Traveling Salesman Problem (TSP)
&
Hamiltonian Cycle

Problem Statement

Given the set of distances, order n cities in a tour $V_{i_1}, V_{i_2}, \ldots V_{i_n}$ with $i_1 = 1$, so it minimizes

$$\underbrace{\sum d(V_{i_j}, V_{i_{j+1}})}_{\text{cost to go from 1st city to last city}} + \underbrace{d(V_{i_n}, V_{i_1})}_{\text{cost to go from last city to first city}}$$

Decision version of TSP:

Given a set of distances on n cities and a bound D, is there a tour of length/cost at most D? (Decision version)

Def. A cycle C in G is a Hamiltonian Cycle, if it visits each vertex exactly once.

Problem Statement:

Given an undirected graph G, is there a Hamiltonian cycle in G?

Aim: Prove Hamiltonian cycle is NP complete
Then from that show Travelling Salesman prob is NP complete

Show that the Hamiltonian Cycle Problem is NP-Complete

1. Show that Hamiltonian cycle is NP

a. Certificate: ordered list of nodes on the HC

b. Certifier: 1) Check to make sure there is an edge between each pair of adj nodes in the list

2) All nodes are visited

3) No repeated nodes

4) Edge between last and first nodes in the list

All this can be done in polynomial time

2. Choose a problem already known to be NP complete $\boxed{\text{vertex cover}}$

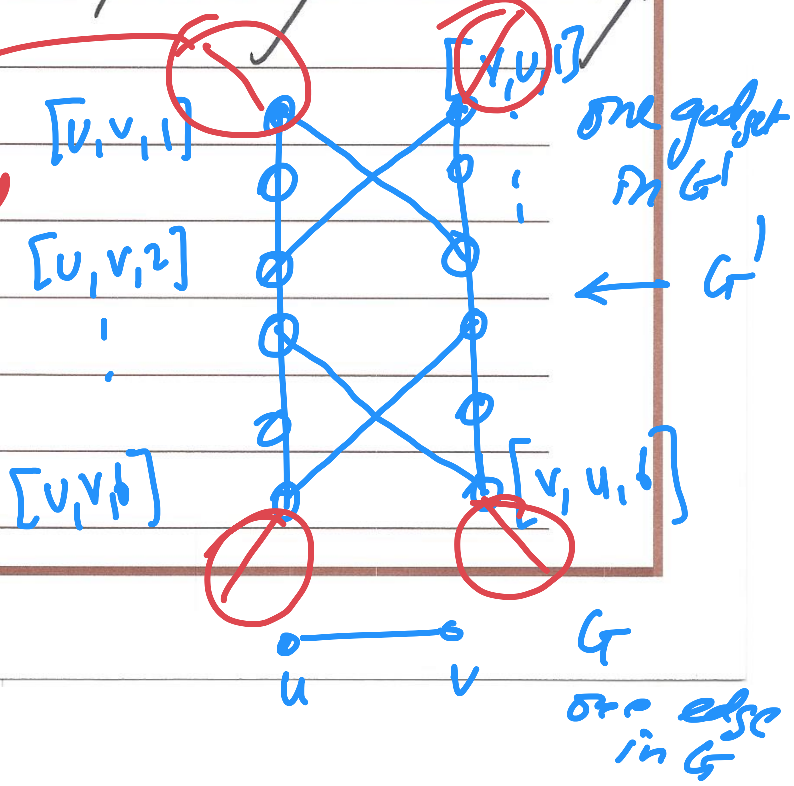
3. Prove vertex Cover $\leq_p$ HC

Plan: Given an undirected graph $G = (V, E)$ and an integer $\underline{k}$, we construct $G' = (V', E')$ that has a Hamiltonian Cycle iff G has a vertex cover of size at most $\underline{k}$.

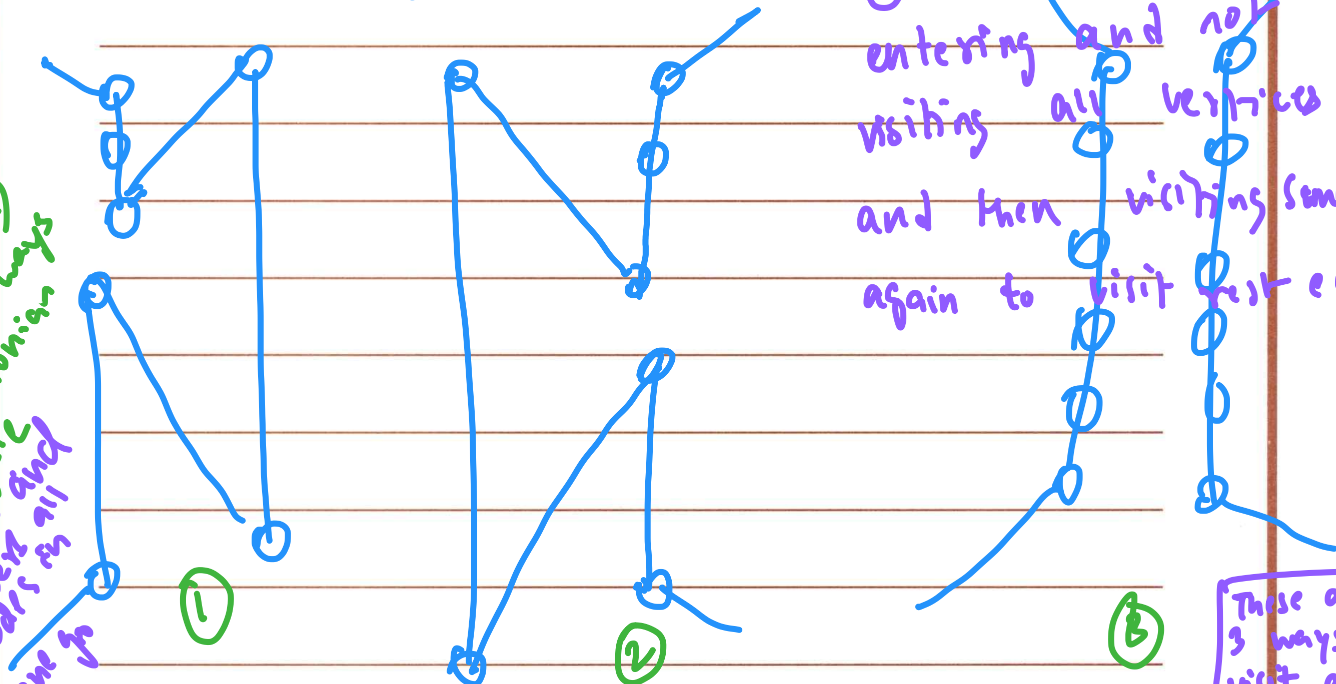
Construction of G'

