Contents

Chapter 1

Algebra

1.1 Summation of Series

Theorem 1.1.1.

$$\sum_{i=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof.

$$\sum_{i=1}^{n} \left((i+1)^3 - i^3\right) = (n+1)^3 - 1$$

Using,

$$a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$$

We get,

$$\sum_{i=1}^{n} \left(3i^2 + 3i + 1\right) = (n+1)^3 - 1$$

Now, solve for $\sum_{i=1}^{n} i^2$ in the above equation. □

1.2 Involving Integration

Theorem 1.2.1.

$$\int_0^{\infty} \frac{x^n}{e^x} = n!$$

Proof. Applying By parts we get,

$$I_n = -\frac{x^n}{e^x}\bigg|_0^{\infty} + n\int_0^{\infty} x^{n-1}e^{-x}$$

$$I_n = nI_{n-1}$$

$$I_0 = \int_0^{\infty} e^{-x} = 1$$

$$I_n = n!$$

□

Chapter 2

Inner Product and Orthogonality

2.1 Independence

Definition 2.1.1. Let V be a vector space. A subset S(not necessarily atmost countable) of V is said to be *dependent* if $\exists$ finitely many distinct elements $x_0, x_1 \ldots x_n$ in S such that:

$$\sum_{i=0}^{n} c_i x_i = 0$$

and not all the c_i's are 0. A set is S is called *independent* if it is not dependent, that is, if any finitely many distinct elements $x_0, x_1 \ldots x_n$ are taken such that:

$$\sum_{i=0}^{n} c_i x_i = 0$$

implies that all the c_i's are 0.

Definition 2.1.2. If V is a vector space, a subset B of V is called *basis* of V, if the set B is independent

Theorem 2.1.1. Let V be a vector space. Let $S = \{x_1, x_2, \ldots x_k\}$ set of k independent vectors, let $L(S)$ be the span of S, then any set with atleast $k+1$ elements of $L(S)$ is dependent.

Proof. We use induction on k. Proving base case $(k = 1)$ is easy. Assume the above statement is true for $n = k - 1$. Now, let $k = n$, choose a set in $L(S)$ with $k + 1$ elements, say, $\{y_1, y_2 \ldots y_{k+1}\}$. Then, express each of the y_i as linear combination of x_j. Now assume in the expansion of y_1 there exists a j such that the coefficient of x_j is not 0. Then, construct $c_i y_1 - y_i$ where $c_i = \frac{\text{coefficient of } x_j \text{ in the expansion of } y_i}{\text{coefficient of } x_j \text{ in the expansion of } y_1}$. Now, note that, the vectors so constructed are k in number and are in the span of $k - 1$ vectors of S, hence by induction hypothesis, dependent. Finish the proof now. $\square$

Corollary 2.1.1.1. If V is finite dimensional, then any two bases of V have the same size.

Proof. If two bases have sizes m and n, then it follows from above theorem that $m \leq n$ and $n \leq m$. $\square$

Theorem 2.1.2. In any finite dimensional vector space V with dimension n, we have:-

1. Any independent subset K is a subset of some basis.

   Proof. Keep adding element which lie outside the span of K to the set K. □

2. Any independent set with n elements is the basis.

   Proof. This set will be a part of some basis by the first part of this theorem, and the cardinality of that basis set must be n. □

2.2 Nontrivial examples of Linear Spaces

1. Space of all real-valued functions on the real line.

   (a) The set $\{cos^2(t), sin^2(t), 1\}$ is dependent.

   (b) The collection of function $u_k(t) = t^k,\ k \in \mathbb{N} \cup \{0\}$ is independent. For any positive integer n and n choices of functions above, let $\sum_{i=1}^{n} c_i u_{k_i} = 0$. Now, put $t = 0$ to conclude c_0 equals 0. For the rest of the c_i, keep differentiating and substitute $t = 0$.

   (c) Let S be the collection of functions such that $\{e^a x | a \in \mathbb{R}\}$, then S is an independent set. To see this choose n many different functions from S and consider $\sum_{i=1}^{n} c_i e^{a_i x} = 0$, let a_m be the biggest real number among a_i's. Then divide both sides by $e^{a_m} x$ and let x go to ∞, which will give c_m to be equal to 0. Repeat the above.

