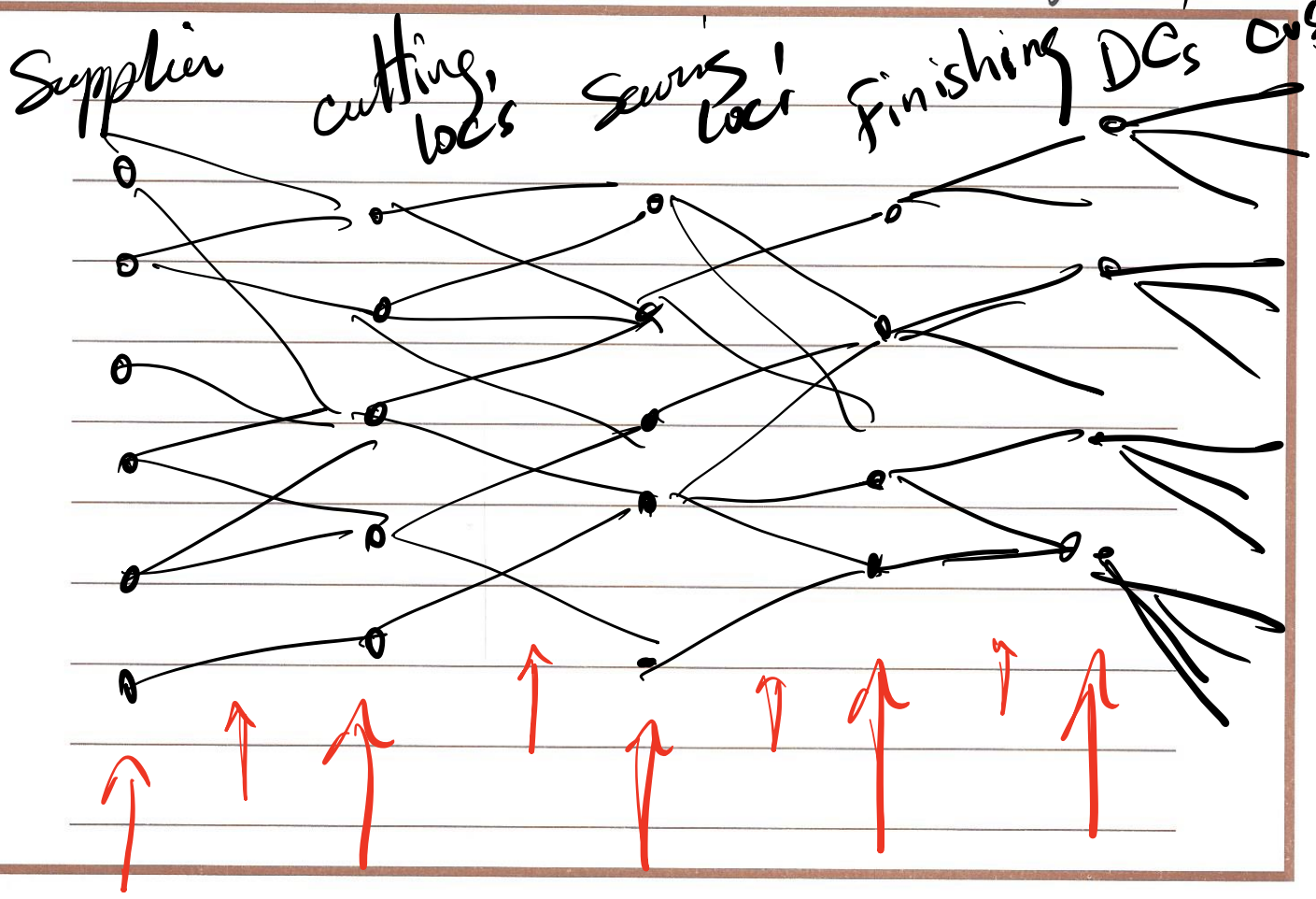
Network Flow

Examples of network flow problems:

- How much data can we send from one point in the network to another point?

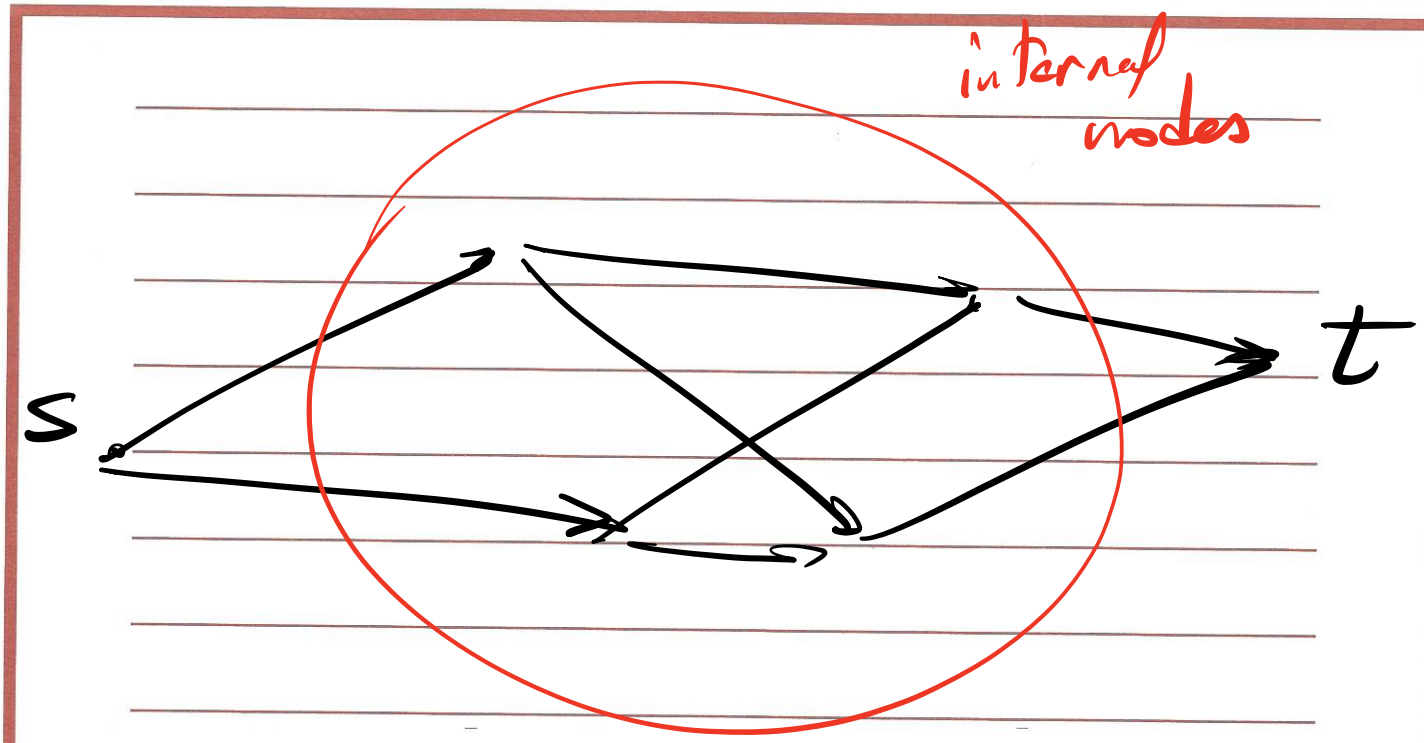
- How much traffic does the freeway system sustain for travel between two cities?

- How profitable could a manufacturing supply chain be?



