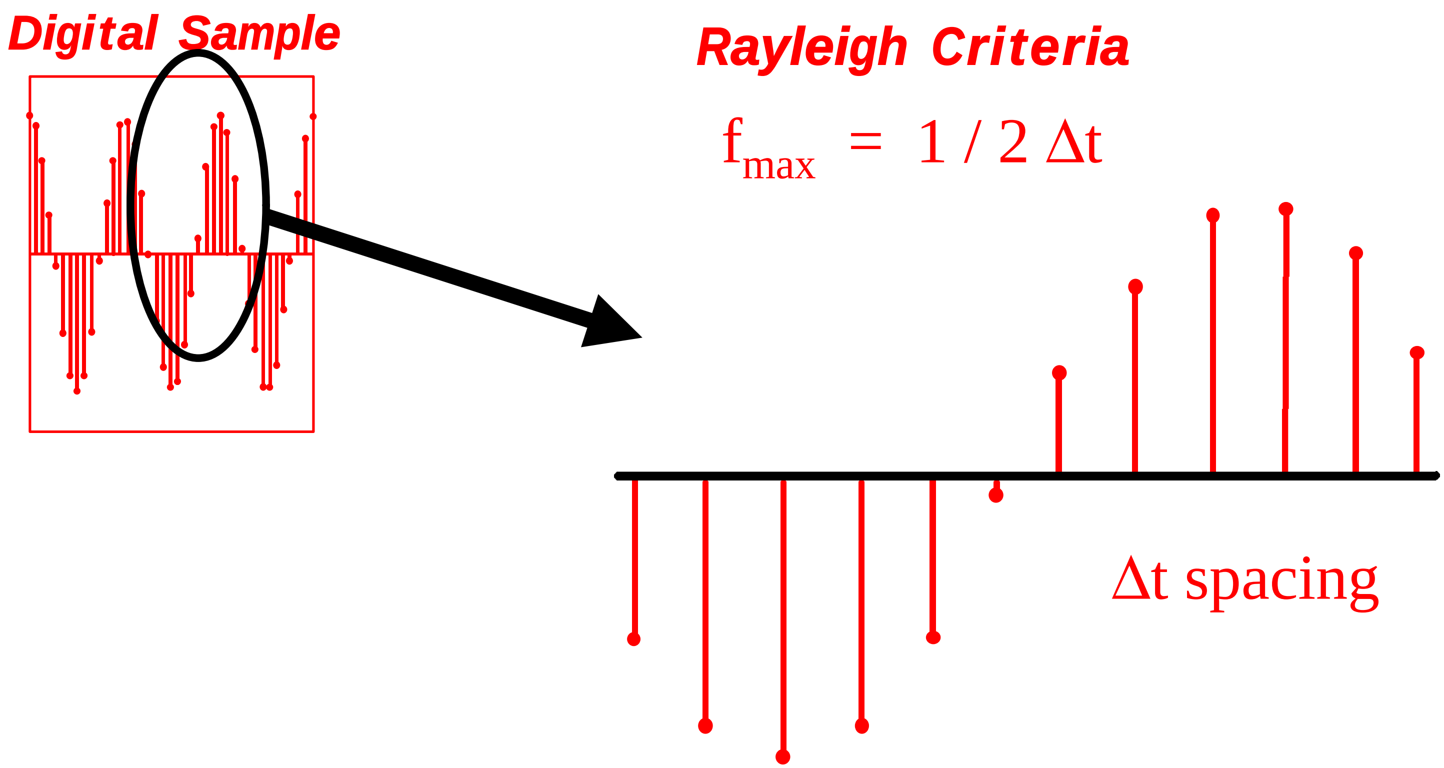


Digital Representation



Sampling

Each sample is spaced delta t seconds apart. Sufficient sampling is needed in order to assure that the entire event is captured. The maximum observable frequency is inversely proportional to the delta time step used



Sampling Theory

In order to extract valid frequency information, digitization of the analog signal must occur at a certain rate.

Shannon's Sampling Theorem states $f_s > 2 f_{max}$

That is, the sampling rate must be at least twice the desired frequency to be measured.

For a time record of T seconds, the lowest frequency component measurable is $\Delta f = 1 / T$

With these two properties above, the sampling parameters can be summarized as

$$f_{max} = 1 / 2 \Delta t$$

$$\Delta t = 1 / 2 f_{max}$$

Sampling Parameters

Due to the Rayleigh Criteria and Shannon's Sampling Theorum, the following sampling parameters must be observed.

With respect to the number of sample increments per period N

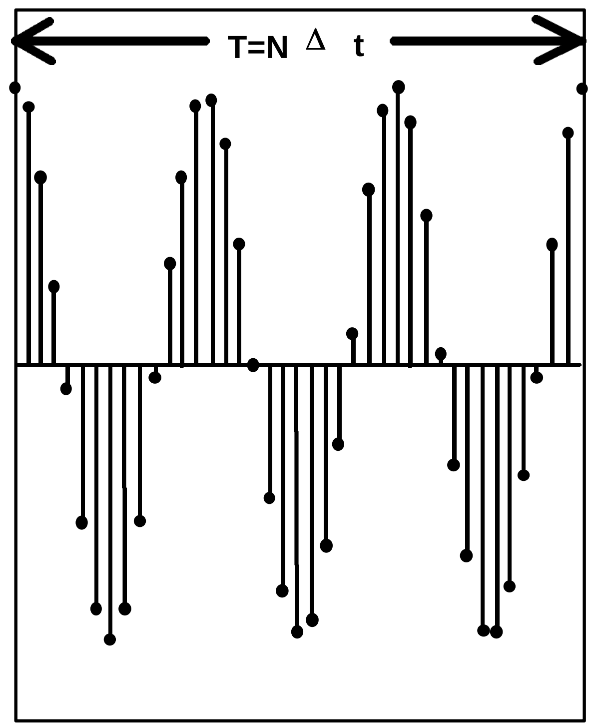
$$T = N\,\Delta t \qquad \text{and } BW = N\,\Delta f / 2$$

where

- Δt - sample interval; time resolution
- N - # of data points
- T - sample record length
- f_{max} - highest desired frequency - BW
- f_s - sampling frequency
- Δf - frequency resolution

Note:

Time Domain	N real data points
Frequency Domain	N/2 real data points
	N/2 imaginary points



Sampling Parameters

Due to the Rayleigh Criteria and Shannon's Sampling Theorum, the following sampling parameters must be observed.

PICK	THEN	AND
Δt	$fmax = 1 / (2 \Delta t)$	$T = N \Delta t$
fmax	$\Delta t = 1 / (2\, fmax)$	$\Delta f = 1/(N \Delta t)$
Δf	$T = 1 / \Delta f$	$\Delta t = T / N$
T	$\Delta f = 1 / T$	$fmax = N \Delta f / 2$

If we choose $\Delta f = 5$ Hz and $N = 1024$

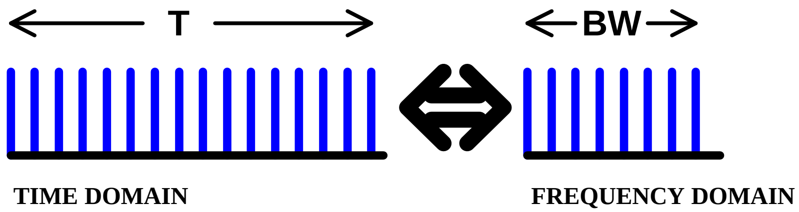
Then $T = 1 / \Delta f = 1 / 5$ Hz $= 0.2$ sec

$f_s = N \Delta f = (1024)(5\text{ Hz}) = 5120$ Hz

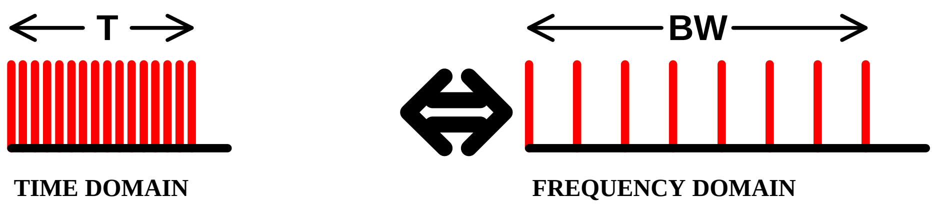
$f_{max} = f_s = (5120\text{ Hz}) / 2 = 2560$ Hz

Sampling Relationship

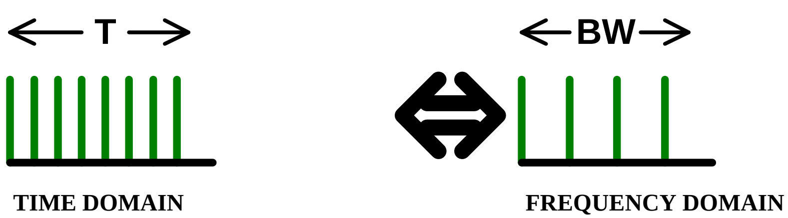
An inverse relationship between time and frequency exists



Given delta t = .0019531 and N = 1024 time points, then T = 2 sec and BW= 256 Hz and delta f = 0.5 Hz



Given delta t = .000976563 and N = 1024 time points, then T = 1sec sec and BW = 512 Hz and delta f = 1 Hz



Given delta t = .0019531 and N = 512 time points, then T = 1 sec and BW = 256 Hz and delta f = 1 Hz

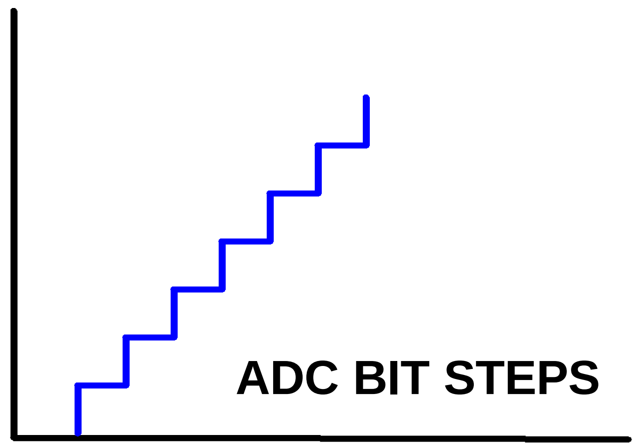
Quantization Error

Sampling refers to the rate at which the signal is collected.
Quantization refers to the amplitude description of the signal.

A 4 bit ADC has 2^4 or 16 possible values

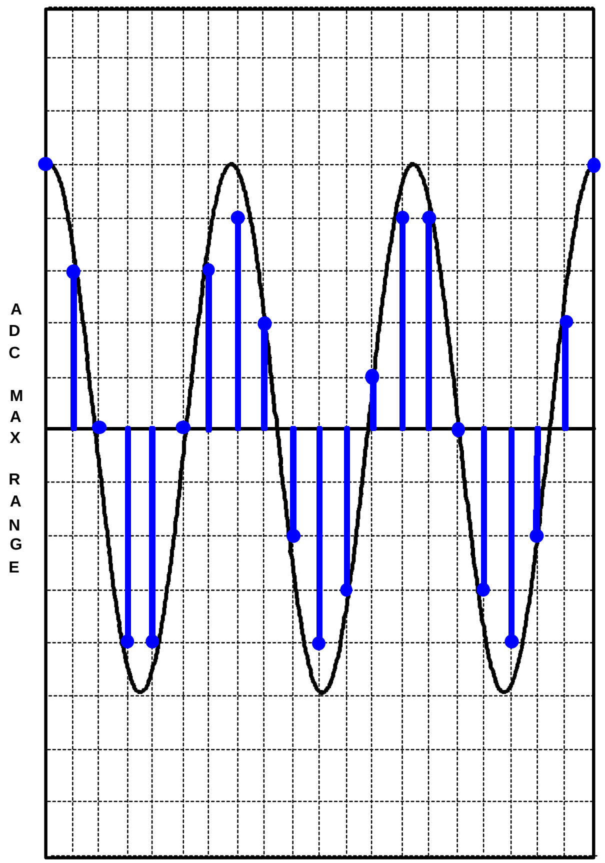
A 6 bit ADC has 2^6 or 64 possible values