2.3 Inner Product

An inner product is a map from $V \times V$ to the underlying field $(\mathbb{R}/\mathbb{C})$ has the following four properties. :- S,L,H,P

1. **Symmetry:** $(x, y) = (y, x)$ if $\mathbb{F} = \mathbb{R}$ and $(x, y) = \overline{(y, x)}$, if $\mathbb{F} = \mathbb{C}$
2. **Linearity:** $(x, y + z) = (x, y) + (x, z)$
3. **Homogeneity** $(cx, y) = c(x, y)$
4. **Positivity** $(x, x) \geq 0$, equality holds iff x is 0.

2.4 Norm

Let V be a *Euclidean Space*,

$$||x|| = (x, x)^{\frac{1}{2}}$$

. In general norm is any function from $V \times V$ to the underlying field $(\mathbb{R}/\mathbb{C})$ satisfying:- H,P,T

1. **Homogeneity:** $||cx|| = |c| ||x||\ c \in \mathbb{R}/\mathbb{C}$
2. **Positivity:** $||x|| \geq 0$, equality holds iff x is 0.
3. **Triangle Inequality:** $||x + y|| \leq ||x|| + ||y||$

2.5 C(a,b)

Theorem 2.5.1. *C(a,b)* : Linear space of all real-valued functions *continuous* on an interval [a,b]. The following are common inner product defined on C(a,b).

$$(f, g) = \int_a^b f(t)g(t)dt$$

$$(f, g) = \int_a^b w(t)f(t)g(t)dt$$

where w(t) is a fixed positive function in C(a,b).

Corollary 2.5.1.1. Applying $CauchySchwarz$ inequality to the above space, we get:

$$\left| \int_a^b f(t)g(t) \right|^2 \leq \int_a^b f^2(t)dt \int_a^b g^2(t)dt$$

Example 1.

$$(f, g) = \int_0^\infty e^{-t} f(t)g(t)dt$$

converges for every choice of *polynomial* f(t), g(t).

Theorem 2.5.2 (Cauchy-Schwarz Inequality). In a *Euclidean Space* V:

$$|(x, y)|^2 \leq (x, x)(y, y) \ \forall x,\ y \in \ V$$

Equality sign hold iff x and y are dependent.

Proof. Let $z = ax + by$, then from the axiom of inner product we get $(z, z) \geq 0$, now take $a = (y, y)$ and $b = -(x, y)$. Observe that, if z is zero, x and y are dependent and equality holds, conversely, let $y = \alpha x$, and put it into the cauchy-shwarz inequality. □

Definition 2.5.1.
- A *Euclidean Space* is a real/complex linear space with an inner product.
- In Euclidean Space V, $|x| = (x, x)^{\frac{1}{2}}$
  1. $||x|| = 0$ if x = 0
  2. $||x|| > 0$ if x $\neq$ 0
  3. $||cx|| = |c|x$
  4. $||x + y|| \leq ||x|| + ||y||$
- In a *real* Euclidean Space, the angle between two non-zero elements x and y is $\theta \ni \theta \in [0, \pi]$ and $cos\theta = \frac{(x,y)}{||x||||y||}$

2.6 Orthogonality

Definition 2.6.1. In a Euclidean Space V,

- x and y are called *Orthogonal* if their inner product is zero. S $\subset$ V is called an *Orthogonal Set* if $\forall$ $(x, y) \in$ S, $(x, y) = 0$ $x \neq y$. *OrthonormalSet* is a set which is Orthogonal and the *norm* of each element is 1.

- A positive number T is called *Period* of a function f if $f(x+T) = f(x)$ and T is the smallest positive number as such. If $g(x) = f(ax+b)$, then Period of g is $\frac{T}{|a|}$

Theorem 2.6.1.

- In a Euclidean space V, every orthogonal set of non-zero elements is independent.

- In a finite dimensional Euclidean Space with dimension n, every orthogonal set with n non zero elements is *basis*.

Example 2 ($C(0, 2\pi)$).

