Final Projects

Directions: Work together in groups of 1-5 on the following four projects. We suggest, but do not insist, that you work in a group of > 1 people. For projects 1,2, and 3, your group should write a text file with (working!) code which answers the questions. Please give at least a tiny bit of documentation for your code (at the very least indicate which project your code goes with). For project 4, your group should turn in a summary of observations and/or conjectures, but not necessarily code. Send your files via e-mail to peterson@math.colostate.edu *and* michael.dipasquale@colostate.edu. Please send your work no later than 12:00 p.m. (noon) on Friday, August 16.

1 Envelopes, Evolutes, and Euclidean Distance Degree

Read the section in Cox-Little-O'Shea's *Ideals, Varieties, and Algorithms* on *envelopes* of families of curves - this is in Chapter 3, section 4. We will expand slightly on the setup in Chapter 3 as follows. A *rational family of lines* is defined as a polynomial $L_t(x, y) \in \mathbb{Q}(t)[x, y]$ of the form

$$L_t(x, y) = \frac{1}{G(t)}(A(t)x + B(t)y + C(t)),$$

where $A(t), B(t), C(t)$, and $G(t)$ are polynomials in t. The envelope of $L_t(x, y)$ is the locus of x and y so that there is some t satisfying $L_t(x, y) = 0$ and $\frac{\partial L_t(x,y)}{\partial t} = 0$. As usual, we can handle this by introducing a new variable s which plays the role of $1/G(t)$. Now replace $1/G(t)$ by s in $L_t(x, y)$ and $\frac{\partial L_t(x,y)}{\partial t}$ to get the ideal generated by

$$\begin{gathered} s(A(t)x + B(t)y + C(t)) \\ s^2\left[(A'(t)G(t) - A(t)G'(t))x + (B'(t)G(t) - B(t)G'(t))y + (C'(t)G(t) - C(t)G'(t))x\right] \\ sG(t) - 1 \end{gathered}$$

in the polynomial ring $\mathbb{Q}[s, t, x, y]$. You can obtain an equation for the envelope by eliminating s and t from this ideal.

Create a script in Macaulay2 which takes as input a rational family of lines and gives the equation of the envelope in $\mathbb{Q}[x, y]$ as output. One suggestion is to take as input the list $\{A(t), B(t), C(t), G(t)\}$ of polynomials in t where $L_t(x) = 1/G(t)(A(t)x + B(t)y + C(t))$. This will make it easier to define the list of equations above.

Now, consider a curve C in $\mathbb{R}^2$ parametrized by $(x(t), y(t))$. The rational family of lines

$$f_t(x, y) = (x'(t), y'(t)) \cdot (x - x(t), y - y(t)),$$

where $\cdot$ represents dot product, satisfies that $f_{t_0}(x, y)$ is the equation of the *normal line* to C at the point $(x(t_0), y(t_0))$. The envelope of $f_t(x, y)$ is called the *evolute* of C.

Use your script for computing the envelope of a rational family of lines to compute the equation of the evolute of the following curves:

1. The parabola $y = x^2$ parametrized as $x(t) = t, y(t) = t^2$
2. The ellipse $x^2 + 4y^2 = 4$ parametrized as

$$x(t) = \frac{8t}{1 + 4t^2} \qquad y(t) = \frac{4t^2 - 1}{1 + 4t^2}$$

3. The cardioid $(x^2 + y^2 + x)^2 = x^2 + y^2$ parametrized as

$$x(t) = \frac{2t^2 - 2}{(1 + t^2)^2} \qquad y(t) = \frac{-4t}{(1 + t^2)^2}$$

Plot these evolutes in Sage along with their corresponding curves.

Finally, let C be the ellipse $x^2+4y^2 = 4$ parametrized above. Given a point $(p, q) \in \mathbb{R}^2$, consider the squared distance function $D(p, q) = (x - p)^2 + (y - q)^2$. Select several points (p, q) inside the evolute of the ellipse and find the number of *real* critical points of $D(p, q)$ restricted to C (use Lagrange multipliers and the Numerical Algebraic Geometry package for this). Now do the same for several points (p, q) *outside* of the evolute. Do you notice anything? Can you make some conjecture in the case of an ellipse?