A 12 bit ADC has 2^{12} or 4096 possible values



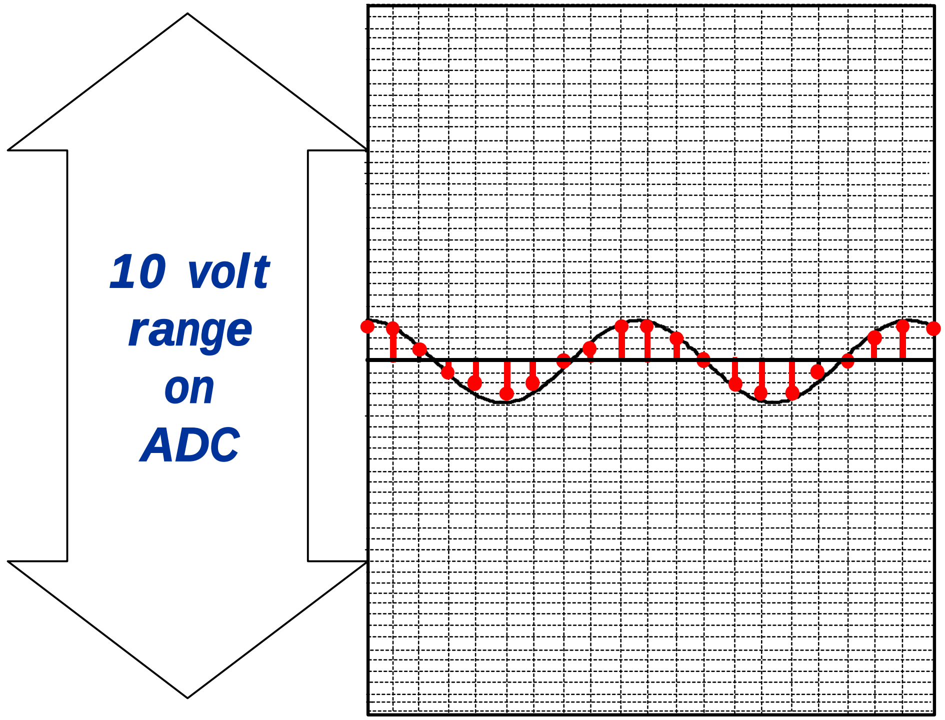
Quantization Error

Quantization errors refer to the accuracy of the amplitude measured. The 6 bit ADC represents the signal shown much better than a 4 bit ADC



Quantization Error

Underloading of the ADC causes amplitude errors in the signal



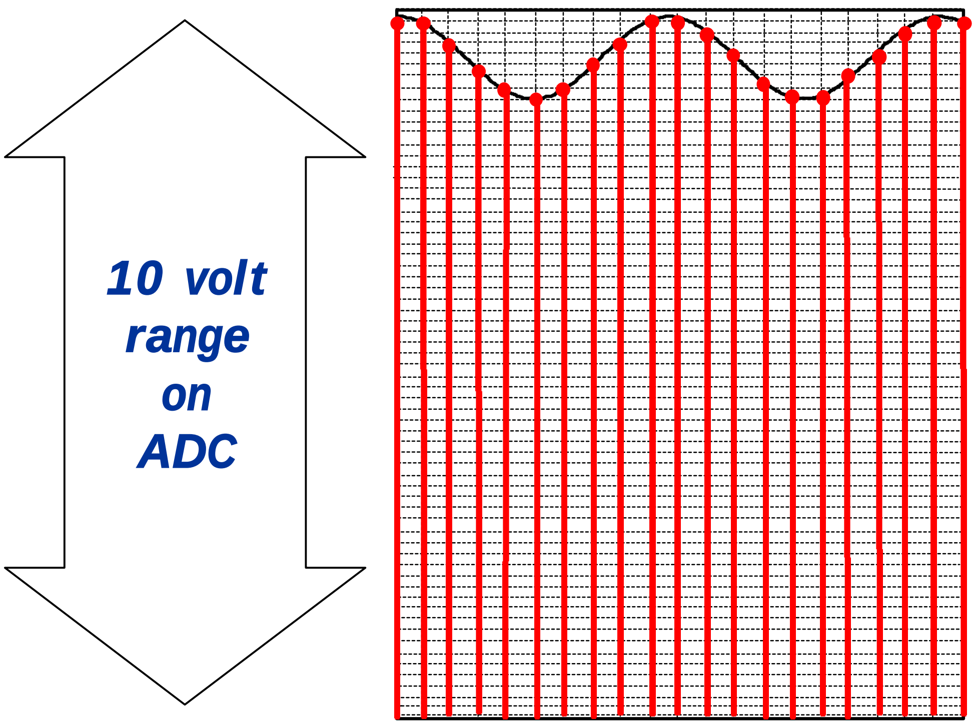
All of the available dynamic range of the analog to digital converter is not used effectively

0.5 volt signal

This causes amplitude and phase distortion of the measured signal in both the time and frequency domains

AC Coupling

A large DC bias can cause amplitude errors in the alternating part of the signal. AC coupling uses a high pass filter to remove the DC component from the signal



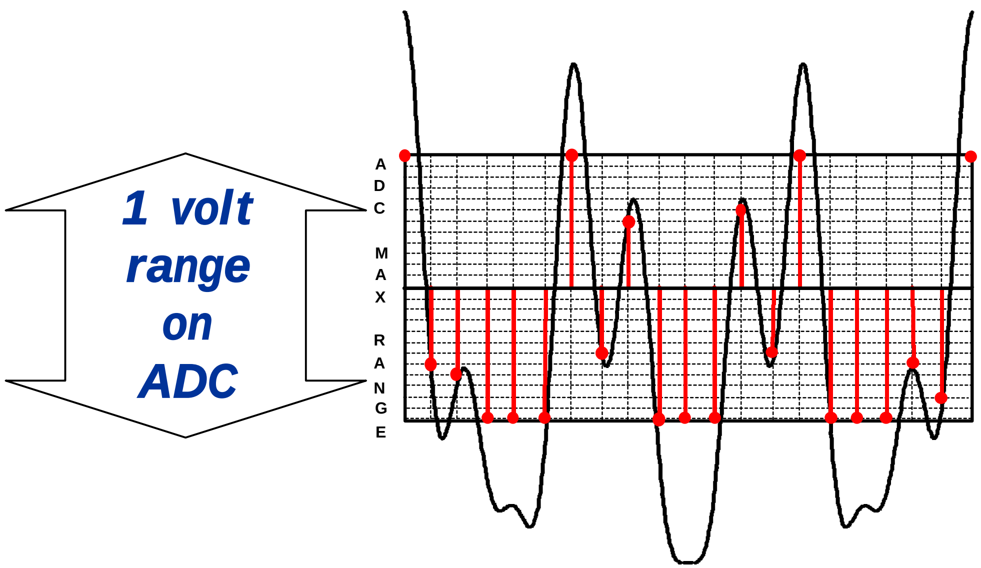
All of the available dynamic range of the analog to digital converter is dominated by the DC signal

The alternating part of the signal suffers from quantization error

This causes amplitude and phase distortion of the measured signal

Clipping and Overloading

Overloading of the ADC causes severe errors also



The ADC range is set too low for the signal to be measured and causes clipping of the signal

1.5 volt signal

This causes amplitude and phase distortion of the measured signal in both the time and frequency domains

The Fourier Transform

Forward Fourier Transform

$$S_x(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$

and Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{+\infty} S_x(f) e^{j2\pi ft} df$$

Discrete Fourier Transform

Even though the actual time signal is continuous, the signal is discretized and the transformation at discrete points is

$$S_x(m\Delta f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi m\Delta f t} dt$$

This integral is evaluated as

$$S_x(m\Delta f) \approx \Delta t \sum_{n=-\infty}^{+\infty} x(n\Delta t) e^{-j2\pi m\Delta f n\Delta t}$$

However, if only a finite sample is available (which is generally the case), then the transformation becomes

$$S_x(m\Delta f) \approx \Delta t \sum_{n=0}^{N-1} x(n\Delta t) e^{-j2\pi m\Delta f n\Delta t}$$