For each edge (v, u) in G, G' will have one gadget W_{vu} with following node labeling:

These are only nodes that gets connected to other gadget in G'



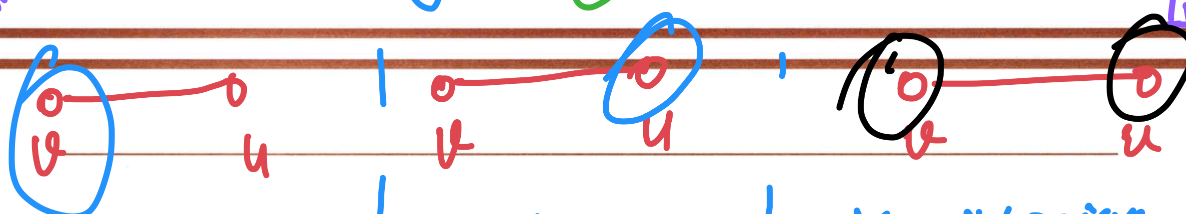
we will send G' to blackbox that solves Hamiltonian cycle



(1) (2) show ways Hamiltonian cycle enters and covers all nodes in one go

(3) represents Hamiltonian entering and not visiting all vertices and then visiting sometime again to visit rest edges

These are only 3 ways to visit all nodes



Hamiltonian cycle enters the gadget from v | Hamiltonian cycle enters the gadget from u | Hamiltonian cycle entering the gadget from both sides

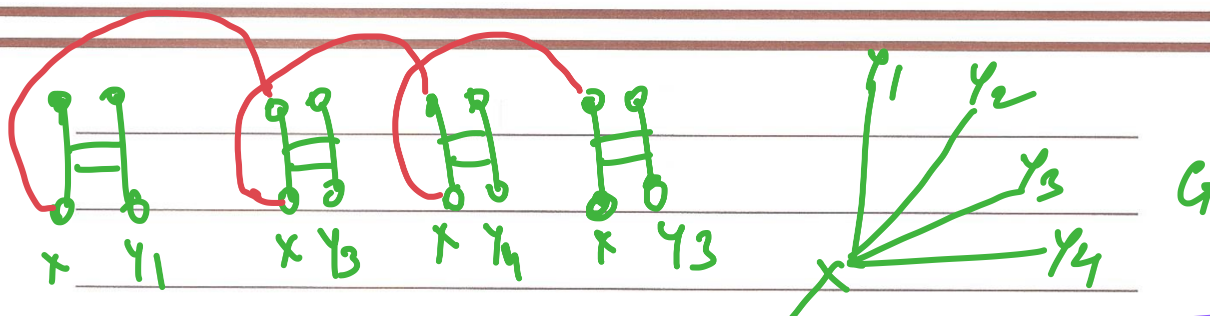
Other vertices in G'

- Selector vertices: There are $\underline{k}$ selector vertices in G', $s_1, \ldots, s_k$

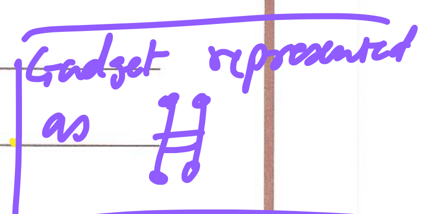
From vertex cover problem statement

Other edges in G'

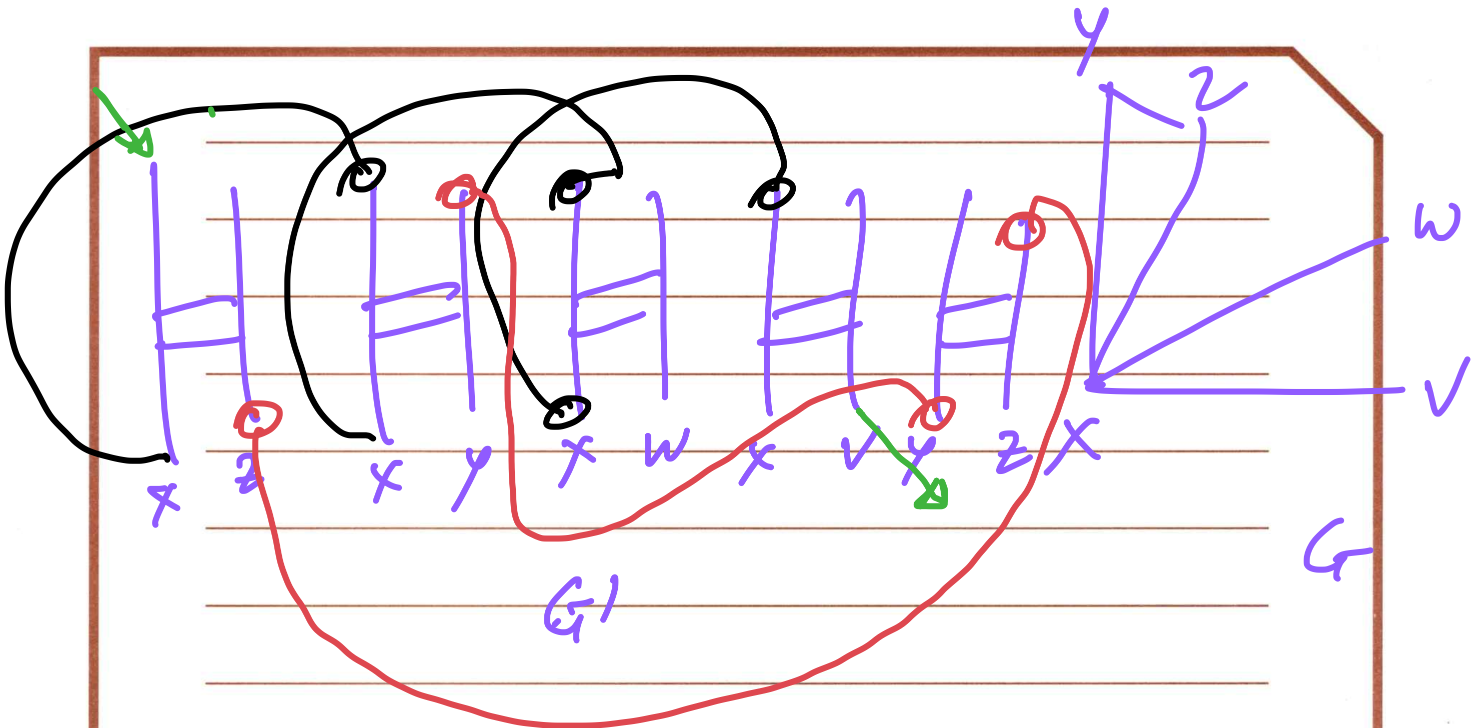
1. For each vertex $v \in V$ we add edges to join pairs of gadgets in order to form a path going through all the gadgets corresponding to edges incident on v in G.



