APPROXIMATION THEORY FOCUS GROUP - THE FIELDS INSTITUTE 2025 - A RESTRICTION MAP AND APPLICATIONS TO FREENESS, WITH EMPHASIS ON HYPERPLANE ARRANGEMENTS

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1. Setup

Notation:

(1) Σ: a pure, hereditary, n-dimensional polyhedral fan in $\mathbb{R}^n$.
(2) Σ_i: i-dimesnional cones of Σ
(3) $|\Sigma| = \bigcup_{\sigma\in\Sigma_n}\sigma$. Σ is *complete* if $|\Sigma| = \mathbb{R}^n$
(4) R: The polynomial ring $\mathbb{R}[x_1,\dots,x_n]$
(5) Smoothness distribution or smoothness parameters $\mathbf{r} : \Sigma_{n-1} \to \mathbb{Z}_{\geq -1}$. If $G = (V,E)$ is a graph we will also use $\mathbf{r}$ to denote a map $\mathbf{r} : E \to \mathbb{Z}_{\geq -1}$. We choose this notation so that it matches when G is the *dual graph* of Σ.
(6) $S^{\mathbf{r}}(\Sigma)$: the S-algebra of functions $F : |\Sigma| \to \mathbb{R}$ that are piecewise polynomial on Σ_n and differentiable to order $\mathbf{r}(\tau)$ across the codimension one face τ, for all $\tau\in\Sigma_{n-1}$.
(7) H: A hyperplane, the zero locus of a linear form $\alpha_H \in R_1$.
(8) $\mathcal{A}$: a *central* hyperplane arrangement $\mathcal{A} = \cup_{i=1}^k \mathbb{V}(\alpha_i)$, where $\alpha_i \in R_1$ is a linear form for $i = 1,\dots,k$.
(9) $\Sigma^{\mathcal{A}}$: the complete fan induced by the hyperplane arrangement $\mathcal{A}$, whose maximal cones are the closure (in the Euclidean topology) of the connected components of the complement $\mathbb{R}^n \setminus \mathcal{A}$.
(10) $\mathcal{R}[\Sigma]$: the cellular chain complex (with coefficients in R) for Σ. The homology is the *Borel-Moore* homology with coefficients in R. This is equivalent to the homology of $B^n\cap\Sigma$ relative to $\partial B^n = \mathbb{S}^n$, where B^n is the unit ball in $\mathbb{R}^n$.
(11) $J^{\mathbf{r}}(\tau)$: Here $\tau\in\Sigma_{n-1}$, and the linear span of τ is defined by the linear form $\alpha_\tau \in R_1$. By definition, $J^{\mathbf{r}}(\tau) = \langle \alpha_\tau^{\mathbf{r}(\tau)+1}\rangle$.
(12) $J^{\mathbf{r}}(\gamma)$: Here $\gamma\in\Sigma_i$, $0\leq i\leq n-1$. By definition, $J^{\mathbf{r}}(\gamma) = \sum_{\tau\ni\gamma} J^{\mathbf{r}}(\tau) = \langle \alpha_\tau^{\mathbf{r}(\tau)+1} : \gamma\in\tau\rangle$.
(13) $\mathcal{J}^{\mathbf{r}}[\Sigma]$: Abbreviated $\mathcal{J}$ if $\mathbf{r}$ and Σ are understood. The sub-complex of $\mathcal{R}$ with modules $\mathcal{J}_n = 0$ and $\mathcal{J}_i = \bigoplus_{\gamma\in\Sigma_i} J^{\mathbf{r}}(\gamma)$ for $0\leq i\leq n$.
(14) $\mathcal{R}/\mathcal{J}^{\mathbf{r}}[\Sigma]$: Abbreviated $\mathcal{R}/\mathcal{J}$ if $\mathbf{r}$, Σ understood. The quotient of the chain complex $\mathcal{R}[\Sigma]$ by the subcomplex $\mathcal{J}^{\mathbf{r}}[\Sigma]$.
(15) $G = (V,E)$: A graph with vertices V and edges E
(16) $\ell : E \to R$: A map associating each edge of G to a linear form of R.
(17) $S^{\mathbf{r}}(G,\ell)$ (simply $S^{\mathbf{r}}(G)$ if ℓ is understood): the R-module of *generalized splines* on the edge-labeled graph $(G,\mathcal{I})$ with map $\mathcal{I} : E \to \{\text{ideals of } R\}$ defined by $\mathcal{I}(e) = \langle \ell(e)^{\mathbf{r}(e)+1}\rangle$, as defined in [4].