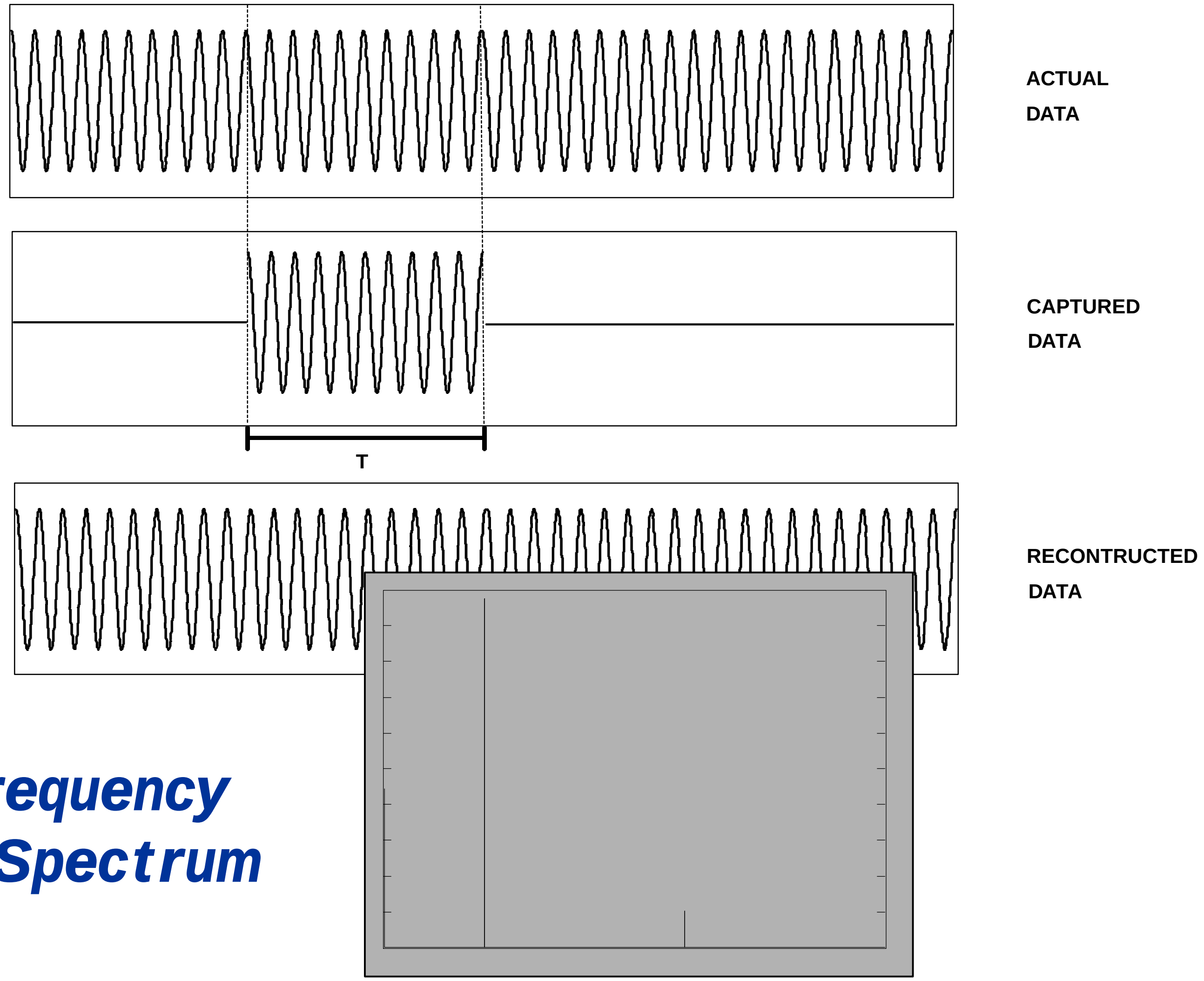
Fourier Transform and FFT

Actual Time Signal

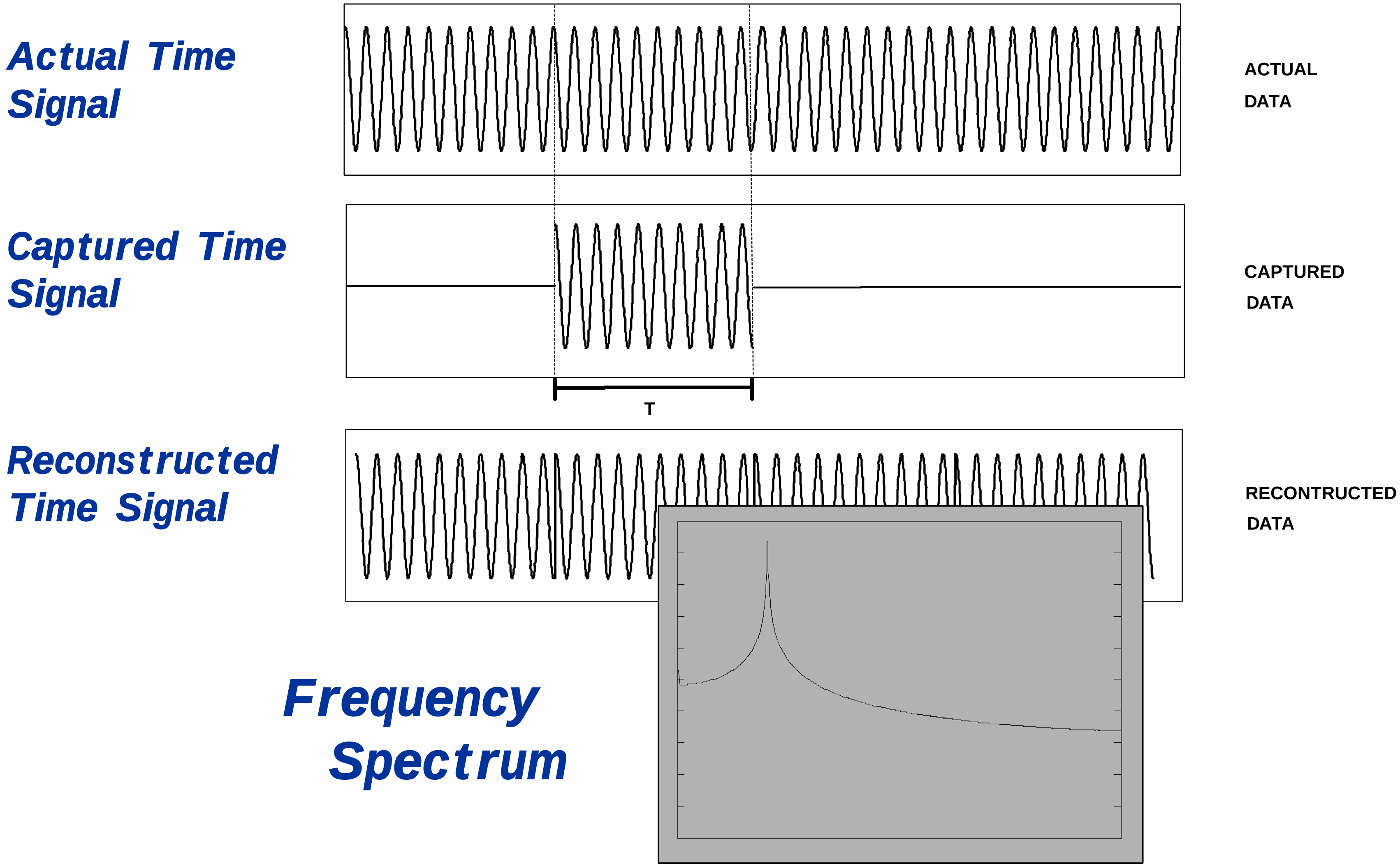
Captured Time Signal

Reconstructed Time Signal

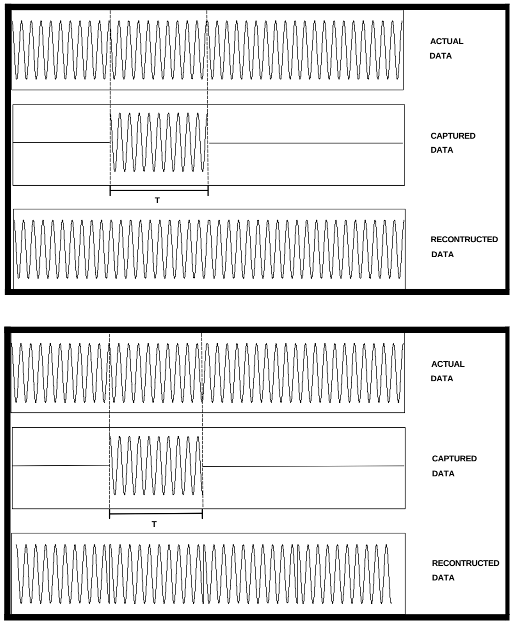
Frequency Spectrum



Fourier Transform and FFT



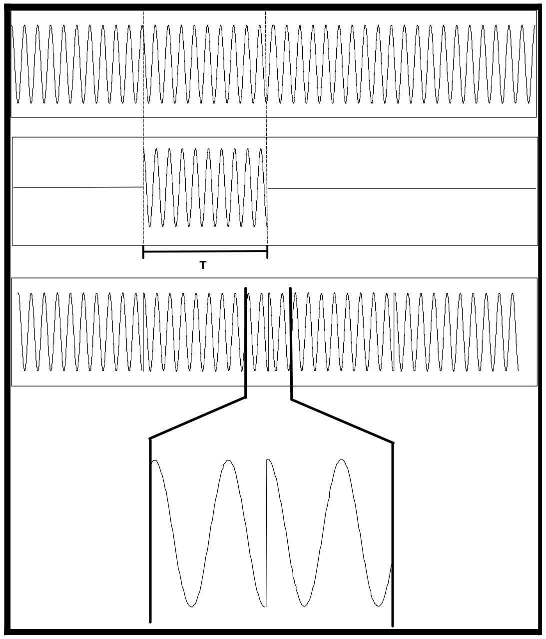
Leakage

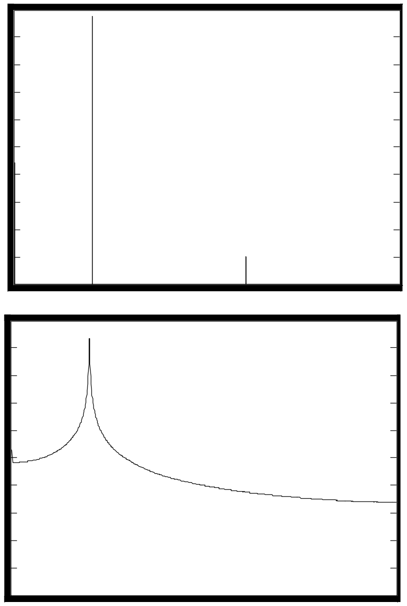


TIME

Periodic Signal

Non-Periodic Signal





FREQUENCY

Leakage due to signal distortion

Leakage

When the measured signal is not periodic in the sample interval, incorrect estimates of the amplitude and frequency occur. This error is referred to as leakage.

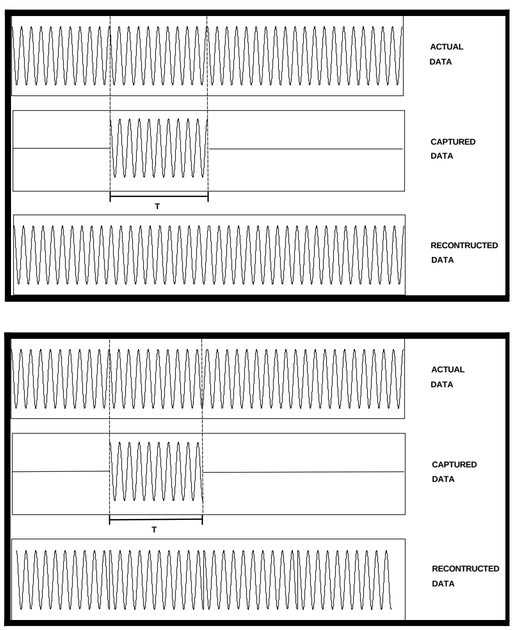
Basically, the actual energy distribution is smeared across the frequency spectrum and energy leaks from a particular Δf into adjacent Δf s.

Leakage is probably the most common and most serious digital signal processing error. Unlike aliasing, the effects of leakage can not be eliminated.

Windows - Minimize Leakage

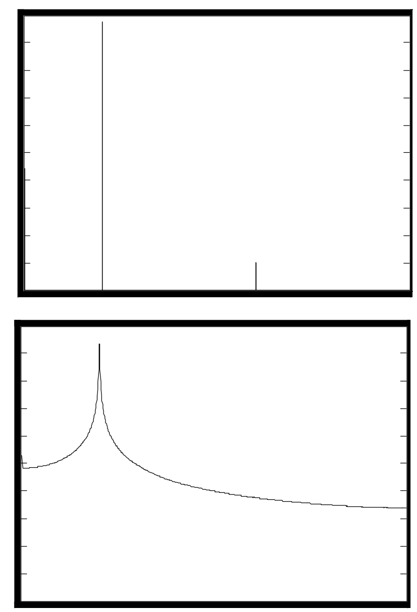
In order to better satisfy the periodicity requirement of the FFT process, time weighting functions, called windows, are used. Essentially, these weighting functions attempt to heavily weight the beginning and end of the sample record to zero - the middle of the sample is heavily weighted towards unity

T I M E



Periodic Signal

Non-Periodic Signal



F R E Q U E N C Y

Windows - Rectangular/Hanning/Flattop

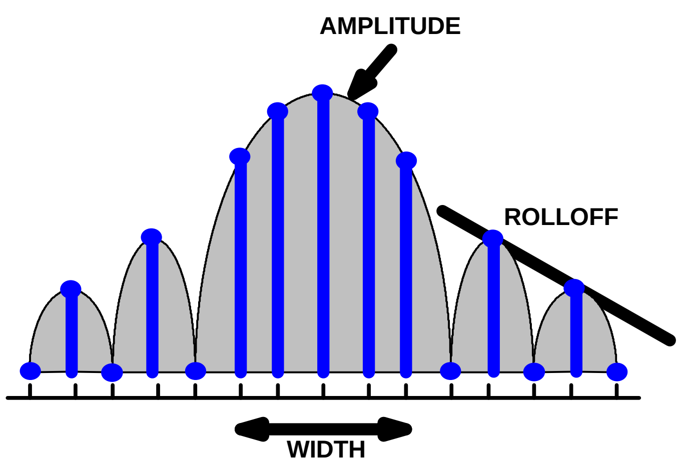
In order to better satisfy the periodicity requirement of the FFT process, time weighting functions, called windows, are used. Essentially, these weighting functions attempt to heavily weight the beginning and end of the sample record to zero - the middle of the sample is heavilty weighted towards unity

***Rectangular* - Unity gain applied to entire sample interval; this window can have up to 36% amplitude error if the signal is not periodic in the sample interval; good for signals that inherently satisfy the periodicity requirement of the FFT process**

***Hanning* - Cosine bell shaped weighting which heavily weights the beginning and end of the sample interval to zero; this window can have up to 16% amplitude error; the main frequency will show some adjacent side band frequencies but then quickly attenuates; good for general purpose signal applications**

***Flat Top* - Multi-sine weighting function; this window has excellent amplitude characteristics (0.1% error) but very poor frequency resolution; very good for calibration purposes with discrete sine**

Windows - Rectangular/Hanning/Flattop



General window frequency characteristics

Time weighting functions are applied to minimize the effects of leakage

Rectangular

Hanning

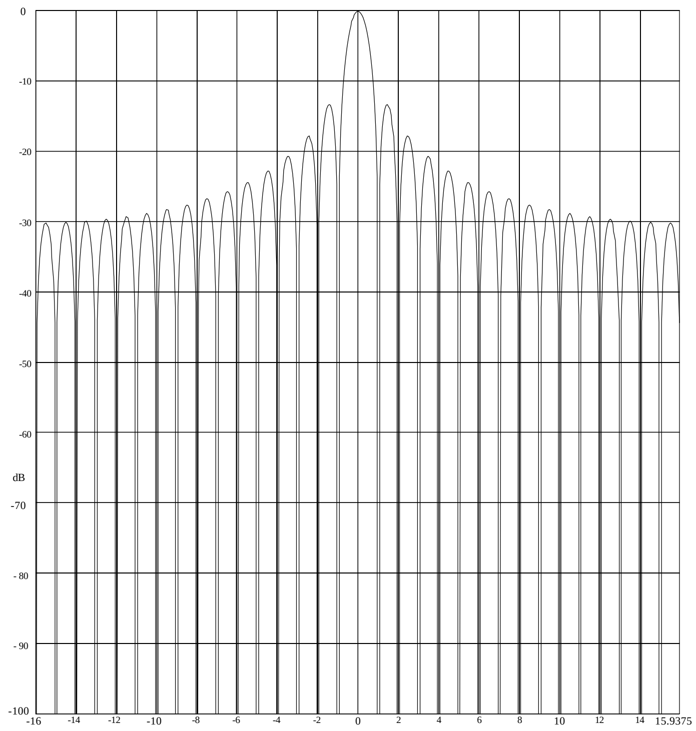
Flat Top

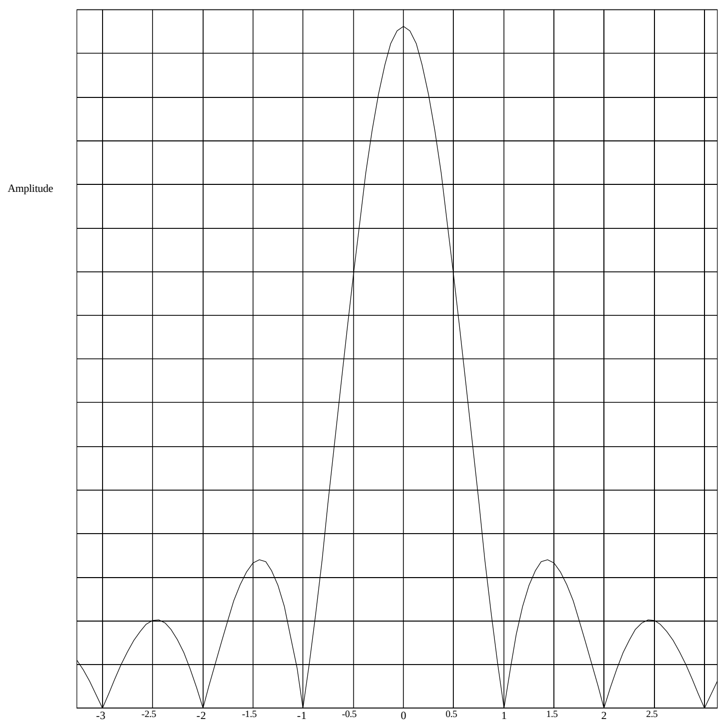
and many others

Windows DO NOT eliminate leakage !!!