Def. A flow network is a directed graph $G = (V, E)$ with following features:

- Each edge e has a non-negative capacity c_e
- Has a single source node $s \in V$
- Has a single sink node $t \in V$



Assumptions:

- no edges enter s or leave t
- at least one edge is connected to each node $O(m+n) \rightarrow O(m)$
- all capacities are integers

Notation

We will call $f(e)$ flow through edge e. $f(e)$ has the following properties:

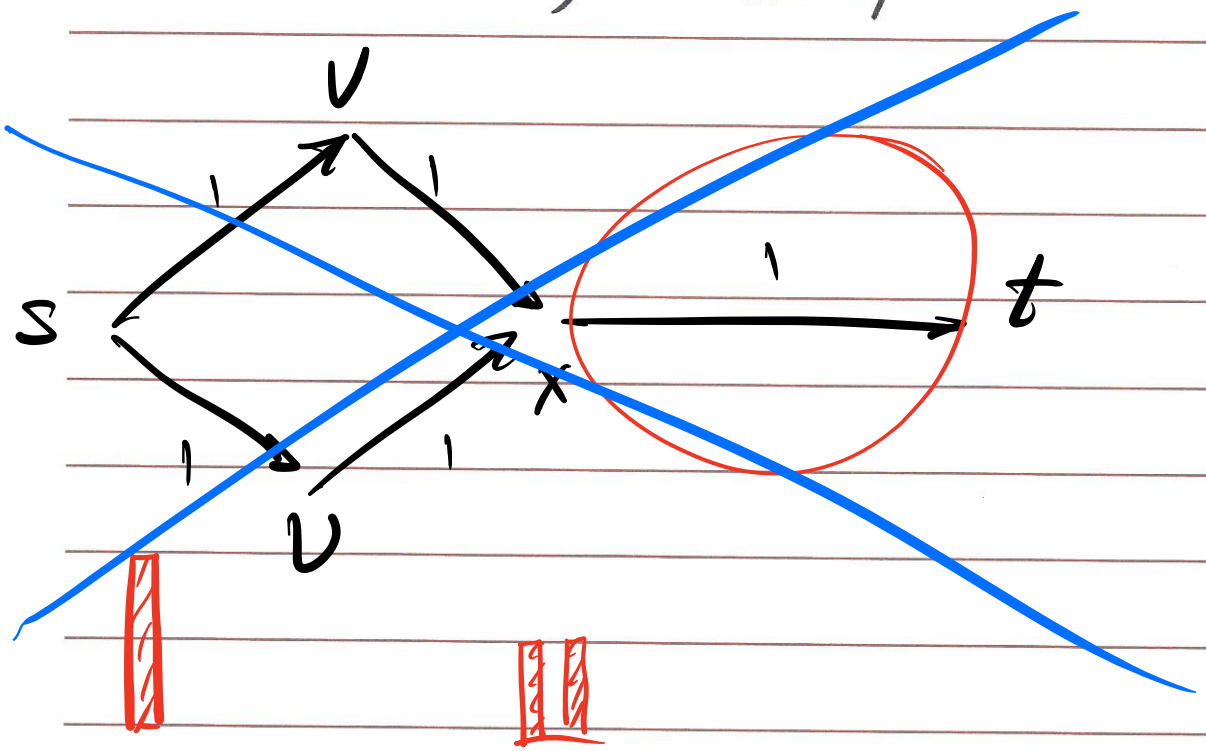
1- Capacity Constraint:

for each edge $e \in E$, $0 \leq f(e) \leq c_e$

2- Conservation of flow:

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e), \text{ except for } s \& t$$

We are looking for steady state flow.

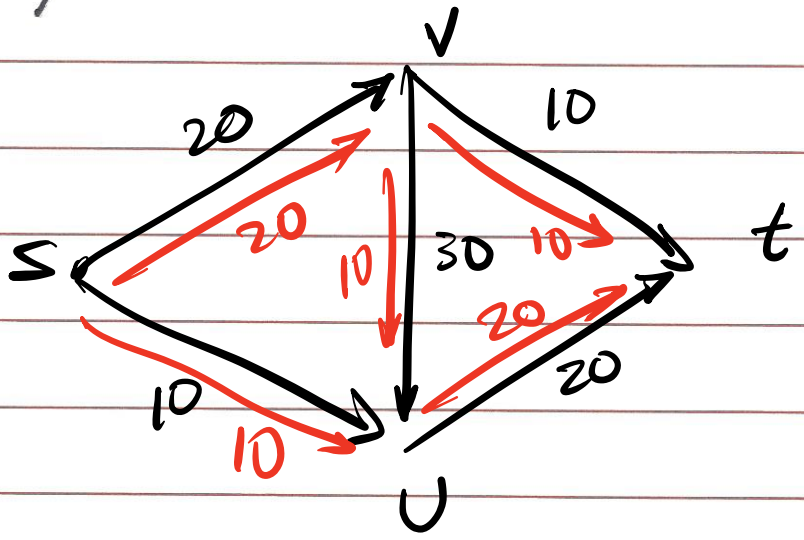


Def. For a steady state flow, the value of flow $v(f)$ is defined as follows:

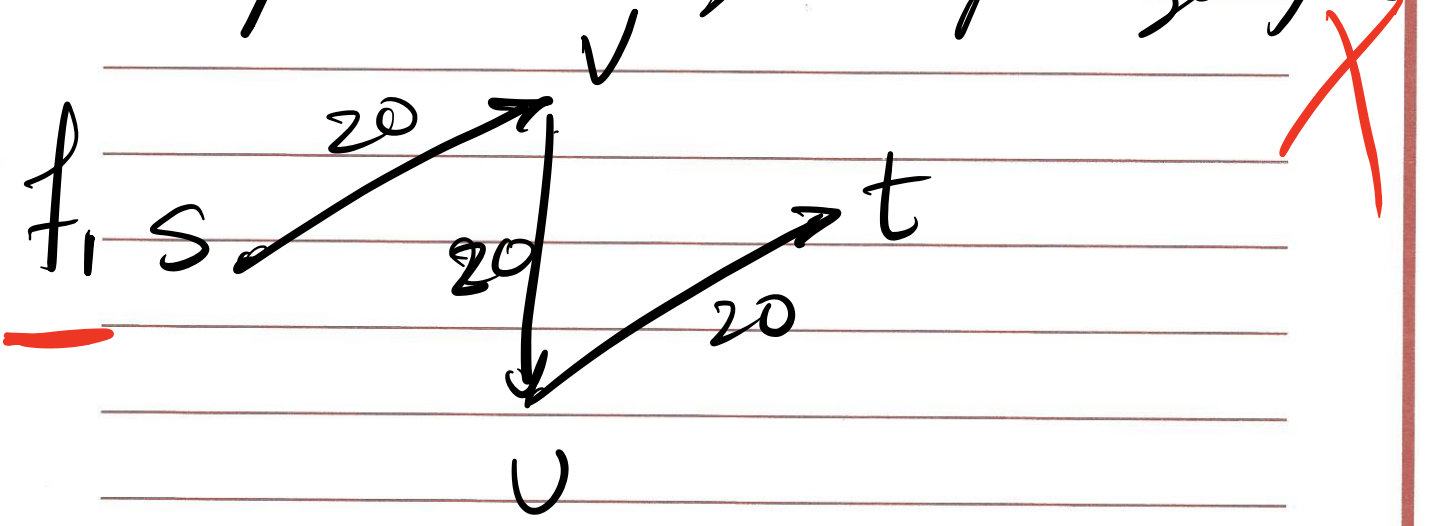
$$v(f) = \sum_{e \text{ out of } s} f(e)$$

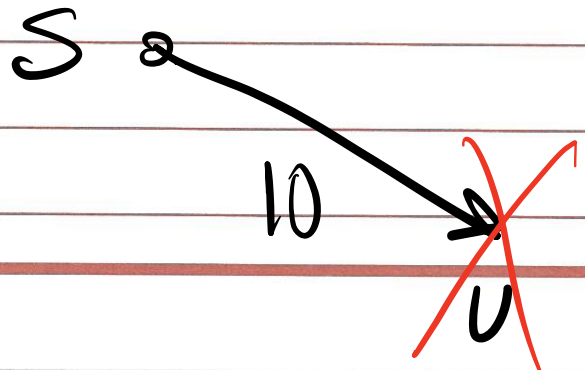
Max Flow problem

Given a flow network G, find an s-t flow with max. value.