G'
(4 gadgets)

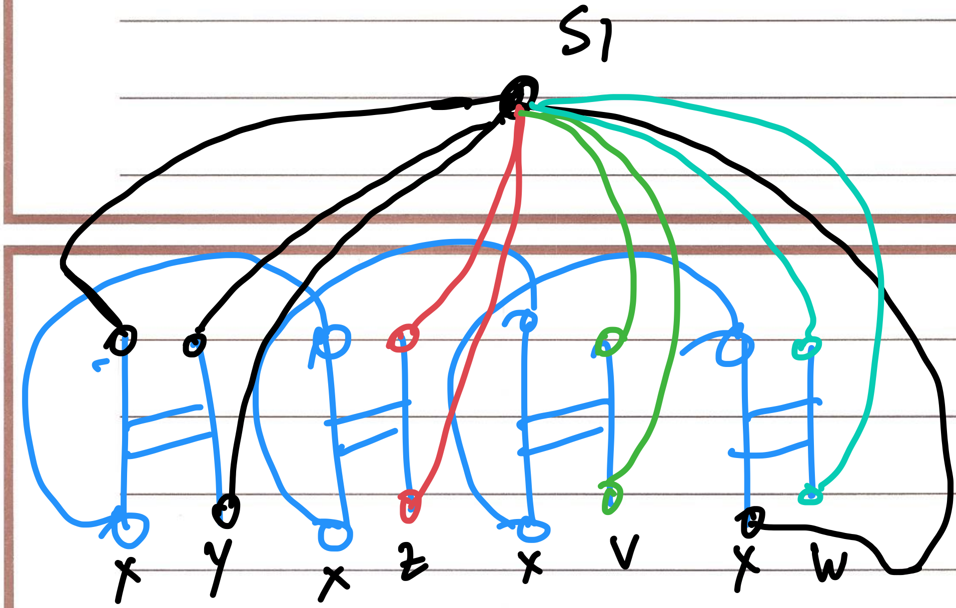


By adding we have created a path to enter gadget 1 and get out from last gadget

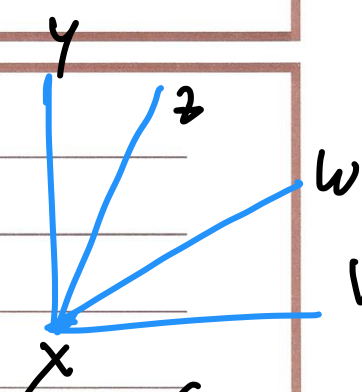


3 paths total

2- Final set of edges in G' join the first vertex $[X, Y, 1]$ and last vertex $[X, Y_{(deg(x))}, 6]$ of each of these paths to each of the selector vertices.