Windows - Rectangular

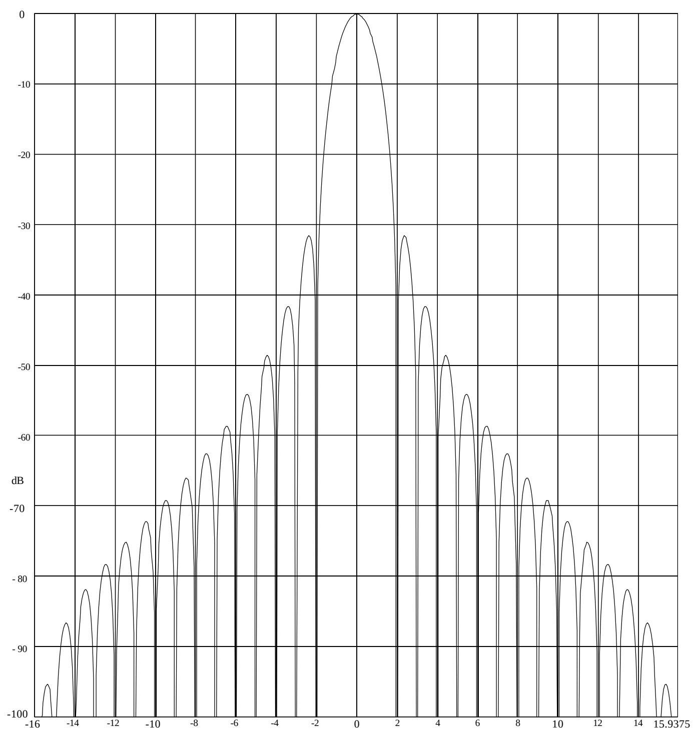
The rectangular window function is shown below. The main lobe is narrow, but the side lobes are very large and roll off quite slowly. The main lobe is quite rounded and can introduce large measurement errors. The rectangular window can have amplitude errors as large as 36%.

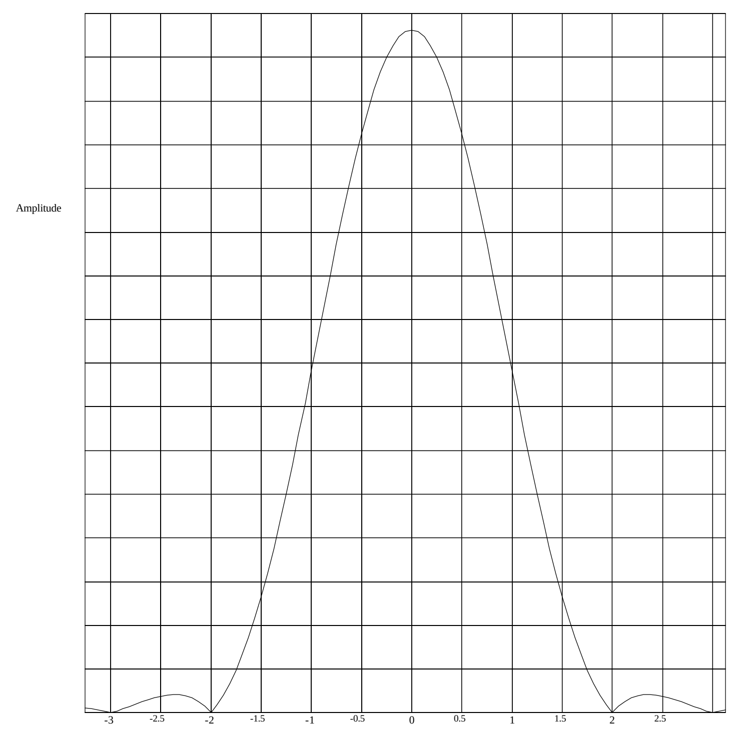




Windows - Hanning

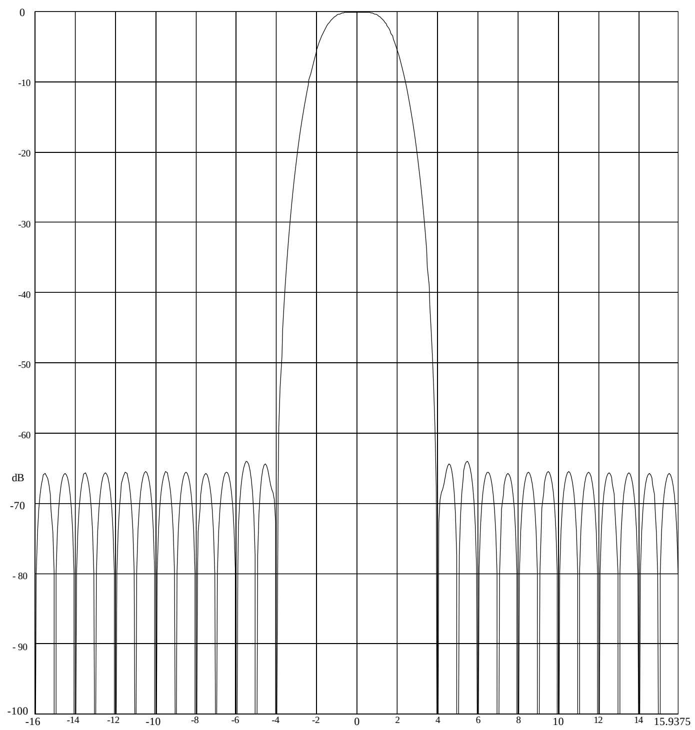
The hanning window function is shown below. The first few side lobes are rather large, but a 60 dB/octave roll-off rate is helpful. This window is most useful for searching operations where good frequency resolution is needed, but amplitude accuracy is not important; the hanning window will have amplitude errors of as much as 16%.

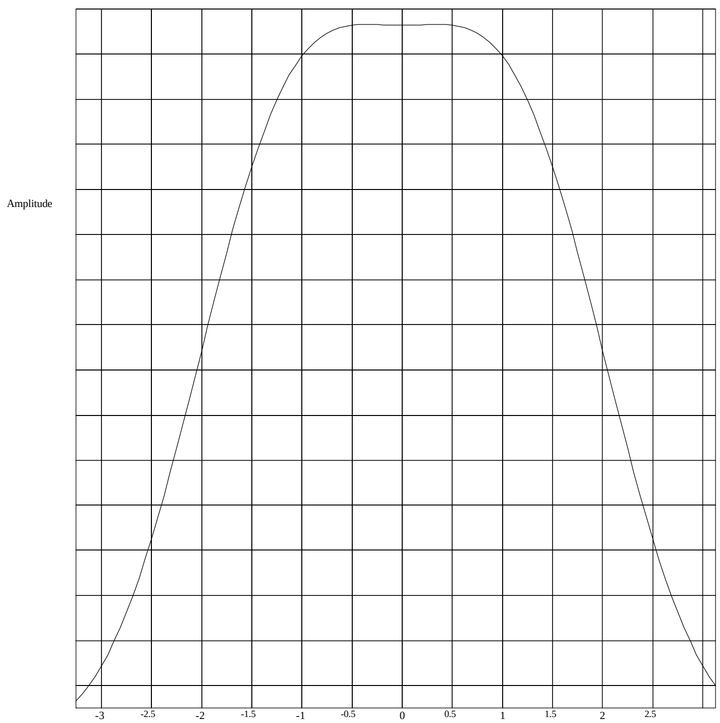




Windows - Flat Top

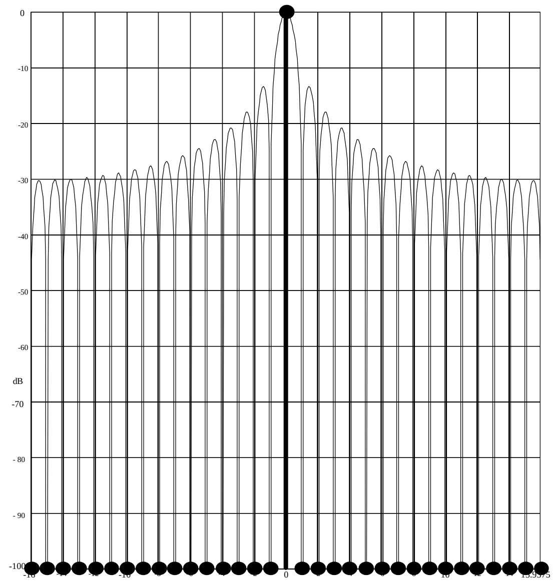
The flat top window function is shown below. The main lobe is very flat and spreads over several frequency bins. While this window suffers from frequency resolution, the amplitude can be measured very accurately to 0.1%.



