APPROXIMATION THEORY FOCUS GROUP - THE FIELDS INSTITUTE 2025 - PRESENTING H1 FOR COMPLETE FANS AND HYPERPLANE ARRANGEMENTS, WORKING NOTES.

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The following lemma is a useful presentation for $H_1(\mathcal{J}[\Sigma])$ when Σ is complete. This is the analogue of [4, Lemma 3.8] for complete fans.

Lemma 0.1 ([1, Lemma 9.12])**.** *Let $\Sigma \subset \mathbb{R}^3$ be a hereditary, complete fan. Define $K^r \subset \bigoplus_{\tau \in \Sigma_2} R(-\mathbf{r}(\tau))$ by*

$$K^{\mathbf{r}} = \{\sum_{\tau \ni \gamma} f_\tau e_\tau | \gamma \in \Sigma_1, \sum f_\tau \alpha_\tau^{\mathbf{r}(\tau)} = 0\}.$$

Also define $V^r \subset \bigoplus_{\tau \in \Sigma_2} R(-\mathbf{r}(\tau))$ by

$$V^{\mathbf{r}} = \{\sum_{\tau \in \Sigma_2} f_\tau e_\tau | \sum f_\tau \alpha_\tau^{\mathbf{r}(\tau)} = 0\}.$$

Then $K^{\mathbf{r}} \subset V^{\mathbf{r}}$ and $H_1(\mathcal{J}[\Sigma]) \cong V^{\mathbf{r}}/K^{\mathbf{r}}$ as R-modules.

Proof. The proof is similar to the proof of [4, Lemma 3.8]. Let $K_\gamma^{\mathbf{r}} \subset \bigoplus_{\gamma \in \tau} R(-\mathbf{r}(\tau)) e_\tau$ be the module of relations around the ray $\gamma \in \Sigma_1$, namely

$$K_\gamma^{\mathbf{r}} = \{\sum_{\tau \ni \gamma} f_\tau e_\tau | \sum f_\tau \alpha_\tau^{\mathbf{r}(\tau)} = 0\}.$$

Furthermore, let $J(\mathbf{0})$ be the ideal of the central vertex of Σ. Set up the following diagram with exact rows, whose first row is the complex $J[\Sigma]$.

$$\begin{tikzcd}
0 & 0 & 0 \\
\bigoplus_{\tau \in \Sigma_2} J^{\mathbf{r}}(\tau) \arrow[r] \arrow[u] & \bigoplus_{\gamma \in \Sigma_1} J^{\mathbf{r}}(\gamma) \arrow[r] \arrow[u] & J^{\mathbf{r}}(\mathbf{0}) \arrow[u] \\
\bigoplus_{\tau \in \Sigma_2} R(-\mathbf{r}(\tau)) \arrow[r] \arrow[u] & \bigoplus_{\substack{\gamma \in \Sigma_1, \tau \in \Sigma_2 \\ \gamma \in \tau}} R(-\mathbf{r}(\tau)) \arrow[r] \arrow[u] & \bigoplus_{\tau \in \Sigma_2} R(-\mathbf{r}(\tau)) \arrow[u] \\
0 \arrow[r] \arrow[u] & \bigoplus_{\gamma \in \Sigma_1} K_\gamma^{\mathbf{r}} \arrow[r, "\iota"] \arrow[u] & V^{\mathbf{r}} \arrow[u] \\
 & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

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The middle row is in fact exact because the inclusion on the left hand side has the effect of gluing together copies of $R(-\mathbf{r}(\tau))$ that correspond to different rays in Σ_1, leaving a copy of $R(-\mathbf{r}(\tau))$ for every codimension one face $\tau \in \Sigma_2$ in the cokernel. Now the long exact sequence in homology yields the isomorphisms $H_2(\mathcal{J}[\Sigma]) \cong \ker(\iota)$ and $H_1(\mathcal{J}[\Sigma]) \cong \operatorname{coker}(\iota)$. The image of $\bigoplus_{\gamma \in \Sigma_1} K^{\mathbf{r}}_\gamma$ under ι is precisely $K^{\mathbf{r}}$, so we are done. $\square$