Remark 1.1. Suppose Σ is a pure, hereditary, n-dimensional fan in $\mathbb{R}^n$, and G is its dual graph. That is, $G = (V,E)$ is the graph with a vertex v_σ for each $\sigma\in\Sigma_n$ and $v_\sigma, v_{\sigma'}$ are connected by an edge if and only if $\sigma\cap\sigma' = \tau \in \Sigma_{n=1}$. By definition each edge e of G corresponds to a codimension one cone $\tau\in\Sigma_{n-1}$.

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Let $\mathbf{r} : \Sigma_{n-1} \to \mathbb{Z}_{\geq -1}$ be a smoothness distribution. By a slight abuse of notation we identify Σ_{n-1} with E, and so also regard $\mathbf{r}$ as a map $\mathbf{r} : E \to \mathbb{Z}_{\geq -1}$. Then $S^{\mathbf{r}}(\Sigma) = S^{\mathbf{r}}(G)$.

Remark 1.2. It is well-known that the spline module $S^{\mathbf{r}}(\Sigma)$ has a similar structure as the *module of derivations* $D(\mathcal{A})$ of a hyperplane arrangement (and the module $D(\mathcal{A}, \mathbf{m})$ of multi-derivations, where $\mathbf{m}$ assigns a multiplicity to each hyperplane – playing the analogous role to $\mathbf{r}$ for splines). The module of derivations was introduced by Saito (citation needed), while Ziegler showed in [9] that multi-derivations show up naturally in the study of derivations via restriction.

The study of derivations and multi-derivations is much more developed from an algebraic point of view than is the study of splines.

Remark 1.3. A difficult and subtle question is to determine when the spline module $S^{\mathbf{r}}(\Sigma)$ is *free*. When Σ is *simplicial*, Schenck shows in [6] that $S^{\mathbf{r}}(\Sigma)$ is free if and only if $H_i(\mathcal{R}/\mathcal{J}[\Sigma]) = 0$ for $0 \leq i \leq n$. This criterion can be extended to non-simplicial fans that meet a certain (somewhat technical) hypothesis. In particular, the criterion holds for the fan induced by a hyperplane arrangement (need notes on this).

Remark 1.4. Freeness of the spline module turns out to be related to resolving ideals generated by powers of linear forms (sometimes called *power ideals* in the literature). A first glimpse of this may be seen in [2, Lemma 9.12], where it is shown that, for a complete fan $\Sigma \subseteq \mathbb{R}^3$, $H_2(\mathcal{R}/\mathcal{J}[\Sigma])$ is isomorphic to the syzygy module of the ideal $J^{\mathbf{r}}(\mathbf{0})$ modulo the sum of syzygy modules of the ideals $J^{\mathbf{r}}(\gamma)$ for $\gamma \in \Sigma_1$.

2. Saito-Rose determinantal criterion for freeness

A well-known criterion for the freeness of the module of derivations $D(\mathcal{A})$ and the module of multi-derivations $D(\mathcal{A}, m)$ is a determinantal criterion known as *Saito's criterion*. In [5], Rose gives an analogous criterion for the freeness of the spline module $S^{\mathbf{r}}(\Delta)$. (There is an interesting result in [3] which extends the Saito-Rose criterion to generalized splines on graphs $S^{\mathbf{r}}(G)$ over a factorial domain).