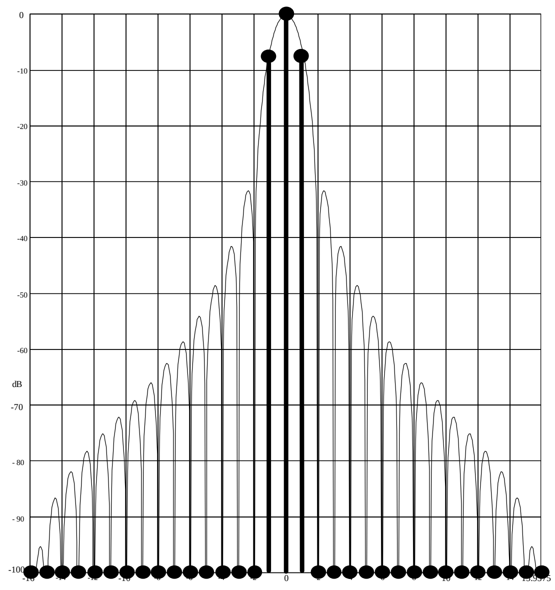


Windows

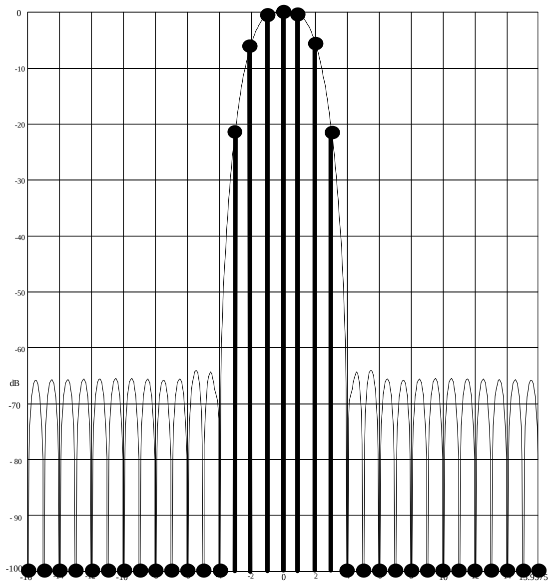
Rectangular



Hanning



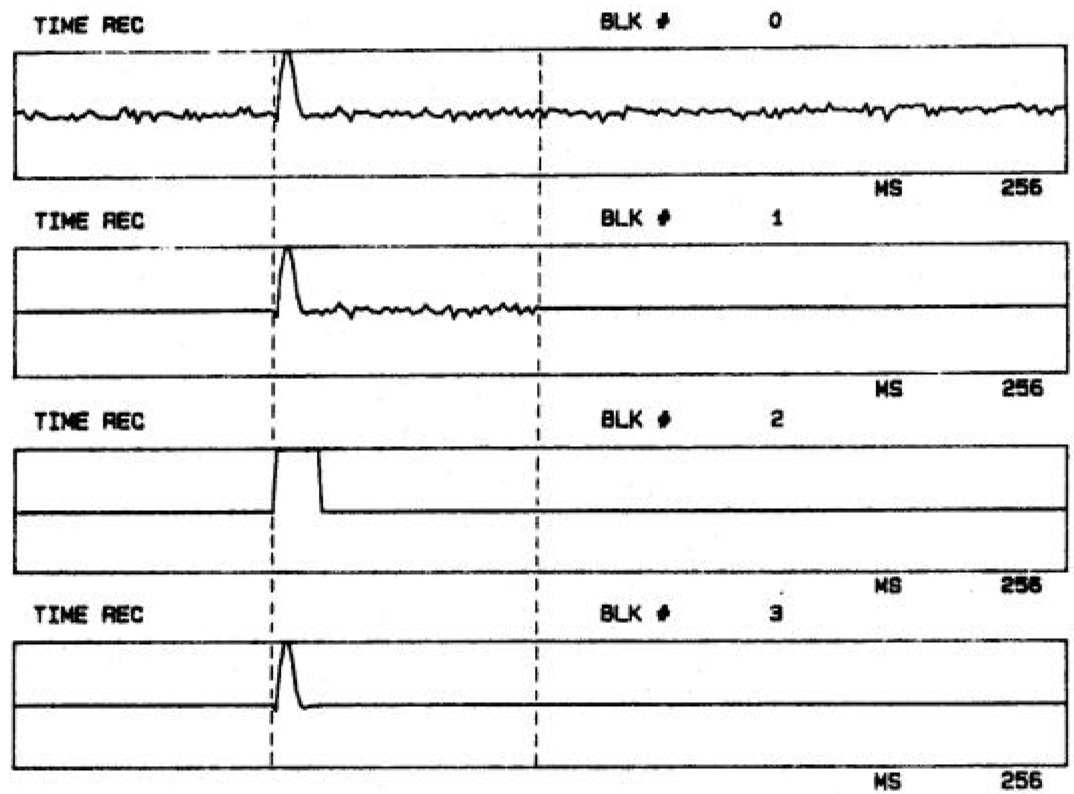
Flat Top



Windows - Force/Exponential for Impact Testing

Special windows are used for impact testing

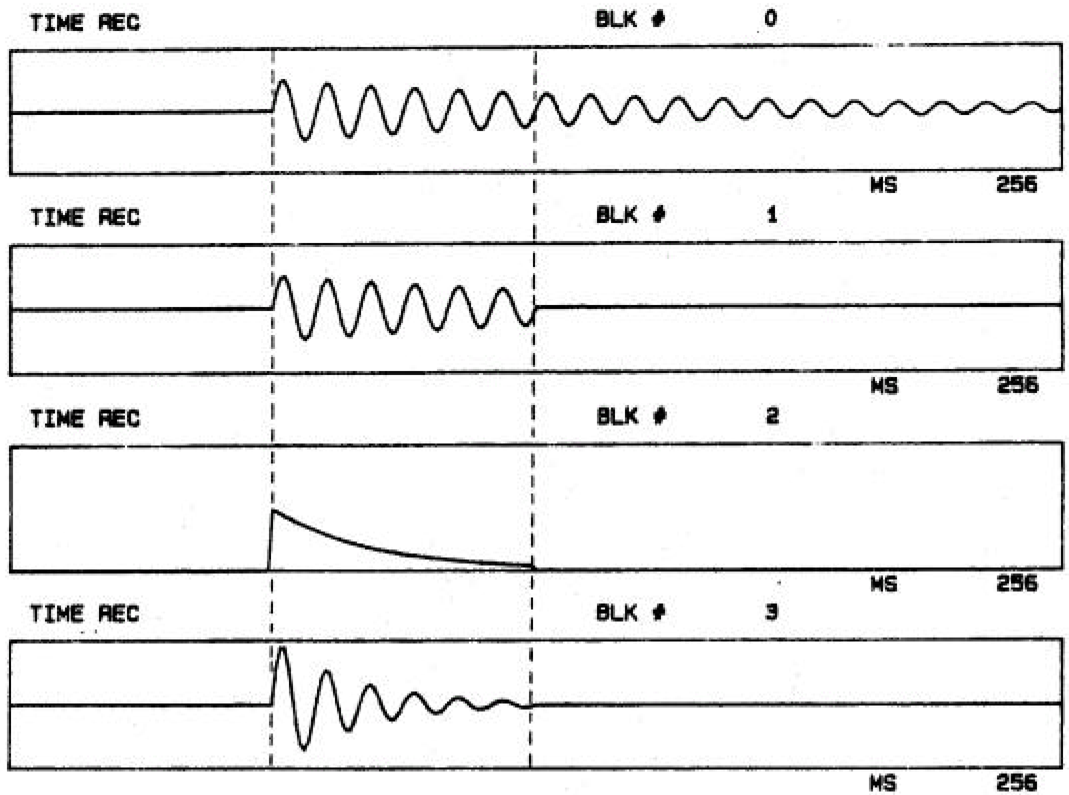
Force window



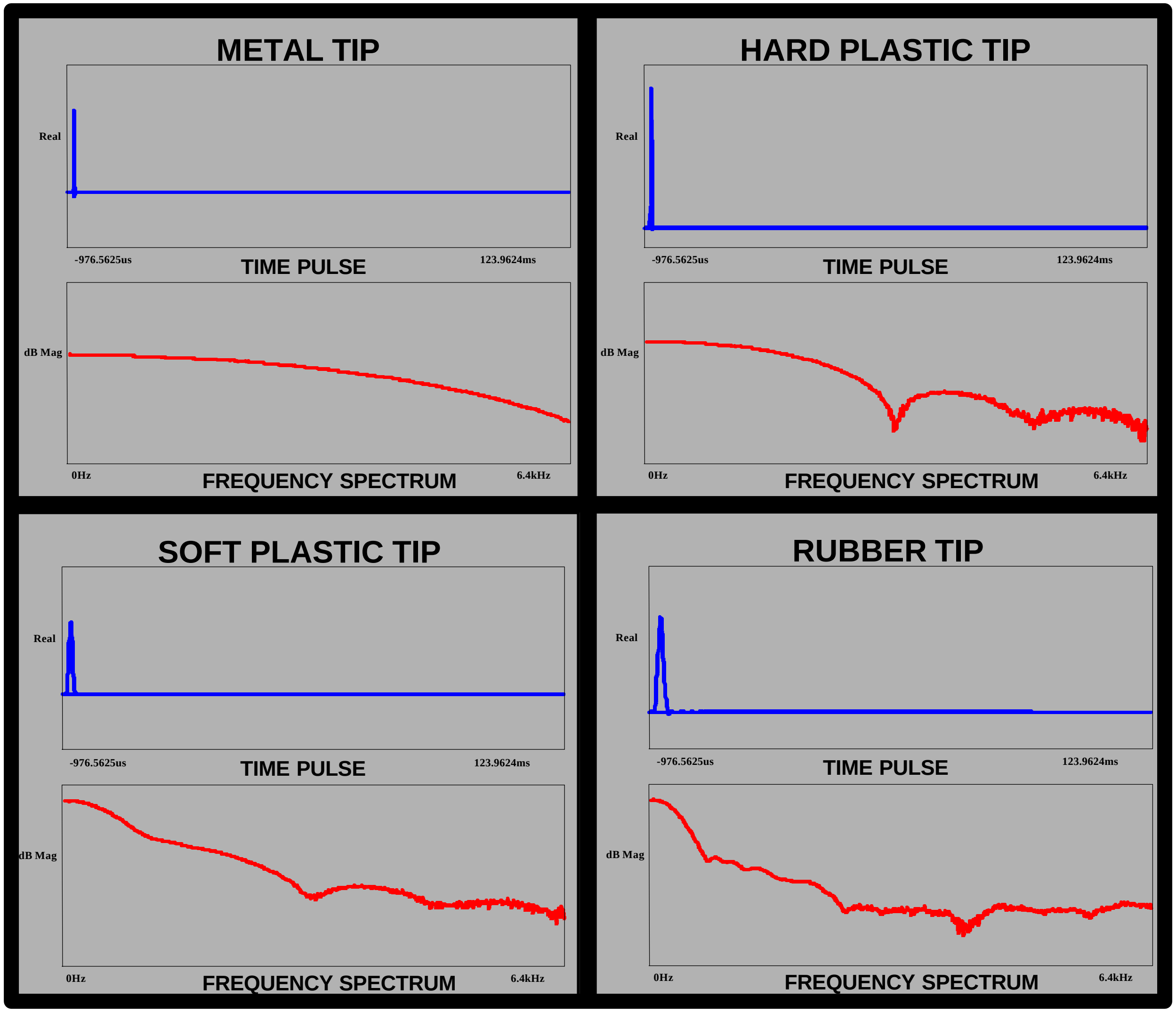
Windows - Force/Exponential for Impact Testing

Special windows are used for impact testing

Exponential window

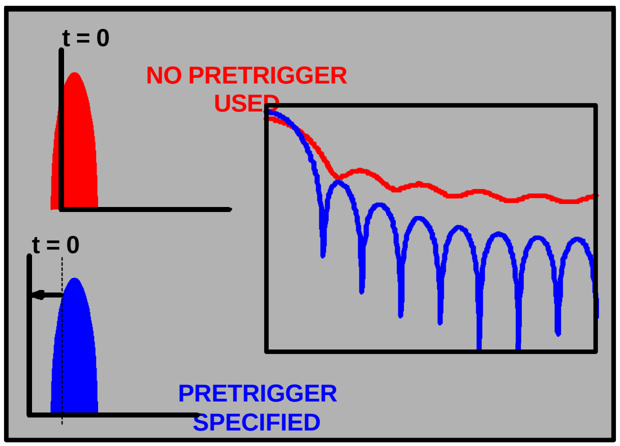


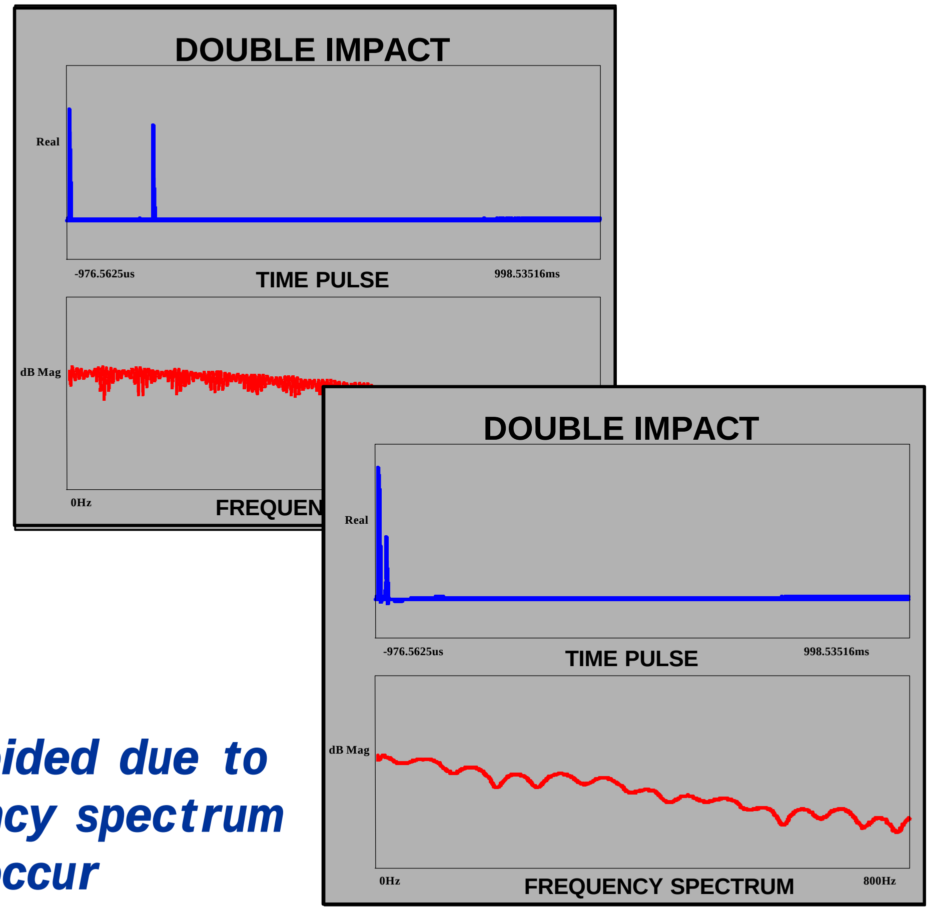
Hammer Tips for Impact Testing



Pretrigger Delay and Double Impacts

Pretrigger delay used to reduce the amount of frequency spectrum distortion



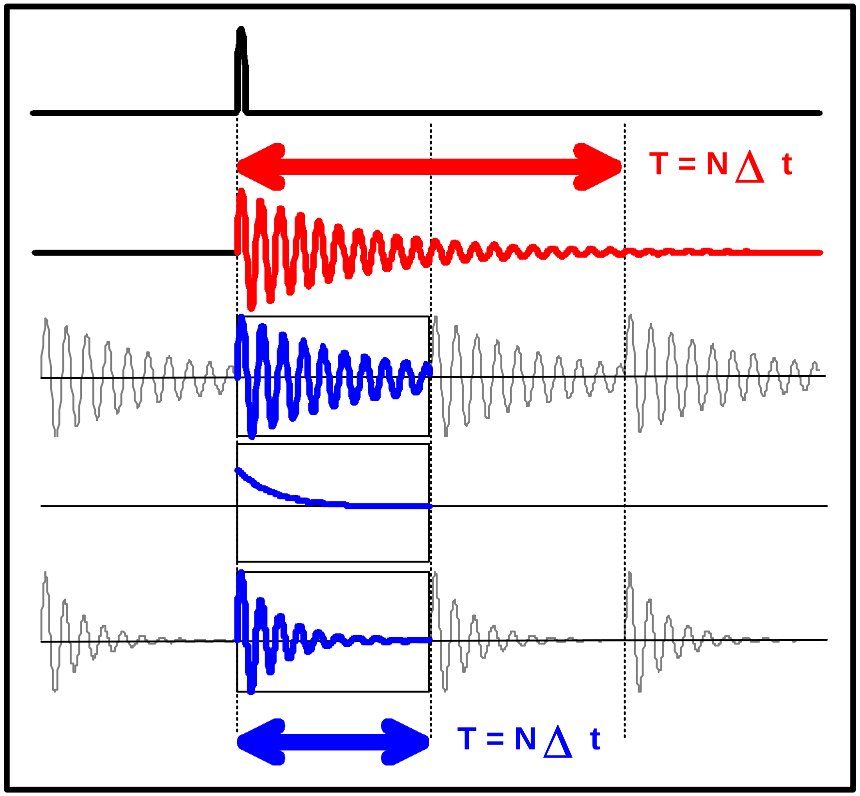


Double impacts should be avoided due to the distortion of the frequency spectrum and force dropout that can occur