2 Lines on Cubics

A classical result in Algebraic Geometry states that over an algebraically closed field, exactly 27 lines lie on a smooth projective cubic surface in $\mathbb{P}^3$. In this course, we have not discussed projective varieties or projective space. Thus, in this problem, we will consider how to find 27 lines lying on a "typical" smooth affine cubic surface. Let $F(x, y, z) \in \mathbb{Q}[x, y, z]$ be a cubic polynomial. Let V be the set of zeros of F in $\mathbb{C}^3$. Most lines in $\mathbb{C}^3$ have an ideal of the form $I = (x - (az + b), y - (cz + d))$ with $a, b, c, d \in \mathbb{C}$. We would like to find all conditions on a, b, c, d such that $F \in (x - (az + b), y - (cz + d))$. This amounts to making the substitution $x \to az + b, y \to cz + d$ and requiring that the result is zero. Making this substitution leads to a polynomial of the form $Az^3 + Bz^2 + Cz + D$ with $A, B, C, D \in \mathbb{Q}[a, b, c, d]$. The ideal $(A, B, C, D) \subseteq \mathbb{Q}[a, b, c, d]$ determines 27 points in $\mathbb{C}^4$. The coordinates of the points give values for a, b, c, d corresponding to the various lines.

- Write a Macaulay2 script that finds the 27 lines on a given cubic surface.

3 Homomorphisms of Polynomial Rings

The first part of this project is to code membership in the image of ring homomorphisms using elimination theory. Suppose $\phi : \mathbb{K}[w_1, \ldots, w_m] \to \mathbb{K}[x_1, \ldots, x_n]$ is a ring homomorphism defined by $\phi(w_i) = f_i$ for some polynomials $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$.

Using Problem 7 on Problem Set 5 as a guide, write a Macaulay2 script which accepts as input a polynomial f and a list of polynomials $\{f_1, \ldots, f_m\}$, all taken from the ring $S = \mathbb{Q}[x_1, \ldots, x_n]$ which the user has previously defined. The $f_1, \ldots, f_m$ are the images of $w_1, \ldots, w_m$. From this input, your program should output a polynomial F in the w_i variables so that $\phi(F) = f$, if one exists, and otherwise should output the string "Not in the image of the ring map". Apply your code to answer the following:

- Write $x^{10} + y^{10} + z^{10}$ as a polynomial in the elementary symmetric functions $w_1 = x + y + z, w_2 = xy + xz + yz$ and $w_3 = xyz$.

- Decide whether the vector $v = (94, 144, 190)$ can be written as a linear combination $A(2, 3, 5)+B(3, 5, 7)+C(5, 7, 11)+D(7, 11, 13)$ with $A, B, C,$ and D all positive integers. If so, find such an $A, B, C,$ and D. Repeat with the vectors $v = (96, 146, 190)$ and $v = (95, 142, 190)$.

- Write the polynomial $f(x,y) = x^8 + 2x^6y^2 - x^5y^3 + 2x^4y^4 + x^3y^5 + 2x^2y^6 + y^8$ as a polynomial in $w_1 = x^2 + y^2, w_2 = x^3y - xy^3$, and $w_3 = x^2y^2$.

4 Singularities of Trigonometric curves

Suppose f is the implicit equation of a trigonometric rose with polar equation $r = \cos(\frac{n}{d}\theta)$ and $I = \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ is the ideal defining the singular locus of f. Make some observations and attempt a conjecture in answer to the following questions.

- Can you describe the singular points of $V(f)$ in terms of the parameters n and d?
- Can you describe the prime decomposition of $\sqrt{I}$ over $\mathbb{Q}$ in terms of the parameters n and d? What degrees does each prime have? In other words, how many points does each prime ideal define? (You can use the 'degree' command in Macaulay2 to compute this.)
- Can you describe the primary components of I over $\mathbb{Q}$ in terms of the parameters n and d?