In the following theorem, suppose $F_1, \ldots, F_k$ are splines in $S^{\mathbf{r}}(\Sigma)$. Write F_{ij} for $(F_j)|_{\sigma_i}$, where $\sigma_1, \ldots, \sigma_n$ is an enumeration of the full-dimensional cones of Σ. We write $\left[F_1 \cdots F_n\right]$ for the $n \times k$ matrix whose entry in row i and column j is F_{ij}. The following theorem is [5, Theorem 2.3], stated in slightly more generality.

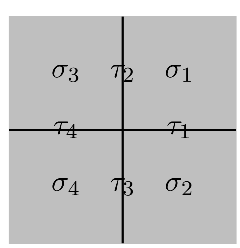
Theorem 2.1 ([5, Theorem 2.3])**.** *Suppose $\Sigma \subset \mathbb{R}^n$ is a pure, hereditary, n-dimensional fan with ℓ full-dimensional cones. Let $\mathbf{r} : \Sigma_{n-1} \to \mathbb{Z}_{\geq 0}$ be a smoothness distribution. Then a collection $\{F_1, \ldots, F_\ell\}$ of splines in $S^{\mathbf{r}}(\Sigma)$ is a free basis for $S^{\mathbf{r}}(\Sigma)$ if and only if* $\det\left[F_1 \cdots F_\ell\right] = cQ$ *for some non-zero constant c, where $Q = \prod_{\tau \in \Sigma_{n-1}} \alpha_\tau^{\mathbf{r}(\tau)+1}$.*

As an immediate corollary we obtain

Corollary 2.2 ([5, Corollary 2.4])**.** *Let $\Sigma \subset \mathbb{R}^n$ and $\mathbf{r}$ be as in the statement of Theorem 2.1. A collection $\{F_1, \ldots, F_n\}$ of S-linearly independent homogeneous elements of $S^{\mathbf{r}}(\Sigma)$ form a basis over S if and only if $\sum_{i=1}^{n} \deg(F_i) = \sum_{\tau \in \Sigma_{n-1}} (\mathbf{r}(\tau) + 1)$.*

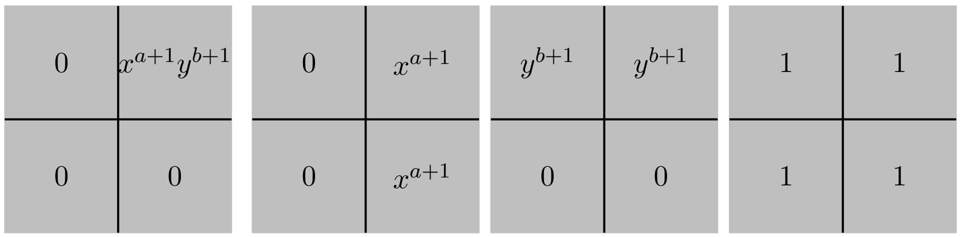
Example 2.3. We illustrate Let Σ be the fan in $\mathbb{R}^2$ with the four cones shown below. The one-dimensional cones $\tau_1, \tau_2, \tau_3, \tau_4$ are also labeled.

$$\sigma_1 = \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\right\}$$
$$\sigma_2 = \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \leq 0\right\}$$
$$\sigma_3 = \left\{(x, y) \in \mathbb{R}^2 : x \leq 0, y \geq 0\right\}$$
$$\sigma_4 = \left\{(x, y) \in \mathbb{R}^2 : x \leq 0, y \leq 0\right\}$$



Let us choose smoothness distribution $\mathbf{r}(\tau_2) = \mathbf{r}(\tau_3) = a \in \mathbb{Z}_{\geq 0}$ and $\mathbf{r}(\tau_1) = \mathbf{r}(\tau_4) = b \in \mathbb{Z}_{\geq 0}$. Put $R = \mathbb{R}[x, y]$. This is one case where we can easily write down generators for $S^{\mathbf{r}}(\Sigma)$. We have $\alpha_{\tau_2} = \alpha_{\tau_3} = x$ and $\alpha_{\tau_1} = \alpha_{\tau_4} = y$.

