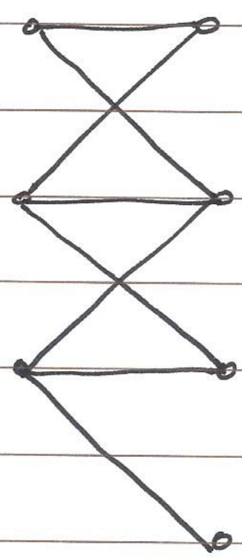
Network Flow

Bipartite Matching Problem

Def. A bipartite graph $G=(V,E)$ is an undirected graph whose node set can be partitioned as $V = X \cup Y$ with property that every edge $e \in E$ has one end in X and the other in Y.

Def. A matching M in G is a subset of the edges $M \subseteq E$ such that each node appears in at most one edge in M.



G

Problem Statement:

Find a matching M of largest possible size in G.

General Plan:

Design a flow network G' that will have a flow value $v(f) = k$ iff there is a ~~max.~~ ~~size~~ matching of size k in G. Moreover, flow f in G' should identify the matching M in G.

Construction of G'

G

Solution

Run Max Flow on G'. Say max. flow is f.

Edges carrying flow between sets X & Y will correspond to our max. size matching in G.

To prove this, we will show that G' will have a max. flow of value $\underline{k}$ iff G has a max. size matching of size $\underline{k}$.

Proof:

A- If we have a matching of size k in G, we can find an s-t flow f of value k in G.

B- If we have an s-t flow of value k in G', we can find a matching of size k in G.

Network Flow

Edge-Disjoint Paths

Def. A set of paths is edge-disjoint if their edge sets are disjoint

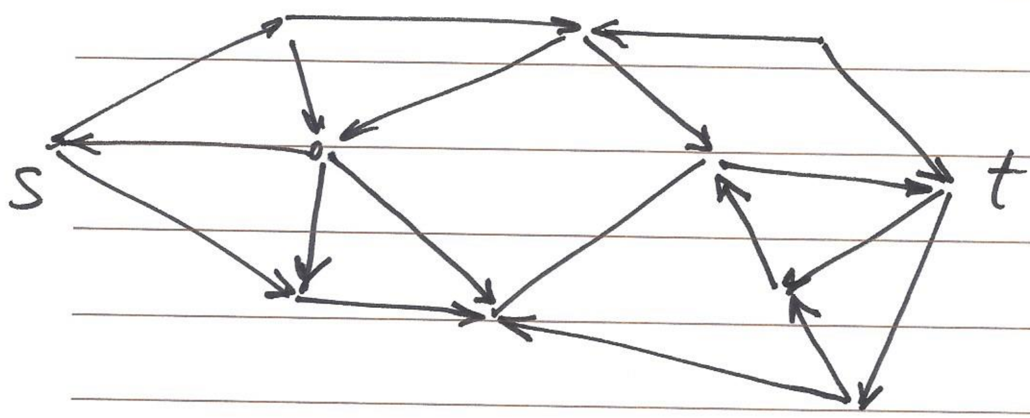
Problem Statement

Given a directed graph G with $s \& t \in V$, find max. number of edge-disjoint s-t paths in G.

General Plan:

Design a flow network G' that will have a flow value $v(f) = k$ iff there are $\underline{k}$ edge-disjoint s-t paths in G.
Moreover, flow f in G' should identify the set of edge-disjoint paths in G.

Construction of G'



Solution:

- Run Max flow in G'
- v(f) will equal the max. number of edge-disjoint s-t paths
- f will identify edges on these paths

To prove this, we will show that there are $\underline{k}$ edge disjoint paths in G iff there is a flow of value $\underline{k}$ in G'!

<u>Proof:</u>

A) If we have $\underline{k}$ edge disjoint s-t paths in G, we can find a flow of value $\underline{k}$ in G'.

B) If we have a flow of value $\underline{k}$ in G', we can find $\underline{k}$ edge-disjoint s-t paths in G.

Network Flow

Node-disjoint Paths

Def. A set of paths is node-disjoint if their node sets (except for starting & ending nodes) are disjoint

Problem Statement

Given a directected graph G with $s \& t \in V$, find the max. number of node-disjoint s-t paths in G.

Plan: As usual...

Construction of G':