G1



$(G, \underline{K=1})$

given

∴ 1 selector vertex

connected to all nodes
same as S_1

$(G,$

$\underline{k=1}$

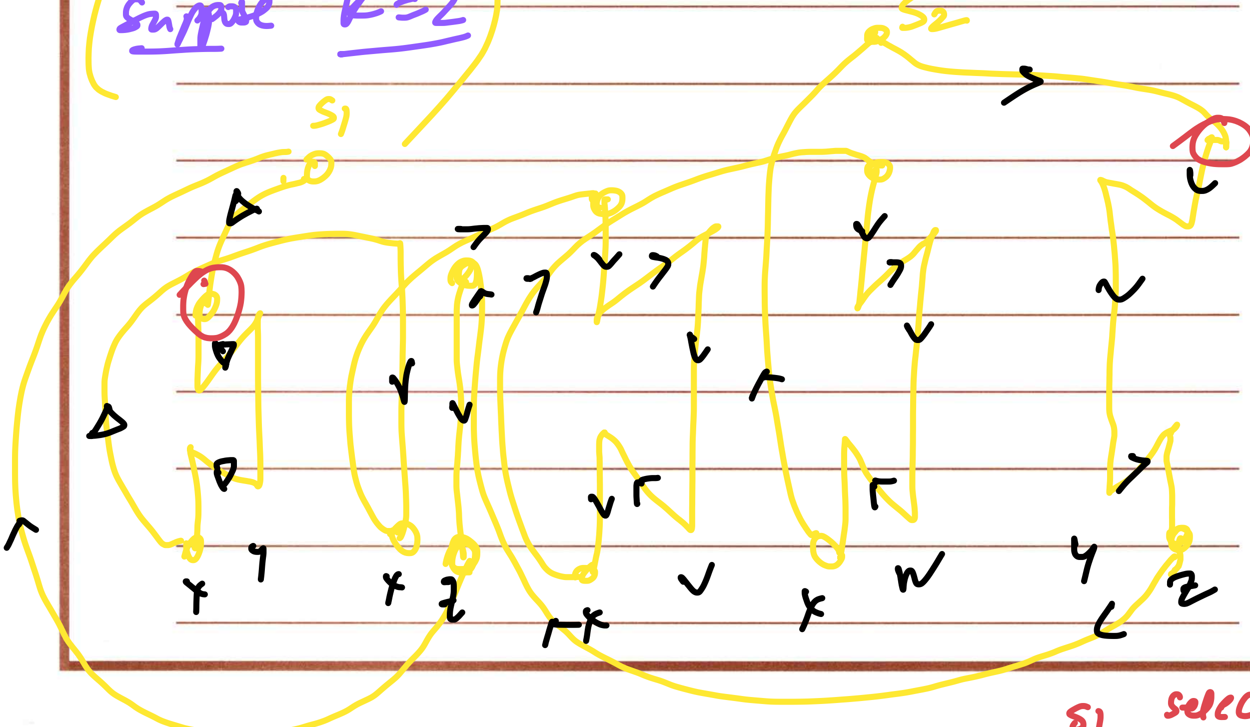
suppose we take $\underline{k=1}$

$\therefore$ no vertex covers of size $k=1$

is there a hamiltonian cycle in G?

No

(suppose $\underline{k=2}$)



hamiltonian cycle

$\therefore$ s_1 selected x to be vertex cover set

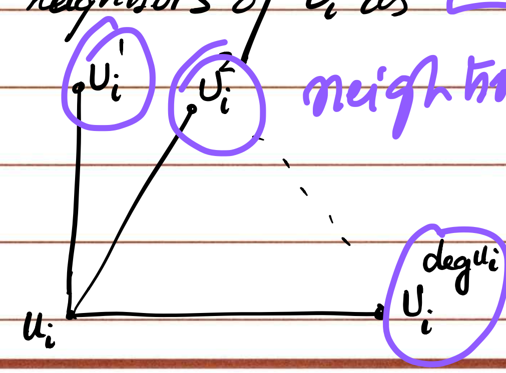
s_2 selected z to be in vertex cover set

Proof: A) Suppose that $G=(V,E)$ has a vertex cover of size $\underline{k}$. Let the vertex cover set be

$$S=\{u_1, u_2, \ldots, u_k\}$$

If this is vertex cover set I can find Hamiltonian cycle

We will identify neighbors of u_i as shown here:



neighbors of u_i

Form a Ham. Cycle in G' by following the nodes in G in this order:

start at s_1 and go to

s_1

$[u_1, u_1^1, 1]$ ---- $[u_1, u_1^1, 6]$

$[u_1, u_1^2, 1]$ ---- $[u_1, u_1^2, 6]$

$\vdots$

$[u_1, u_1^{\deg u_1}, 1]$ ---- $[u_1, u_1^{\deg u_1}, 6]$

s_2

Then go to S_2 and follow the nodes

$[U_2, U_2^1, 1]$ ---→ $[U_2, U_2^1, 6]$

$[U_2, U_2^2, 1]$ ---→ $[U_2, U_2^2, 6]$

⋮

$[U_2, U_2^{\deg U_2}, 1]$ ---→ $[U_2, U_2^{\deg U_2}, 6]$

Then go to S_3 . . . .

S_3

⋮

$[U_k, U_k^1, 1]$ --- $[U_k, U_k^1, 6]$

$[U_k, U_k^2, 1]$ --- $[U_k, U_k^2, 6]$

⋮

$[U_k, U_k^{\deg U_k}, 1]$ --- $[U_k, U_k^{\deg U_k}, 6]$

Then return back to S_1.

This will be our Hamiltonian cycle

as suppose vertex cover is $\{a, b, c\}$

∴ Step1 S_1 we cover all adj edges of $a \to S_2$

Step2 S_2 cover all adj edges of $b \to S_3$

(S1) Step 3 (S3) → cover all edges adj to C

B) Suppose G' has a Hamiltonian cycle C, then the set

$$S = \{ v_j \in V : (s_j, [v_j, v_j', 1]) \in C \text{ for some } 1 \le j \le k \}$$

selector vertices selects which nodes belongs to vertex cover set

will be a vertex cover set in G.

$\therefore$ Hamiltonian Cycle is NP Complete

We Prove that TSP is NP-Complete

1- Show that TSP $\in$ NP

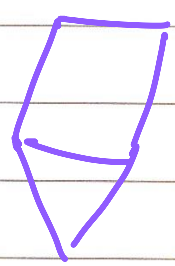
a- Certificate : tour of cost no more than $\underline{\underline{C}}$

b- Certifier : Everything in HC
+
check total cost $\leq$ C

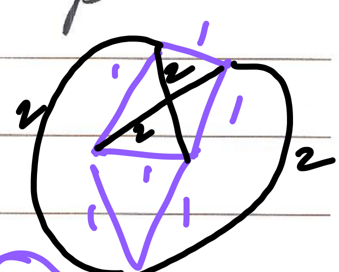
2- Choose an NP-Complete problem:
Hamiltonian Cycle.

