

## Problem Set 5

Many of these problems are taken from the 4th edition of Cox-Little-O'Shea.

1. (There was a typo on this problem in the last worksheet) GRevLex is not always better than Lex, but it often is. Use Macaulay2 to compute Gröbner bases for  $I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1 \rangle$  with respect to GRevLex, with respect to Lex, and with respect to the monomial order defined by the weight vectors  $[1, 0, 0]$ ,  $[1, 1, 1]$ ,  $[1, 1, 0]$ . Compare your results.
2. If  $\alpha$  is an element of the field  $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ , then  $\alpha = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ . Find the minimal polynomial of  $\alpha$  in Macaulay2 as follows:
  - Define  $R = \mathbb{Q}[a, b, c, d, x, y, t]$
  - Define  $I = \langle x^2 - 2, y^2 - 3, t - (a + bx + cy + dxy) \rangle$
  - Eliminate  $x$  and  $y$  from  $I$ . (use the command 'eliminate(I, {x,y})' to do this efficiently)

Your result should be a polynomial in  $t$  with coefficients which are polynomials in  $a, b, c, d$ .

3. (4-bar linkage) In Figure 1 a 4-bar linkage is shown with lengths assigned to the edges (these are called *bars*). The points  $(0, 0)$  and  $(5, 0)$  are fixed, but the bars are allowed to turn at the joints where they are attached. The *coupler curve* of the linkage is the curve traced out by the point with coordinates  $(s, t)$  as the linkage moves. Check out the following website for some nice visualizations of coupler curves of 4-bar linkages: <https://saltire.com/HTML5/Mechanisms/linkages.html>.

In this problem you will use Macaulay2 to find the implicit equation of the coupler curve for the 4-bar linkage with lengths indicated in Figure 1. Do this as follows:

- Set up a polynomial ring in the six variables corresponding to the coordinates of the three movable joints, as in Figure 1.
- Set up an ideal containing expressions corresponding to bar lengths.
- Eliminate all variables except  $s$  and  $t$  (you should get a single polynomial)

Once you get the equation in  $s$  and  $t$ , open up a Sage worksheet in CoCalc and plot the curve in  $\mathbb{R}^2$  defined by this equation. Here are the Sage commands you will need: Suppose your polynomial in  $s$  and  $t$  is  $3s + 5t$ . In Sage, the command " $F(s, t) = 3s + 5t$ " defines a function, and the command "`implicit_plot(F(s,t),(s,a,b),(t,c,d))`" plots the curve defined by  $F(s, t) = 0$  over the window  $a \leq s \leq b, c \leq t \leq d$ .

4. (Lagrange multipliers) In this problem you will use both symbolic and numerical methods to explore the critical points of  $g(x, y) = x^3 + 3xyz - 2z^2$  restricted to the sphere defined by  $x^2 + y^2 + z^2 = 1$ . Lagrange multipliers yields the equations

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ 3x^2 + 3yz &= 2\lambda x \\ 3xz &= 2\lambda y \\ 3xy - 4z &= 2\lambda z \end{aligned}$$

for the critical points of  $g$  restricted to the sphere.

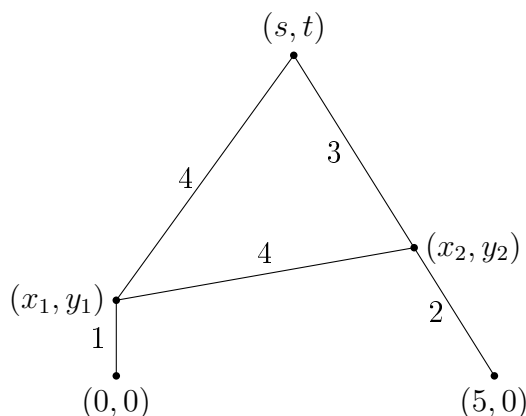


Figure 1: 4-bar linkage for problem 3

- (a) Let  $S = \mathbb{Q}[x, y, z, \lambda]$  (feel free to use a different variable than  $\lambda$ ). Use Macaulay2 to find a Gröbner basis for the ideal  $I \subset S$  defined by these equations and find the dimension of  $S/I$  as a vector space.
  - (b) Load the Numerical Algebraic Geometry package in Macaulay2 using the command: `loadPackage "NumericalAlgebraicGeometry"`  
Now use the command ‘`solveSystem`’ from the Numerical Algebraic Geometry package to find all the solutions of this system of equations over  $\mathbb{C}$ . Compare the number of solutions to the dimension of  $S/I$  as a vector space that you found in (a). Then use the command ‘`realSolutions`’ to find the real solutions of this system of equations.
5. (Singular Locus) Suppose  $f \in \mathbb{K}[x, y]$  defines a curve  $C = V(f) \subset \mathbb{K}^2$ . The *singular locus* of  $C$  consists of those points on  $C$  where a tangent line is not well-defined. These are places where  $f$  vanishes *and* both of its derivatives vanish. In other words the singular locus is  $X_{\text{sing}} = V(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ . If  $X_{\text{sing}} = \emptyset$ , then  $V(f)$  is *smooth*.
- (a) If  $f = y^2 - x(x-1)(x-3)$ , verify that the curve  $C = V(f)$  is smooth. Hint: show that  $1 \in \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ . Plot the curve  $C$  in Sage on CoCalc and verify that it looks smooth.
  - (b) If  $f = x^4 + 2x^3y - x^2y^2 + 2xy^3 - 2y^4 - 4x^3 - 4xy^2 + 4y^3 + 2x^2 + 2xy - 2y^2$ , find the singular locus of  $V(f)$ . Do this in Macaulay2 by defining the ideal  $I = \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ . (The command ‘`diff(x,f)`’ takes the derivative of  $f$  with respect to  $x$ .) Then find a Gröbner basis of  $I$  (use the command ‘`gens gb I`’) and solve these equations by hand. Plot the curve  $f = 0$  in Sage on CoCalc to see that your answer makes sense. Make sure that your viewing window includes the singular locus.
6. (Implicitization) The *Enneper surface* in  $\mathbb{R}^3$  is parametrized by
- $$\begin{aligned} x &= 3u + 3uv^2 - u^3 \\ y &= 3v + 3u^2v - v^3 \\ z &= 3u^2 - 3v^2 \end{aligned}$$
- (a) Define an appropriate ideal and use elimination to obtain the equation of this surface in the variables  $x, y, z$ .