1. Inner Product: $\int_0^{2\pi} f(t)g(t)dt$
2. We have
   (a) $\int_0^{2\pi} cosnx\ sinmx \ = \ 0$ if $n \neq m$ (use integration by parts)
   (b) $\int_0^{2\pi} cosnx\ sinmx \ = \ \pi$ if $n = m$
3. The sequence $\{1,\ cosx,\ sin2x,\ cos3x,\ sin4x,\ cos5x...\}$ form an Orthogonal Set.

Chapter 3

Trigonometry

3.1 Product and Sum

Theorem 3.1.1. $sin(A \pm B) \;=\; sin(A)Cos(B) \pm cos(A)sin(B)$
$cos(A \pm B) \;=\; cos(A)cos(B) \mp sin(A)sin(B)$
$sin(A+B) + sin(A-B) = 2sin(A)cos(B)$
$sin(A+B) - sin(A-B) = 2cos(A)sin(B)$
$cos(A+B) + cos(A-B) = 2cos(A)cos(B)$
$cos(A+B) - cos(A-B) = -2sin(A)sin(B)$

Theorem 3.1.2. $sin^2(x) = \frac{(1-cos2x)}{2}$
$cos^2(x) = \frac{(1+cos2x)}{2}$

Chapter 4

Topology

4.1 Product Topology

Definition 4.1.1.
- **Basis:** A *basis* is a collection $\mathbb{B}$ of subsets of a set X such that the following holds:
  1. If $B_1,\ B_2\ \in\ \mathbb{B}$, then $\exists$ a $B_3\ \ni\ B_3 \subseteq\ B_1 \cap B_2$.
  2. $\forall\ x \in\ X\ \exists\ B_j \in \mathbb{B}\ \ni x \in B_j$, that is, $\mathbb{B}$ covers X.
- **Topology generated by** $\mathbb{B}$: A topology τ is said to be generated by $\mathbb{B}$ if $\forall\ U \in \tau,\ \forall\ x\ \in\ U\ \exists\ B\ \in\ \mathbb{B} \ni\ x\ \in\ B \subset U.$
- **Product Topology**: Let (X, τ) and (Y, γ) be two topological space with $\mathcal{T}$ and $\mathcal{Y}$ as basis. Then the product topology is defined on the set $X \times Y$ as the topology whose basis is the collection of sets of form $(U \times V | U \in \tau, V \in \gamma)$

  Lemma 4.1.1. if $\mathcal{T}$ and $\mathcal{Y}$ is the collection of basis of X and Y, then the basis of the product topology on $X \times Y$ is the collection of sets of form $(B \times C | B \in \mathcal{T},\ C \in \mathcal{Y})$

Theorem 4.1.2. **TFAE**.

- Let (X, τ) be a topological space. Let $\mathbb{B}$ be the set of basis for the topology τ. Then τ is the collection of arbitrary unions of sets in $\mathbb{B}$.
- τ is the topology generated by $\mathbb{B}$.

Theorem 4.1.3 (Tychnoff's Theorem). A product of compact topological spaces is compact with respect to product topology.

4.2 Continuity

Chapter 5

Complex Analysis

5.1 Complex Functions

5.1.1 Modulus

$|x + iy| = \sqrt{x^2 + y^2}$
$|z_1 z_2| = |z_1||z_2|$
$|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$ Given that $|z_2| \neq 0$
$|x + y| \leq |x| + |y|$

5.1.2 Complex Exponential

Definition 5.1.1. If $z = x+iy$ then e^z is defined to be equal to $e^x(cos(y)+isin(y))$

Theorem 5.1.1.

- $e^{z_1}e^{z_2} = e^{z_1+z_2}$
- e^z is never 0.
- if x is real then $|e^{ix}| > 0$ and $|e^{ix}|^2 = 1$.
- $e^z = 1$ iff z is an integer multiple of $2\pi i$.
- $e^{z_1} = e^{z_2}$ iff $z_1 - z_2 = 2in\pi$

5.1.3 Argument

Definition 5.1.2. If $z = x + iy$ be a non-zero complex number. The unique real number θ which satisfies the following is called the *principal argument* of z, denoted by $\theta = arg(z)$.

$$x = |z|cos\theta$$

$$y = |z|sin\theta$$

$$-\pi < \theta \leq \pi$$

Theorem 5.1.2. If $z_1 z_2 \neq 0$ then $arg(z_1 z_2) = arg(z_1) + arg(z_2) + \alpha 2\pi$ where $\alpha \in \{0, 1, -1\}$ such that the right hand side lies in $(-\pi, \pi]$