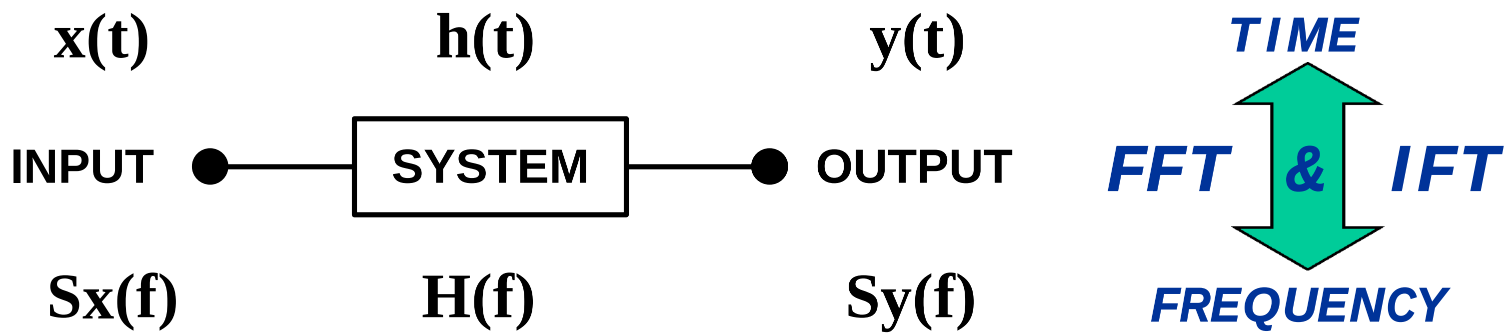
Exponential Window

If the signal does not naturally decay within the sample interval, then an exponentially decaying window may be necessary.

However, many times changing the signal processing parameters such as bandwidth and number of spectral lines may produce a signal which requires less window weighting



Measurement - Linear Spectra



x(t) - time domain input to the system

y(t) - time domain output to the system

Sx(f) - linear Fourier spectrum of x(t)

Sy(f) - linear Fourier spectrum of y(t)

H(f) - system transfer function

h(t) - system impulse response

Measurement - Linear Spectra

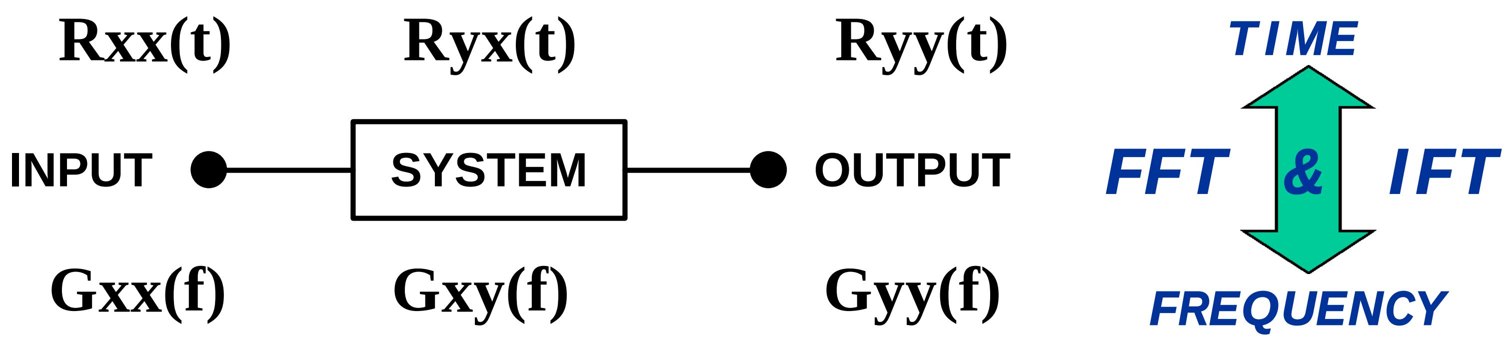
$$x(t)=\int_{-\infty}^{+\infty} S_x(f)e^{j2\pi ft}df \qquad S_x(f)=\int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt$$

$$y(t)=\int_{-\infty}^{+\infty} S_y(f)e^{j2\pi ft}df \qquad S_y(f)=\int_{-\infty}^{+\infty} y(t)e^{-j2\pi ft}dt$$

$$h(t)=\int_{-\infty}^{+\infty} H(f)e^{j2\pi ft}df \qquad H(f)=\int_{-\infty}^{+\infty} h(t)e^{-j2\pi ft}dt$$

Note: S_x and S_y are complex valued functions

Measurement - Power Spectra



$R_{xx}(t)$ - autocorrelation of the input signal $x(t)$

$R_{yy}(t)$ - autocorrelation of the output signal $y(t)$

$R_{yx}(t)$ - cross correlation of $y(t)$ and $x(t)$

$G_{xx}(f)$ - autopower spectrum of $x(t)$ $\quad G_{xx}(f) = S_x(f) \bullet S_x^*(f)$

$G_{yy}(f)$ - autopower spectrum of $y(t)$ $\quad G_{yy}(f) = S_y(f) \bullet S_y^*(f)$

$G_{yx}(f)$ - cross power spectrum of $y(t)$ and $x(t)$ $\quad G_{yx}(f) = S_y(f) \bullet S_x^*(f)$

Measurement - Power Spectra

$$R_{xx}(\tau)=E[x(t),x(t+\tau)]=\lim_{T\to\infty}\frac{1}{T}\int_T x(t)x(t+\tau)dt$$

$$G_{xx}(f)=\int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-j2\pi ft}d\tau=S_x(f)\bullet S_x^*(f)$$

$$R_{yy}(\tau)=E[y(t),y(t+\tau)]=\lim_{T\to\infty}\frac{1}{T}\int_T y(t)y(t+\tau)dt$$

$$G_{yy}(f)=\int_{-\infty}^{+\infty} R_{yy}(\tau)e^{-j2\pi ft}d\tau=S_y(f)\bullet S_y^*(f)$$

$$R_{yx}(\tau)=E[y(t),x(t+\tau)]=\lim_{T\to\infty}\frac{1}{T}\int_T y(t)x(t+\tau)dt$$

$$G_{yx}(f)=\int_{-\infty}^{+\infty} R_{yx}(\tau)e^{-j2\pi ft}d\tau=S_y(f)\bullet S_x^*(f)$$

The Frequency Response Function and Coherence

$$S_y = H S_x$$

H_1 formulation
- **susceptible to noise on the input**
- **underestimates the actual H of the system**

Other formulations for H exist

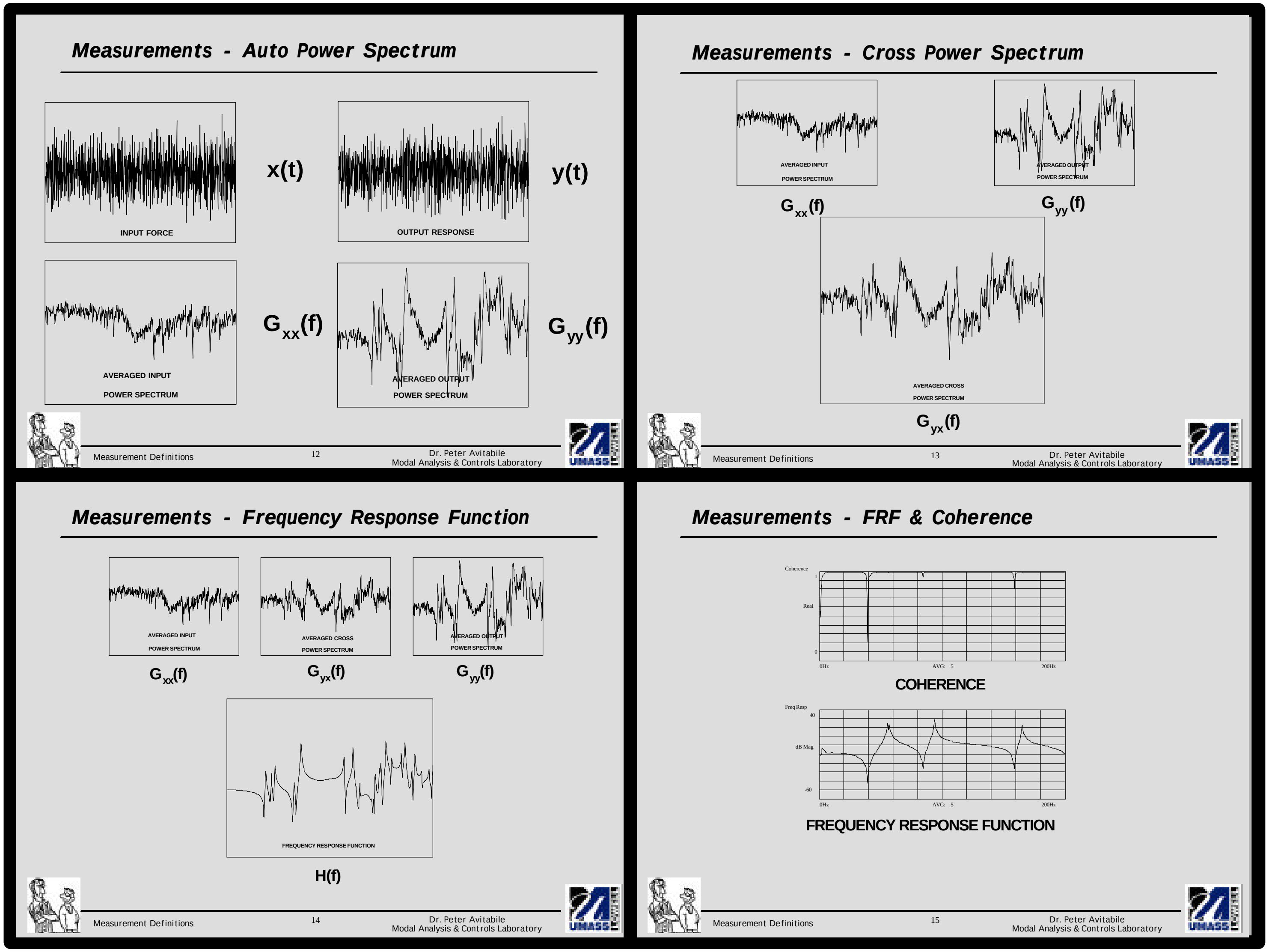
$$S_y \bullet S_x^* = H S_x \bullet S_x^*$$