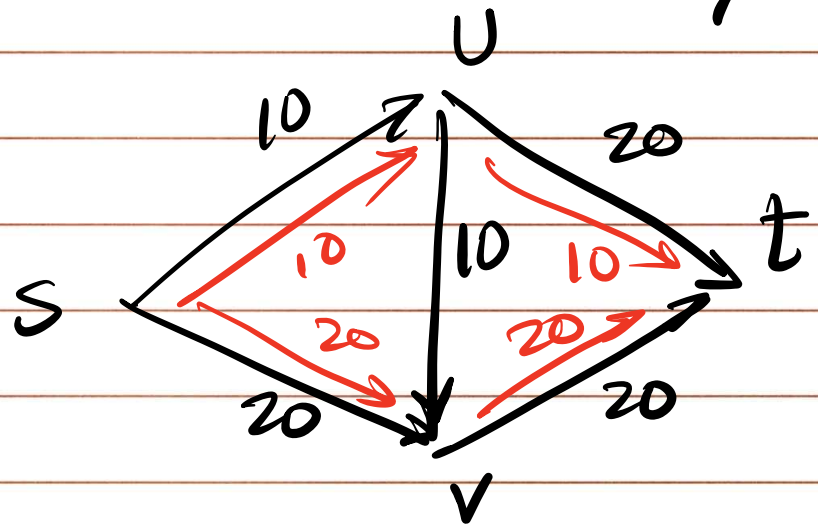


try #1 use highest cap. edges first.

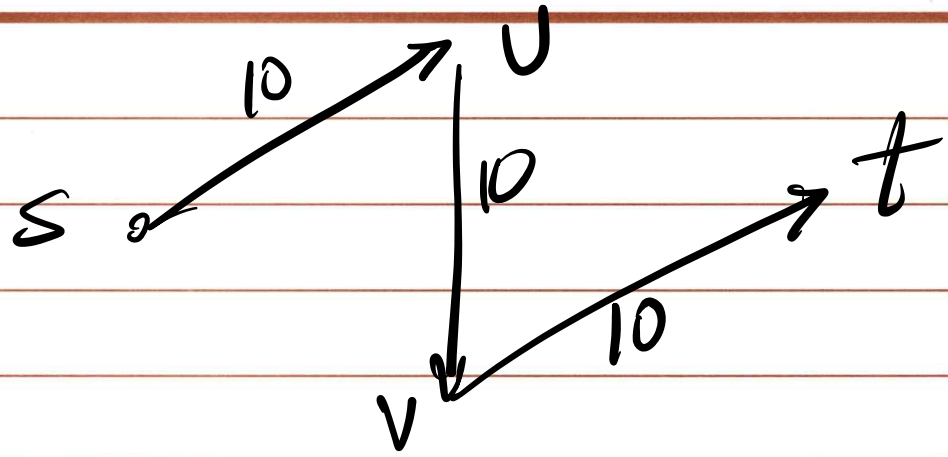




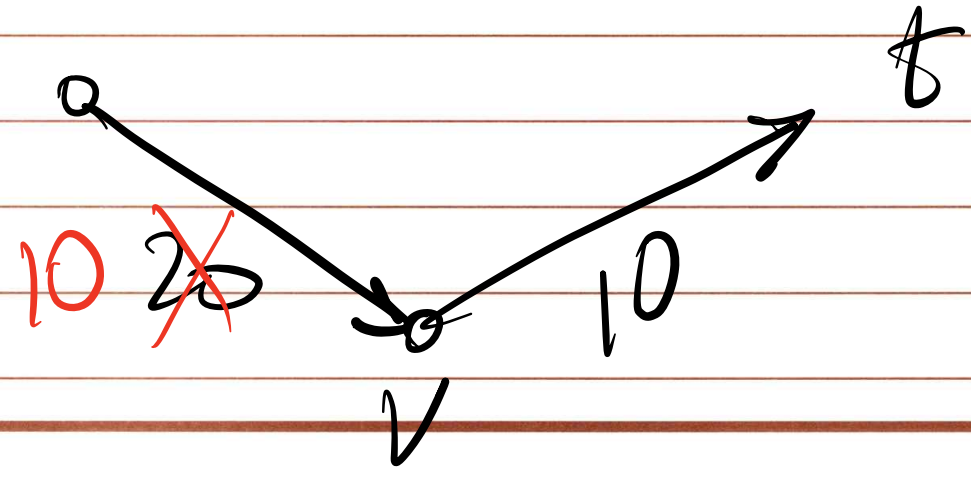
try #2 use smallest cap. edges first



f_1



f_2

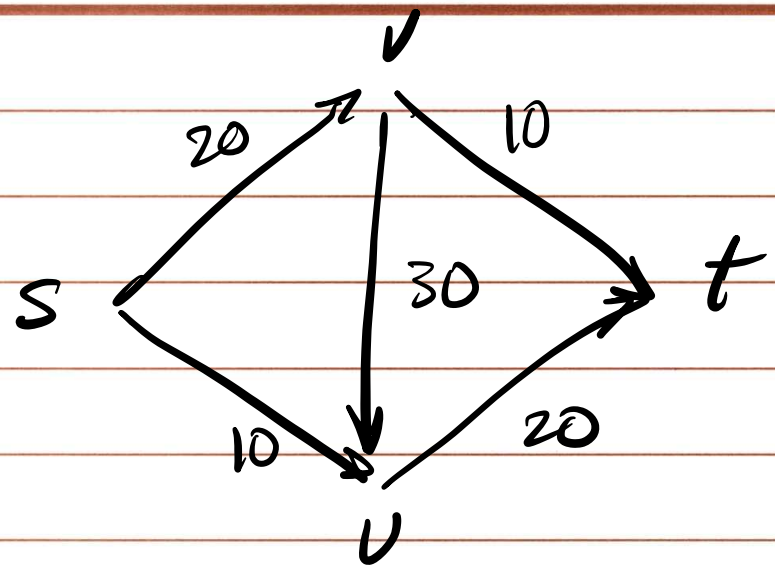


Def. G_f is the residual graph of G with the following definitions:

- G_f has the same set of nodes as G
- forward edge → for each edge e w/ $f(e) < c_e$, we include e in G_f with capacity $c_e - f(e)$
- backward edge → for each edge e w/ $f(e) > 0$, we include edge e' (opposite direction to e) in G_f with $f(e)$ units of capacity

To create G_f:

- if $f(e) = 0$
  - one forward edge w/ Cap. c_e
- if $f(e) = c_e$
  - one backward edge w/ Cap. c_e
- if $0 < f(e) < c_e$
  - one forward edge w/ Cap. $c_e - f(e)$
  - one " " w/ Cap. $f(e)$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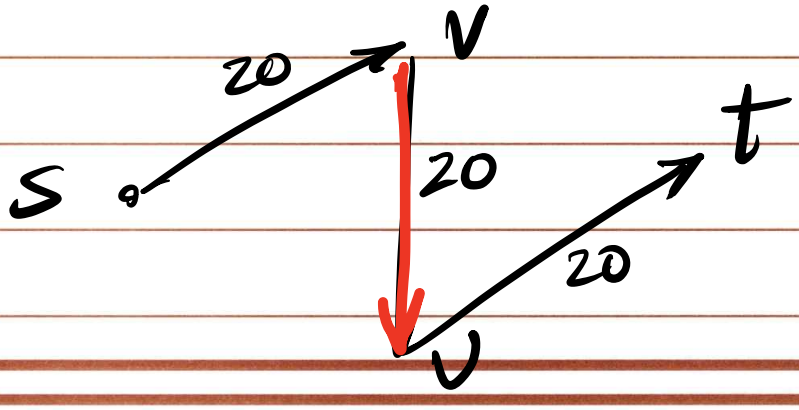
f_1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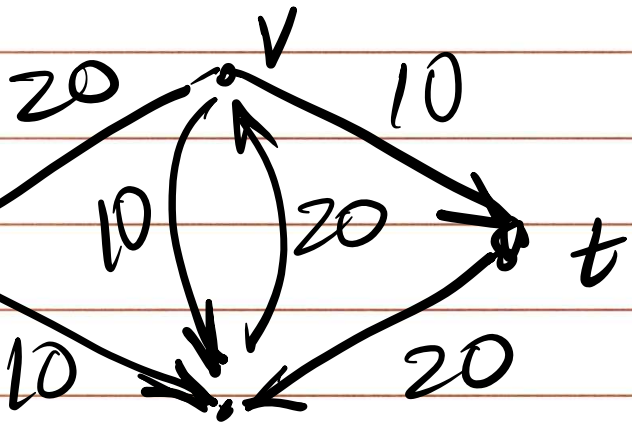
G_{f_1}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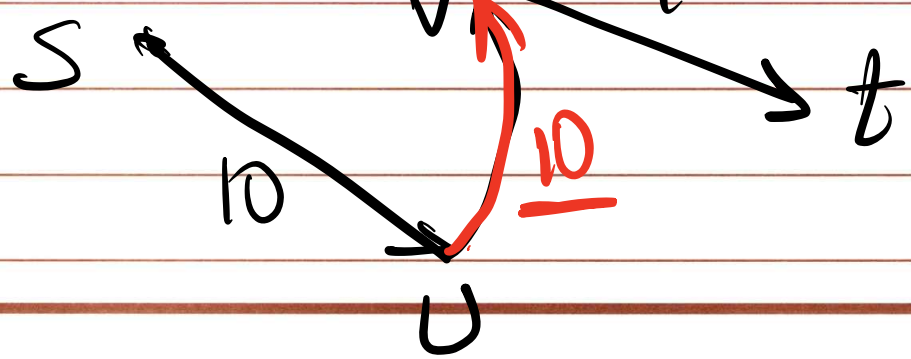
f_2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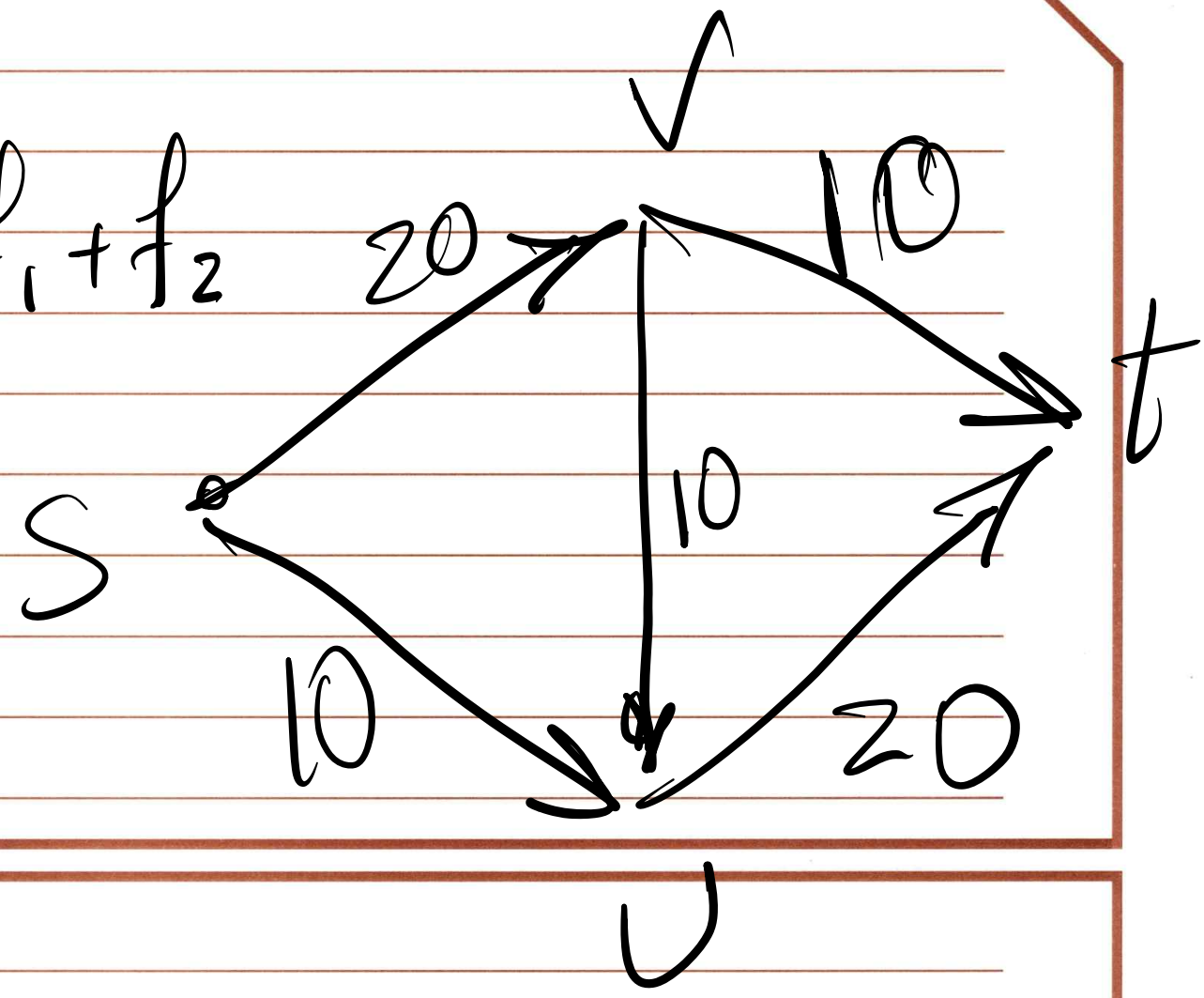
$f = f_1 + f_2$



Def. If P is a simple path from s to t in G_f, then bottleneck(P) is the minimum residual capacity of any edge on P.

Overall strategy to find Max Flow

- Find a path from s to t
- Find the bottleneck value for this path
- Push flow through this path with value equal to bottleneck value
- Repeat

```
Augment (f, c, P)
  Let b = bottleneck(P)
  for each edge (v,u) ∈ P
    if e=(v,u) is a forward edge,
      then increase f(e) in G by b
    else (v,u) is a backward edge
      and let e=(u,v)
      then decrease f(e) in G by b
    end if
  endfor
```

Return (f')

If f is flow before augmentation, and f' is flow after augmentation, we need to show that if f is a valid flow, then f' will also be a valid flow.

Proof: 1- Check capacity condition

Need to show that for each edge $e \in E$, we have $0 \leq f'(e) \leq c_e$

A- If e is a forward edge,

$$\text{bottleneck}(P) \leq c_e - f(e)$$

$$\underbrace{f(e) + \text{bottleneck}(P)}_{0 \leq f'(e) \leq c_e} \leq c_e - f(e) + f(e)$$

B- If e is a backward edge,

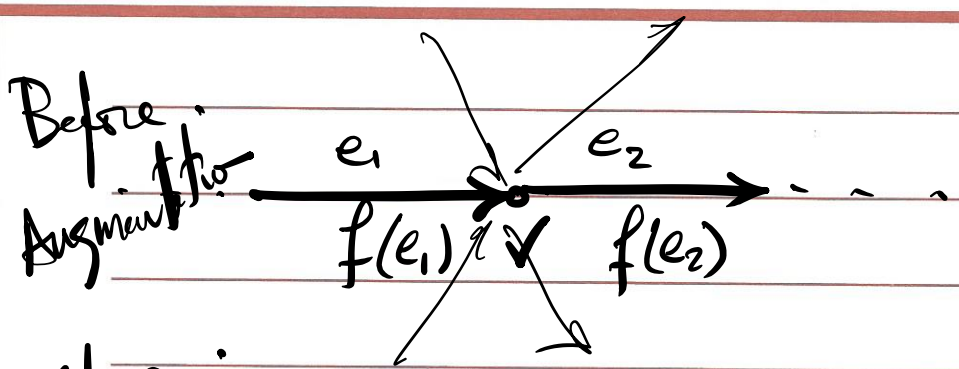
$$\text{bottleneck}(P) \leq f(e)$$

$$\underbrace{f(e) - \text{bottleneck}(P)}_{c_e \geq f'(e) \geq 0} \geq f(e) - f(e)$$

2- Check conservation of flow

Since f is a valid flow, for each node v other than s & t we have:

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$



Before Aug.

Case 1 both e_1 & e_2 correspond to forward edges of G_f

Three Other Cases to examine

Ford-Fulkerson algorithm for Max Flow.