5.1.4 n-th root

Theorem 5.1.3. If z is a given complex number, then there are exactly n different complex numbers such that $(z_k)^n = z$, where $z_k = Re^{i\phi_k}$ where $R = |z|^{\frac{1}{n}}$ and $\phi_k = \frac{arg(z)}{n} + \frac{2\pi k}{n}$ where $k \in \{0, 1, \dots n-1\}$

The $n - th$ roots of $|z|$ are n equally placed complex number on circle having radius $|z|^{\frac{1}{n}}$

5.1.5 Complex Logarithm

Theorem 5.1.4. If z is a non-zero complex number, then $\exists\ w \ni\ e^w = z$. One such w is $Log(z) = log|z| + iarg(z)$ any other such w have the form $log|z| + iarg(z) + 2i\pi n$

Definition 5.1.3. If $z \neq 0$, then we define $z^w = e^{wLogz}$, where z and w are complex numbers.

Theorem 5.1.5.
1. $z^{w_1+w_2} = z^{w_1} z^{w_2}\ \ z \neq 0$
2. $(z_1 z_2)^w = {z_1}^w {z_2}^w e^{iw2\pi\alpha}$

5.1.6 Complex Sines and Cosines

Definition 5.1.4.

$$sinz = \frac{e^{iz} - e^{-iz}}{2i}$$
$$cosz = \frac{e^{iz} + e^{-iz}}{2}$$

5.2 Differentiability

Definition 5.2.1. Let f be a complex valued function defined on an open set S in $\mathbb{C}$. Let $c\ \in\ S$. Then f is said to be differentiable at c, if :

$$f^{'}(c) = \lim_{z \to c} \frac{f(z) - f(c)}{z - c}$$

exists at c.

Theorem 5.2.1.
1. f is differentiable at c, iff, $\exists$ a function $f^*(z)\ \ni$
$$f(z) - f(c) = (z - c)f^*(z)$$
$\ni\ f^*(z)$ is continuous at c and $f^*(c) = f^{'}(c)$
2. If f is differentiable at c, then f is continuous at c.
3. If $f,\ g$ are complex functions defined on $S\ \ni f^{'}(c)$ and $g^{'}(c)$ exists, then, :
   - $(f \pm g)^{'}(c) = f^{'}(c) \pm g^{'}(c)$
   - $(fg)^{'}(c) = f^{'}(c)g(c) + f(c)g^{'}(c)$
   - $\left(\frac{f}{g}\right)^{'}(c) = \frac{g(c)f^{'}(c) - g^{'}(c)f(c)}{(g(c))^2}$. Assuming $g(c) \neq 0$.
4. The chain rule is valid. If the domain of g contains a neighborhood of $f(c)$ and if $f^{'}(c)$ and $g^{'}[f(c)]$ exists.
$$(g \circ f)^{'}(c) = g^{'}[f(c)]f^{'}(c)$$

5.3 Cauchy Riemann Equations

Theorem 5.3.1. Let $f = u + iv$ be defined on an open set S in $\mathbf{C}$. If $f^{'}(c)$ exists, then the following holds:

$$f^{'}(c) = \frac{\partial u}{\partial x}(c) + i\frac{\partial v}{\partial x}(c)$$
$$f^{'}(c) = \frac{\partial u}{\partial y}(c) - i\frac{\partial v}{\partial y}(c)$$
$$\frac{\partial v}{\partial x}(c) = \frac{\partial v}{\partial u}(c)$$
$$-\frac{\partial u}{\partial y}(c) = \frac{\partial v}{\partial x}(c)$$

Proof. Let $c = a + ib$, then find out $f^{'}(c)$ by approaching c from two different directions, one parallel to y axis another parallel to x axis, then compare the real and complex part of $f^{'}(c)$. □

Theorem 5.3.2. Let $f = u + iv$ be a complex function defined on an open disk D centered around (a, b). Then,

1. if $u/v/|f|$ is constant on D implies, f is constant on D.
2. if $f^{'} = 0$ implies, f is constant on D.