3- Prove that Ham. Cycle $\leq_p$ TSP

G



instance of HC



G'

- weighted
- undirected
- fully connected

Is there a TSP of size n is G'

no. of nodes in G

There is a tour of TSP of size n in G' if and only if there is HC in G

Proof

(a) If there is a HC prove there is TSP of size n

Use the same HC in G'

n*1 = n

(b) If there is a TSP of size n in G' prove there is HC in G

If TSP have path of cost = $\underline{\underline{n}}$

and there are 'n' edges

$\therefore$ each edge is of cost 1

$\therefore$ all cost 1 edges select

$\therefore$ This will be $\underline{\underline{HC}}$ in G

NP complete problems we know:

3 SAT, vertex cover, indp set, set cover,

Hamiltonian cycle, TSP

proved in class

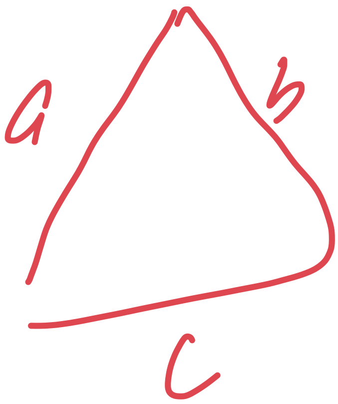
0/1 Knapsack, subset sum,
graph 3-coloring

Not proved in class

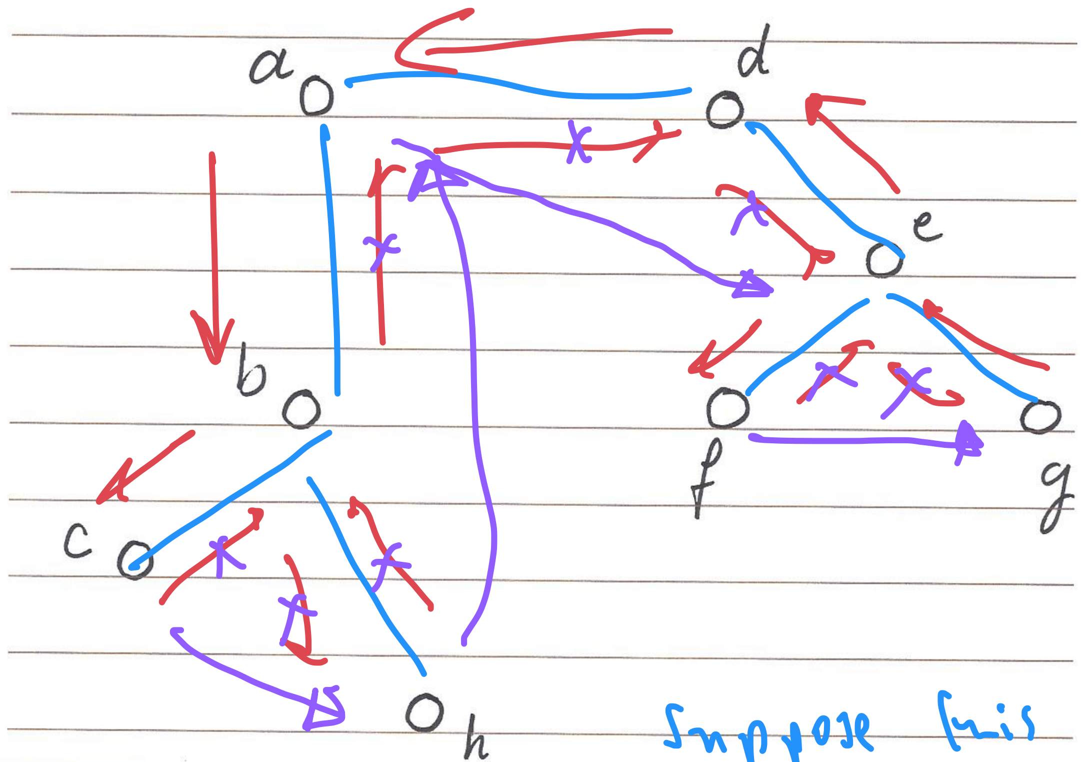
But these can also be used in proofs for proving some problem to be NP-complete

Traveling Salesman Problem

(w/ triangle inequalities)



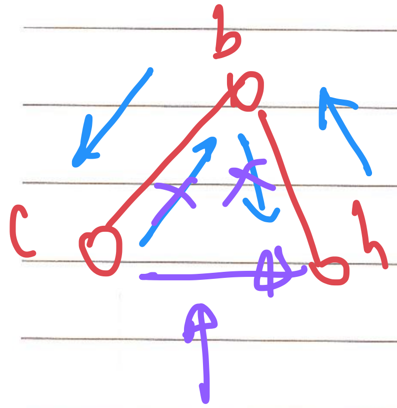
$$(a+b) > c$$



Suppose this is MST

Initial Tour

cost : 2 x cost of MST



Δ ineq

$(ch) < (cb + bh)$

My approx tour

Cost $\leq$ 2 * cost of MST

Cost of Optimal tour $\geq$ cost of MST (As if not then tour would have been MST)

Cost of our approx sol $\leq$ 2 * cost of opt solution

This is a 2 - approximation

called "2 - approximation" as we are in range of factor of 2 to the optimal solution

General TSP

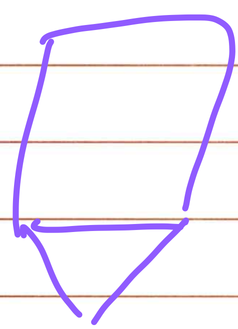
Theorem: if $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial time approximation algorithm with approximation ratio ρ for the general TSP

if I say my soln in 1/1000 in range of optimal soln then I proved $P = NP$

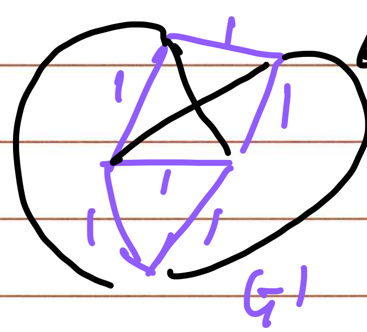
Plan: We will assume that such an approximation algorithm exists. We will then use it to solve the HC problem.