We record a spline $F \in S^{\mathbf{r}}(\Sigma)$ as a tuple $(F_{\sigma_1}, F_{\sigma_2}, F_{\sigma_3}, F_{\sigma_4})$ or in transposed form as a column vector. Put $F_{\sigma_i} = F_i$ for $i = 1, 2, 3, 4$. There are four natural splines: $F_1 = (x^{a+1}y^{b+1}, 0, 0, 0), F_2 = (x^{a+1}, x^{a+1}, 0, 0), F_3 = (y^{b+1}, 0, y^{b+1}, 0)$, and $F_4 = (1, 1, 1, 1)$ (check the spline conditions):



Putting these as column vectors into a matrix we get:

$$\begin{array}{c c} & \begin{array}{cccc} F_1 & F_2 & F_3 & F_4 \end{array} \\ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} & \begin{bmatrix} x^{a+1}y^{b+1} & x^{a+1} & y^{b+1} & 1 \\ 0 & x^{a+1} & 0 & 1 \\ 0 & 0 & y^{b+1} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Observe that the determinant of the above matrix is $x^{2(a+1)}y^{2(b+1)}$, which means by the Saito-Rose criterion that these are a free basis for the spline module $S^{\mathbf{r}}(\Sigma)$.

There is a nice generalization of this basis to the fan whose cones are the 2^n orthants of $\mathbb{R}^n$.

Example 2.4.

3. A Restriction Map

A standard operation for sheaves on projective space is the restriction to a hyperplane. This operation is of fundamental importance in the theory of hyperplane arrangements, where it gives rise to addition-deletion theorems for (simple) arrangements [7, 8]. There are also addition-deletion theorems for multi-arrangements that were discovered much later [1]. One reason that it took so long (almost 30 years after Terao's addition-deletion theorems and 20 years after Ziegler defined multi-arrangements) to generalize addition-deletion methods to multi-arrangements is that these methods required a technical tool developed by the authors of [1] called the *Euler multiplicity.* It is not clear if there is an anologous technique that will lead to addition-deletion techniques for splines. We define a natural restriction of the spline module that allows inductive arguments to work on some classes of subdivision.

In this section, we discuss what restriction to a hyperplane looks like for splines on fans.

We first describe a natural restriction of the module of splines $S^{\mathbf{r}}(\Sigma)$ which turns out, in general, not to have the good properties necessary to formulate addition-deletion theorems. We then conjecture that a slight modification of this restriction does have the right properties. We also consider circumstances in which no alteration is necessary.

If $H \subset \mathbb{R}^n$ is a hyperplane, we write $\mathbf{1}_H$ for the function $\mathbf{1}_H : \Sigma_{n-1} \to \mathbb{Z}_{\geq 0}$ defined by

$$\mathbf{1}_H(\tau) = \begin{cases} 1 & \tau \subset H \\ 0 & \text{otherwise} \end{cases}.$$

We write α_H for a choice of linear form vanishing on H.

Definition 3.1. For a fan $\Sigma \subset \mathbb{R}^n$, a hyperplane $H \subset \mathbb{R}^n$, and a smoothness distribution $\mathbf{r} : \Sigma_{n-1} \to \mathbb{Z}_{\geq -1}$ which is non-negative on every $\tau \in \Sigma_{n-1}$ so that $\tau \subset H$, the pair $(\Sigma, \mathbf{r} - \mathbf{1}_H)$ is called the *deletion* of $(\Sigma, \mathbf{r})$ along H.

For a fan $\Sigma \subset \mathbb{R}^n$ we write G_Σ for the dual graph. Given a hyperplane $H \subset \mathbb{R}^n$ with linear form α_H vanishing on H, we write $\ell|_H$ for the map $\ell|_H : E \to R/\langle \alpha_H \rangle$ for the map that sends an edge e (dual to a codimension one cone τ) to the coset defined by the linear form $\alpha_\tau + \langle \alpha_H \rangle$ in the quotient $R/\langle \alpha_H \rangle$. Then $S^{\mathbf{r}}(G, \ell|_H)$ is the module of generalized splines on the edge labeled graph $(G_\Sigma, \mathcal{I})$, where $\mathcal{I}(e) = \alpha_\tau^{\mathbf{r}(\tau)+1} + \langle \alpha_H \rangle = J^{\mathbf{r}}(\tau) + \langle \alpha_H \rangle$ in the ring $R/\langle \alpha_H \rangle$.

