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# ADA week 12 Notes

## Independent set Problem:

Given $G = (V, E)$, we say a set of nodes $S \subseteq V$ is independent if no two nodes in $S$ are joined by an edge.

## Vertex Cover Problem:

Given $G = (V, E)$, we say a set of nodes $S \subseteq V$ is a vertex cover if every edge $e \in E$ has atleast one end in $S$.

## Set Cover Problem:

Given a set $U$ of $n$ elements, a collection $S_1, \ldots S_m$ of subsets of $U$, and a number $k$, does there exist a collection of atmost $k$ of these sets whose union is equal to all of $U$.

## 3 SAT Problem:

SAT problem: Given a set of clauses $C_1, \ldots C_k$ over a set of variables $X = \{x_1, \ldots x_n\}$ does there exist a satisfying truth assignment?

eg: $X = \{x_1, x_2, x_3\}$

$C_1 = (x_1 \vee \overline{x_2})$

$C_2 = (\overline{x_1} \vee \overline{x_3})$

$C_3 = (x_2 \vee \overline{x_3})$

$C_1 \wedge C_2 \wedge C_3$

does there exist truth assign of $x_1, x_2, x_3$ such that $C_1 \wedge C_2 \wedge C_3$ is True.

(*) A large set of problems is in the "gray area".
A poly time solution to anyone would imply poly time algo for all of them.

(*) Poly Time Reductions.

$Y \leq_p X$ (Y is poly time reducible to X)

If Y can be solved using poly number of standard computational steps plus a poly number of calls to a black box that solves X.

$Y \leq_p X$ (Problem X is atleast as hard as Problem Y)

In other words X is powerful enough to let us solve Y.

(8.1) Suppose $Y \leq_p X$. If X can be solved in poly time, then Y can be solved in poly time.

(8.2) Suppose $Y \leq_p X$. If Y cannot be solved in poly time, then X cannot be solved in poly time {As X is atleast as hard as Y}

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# Reductions

## #1 Indp Set and Vertex cover

(1) Indp set :- Given graph G and number k does G contain an indp set of size atleast k?
(decision version)

- Find largest indp set in graph G
(optimization version)

Optimization prob is atleast as hard as decision prob.

(2) Vertex Cover :- Find smallest vertex cover in G
(opt version)

- Given graph G and number k, does G contain a vertex cover set of size atmost k
(decision version)

FACT : let $G = (V, E)$ be a graph, then S is an indp set if and only if its complement $V-S$ is a vertex cover set.

Proof :

(A) Suppose S is an indp set

$u \overset{e}{\text{———}} v$ Consider an edge e

Case 1 : u is in S and v is not
$V-S$ will have v and not u.

Case 2 : v is in S and u is not
$V-S$ will have u and not v.

Case 3 : Neither v and u in S
$V-S$ will have both v and u.

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(B) Suppose V−S is a vertex cover set then S in indp set:
Similar proof.

(8.4) Indp set $\leq_p$ vertex cover

proof: If we have a blackbox to solve vertex cover, then we can decide whether G has an indp set of size atleast k by asking the blackbox whether G has a vertex cover of size atmost n−k.

(8.5) Vertex Cover $\leq_p$ Indp set
Similar proof.

#2 vertex cover to set cover

Note: Indp set and vertex cover are two diff types of prob
Indp set is a type of packing problem:
The goal is to "pack in" as many vertices as possible, subject to conflicts (the edges) that try to prevent one from doing this.
Vertex cover is a type of covering problem:
The goal is to cover all the edges in the graph using as few vertices as possible.

Set Cover is also a covering problem

Intuitively it feels vertex cover is a special case of set cover

(S.G) $\boxed{\text{Vertex Cover} \le_p \text{set cover}}$

| | |
|---|---|
| Vertex cover has $G=(V,E)$ set $S \to$ vertex cover such that every edge has atleast one end in set S. | Set cover $G=(V,E)$ $S \to$ set cover such that union of all sets in S gives V. |

∴ To convert vertex cover to set cover we need some number of sets, and select 'k' of the sets such that their union covers all nodes

eg : $U = \{x_1, x_2, x_3\}$

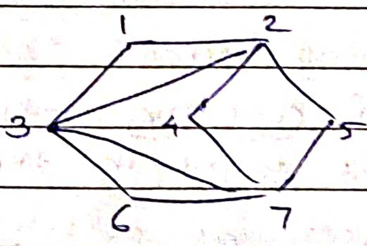
$S_1 = \{x_1, x_2\}$ $S_2 = \{x_3, x_1\}$

$S_3 = \{x_2, x_3\}$ $S_4 = \{x_1\}$

If we select ($S_1$ and $S_2$) $\boxed{S_1 \cup S_2 = U}$

∴ $K = 2$

min number of subsets chosen.