Given an instance of the HC problem on graph G, we will construct G' as follows:

- G' has the same set nodes as in G
- G' is a fully connected graph.
- Edges in G' that are also in G have a cost of 1.
- Other edges in G' have a cost of $\rho|V|+1$



G



G'

very large cost

$\rho|v|+1$

anything higher than this number

- If there is a HC in G

Cost of optimal solution in $G' = |V|$

Red 10C

- If G' has a tour of cost $\leq \rho|V|$

$\therefore$ It does not have any high cost edge

$\therefore$ all edge cost $= 1$

$\therefore$ There is HC in G

Discussion 11

1. In the *Min-Cost Fast Path* problem, we are given a directed graph $G=(V,E)$ along with positive integer times t_e and positive costs c_e on each edge. The goal is to determine if there is a path P from s to t such that the total time on the path is at most T and the total cost is at most C (both T and C are parameters to the problem). Prove that this problem is **NP**-complete.

2. We saw in lecture that finding a Hamiltonian Cycle in a graph is **NP**-complete. Show that finding a Hamiltonian Path -- a path that visits each vertex exactly once, and isn't required to return to its starting point -- is also **NP**-complete.

3. Some **NP**-complete problems are polynomial-time solvable on special types of graphs, such as bipartite graphs. Others are still **NP**-complete.
Show that the problem of finding a Hamiltonian Cycle in a bipartite graph is still **NP**-complete.

② **Step1** Show HP is in NP

Certificate

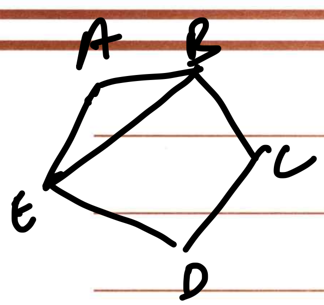
Certifier

Step2 Select problem already known to be NP-complete

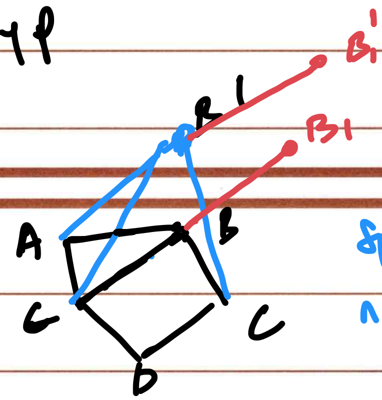
HC

Step3 Prove $HC \le_p HP$

Method1



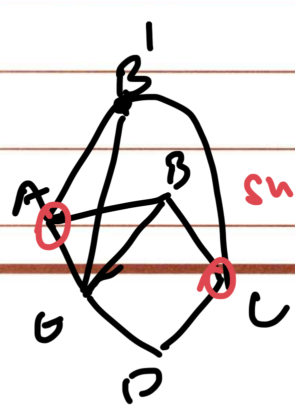
G



split any one node into two

G'

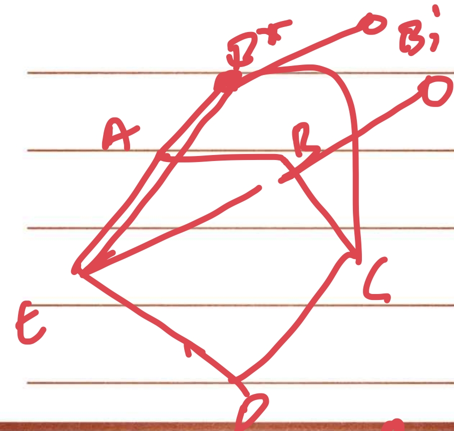
Construct G' such that G' has HP if and only if G has HC



suppose we have HP starting from A and ending at C

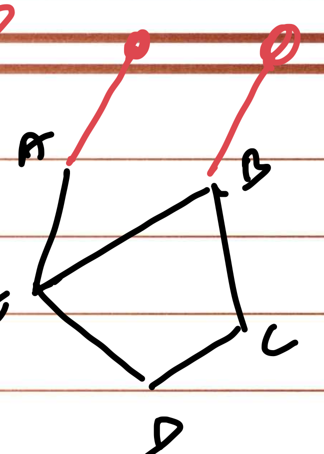
so from this HP how will we find HC

∴ we have to restrict blackbox to answer { Is there is HP starting at B and ending at Bi or vice versa }



∴ This modification

method2



Check if there is HP

But we have to do this for all edges

[because there might be another HP including AB]

Repeat for all edges

if any one call to blackbox returns "Yes"