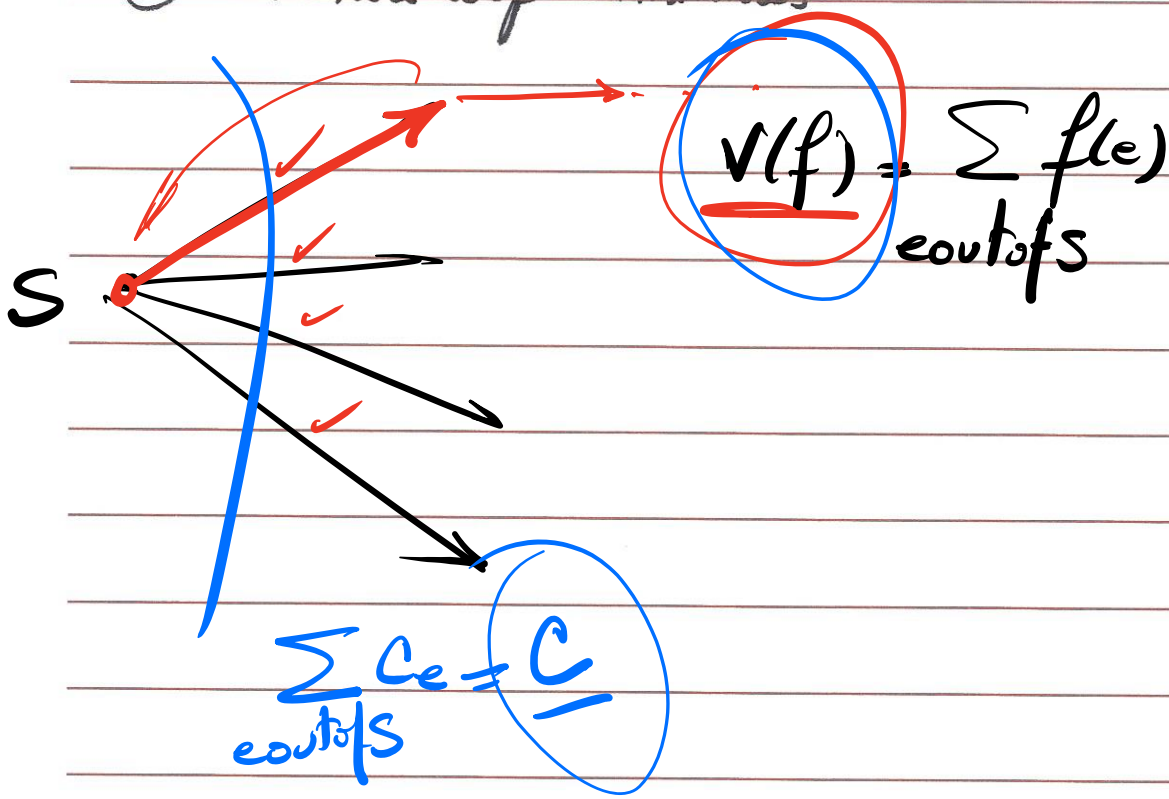
```
MaxFlow (G, s, t, c)
    Initially f(e) = 0 for all e in G
    While there is an s-t path in Gf
        let P be a simple s-t path in Gf
        f' = augment(f, c, P)
        f = f'
        update Gf
    endwhile
```

Return f.

Proof of correctness should include:

- Proof of termination
- Proof that f is a Max Flow.

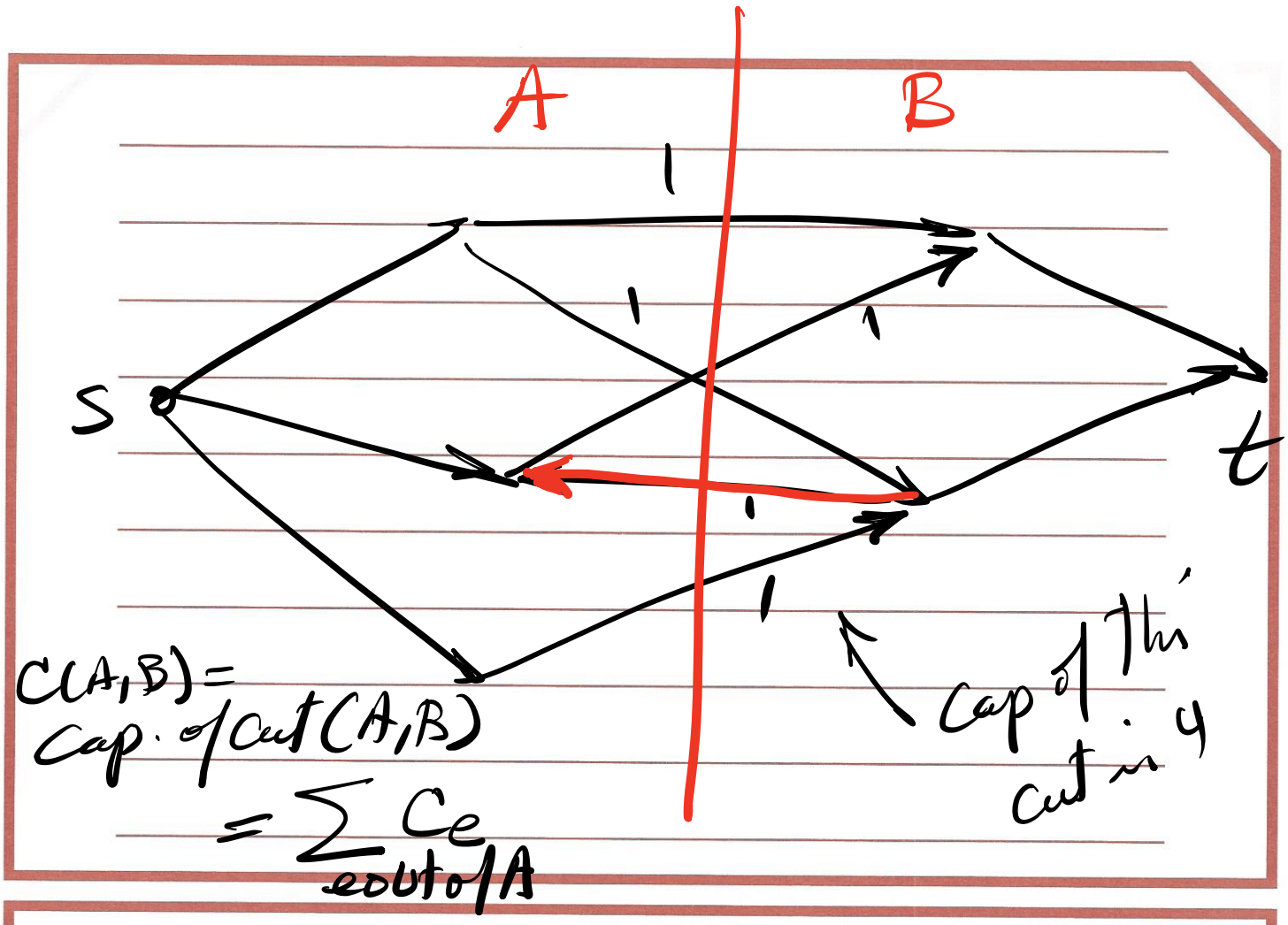
① while loop terminates



② f is a Max Flow

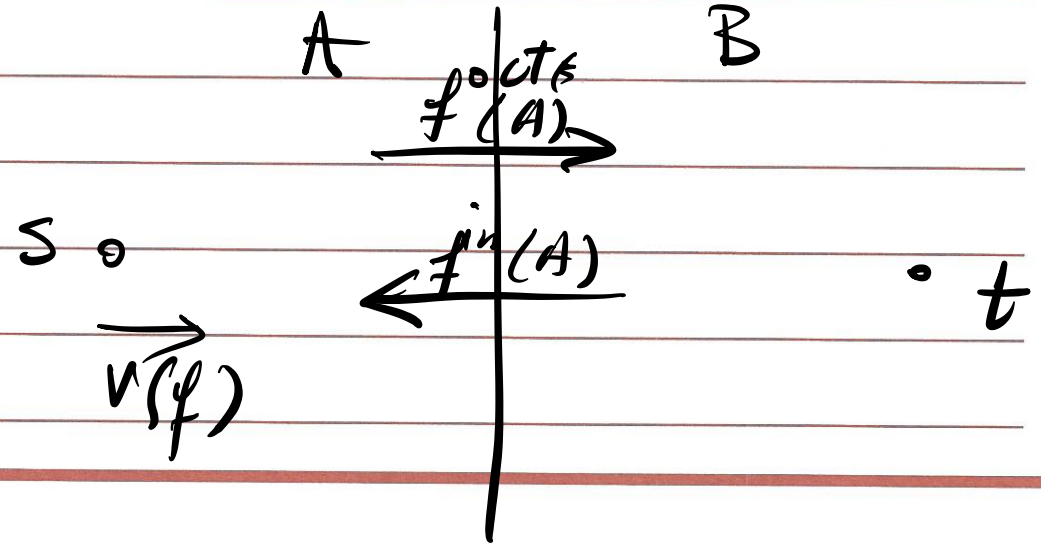
We first need some definitions

Define a cut: A cut divides nodes in the graph into 2 sets A & B such that $s \in A$ and $t \in B$



FACT: Let f be any s-t flow and (A,B) any s-t cut, then

$$v(f) = f^{out}(A) - f^{in}(A)$$



Max. value of the flow $\leq$ Cap. of the (A,B) cut

Proof: $v(f) = f^{out}(A) - f^{in}(A)$ ≥ 0

$$v(f) \leq f^{out}(A) \leq \sum_{\text{edges out of } A} c_e$$

$$v(f) \leq c(A,B)$$

So, the MaxFlow is bounded by the cap. of every s-t cut.