$$H = \frac{S_y \bullet S_x^*}{S_x \bullet S_x^*} = \frac{G_{yx}}{G_{xx}}$$

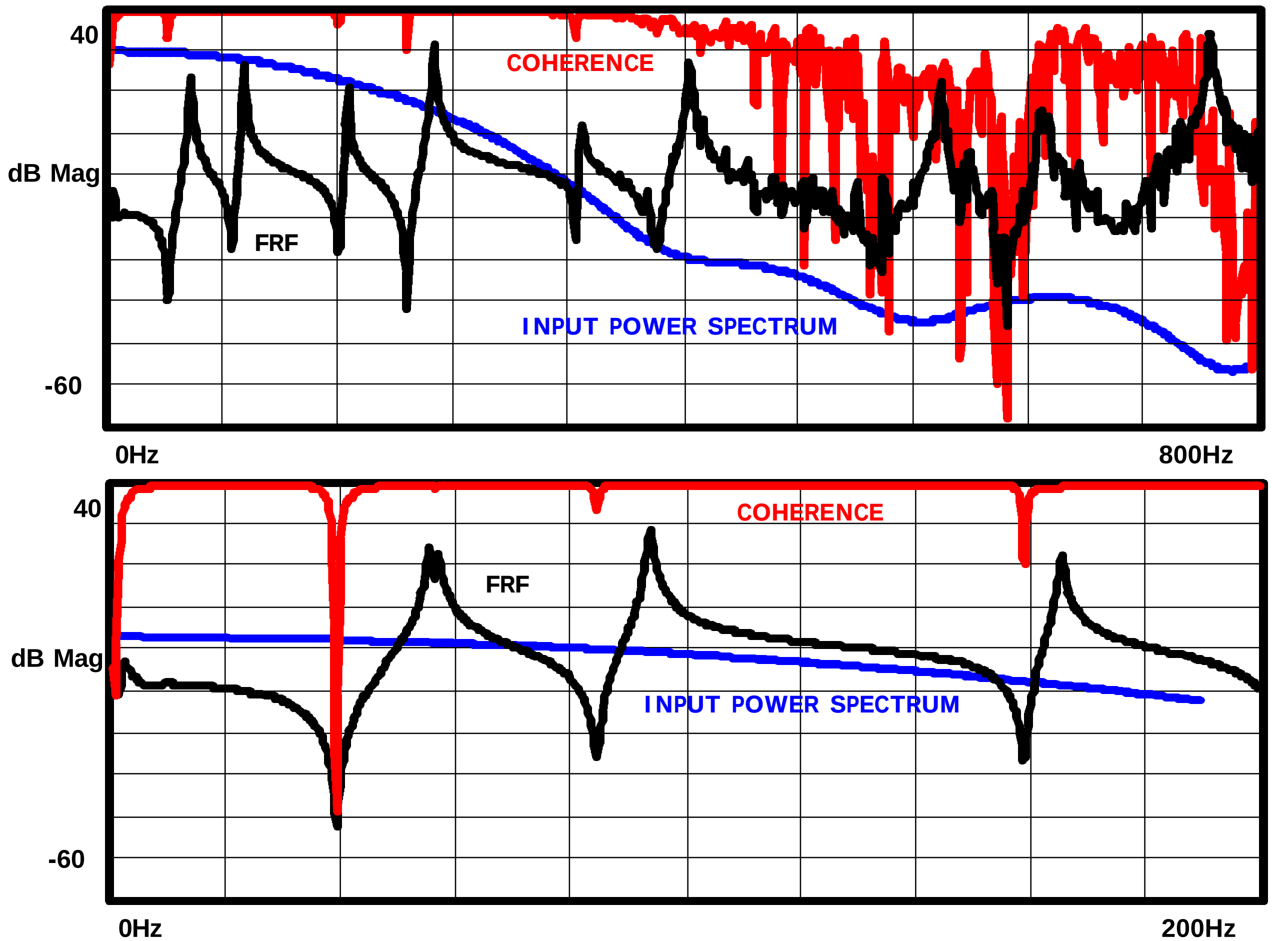
COHERENCE

$$\gamma_{xy}^2 = \frac{(S_y \bullet S_x^*)(S_x \bullet S_y^*)}{(S_x \bullet S_x^*)(S_y \bullet S_y^*)} = \frac{G_{yx} / G_{xx}}{G_{yy} / G_{xy}} = \frac{H_1}{H_2}$$