Circulation &

Circulation with Lower Bounds

Circulation Network

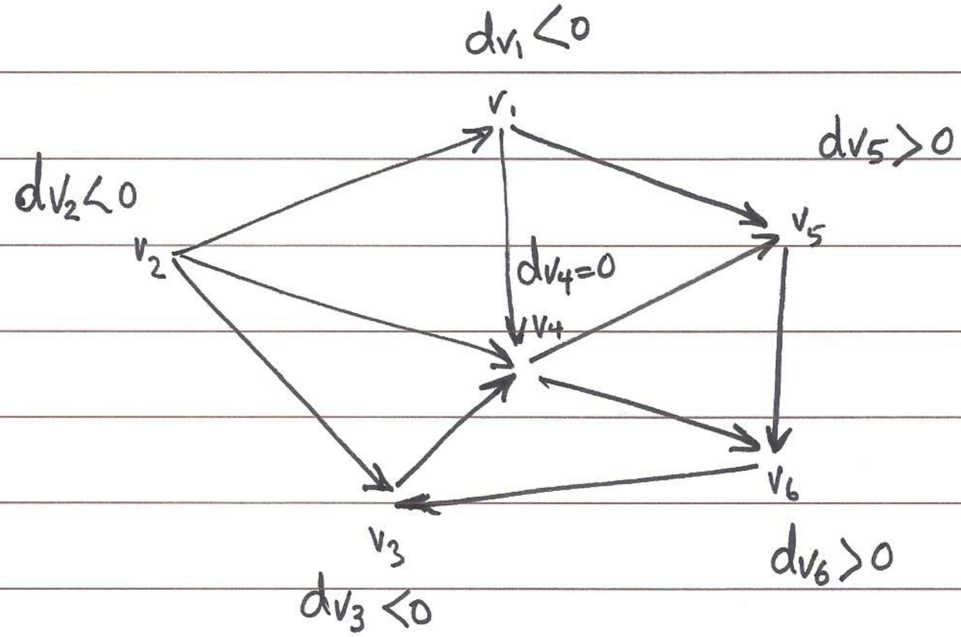
- We are given a directed graph $G=(V,E)$ with capacities on the edges.
- Associated with each node $v \in V$ is a demand d_v
  - if $d_v > 0$, node $\underline{v}$ has demand of d_v for flow (Sink)
  - if $d_v < 0$, node $\underline{v}$ has a supply of $|d_v|$ for flow (Source)
  - if $d_v = 0$ $\underline{v}$ is neither a sink nor a source

Def. A circulation with demand $\{d_v\}$ is a function f that assigns non-negative real numbers to each edge and satisfies:

1) Capacity condition
   for each edge $e \in E$ $0 \leq f(e) \leq C_e$

2) Demand condition
   for each node $v \in V$,

FACT: If there is a feasible circulation w/ demands $\{d_v\}$ then $\sum_v d_v = 0$



dv1 < 0
v1
dv5 > 0
dv2 < 0
v2
v5
dv4 = 0
v4
v6
v3
dv6 > 0
dv3 < 0

Proof:

A) If there is a feasible circulation f w/ demand values $\{d_v\}$ in G, we can find a Max Flow in G' of value D.

B) If there is a Max Flow in G' of value D, we can find a feasible circulation in G.

Circulation with Demands & Lowerbounds

Conditions:

1) Capacity condition
   for each edge $e \in E$, $l_e \leq f(e) \leq c_e$

2) Demand condition
   for every node $v \in V$, $f^{in}(v) - f^{out}(v) = d_v$

Solution:
Find feasible circulation (if it exists) in two passes.

Pass #1. find f_0 to satisfy all l_e's

Pass #2. Use remaing capacity of the network to find a feasible circulation f_1 (if it exists)

Combine the two flows: $f = f_0 + f_1$

Survey Design Problem

Survey Design

Input:
- Information on who purchased which products
- Maximum and Minimum number of questions to send to customer i
- Maximum and minimum number of questions to ask about product j

Min Flow Problem

Input: Directed graph $G=(V,E)$
Source $s \in V$
Sink $t \in V$
l_e for each edge $e \in E$
There are no c_e's

Objective: Find a feasible flow of minimum possible value

Solution:

- Assign "large" capacities to all edges and find a feasible flow f
- Construct G', where all the edges are reversed and the reversed edge e has capacity $= f_e - l_e$
- Find maximum flow from $\underline{t}$ to $\underline{s}$ in G'
- Min flow $= f - f'$

Discussion 9

1. We're asked to help the captain of the USC tennis team to arrange a series of matches against UCLA's team. Both teams have *n* players; the tennis rating (a positive number, where a higher number can be interpreted to mean a better player) of the *i*-th member of USC's team is t_i and the tennis rating for the *k*-th member of UCLA's team is b_k. We would like to set up a competition in which each person plays one match against a player from the opposite school. Our goal is to make *as many matches as possible* in which the USC player has a higher tennis rating than his or her opponent. Use network flow to give an algorithm to decide which matches to arrange to achieve this objective.

2. CSCI 570 is a large class with *n* TAs. Each week TAs must hold office hours in the TA office room. There is a set of *k* hour-long time intervals $I_1, I_2, \ldots I_k$ in which the office room is available. The room can accommodate up to 3 TAs at any time. Each TA provides a subset of the time intervals he or she can hold office hours with the minimum requirement of l_j hours per week, and the maximum m_j hours per week. Lastly, the total number of office hours held during the week must be *H*. Design an algorithm to determine if there is a valid way to schedule the TA's office hours with respect to these constraints.

3. There are n students in a class. We want to choose a subset of k students as a committee. There has to be m1 number of freshmen, m2 number of sophomores, m3 number of juniors, and m4 number of seniors in the committee. Each student is from one of k departments, where k = m1 +m2 +m3 +m4. Exactly one student from each department has to be chosen for the committee. We are given a list of students, their home departments, and their class (freshman, sophomore, junior, senior). Describe an efficient algorithm based on network flow techniques to select who should be on the committee such that the above constraints are all satisfied.

4. Given a directed graph G=(V,E) a source node s ϵ V, a sink node t ϵ V, and lower bound ℓ_e for flow on each edge e ϵ E, find a feasible s-t flow of minimum possible value.
Note: there are no capacity limits for flow on edges in G.