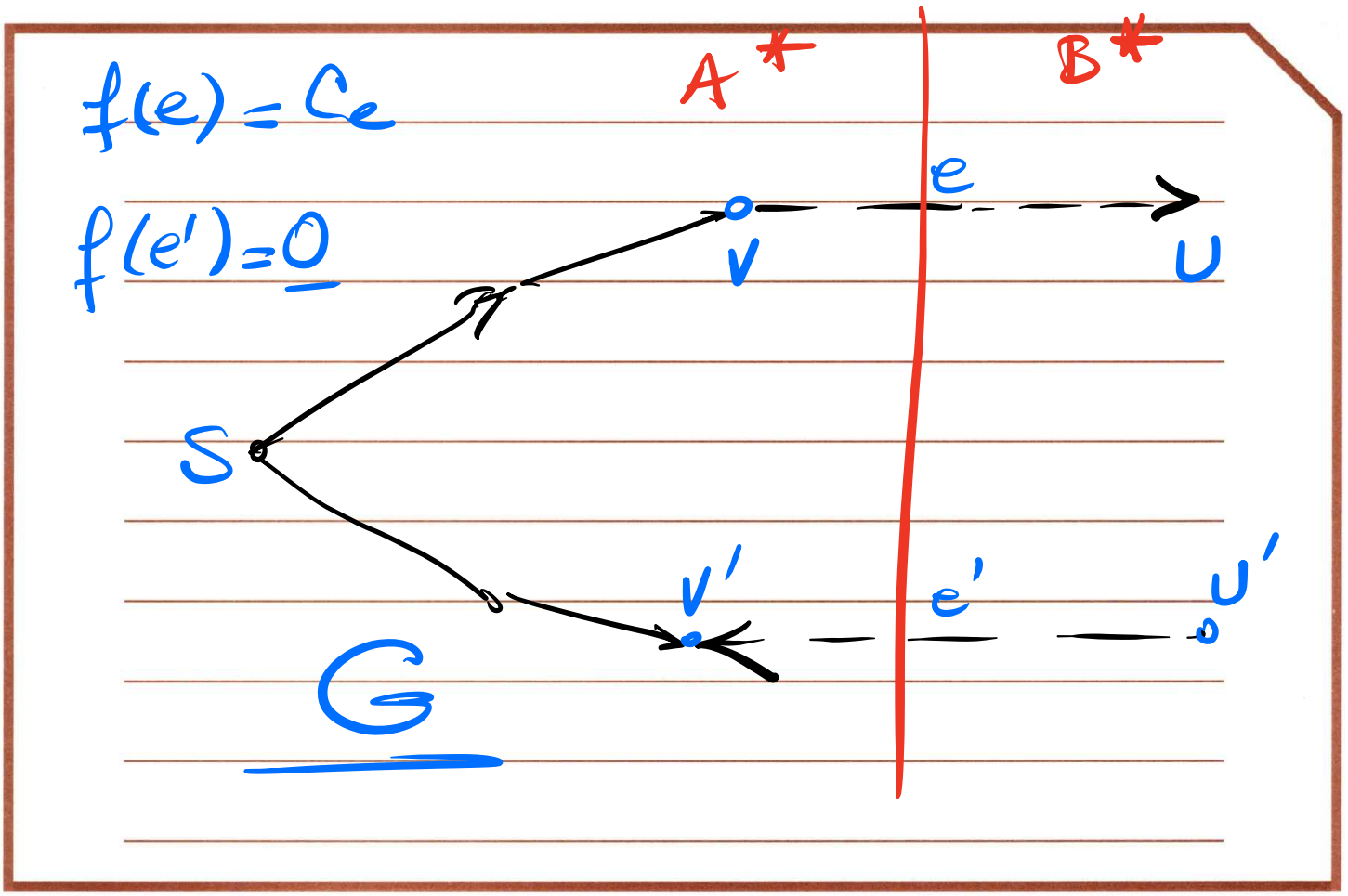
Ford-Fulkerson terminates when the flow f has no s-t paths in G_f.

Claim: If there is no s-t path in G_f, then there is an s-t cut (A^*, B^*) where

$$v(f) = c(A^*, B^*)$$

Proof: Create sets A^* & B^* such that A^* includes all nodes v where there is an s-v path in G_f.

$$B^* = V - A^*$$



$$v(f) = f^{out}(A^*) - f^{in}(A^*)$$

$$= \sum_{e \text{ out of } A^*} c_e - \emptyset$$

$$v(f) = c(A^*, B^*)$$

(A^*, B^*) has the Min Cap. of any s-t cut in G.

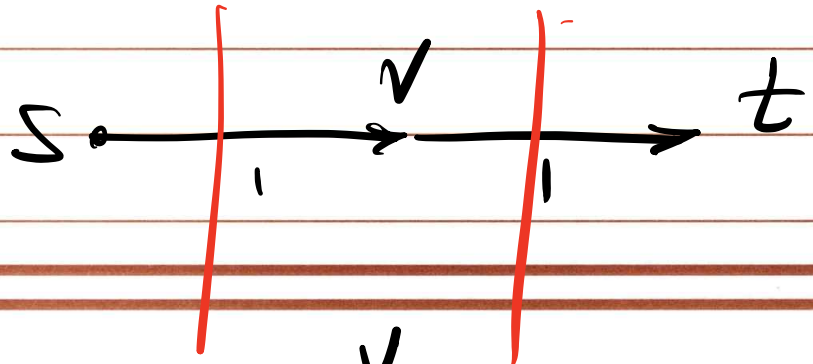
- How do we find min cut if Max Flow is given to us?

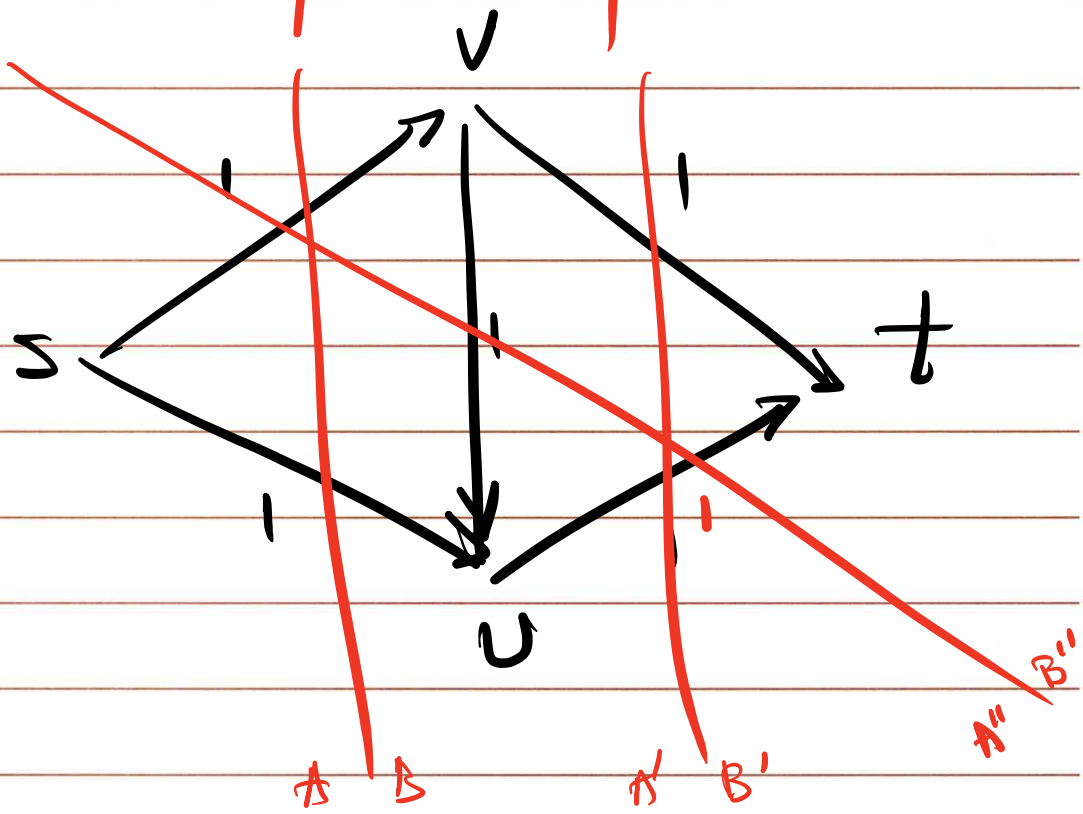
Run BFS/DFS in G_f from s

all nodes reachable from $s \longrightarrow A^*$

other nodes $\longrightarrow B^*$

(A^*, B^*) is our min cut.