Now suppose $\mathcal{A} = \bigcup_{i=1}^k H_i \subset \mathbb{R}^3$ is a hyperplane arrangement with associated complete fan $\Sigma^{\mathcal{A}}$. Let $\mathbf{r} : \Sigma_2^{\mathcal{A}} \to \mathbb{Z}_{\geq -1}$ be a smoothness distribution that is constant on hyperplanes (that is, if $\tau, \tau' \subset H \in \mathcal{A}$, then $\mathbf{r}(\tau) = \mathbf{r}(\tau')$). In this case, we also regard $\mathbf{r}$ as a map from $\mathcal{A} \to \mathbb{Z}_{\geq -1}$. Let $M^{\mathbf{r}} = \left[\alpha_1^{\mathbf{r}(H_1)} \cdots \alpha_k^{\mathbf{r}(H_k)}\right]$ be the matrix whose entries are the linear forms defining the hyperplanes of $\mathcal{A}$, raised to the power stipulated by $\mathbf{r}$. Let

$$\operatorname{syz}(M^{\mathbf{r}}) := \left\{ \sum_{i=1}^k f_i e_i : \sum_{i=1}^k f_i \alpha_i^{\mathbf{r}(H_i)} = 0 \right\} \subset \bigoplus_{i=1}^k R(-\mathbf{r}(H_i))$$

be the syzygy module of the matrix $M^{\mathbf{r}}$.

For a given line $\bar{\gamma}$ appearing as the intersection of at least two hyperplanes of $\mathcal{A}$, we write $M^{\mathbf{r}}_{\bar{\gamma}}$ for the matrix with a single row whose entries are $\{\alpha_H^{\mathbf{r}(H)} : \bar{\gamma} \subset H\}$. We similarly have

$$\operatorname{syz}(M^{\mathbf{r}}_{\bar{\gamma}}) := \left\{ \sum_{H \subset \bar{\gamma}} f_H e_H : \sum_{H \subset \bar{\gamma}} f_H \alpha_H^{\mathbf{r}(H)} = 0 \right\} \subset \bigoplus_{H \supset \bar{\gamma}} R(-\mathbf{r}(H)).$$

There is a natural inclusion from $\operatorname{syz}(M^{\mathbf{r}}_{\bar{\gamma}})$ into $\operatorname{syz}(M^{\mathbf{r}})$ by extending the syzygy on $M^{\mathbf{r}}_{\bar{\gamma}}$ by zero to the rest of the entries of $\operatorname{syz}(M^{\mathbf{r}})$.

Corollary 0.2. *If $\mathcal{A} = \bigcup_{i=1}^k H_i \subset \mathbb{R}^3$ is a hyperplane arrangement with associated complete fan $\Sigma^{\mathcal{A}}$, and $\mathbf{r} : \Sigma_2^{\mathcal{A}} \to \mathbb{Z}_{\geq 0}$ is a smoothness distribution, then*

$$H_1(\mathcal{J}[\Sigma^{\mathcal{A}}]) \cong \frac{\operatorname{syz}(M^{\mathbf{r}})}{\sum_{\bar{\gamma} \in L_2(\mathcal{A})} \operatorname{syz}(M^{\mathbf{r}}_{\bar{\gamma}})},$$

where $L_2(\mathcal{A})$ is the collection of lines appearing as intersections of hyperplanes of $\mathcal{A}$.

The case $\mathbf{r} \equiv \mathbf{0}$ of the above proposition deserves special attention.

Corollary 0.3. *If $\mathcal{A} = \bigcup_{i=1}^k H_i \subset \mathbb{R}^3$ is a central and essential hyperplane arrangement with associated complete fan $\Sigma^{\mathcal{A}}$ and $\mathbf{r} \equiv \mathbf{0}$, then $H_1(\mathcal{J}[\Sigma^{\mathcal{A}}])$ is isomorphic to the $\mathbb{R}$-vector space of $\mathbb{R}$-linear relations among the linear forms $\alpha_1, \ldots, \alpha_k$ modulo the $\mathbb{R}$-linear relations among $\alpha_1, \ldots, \alpha_k$ of length three.*

Proof. In this case, $\mathbf{r}(\tau) = 1$ for all $\tau \in \Sigma_2^{\mathcal{A}}$, so $M^{\mathbf{r}} = \left[\alpha_1 \cdots \alpha_k\right]$. Suppose $\alpha_1, \alpha_2, \alpha_3$ are a basis for the $\mathbb{R}$-span of the entries of $M^{\mathbf{r}}$ (this is three dimensional since $\mathcal{A}$ is essential).