Proof.
1. If u is constant, Cauchy riemann $\implies$ $\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$ is 0, $\implies$ v is constant. If $|f|$ is constant, take partial derivatives, and use Cauchy Riemann equations.
2. $f^{'} = 0 \implies \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$ is 0, $\implies$ v is constant.(and u is constant)

□

Remark.
1. The converse of Cauchy Riemann Condition isn't true.

   Example 3.
   $$u(x,y) = \frac{x^3 - y^3}{x^2 + y^2} \quad (x,y) \neq (0,0)$$
   $$v(x,y) = \frac{x^3 + y^3}{x^2 + y^2} \quad (x,y) \neq (0,0)$$
   at $(x, y) = (0, 0)$ both u and v are 0. The function $f = u+iv$ with the above u and v satisfy the Cauchy Riemann equation at $(0, 0)$ but isn't differentiable at $(0, 0)$. Calculate the following quotient in two different directions $x = y$ and $x = 0$ at $(0, 0)$, $lim_{z\to 0}\frac{f(z)-f(0)}{z-0}$

5.4 Analytic function

Definition 5.4.1. Let S be an open set in $\mathbb{C}$, then a function f is said to be analytic on S, if $f^{'}$ exists $\forall$ points in S and $f^{'}$ is continuous over S. If T is some set, then we say f is analytic on T to mean that f is analytic on some open set containing T.

5.4.1 Examples of Analytic functions

It is possible for a function to have derivative at some point p but not analytic at that point.

$$f(z) = |z|^2$$

Check that Cauchy Riemann condition isn't satisfied for any point except 0 implying that the derivative doesn't exist at other points. Also, the function has derivative at 0, but not analytic at 0.

1. For positive integer n, $f(z) = z^n$ is analytic everywhere. Show that $f(z) = z$ has derivative at every point of $\mathcal{C}$, then establish the formula $(fg)^{'} = f^{'}g + fg^{'}$, and hence prove that z^n has derivative everywhere using induction.
2. For negative integer n, $f(z) = z^n$ is anlaytic everywhere except 0. Establish the formula of derivative of $\frac{f}{g}$, then take $f = 1$ and $g = z$.
3. e^z is analytic everywhere. (Read ahead to get the proof.)
4. $f(z) = Logz$ is analytic except at $x \leq 0,\ y = 0$. (Read ahead to see the proof)

5.5 Contour Integral

A path is a continuous function from [a,b] to $\mathbb{C}$

A positively oriented circle with center at a and radius r is a path $\gamma(\theta) = a + re^{i\theta}$ $\theta \in [0, 2\pi]$

Theorem 5.5.1. Let $f = f_1 + if_2$ and $\alpha = \alpha_1 + i\alpha_2$ be complex valued function defined on the interval $[a, b]$. Then we have:

$$\int_a^b f d\alpha = \left(\int_a^b f_1 d\alpha_1 - \int_a^b f_2 d\alpha_2 \right) + i \left(\int_a^b f_1 d\alpha_2 + \int_a^b f_2 d\alpha_1 \right)$$

whenever all the four integrals on the rights exists.

Theorem 5.5.2 (Change of Variable (*true for real/complex Riemann integration*)). Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$, and let g be a monotonic function on an iterval whose endpoints are c and d such that $g(c) = a$ and $g(d) = b$, and let $h = f(g)$ and $\beta = \alpha(g(t))$, then we have the following:

$$\int_a^b f d\alpha = \int_c^d h d\beta$$

Theorem 5.5.3. $\int_a^b f d\alpha = \int_a^b f\alpha^{'} dx$

Definition 5.5.1. Let γ : - A path from [a,b] to $\mathbb{C}$ and f : - A function from $\mathbb{C}$ to $\mathbb{C}$. Then $\int_\gamma f := \int_a^b f[\gamma(t)]\gamma^{'}(t)dt$, whenever the Riemann integration on the right hand side exists.

Two paths $\gamma : [a, b] \to \mathbb{C}$ and $\beta : [c, d] \to \mathbb{C}$ are equivalent if $\exists$ a monotonic function u from $[c, d]$ to $[a, b]$ $\ni$ $\gamma(u(t)) = \beta(t)$. Two paths are equivalent iff their graphs are same.

Theorem 5.5.4. If two paths are equivalent, then we have
$\int_\gamma f = \int_\beta g$ if γ and β have the same orientation, else, one is negative of the other.