How to find out if Min cut is unique?

Find Min cut closest to S
Reverse edges & find Min cut closest to t. If they are the same cut → unique min cut.

Ford-Fulkerson algorithm for Max Flow.

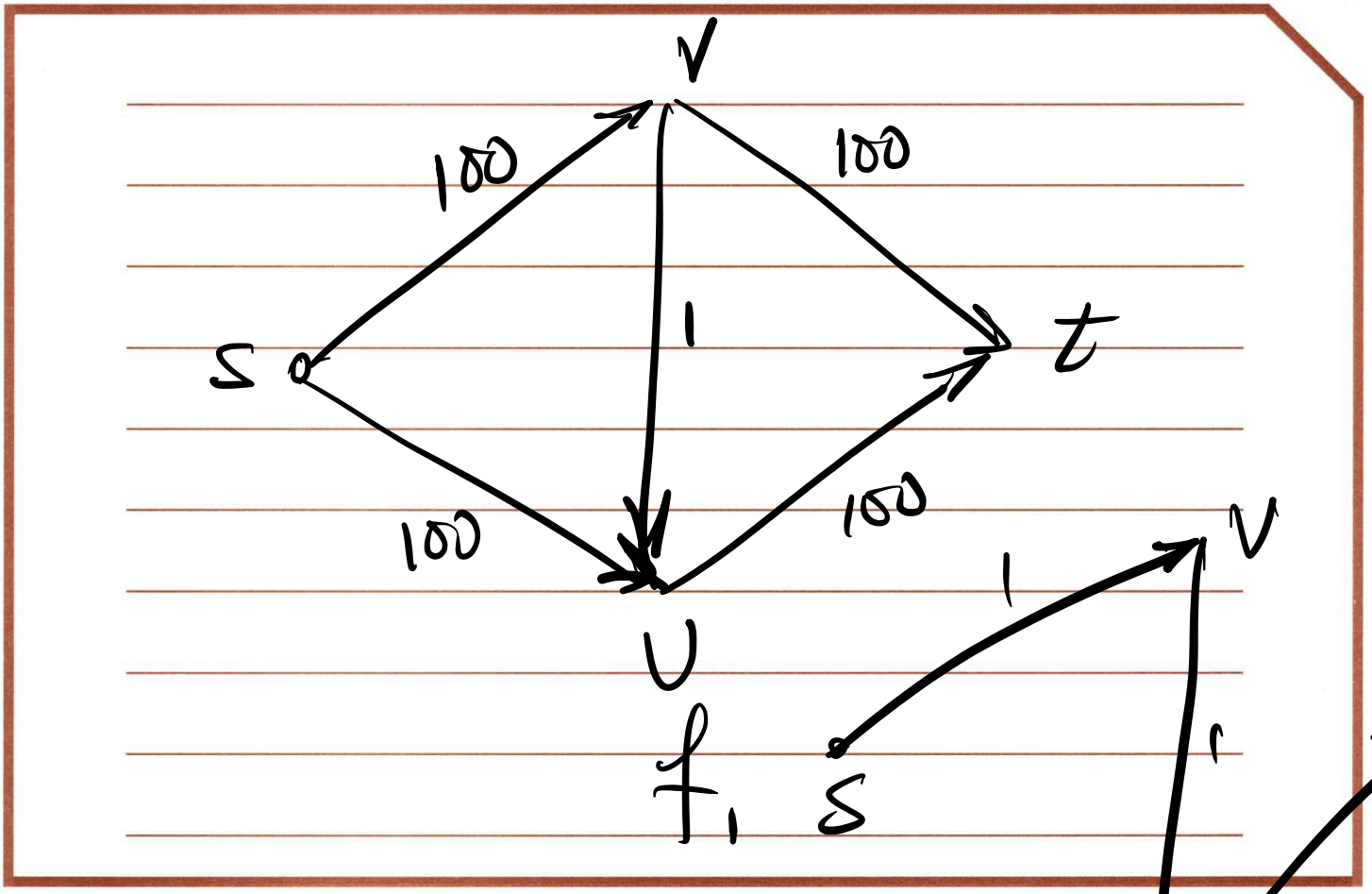
```
MaxFlow(G, s, t, C)
    Initially f(e) = 0 for all e in G
    While there is an s-t path in Gf
        let P be a simple s-t path in Gf
        f' = augment(f, c, P)
        f = f'
        update Gf
    endwhile
```

C(O(m) + O(m) + O(m))

Return f.

pseudo polynomial complexity

takes O(Cm)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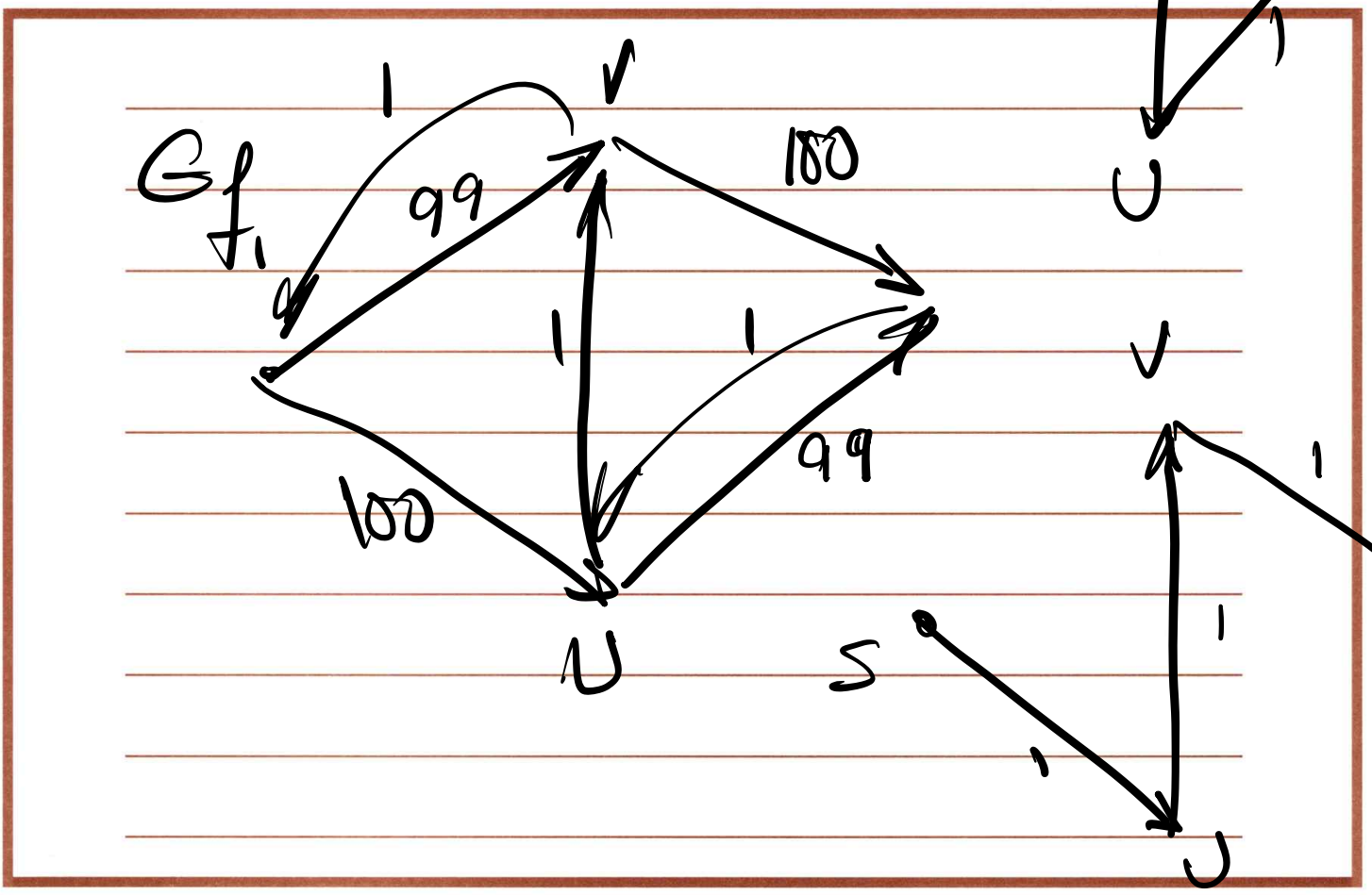
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Scaled version of Ford-Fulkerson