Then $\operatorname{syz}(M^{\mathbf{r}})$ is generated by the Koszul syzygies on $\{\alpha_1, \alpha_2, \alpha_3\}$ along with all the $\mathbb{R}$-linear relations on the entries of $M^{\mathbf{r}}$.

If $\bar{\gamma} \in L_2(\mathcal{A})$, then we can select two linear forms, without loss suppose these are α_1 and α_2, that intersect in the line $\bar{\gamma}$. The syzygy module $\operatorname{syz}(M^{\mathbf{r}}_{\bar{\gamma}})$ is generated by the Koszul syzygy between α_1 and α_2, along with the $\mathbb{R}$-linear relations on $\{\alpha_H\}_{\gamma \in H}$. Since these linear forms effectively live in the two-dimensional vector space spanned by α_1 and α_2, the relations among them all have length three.

From the above descriptions, we see that the Koszul syzygies in $\mathrm{syz}(M^{\mathbf{r}})$ appear also in $\sum\limits_{\bar{\gamma}\in L_2(\mathcal{A})} \mathrm{syz}(M^{\mathbf{r}}_{\bar{\gamma}})$. Thus the presentation in Corollary 0.2 implies that

$$H_1(\mathcal{J}[\Sigma^{\mathcal{A}}]) \cong \frac{\mathrm{syz}_0(M^{\mathbf{r}})}{\sum_{\bar{\gamma}\in L_2(\mathcal{A})} \mathrm{syz}_0(M^{\mathbf{r}}_{\bar{\gamma}})},$$

where syz_0 represents 'syzygies of degree zero' – that is, $\mathbb{R}$-linear relations.

Furthermore, any relation of length three among $\{\alpha_1, \ldots, \alpha_k\}$, without loss suppose $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$, necessarily expresses the fact that α_1, α_2, and α_3 all vanish along a common line $\bar{\gamma} \in L_2(\mathcal{A})$. Thus this relation appears in $\mathrm{syz}(M^{\mathbf{r}}_{\bar{\gamma}})$. It follows that we may recast the above presentation as the space of all $\mathbb{R}$-linear relations on $\alpha_1, \ldots, \alpha_k$ modulo the space of $\mathbb{R}$-linear relations of length three. □

Definition 0.4. If $\mathcal{A} = \bigcup_{i=1}^k H_i$ is a hyperplane arrangement with H_i the vanishing locus of α_i for $i = 1, \ldots, k$, then $\mathcal{A}$ is called 3-generated if the space of all $\mathbb{R}$-linear relations among $\alpha_1, \ldots, \alpha_k$ is generated by the relations of length 3.

Lemma 0.5. *If $\mathcal{A} = \bigcup_{i=1}^k H_i \subset \mathbb{R}^3$ is a hyperplane arrangement with associated complete fan $\Sigma^{\mathcal{A}}$, and $\mathbf{r} : \Sigma_2^{\mathcal{A}} \to \mathbb{Z}_{\geq 0}$ is a smoothness distribution, then $H_1(\mathcal{J}[\Sigma^{\mathcal{A}}])$ has finite length. Furthermore $S^{\mathbf{r}}(\Sigma^{\mathcal{A}})$ is free if and only if $H_1(\mathcal{J}[\Sigma^{\mathcal{A}}]) = 0$.*

Sketch of proof. Show that the localization of the presentation in Corollary 0.2 at all homogeneous prime ideals besides the maximal ideal vanishes. The latter fact (concerning freeness) follows from a seminal result of Schenck [2], generalized in [3, Theorem 3.4]. In the three-dimensional case, this can be argued fairly quickly using Ext. □

Corollary 0.6. *$S^0(\Sigma^{\mathcal{A}})$ is free if and only if $\mathcal{A}$ is 3-generated.*

The subtlety of this can be seen in action with an example that is sometimes called *Ziegler's pair*. There will be a Macaulay demo walking through this example.

References

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