Then there is HC in G

There are atmost O(m) calls to blackbox

(For reductions decision version of prob is used

① **step1** Show prob is NP

step2 Choose a problem known to be NP complete Subset Sum

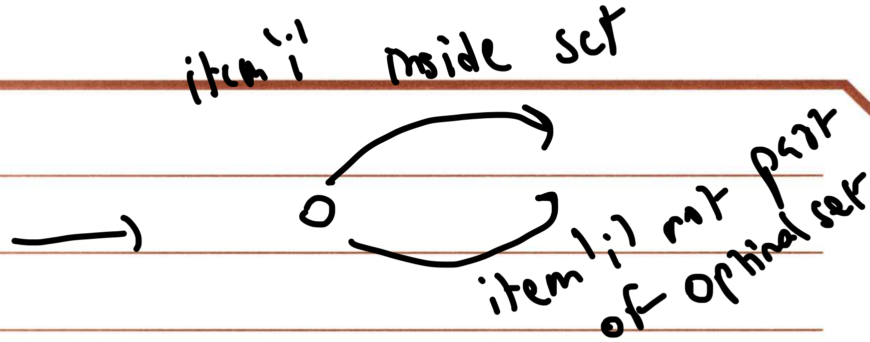
step3 Show Subset Sum $\leq_p$ MCFP

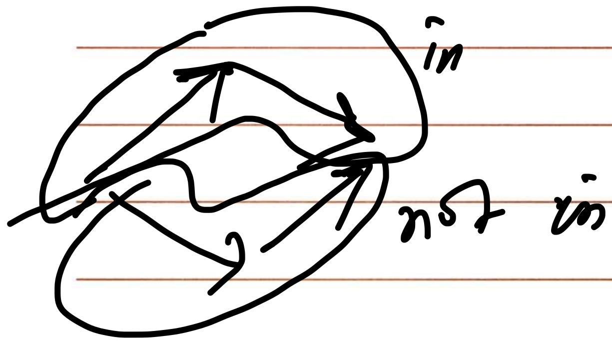
It is opt prob

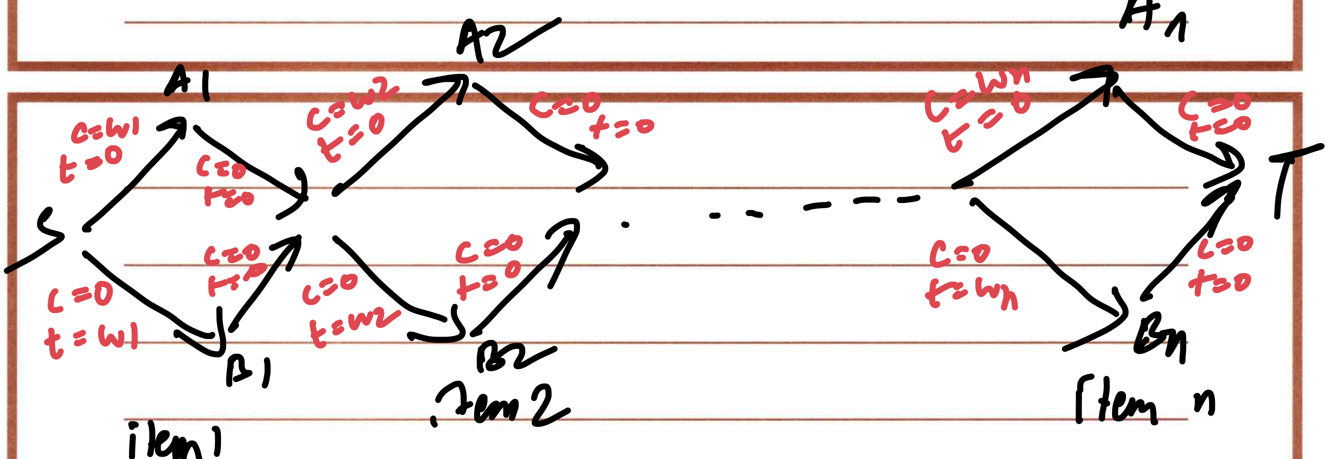
Decision version: Given a set of $\underline{n}$ items each w/ weight $w_i \geq 0$ is there a subset of them with total weight $\leq W$ and $\geq M$

$\therefore$ For each item we want to know whether it is in ~~B~~ the subset or not.

item i





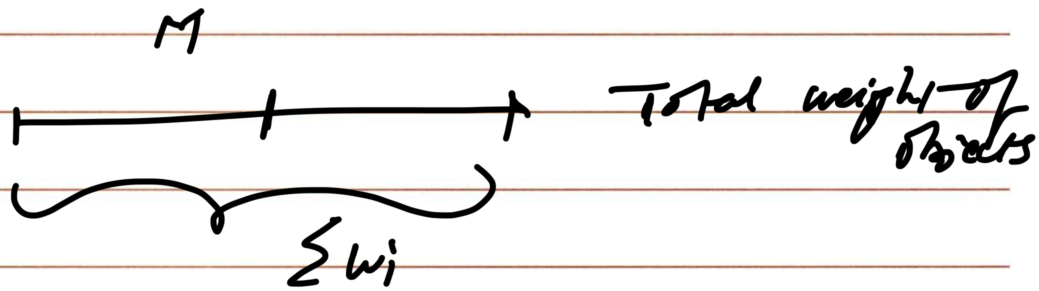


if path goes through A_i
item i is optimal set
if path goes through B_i
item i does not go to optimal set

If there is an ST path in G with

total cost $\leq$ $\underline{\quad W \quad}$

and total time $\leq$ $\underline{(\sum w_i) - M}$



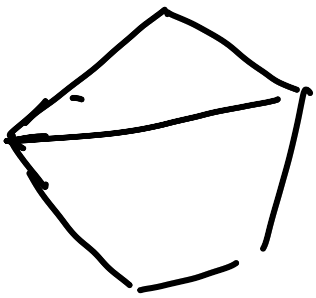
if we want total weight of objects to be M we want weight of excluded obj to be $\sum w_i - M$

ai if we are on time path we are excluding it from optimal set.

③ Step 1 prove it in NP

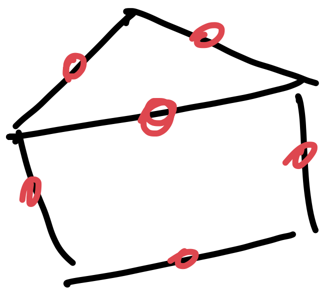
Step 2 choose Hamiltonian cycle

Step 3 show HC $\leq_p$ HC in bipartite graph.



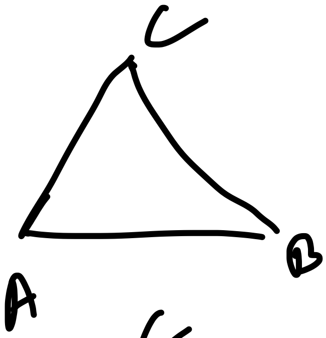
G

will fail in step 4B

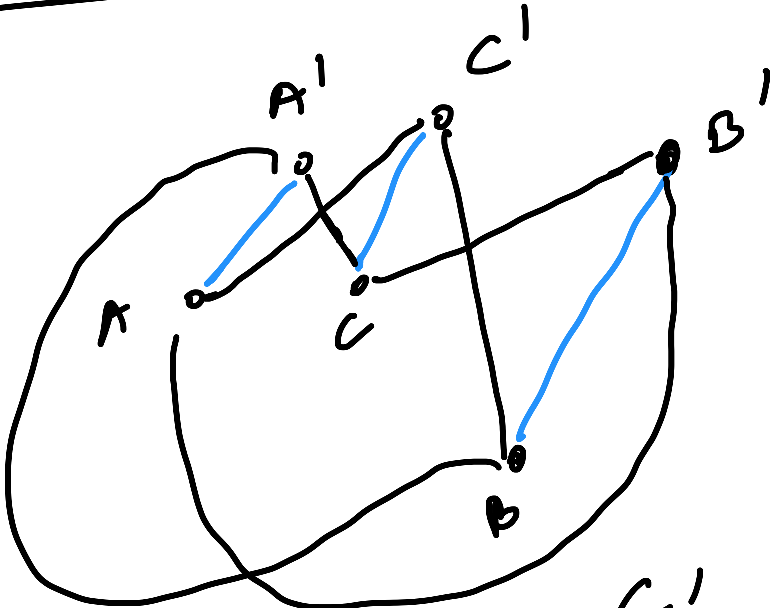


G'

false -ve



G



G'