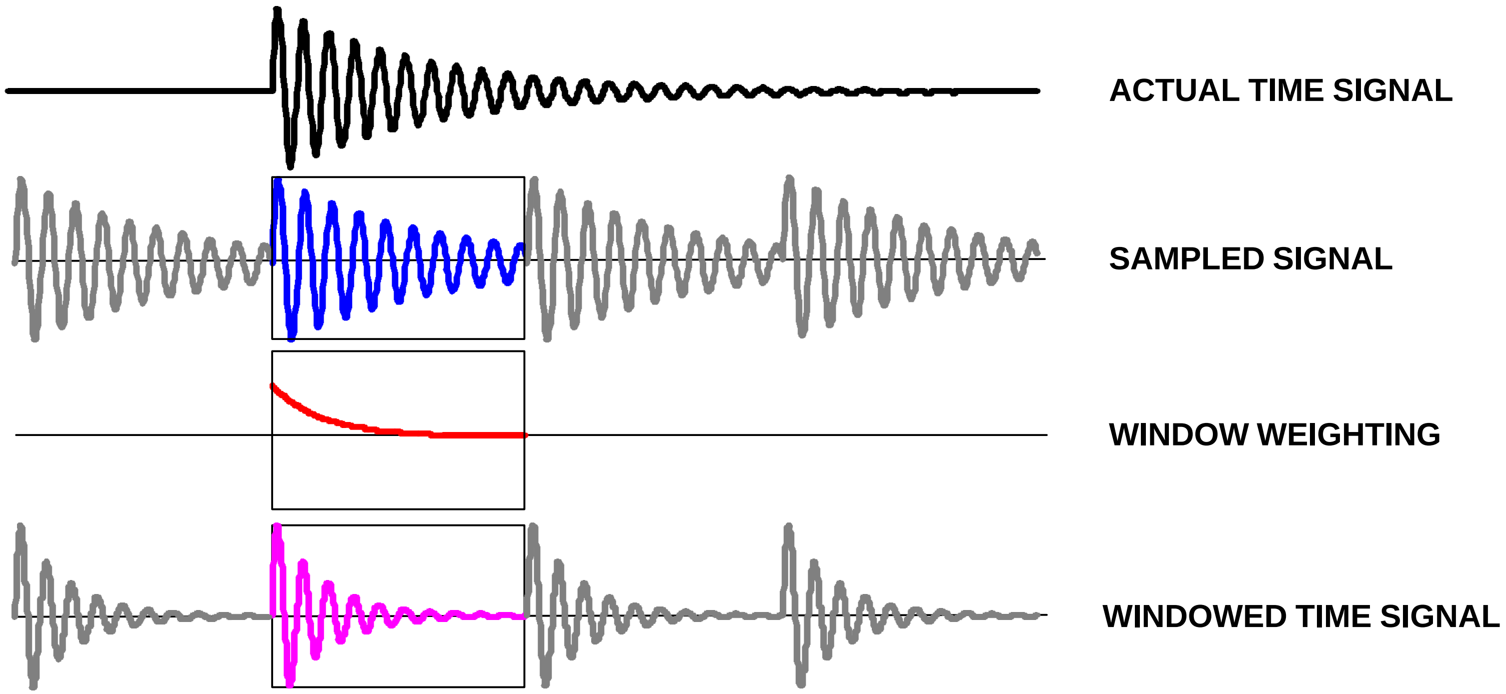
Typical Measurements



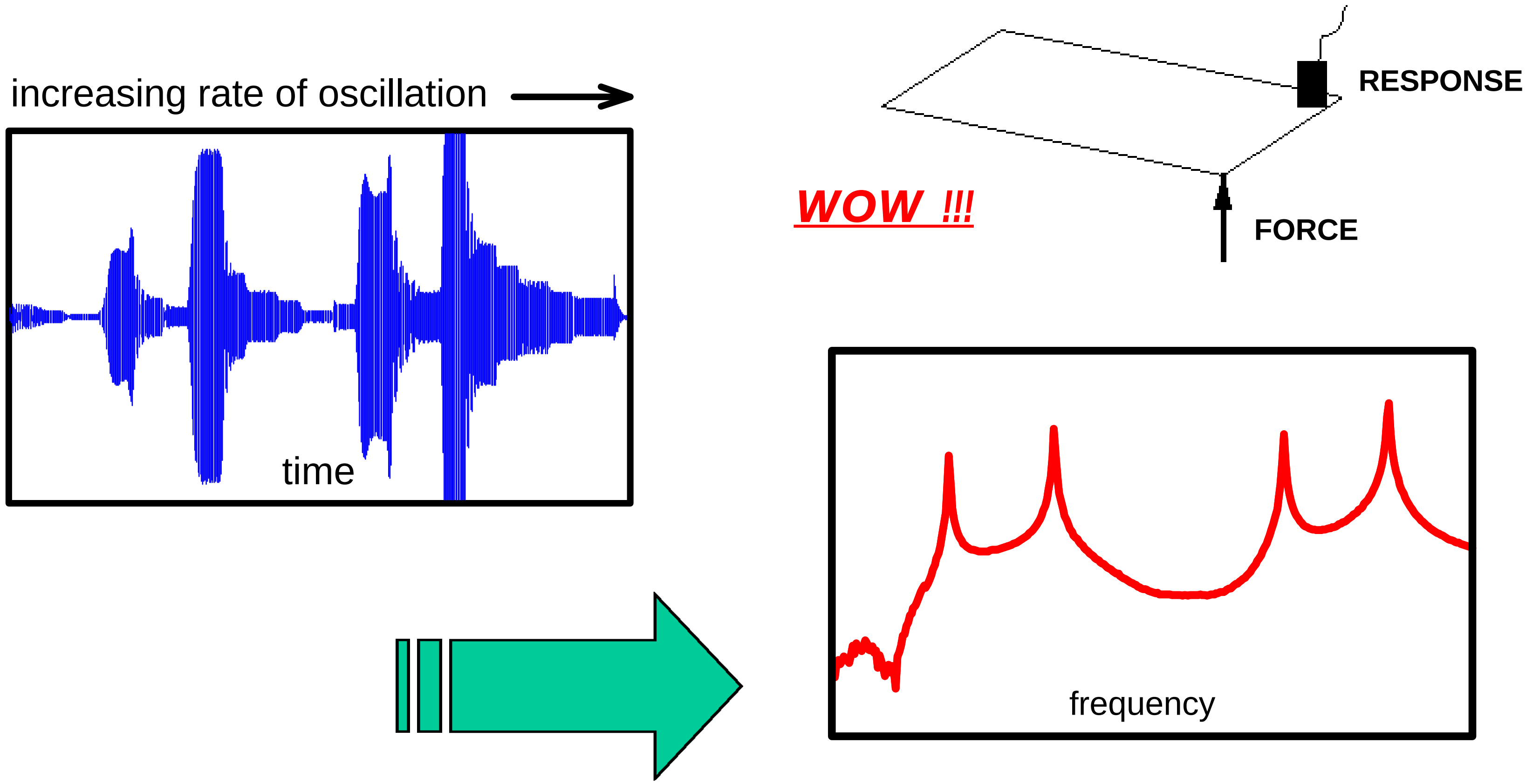
Hammers and Tips



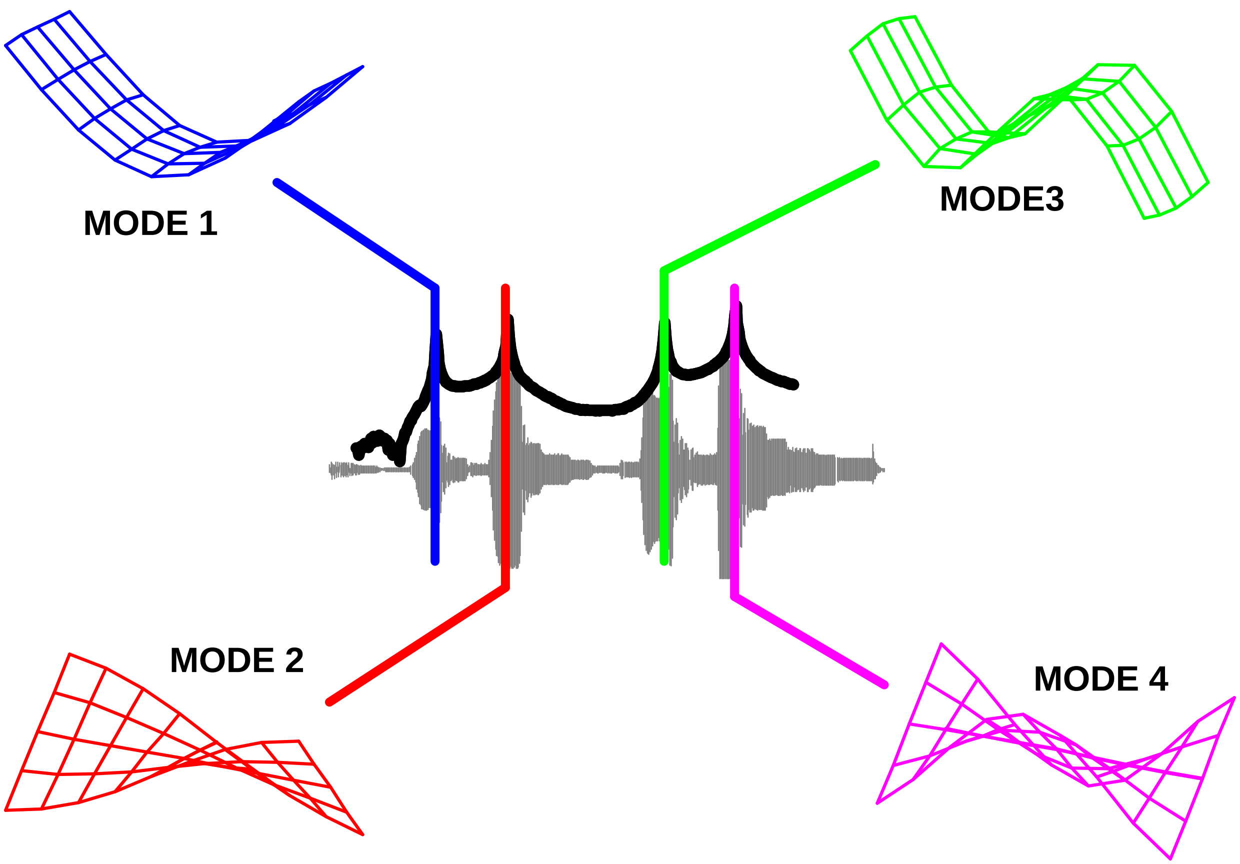
Leakage and Windows for Impact Testing



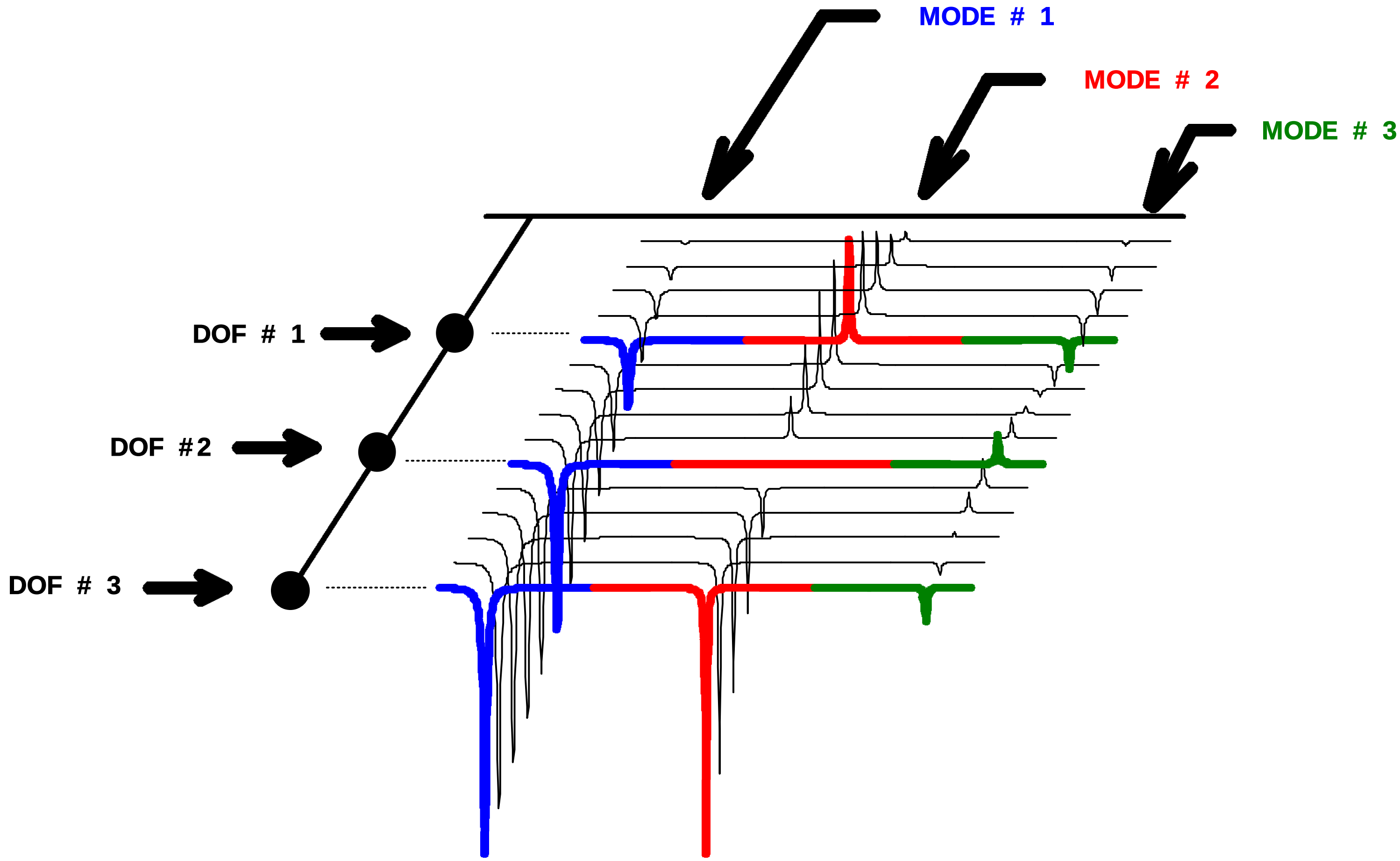
Simple time-frequency response relationship



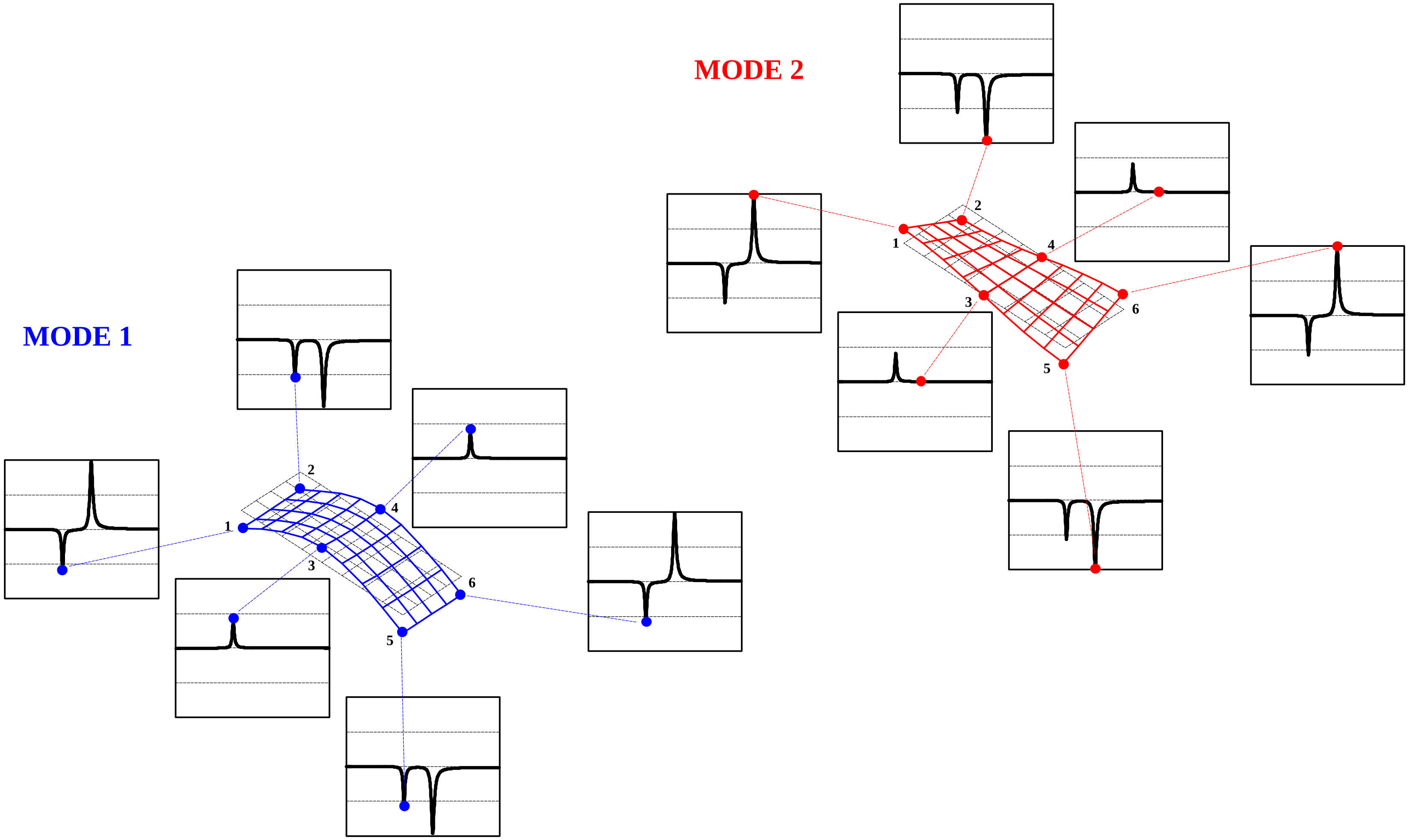
Sine Dwell to Obtain Mode Shape Characteristics



Mode Shape Characteristics for a Simple Beam



Mode Shape Characteristics for a Simple Plate



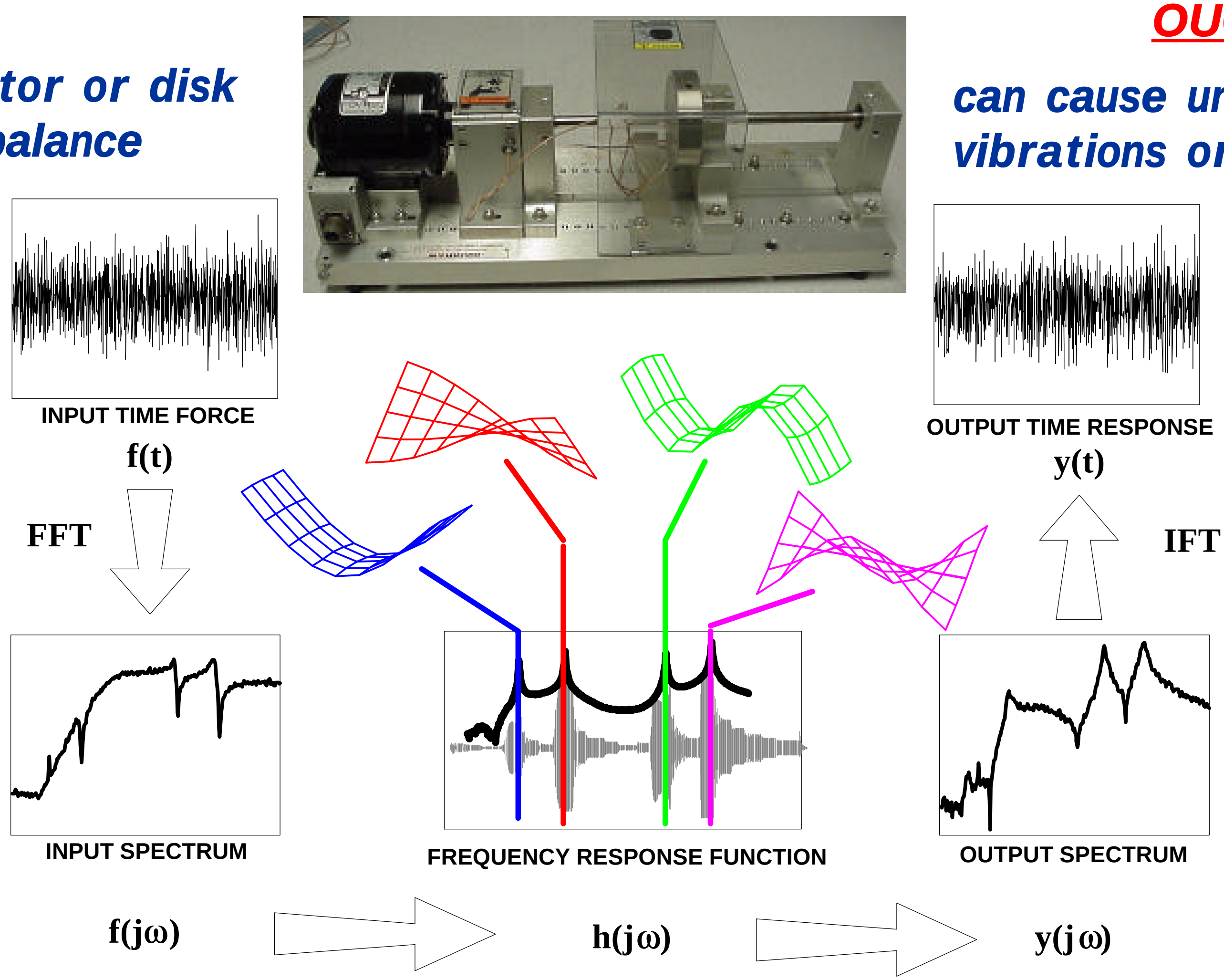
Why and How Do Structures Vibrate?

Motor or disk unbalance

OUCH !!!

can cause unwanted vibrations or worse

OOOPS !!!



HP 35660 FFT Dual Channel Analyzer

HP 35660 FFT Dual Channel Analyzer