Forming sets

$S_1 = \{(1,2)\ (1,3)\}$

$S_2 = \{(2,1)\ (2,3)\ (2,4)\ (2,5)\}$

all edges incident on 2

$S_7 = \{\qquad\}$

all edges incident on 7

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Suppose vertex cover Ans = $\{2, 3, 7\}$

$\therefore$ All edges in graph will have atleast one end from $\{2, 3, 7\}$

$\therefore$ we can say $S_2 \cup S_3 \cup S_7$ = All Edges = $V$

$\therefore$ G has a vertex cover of size $K$, if the corresponding set cover instance has $K$ sets whose union contains all edges in G.

Proof: (A) If I have a vertex cover set of size $k$ in G, I can find a collection of $K$ sets whose union contains all edges in G.

Given $\{2, 3, 7\}$ as vertex cover set.
$S_2 \cup S_3 \cup S_7$ covers all edges.

(B) If I have $K$ sets whose union contains all edges in G, I can find a vertex cover set of size $k$ in G.

SAME AS ABOVE

(*) Set Packing Problem.

Given a set $U$ of $n$ elements, a collection $S_1, .. S_m$ of subsets of $U$, and a number $K$, does there exist a collection of atleast $K$ of these sets with the property that no two of them intersect.

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## (8.7) Indp set $\leq_p$ Set Packing

Imp Note: we transforming $Y \leq_p X$
we transform a single instance of Y to single instance of X

## # Reduction using Gadgets.

Mainly gadgets used ~~when~~ when we want to convert

Non graph problem $\Longrightarrow$ Graph problem.

SAT problem:

Given $\underline{n}$ Boolean variables

A clause is disjunction of terms $t_1 \vee t_2 \cdots \vee t_l$

where $t_i \in \{x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots x_n, \bar{x}_n)$

Find truth assignment for variables $x_1, \ldots x_n$ such that

(Clause 1) $\wedge$ (Clause 2) .. $\wedge$ (Clause k) = T

example:

$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3)$

$x_1 = 1$ $x_2 = 1$ $x_3 = 1$ (Not satisfying constraints)

$x_1 = 0$ $x_2 = 0$ $x_3 = 0$ (✓)

$x_1 = 1$ $x_2 = 0$ $x_3 = 0$ (✓)

SAT problem: Given a set of clauses $C_1 .. C_K$ over a set of variables $X = \{x_1, .. x_n\}$ does there exist a satisfying truth assignment.

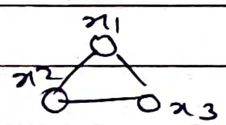
(8.8) $\boxed{3\ SAT \le_p \text{Indp set}}$

Non graph prob — graph prob

selects max nodes such that no two nodes from selected set form a pair.

Plan: Given an instance of 3 SAT with $K$ clauses, build a graph G that has an ~~edge~~ indp set of size $K$ if the 3 SAT ~~[illegible]~~ instance is satisfiable

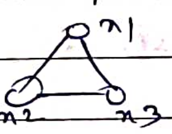
$(x_1 \vee x_2 \vee x_3)$




To make this clause T

we just need to make $x_1$ or $x_2$ or $x_3$ any one of them T.

So Indp set will pick any one vertex from this



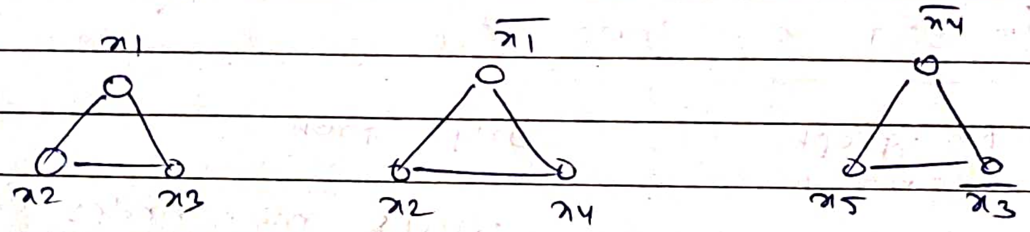

as if two vertex is selected they will have edge between them.

So given $k$ clauses there will $k$ $\Delta$'s.