G

ABCA from this we can get AA'BB'CC'A

False +ve

CC'AA'BB'C

from this we can get G'

CABC

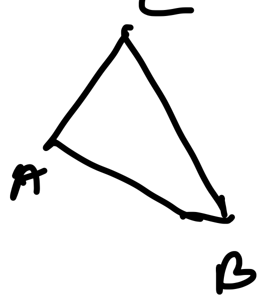
G

But if we ~~the~~ our black box produces

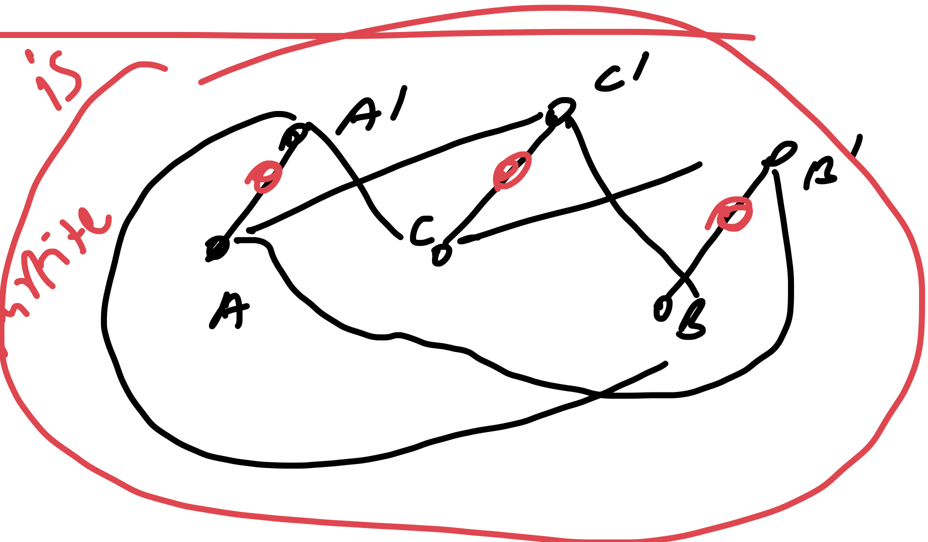
AC'BA'CB'A

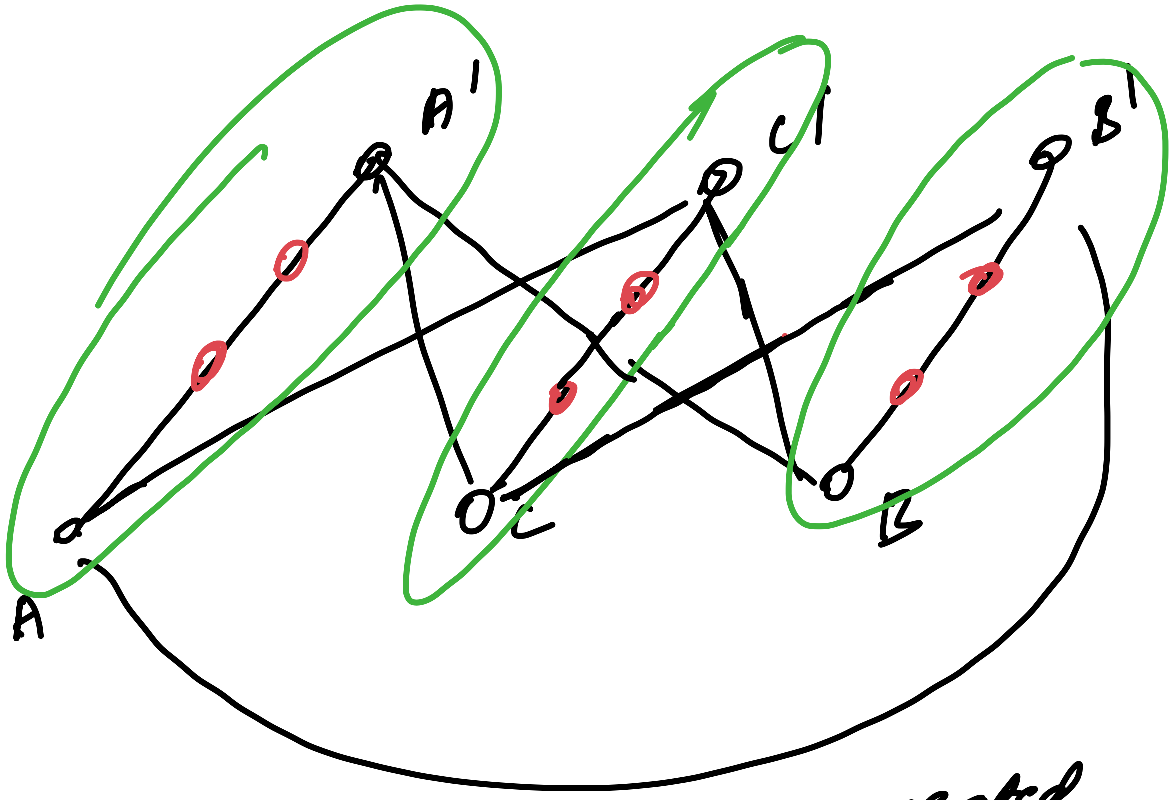
Hamiltonian cycle in G'

But no hamiltonian cycle in G



This is not bipartite





So By trial and error we created gadgets