- (b) Use Sage to plot this surface in two ways: using “implicit\_plot3d” with the equation you found in (a) and using “parametric\_plot3d” with the parametrization given above.
- (c) (optional) The difference between the two plots in (a) and (b) is partially explained by the complicated singular locus of the Enneper surface (notice from the two plots that Sage has trouble on ‘implicit\_plot3d’ near the singular locus). Let  $f$  be the polynomial you found in (a). In Macaulay2, define the ideal  $I = \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$ . As we saw in Problem 5, the variety of this ideal defines the singular locus of  $V(f)$ . The command ‘decompose(I)’ in Macaulay2 “pulls apart” the singular locus into its components (we’ll learn more about this when we learn about primary decomposition). See if you can match at least some of the output of this command with what you see in part (b).
7. (Ring homomorphisms) Suppose  $\phi : \mathbb{K}[w_1, \dots, w_m] \rightarrow \mathbb{K}[x_1, \dots, x_n]$  is a ring homomorphism defined by  $\phi(w_i) = f_i$  for some polynomials  $f_1, \dots, f_n \in \mathbb{K}[x_1, \dots, x_n]$ . Let  $S = \mathbb{K}[x_1, \dots, x_n, w_1, \dots, w_m]$  and define the ideal  $I = \langle w_i - f_i \rangle \subset S$ . Let  $>$  be an  $n$ -elimination order on  $S$  (i.e. a monomial order satisfying that any monomial involving the  $x_i$  variables is bigger than any monomial which involves only the  $w_j$  variables) and let  $G$  be a Gröbner basis for  $I$  under  $>$ . Suppose  $f \in \mathbb{K}[x_1, \dots, x_n]$  and  $R$  is the remainder of  $f$  on division by  $G$ .
- (a) If  $f \in \text{im}(\phi)$ , prove that  $R \in \mathbb{K}[w_1, \dots, w_m]$ ,  $\phi(R) = f$ , and  $R$  is the minimal polynomial in  $\mathbb{K}[w_1, \dots, w_m]$  under  $>$  satisfying these properties.
- (b) Prove that  $f \in \text{im}(\phi)$  if and only if  $R \in \mathbb{K}[w_1, \dots, w_m]$ .
- (c) If  $\phi(w_i)$  is a monomial for  $i = 1, \dots, m$ , prove the following:
- Every polynomial in  $G$  is a difference of monomials.
  - If  $f$  is a monomial, its remainder  $R$  on division by  $G$  is also a monomial.
8. (Ring homomorphisms, continued) Consider the ring map  $\phi : \mathbb{K}[w_1, w_2, w_3] \rightarrow \mathbb{K}[x, y]$  defined by  $\phi(w_1) = x^2y^5$ ,  $\phi(w_2) = x^7y^2$ , and  $\phi(w_3) = x^3y^3$ .
- (a) Using the previous problem, use Macaulay2 to verify that  $x^{200}y^{200}$  is in the image of  $\phi$  and find an explicit monomial  $m = w_1^a w_2^b w_3^c$  so that  $\phi(m) = x^{200}y^{200}$ . You can do this as follows: use the command  
`‘R=QQ[x,y,w_1,w_2,w_3,MonomialOrder=>{Weights=>{1,1,0,0,0}}]’`  
to define a Bayer-Stillman elimination order on  $R$  (Macaulay2 will automatically break ties using GRevLex). Then define the ideal  $I$  as in problem 7 and set  $m = x^{200}y^{200}$ . Then use the command ‘m%I’ to get the remainder of  $m$  on division by a Gröbner basis of  $I$  using the monomial order you defined (Macaulay2 will automatically compute the Gröbner basis for  $I$  with respect to the chosen monomial order when you type in ‘%’).
- (b) Repeat (a) with the monomial order on  $\mathbb{K}[x, y, w_1, w_2, w_3]$  defined by first using the weight vector  $[1, 1, 0, 0, 0]$ , then the weight vector  $[0, 0, 0, 1, 0]$ , and then breaking ties with GRevLex. You can do this with the command  
`‘R=QQ[x,y,w_1,w_2,w_3,MonomialOrder=>{Weights=>{1,1,0,0,0},Weights=>{0,0,0,1,0}}]’`  
Conclude that there are at least two monomials which map to  $x^{200}y^{200}$  under the ring homomorphism  $\phi$ .