Proposition 3.2. *If $\mathbf{r} : \Sigma_{n-1} \to \mathbb{Z}_{\geq 0}$ is non-negative, there is a left exact sequence*

$$0 \to S^{\mathbf{r}-\mathbf{1}_H}(\Sigma) \xrightarrow{\cdot \alpha_H} S^{\mathbf{r}}(\Sigma) \xrightarrow{\pi} S^{\mathbf{r}}(G_\Sigma, \ell|_H),$$

where $\cdot\alpha_H$ is multiplication by α_H (this should be viewed as happening in the free module R^{Σ_n}) and π is the quotient map taking each polynomial constituent modulo α_H.

Remark 3.3. By a change of coordinates, we may assume that $\alpha_H = x_n$ and so $R/\langle \alpha_H \rangle \cong \mathbb{R}[x_1, \ldots, x_{n-1}]$. Then, if $(x_1, \ldots, x_n) \in \mathbb{R}^n$, we can concretely view $\pi(F)$ as the tuple $(F_\sigma(x_1, \ldots, x_{n-1}, 0))_{\sigma \in \Sigma_n} \in (R')^{\Sigma_n}$.

Proof. If $F \in S^{\mathbf{r}}(\Sigma)$, then $F = (F_\sigma)_{\sigma \in \Sigma_n} \in R^{\Sigma_n}$. Put $R' = R/\langle \alpha_H \rangle$. The map π can be described as $\pi(F) = (F_\sigma + \langle \alpha_H \rangle)_\sigma \in (R')^{\Sigma_n}$. Write $\bar{F}_\sigma$ for the coset $F_\sigma + \langle \alpha_H \rangle$.

We first show that if $F \in S^{\mathbf{r}}(\Sigma)$ then $\pi(F) \in S^{\mathbf{r}}(G_\Sigma, \ell|_H)$. The spline conditions in $S^{\mathbf{r}}(G_\Sigma, \ell|_H)$ imply that if $\sigma, \sigma' \in \Sigma_n$ and $\sigma' \cap \sigma = \tau \in \Sigma_{n-1}$, then $F_\sigma - F_\sigma \in \langle \alpha_\tau^{\mathbf{r}(\tau)+1} \rangle = J^{\mathbf{r}}(\tau)$. Thus $\bar{F}_\sigma - \bar{F}_{\sigma'} \in J^{\mathbf{r}}(\tau) + \langle \alpha_H \rangle$, as desired. It follows that if $F \in S^{\mathbf{r}}(\Sigma)$ then $\pi(F) \in S^{\mathbf{r}}(G_\Sigma, \ell|_H)$.

Since $S^{\mathbf{r}-\mathbf{1}_H}(\Sigma)$ is an R-submodule of R^{Σ_n}, $\alpha_H S^{\mathbf{r}-\mathbf{1}_H}(\Sigma)$ is simply given as pointwise multiplication by α_H. This is an injective map. So what is left is to prove that $\ker(\pi) = \alpha_H S^{\mathbf{r}-\mathbf{1}_H}(\Sigma)$.

First we prove that $\alpha_H S^{\mathbf{r}-\mathbf{1}_H}(\Sigma) \subset S^{\mathbf{r}}(\Sigma)$. Suppose $F' \in S^{\mathbf{r}-\mathbf{1}_H}(\Sigma)$. Now suppose $\sigma_1, \sigma_2 \in \Sigma_n$ so that $\sigma_1 \cap \sigma_2 = \tau \in \Sigma_{n-1}$. Then

$$F'_{\sigma_1} - F'_{\sigma_2} = g\alpha_\tau^{(\mathbf{r}-\mathbf{1}_H)(\tau)} \tag{3.1}$$