Proof. The idea is to use change of variable formula for integration. if $c < d$,

$$\int_\gamma f = \int_a^b f[\gamma(t)]d\gamma = \int_c^d f[\gamma(u(t)]d\gamma(u(t)) = \int_c^d f[\beta(t)]d\beta = \int_\beta f$$

if $c > d$, then $\int_c^d f[\beta(t)]d\beta = -\int_\beta f$ □

Theorem 5.5.5. 1. $\int_\gamma (af + bg) = a\int_\gamma f + b\int_\gamma g$

2. Let $\gamma : [a, b] \to \mathbb{C}$, let γ_1 and γ_2 be the restrictions of γ on $[a, c]$ and $[c, b]$ respectively, where $a < c < b$, then we have:

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

Proof. Follows from the properties of riemann integration. □

Theorem 5.5.6. If ϕ is a complex piecewise continuous function defined on the interval $[a, b]$, then:

$$\left|\int_a^b \phi(t)\right| \leq \int_a^b |\phi(t)|$$

Theorem 5.5.7.

$$\left|\int_\gamma f(t)\right| \leq \int_a^b |f[\gamma(t)]\gamma^{'}(t)|dt \leq \int_a^b M|\gamma^{'}(t)|dt \leq M\int_a^b |\gamma^{'}(t)|dt = ML$$

where M is the maximum value that $|f|$ takes on the graph of γ and L is called the length of the curve γ

Chapter 6

Limits

6.1 Standard limits

Theorem 6.1.1. $\lim_{x \to 0} \frac{sinx}{x} = 1$

Proof. Using geometry, we get that $1 > \frac{sinx}{x} > \frac{1}{cosx}$. Now use Sandwich theorem ☐

Corollary 6.1.1.1. $\lim_{x \to 0} \frac{tanx}{x} = 1$

Theorem 6.1.2. $\lim_{x \to 1} \frac{x^n - 1}{x - 1} = n$

Proof. $\frac{x^n-1}{x-1} = (x^{n-1} + x^{n-2} + \ldots + 1)$ when $x \neq 1$. Now, put 1 in the RHS. Here, we are using the fact that the RHS is continuous, which can be shown by first showing that x is continuous and hence the above expression. ☐

Theorem 6.1.3. $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

Proof. Take log sides and then use L'hospital, continuity of log and e. Else, you can use generalized binomial theorem. ☐

Theorem 6.1.4. $\lim_{x \to 0} \frac{a^x - 1}{x} = lna$

Proof. We use taylor expansion of a^x. Some subtle theorems are used like summation of series, convergence of series and continuity of power series. ☐

6.2 Some special sequences

p:- will be used for positive number.
α:- will be used for real number.

1. $\lim_{n \to \infty} \frac{1}{n^p} = 0$.

   By Archimedean property of real number, take $n > \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}$ which will imply $\frac{1}{n^p} < \epsilon$

2. $\lim_{n \to \infty} p^{\frac{1}{n}} = 1$.
   Let $x_n = p^{\frac{1}{n}} - 1$, then use $1 + nx_n \leq (1 + x_n)^n = p$, which implies $0 \leq x_n \leq \frac{p-1}{n} \implies x_n$ goes to 0 as n tends to ∞, implying $p^{\frac{1}{n}}$ goes to 1.

3. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.
   Let $x_n = n^{\frac{1}{n}} - 1$, then use $\frac{n(n-1)}{2}{x_n}^2 \leq (1+x_n)^n = n$, which implies $0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \implies x_n$ goes to 0 as n tends to ∞, implying $p = n^{\frac{1}{n}}$ goes to 1.

4. $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.
   Let $k > \alpha$ and let $n > 2k$. Then, by using binomial theorem, we get $(1+p)^n > \binom{n}{k}p^k > \frac{n^k}{2^k}k!p^k$. Thus, we get $0 \leq \frac{n^\alpha}{(1+p)^n} \leq \frac{2^k}{p^k k!}n^{\alpha-k}$. Now, let n tend to infinity and the result follows as $\alpha - k$ is less than 0.

5. if $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.
   Take $\alpha = 0$ in the above case.

Chapter 7

Rudin