Problem Set 2

Most of these problems are taken from the 4th edition of Cox-Little-O'Shea, § 1.4, § 1.5.

1. Use the Euclidean algorithm to do the following.
   (a) Find $\gcd(112, 84)$ and find integers a, b so that $\gcd(112, 84) = a \cdot 84 + b \cdot 112$.
   (b) Find h so that $\langle x^3 + x + 1, x^2 + 2x + 3 \rangle = \langle h \rangle$.
   (c) Find h so that $\langle x^3 - x, x^2 + 3x + 2 \rangle = \langle h \rangle$.
   (d) Find h so that $\langle x^4 - x, x^5 - x, x^6 - x \rangle = \langle h \rangle$.
2. Find the row reduced echelon form of the matrix M below over $\mathbb{Q}(x)$ and over $\mathbb{F}_2[x]/\langle x^3 + x + 1 \rangle$.
$$\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 + x & 1 + x^2 \end{bmatrix}$$
3. Prove the following equalities of ideals in $\mathbb{Q}[x, y]$:
   (a) $\langle x + y, x - y \rangle = \langle x, y \rangle$
   (b) $\langle x + xy, y + xy, x^2, y^2 \rangle = \langle x, y \rangle$
   (c) $\langle y^2 - xz, xy - z, x^2 - y \rangle = \langle y - x^2, z - x^3 \rangle$
4. A *radical* ideal is an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ satisfying that if $f^k \in I$ for some integer k, then $f \in I$.
   (a) Prove that $I(V)$ is a radical ideal for any set $V \subset \mathbb{K}^n$.
   (b) Prove that $\langle x^2, y^2 \rangle$ is not a radical ideal, so it is not the ideal of any set.
5. Let $V \subset \mathbb{R}^3$ be parametrized by $x = t, y = t^3, z = t^4$.
   (a) Prove that V is an affine variety.
   (b) Determine the ideal $I(V)$.
6. Suppose $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is an ideal and f, g are polynomials.
   (a) If $f^2, g^2 \in I$, show that $(f + g)^3 \in I$.
   (b) More generally, if $f^r, g^s \in I$, show that $(f + g)^{r+s-1} \in I$.
7. Show that the Vandermonde determinant
$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ & \vdots & & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}$$
   is non-zero when all the a_i are distinct. Hint: if the determinant is zero, show that cofactor expansion along a row leads to a polynomial of degree $n - 1$ which has at least n roots.

8. Suppose $f \in \mathbb{C}[x]$.

   (a) If $f = (x-a)^r h$, where $a \in \mathbb{C}$ and $h \in \mathbb{C}[x]$ does not vanish at a, show that $f' = (x-a)^{r-1}h_1$, where h_1 does not vanish at a.

   (b) Let $f = c(x-a_1)^{r_1}\cdots(x-a_\ell)^{r_\ell}$ be the factorization of f, where $a_1, \ldots, a_\ell$ are distinct. Prove that $f' = (x-a_1)^{r_1-1}\cdots(x-a_\ell)^{r_\ell-1}H$, where $H \in \mathbb{C}[x]$ does not vanish at any of $a_1, \ldots, a_\ell$.

   (c) Prove that $\gcd(f, f') = (x-a_1)^{r_1-1}\cdots(x-a_\ell)^{r_\ell-1}$.

9. If $f \in \mathbb{C}[x]$ factors as $f = c(x-a_1)^{r_1}\cdots(x-a_\ell)^{r_\ell}$, then the *squarefree part* of f, denoted f_{red} is defined as $f_{red} = c(x-a_1)\cdots(x-a_\ell)$.

   (a) Use the previous exercise to show that

   $$f_{\text{red}} = \frac{f}{\gcd(f, f')}.$$

   This allows for quick computation of f_{red} *without* factoring f.

   (b) Use Macaulay2 to find f_{red} if

   $$f = x^{11} - x^{10} + 2x^8 - 4x^7 + 3x^5 - 3x^4 + x^3 + 3x^2 - x - 1.$$