Multiplying both sides of (3.1) by α_H, we obtain $\alpha_H F'_{\sigma_1} - \alpha_H F'_{\sigma_2} = g\alpha_H \alpha_\tau^{(\mathbf{r}-\mathbf{1}_H)(\tau)}$. If $\tau \not\subset H$, then

$$\alpha_H F'_{\sigma_1} - \alpha_H F'_{\sigma_2} = g\alpha_H \alpha_\tau^{(\mathbf{r}-\mathbf{1}_H)(\tau)} = (g\alpha_H)\alpha_\tau^{\mathbf{r}(\tau)+1}.$$

If $\tau \subset H$, then $\alpha_\tau = \alpha_H$ and

$$\alpha_H F'_{\sigma_1} - \alpha_H F'_{\sigma_2} = g(\alpha_H \alpha_\tau^{(\mathbf{r}-\mathbf{1}_H)(\tau)}) = g\alpha_\tau^{\mathbf{r}(\tau)+1}.$$

In either case, $\alpha_H F'_{\sigma_1} - \alpha_H F'_{\sigma_2}$ satisfies the spline criterion across τ. Since σ_1, σ_2 were arbitrary, $\alpha_H F' \in S^{\mathbf{r}}(\Sigma)$. It is clear that $\alpha_H F'$ is in the kernel of π, since every polynomial constituent will be mapped to zero in the qotient $R/\langle \alpha_H \rangle$.

Let $F = (F_\sigma)_{\sigma \in \Sigma_n} \in \ker(\pi)$. Then, for every $\sigma \in \Sigma_n$, $F_\sigma + \langle \alpha_H \rangle = \langle \alpha_H \rangle$, so $F_\sigma \in \langle \alpha_H \rangle$. It follows that, for every $\sigma \in \Sigma_n$, $F_\sigma = F'_\sigma \alpha_H$ for some $F'_\sigma \in R$. We prove that $(F'_\sigma)_{\sigma \in \Sigma_n} \in S^{\mathbf{r}-\mathbf{1}_H}$. Suppose $\sigma_1, \sigma_2 \in \Sigma_n$ so that $\sigma_1 \cap \sigma_2 = \tau \in \Sigma_{n-1}$. Then

$$F_{\sigma_1} - F_{\sigma_2} = g\alpha_\tau^{\mathbf{r}(\tau)+1} \tag{3.2}$$

for some $g \in R$. If $\tau \not\subset H$ then $\mathbf{r}(\tau) = (\mathbf{r} - \mathbf{1}_H)(\tau)$ by definition and we can re-write (3.2) as $\alpha_H(F'_{\sigma_1} - F'_{\sigma_2}) = g\alpha_\tau^{(\mathbf{r}-\mathbf{1}_H)(\tau)}$. Since α_τ and α_H are linear forms, they are both prime ring elements of R. They are coprime since they are not multiples of each other. So α_H must divide g, yielding $g = \alpha_H g'$ for some $g' \in R$. Thus

$$F'_{\sigma_1} - F'_{\sigma_2} = g'\alpha_\tau^{\mathbf{r}(\tau)+1} = g'\alpha_\tau^{(\mathbf{r}-\mathbf{1}_H)(\tau)+1},$$

as desired.

If $\tau \subset H$ then $\alpha_\tau = \alpha_H$ and we can re-write (3.2) as $\alpha_\tau(F'_{\sigma_1} - F'_{\sigma_2}) = g\alpha_\tau^{\mathbf{r}(\tau)+1}$. Cancelling α_τ on both sides yields

$$F'_{\sigma_1} - F'_{\sigma_2} = g\alpha_\tau^{\mathbf{r}(\tau)+1-1} = g\alpha_\tau^{(\mathbf{r}-\mathbf{1}_H)(\tau)+1},$$

as desired. Thus, for any $\sigma_1, \sigma_2 \in \Sigma_n$ satisfying that $\sigma_1 \cap \sigma_2 = \tau \in \Sigma_{n-1}$, $F'_{\sigma_1} - F'_{\sigma_2} \in \langle \alpha_\tau^{(\mathbf{r}-\mathbf{1}_H)(\tau)+1} \rangle$. It follows that $F' = (F'_\sigma)_{\sigma \in \Sigma_n} \in S^{\mathbf{r}-\mathbf{1}_H}(\Sigma)$. Thus $F = \alpha_H F \in \alpha_H S^{\mathbf{r}-\mathbf{1}_H}(\Sigma)$, as desired. □