eg: $C_1 = (x_1 \vee x_2 \vee x_3)$

$C_2 = (\bar{x}_1 \vee x_2 \vee x_4)$

$C_3 = (\bar{x}_4 \vee x_5 \vee \bar{x}_3)$

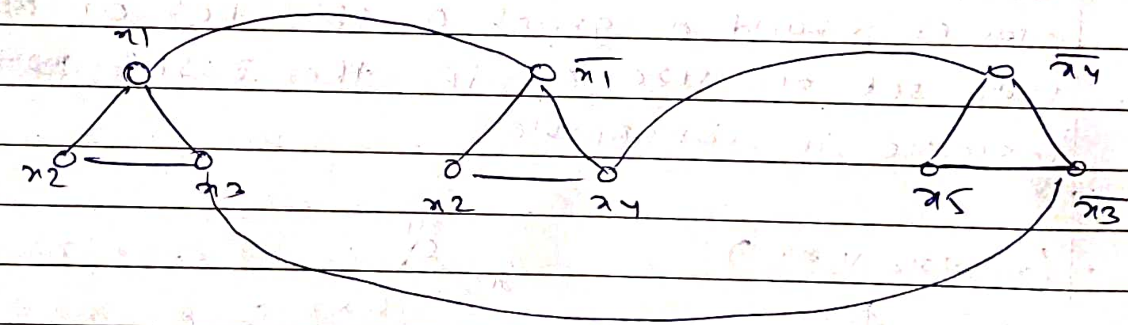



Problem: suppose indp set is

$\{x_1, \bar{x}_1, \bar{x}_4\}$

then $x_1 = T$ and $\bar{x}_1 = T$ (Not possible)

Hence little modification is needed.



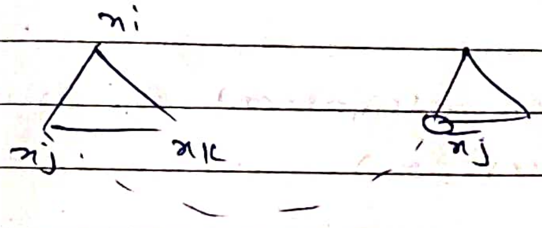

connecting all $x_i$ and $\bar{x}_i$ makes sure that only one of them is picked.

Claim: 3 SAT instance is satisfiable if the graph G has an indp set of size $k$

As from every $\Delta$ if indp set selects one node 3 SAT problem is solved

Proof: (A) If the 3-SAT instance is satisfiable, then there is atleast one node label per triangle that evaluates to 1

Let S be a set containing one such node from each $\Delta$.




If there is a satisfying truth assignment then atleast one of $x_i$, $x_j$, $x_k$ would have value = 1

only problem could be if $x_j$ and $\bar{x}_j$ both = 1
But by adding such edges we have solved that problem.

(B) Suppose G has an indp set S of size atleast K
- if $x_i$ appears as a label in S then set $x_i = 1$
- if $\bar{x}_i$ appears as a label in S then set $x_i = 0$
- if neither $x_i$ nor $\bar{x}_i$ appear as a label in S, then its value does not matter

(*) Transitivity Reductions

If $Z \le_p Y$ and $Y \le_p X$
Then $Z \le_p X$

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(A) Efficient certification.

1. Poly length certificate
2. Poly Time Certificate

3-SAT

Certificate t is an assignment of truth values ~~of~~ to variables ($x_i$)

Certifier : evaluate the clauses if all of them evaluate to 1 then it answers yes.

Indp set

Certificate t in a set of nodes of size atleast k in G.

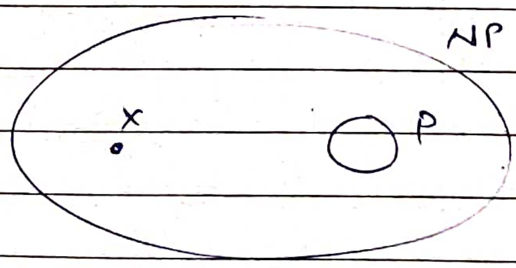
Certifier: Check each edge to make sure no edges have both ends in the set

Check size of set ≥ k

no repeating nodes.

(Class NP)

Is a set of all problems for which there exists an efficient certifier



if $x$ is in NP

and for all $Y \in NP$

$Y \leq_p X$, then $x$ must be hardest problem.

(3-SAT) is the hardest problem.

Such a problem is called NP-complete.

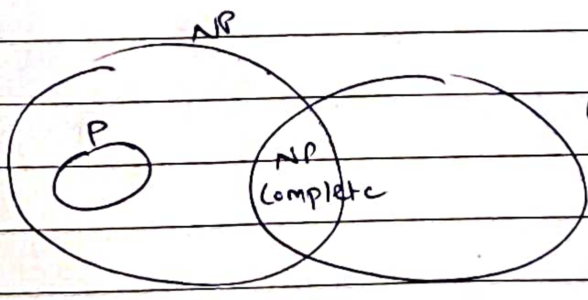
3-SAT $\leq_p$ Indp set $\leq_p$ vertex cover $\leq_p$ set cover

$\therefore$ 3 SAT, Indp set, vertex cover, set cover are all NP complete.

How to show problem is NP-complete?

Basic strategy to prove problem $x$ is NP-complete.

1. we need to prove $x$ is in NP-class
2. choose a problem $Y$ know to be NP complete
3. Prove $Y \leq_p X$ } (This proves $x$ is atleast as hard as $Y$ and $Y$ is NP-complete $\therefore x$ is NP-complete)




NP Hard problems are atleast as hard as NP complete problems.

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If from given 3 steps
we only prove step 3
i.e $Y \leq_p X$

we prove $X$ is NP-Hard
only if $X$ is $\underbrace{(\text{NP and NP-Hard})}_{\text{NP-complete.}}$