$S^{\mathbf{r}}(G_\Sigma, \ell|_H)$ appears to be a good candidate for addition-deletion techniques, but it is typically not. It turns out that a fundamental property must be satisfied in order to have a chance for addition-deletion techniques - this is that the map π needs to be locally surjective in codimension two. That is, upon localizing the above exact sequence at primes of codimension two, it becomes a short exact sequence. If Σ is three-dimensional, this means that π induces a surjective map of sheaves, whence we get a long exact sequence in sheaf cohomology that allows us to determine freeness via Horrock's criterion (via duality theorems - either Serre duality or local duality - this is equivalent to vanishing of Ext^i for $i > 0$). It turns out that the map π in Proposition 3.2 is *not* surjective in codimension two. We illustrate by considering a simple example when Σ itself is two-dimensional, and the map π is not surjective.

4. If both the spline module and its deletion are free

5. Restriction to a generic hyperplane

In this section we will see that, in case H is what we call a *generic* hyperplane, then $S^{\mathbf{r}}(G_\Sigma, \ell|_H)$ is a good restriction in the sense that the map π in Proposition 3.2 is surjective in codimension two. We start with a simple proposition.

Proposition 5.1. *Let $\alpha \in R$ be a polynomial and I an ideal of R. Given a positive integer k, define a map $\phi : \dfrac{S}{\langle \alpha^k \rangle + I} \to \dfrac{S}{\langle \alpha^{k+1} \rangle + I}$ by $\phi(F + \langle \alpha^k \rangle + I) = \alpha \cdot F + \langle \alpha^{k+1} \rangle + I$, where $F \in R$. Then*

- *ϕ is a well-defined homomorphism of R-modules.*
- *If α is a non-zero divisor on R/I, then ϕ is injective.*

Proof. To show ϕ is well-defined, suppose that $F, G \in R$ satisfy that $F + \langle \alpha^k \rangle + I = G + \langle \alpha^k \rangle + I$. Then there exist $f \in R$ and $p \in I$ so that $F - G = f\alpha^k + p$. So $\alpha \cdot F - \alpha \cdot G = f\alpha^{k+1} + \alpha \cdot p \in \langle \alpha^{k+1} \rangle + I$. It follows that ϕ is well-defined. It is routine to check that ϕ is a homorphism of R-modules.

Now suppose that α is a non-zero divisor on R/I. We show ϕ is injective. Suppose $F \in R$ and $\phi(F + \langle \alpha^k \rangle + I) = \alpha \cdot F + \langle \alpha^{k+1} \rangle + I = 0 + \langle \alpha^{k+1} \rangle + I$. Then $\alpha \cdot F \in \langle \alpha^{k+1} \rangle + I$, so there exist $f \in R$, $p \in I$ so that $\alpha \cdot F = f\alpha^{k+1} + p$. Hence $\alpha(F - \alpha^k) \in I$. But α is a non-zero divisor on R/I, hence we must have $F - \alpha^k \in I$, or $F + \langle \alpha^k \rangle + I = 0 + \langle \alpha^k \rangle + I$. Hence ϕ is injective. □

To simplify things, we only make the following definition for $\mathbb{R}^3$ only.

Definition 5.2. Let $\Sigma \subset \mathbb{R}^3$ be a fan, and H a hyperplane in $\mathbb{R}^n$. We say H is *generic* (with respect to Σ) if, for every ray $\gamma \in \Sigma_1$ so that $\gamma \subset H$, there exists a hyperplane H' so that if $\tau \in \Sigma_2$ and $\gamma \subset \tau$ then $\alpha_\tau = \alpha_H$ or $\alpha_\tau = \alpha_{H'}$. In particular, this condition is met if H is chosen so that no $\gamma \in \Sigma_1$ is contained in H.

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