Initially $f(e) = 0$ for all e in G

Set Δ to be the largest power of 2 that is no larger than the Max. cap. out of s.

While $\Delta \geq 1$

  While there is an s-t path in $G_f(\Delta)$

    $O(m)$ Let P be a simple s-t path in $G_f(\Delta)$

    $O(m)$ $f' = \text{augment}(f, P)$

    $f = f'$

    update $G_f(\Delta)$

  endwhile

  $\Delta = \Delta / 2$

endwhile

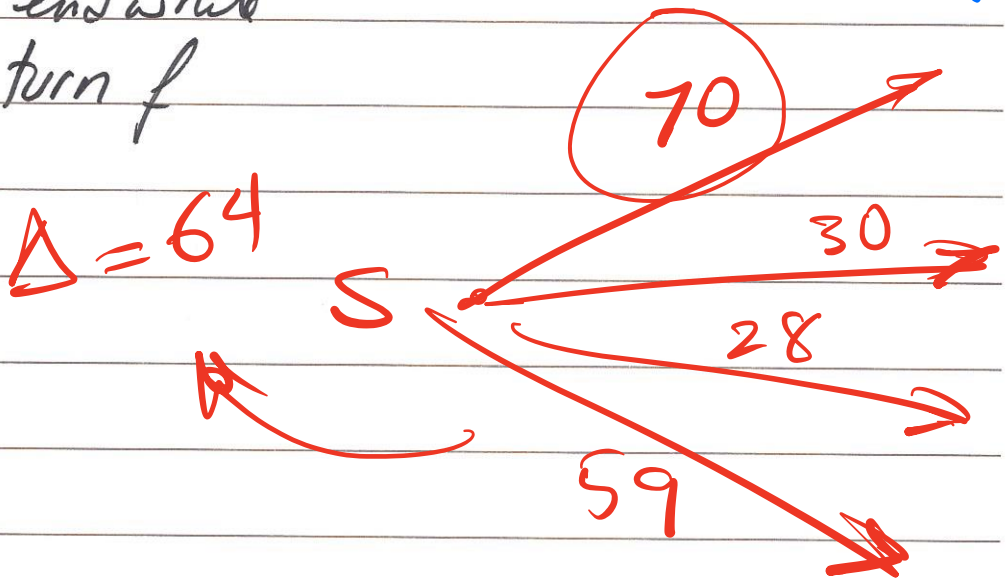
Return f

$O(mC)$ or $O(m^2)$

end of a Δ scaling phase

Δ scaling phase

$\Delta = 64$

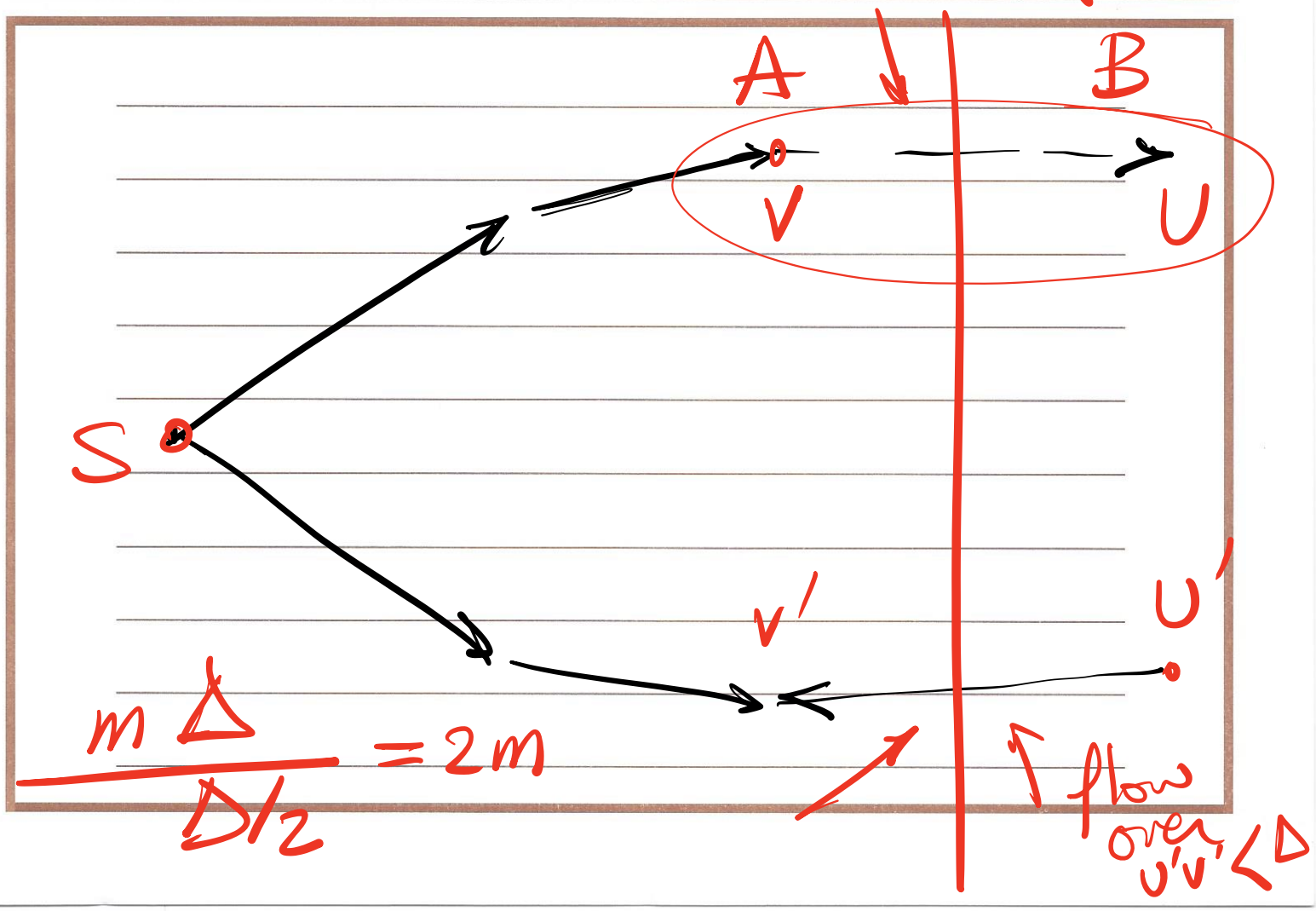


Background:

- During the Δ-scaling phase, each augmentations increases the flow value by at least Δ.

- Let f be the flow at the end of the Δ-scaling phase.

Claim: There is an s-t cut (A, B) in G for which $c(A,B) \leq v(f) + m\Delta$



Claim: The number of augmentations in a scaling phase is at most $2m$.

First scaling phase

How many times can we use each edge going out of s?

Other scaling phases

At the end of the Δ-scaling phase, we have already shown that $c(A,B) \leq v(f) + m\Delta$

If f^* is Max flow, Then $v(f^*) \leq c(A,B)$

$\Rightarrow v(f^*) \leq v(f) + m\Delta$

So, in the next scaling phase, where $\Delta' = \Delta/2$ how many iterations can we have?

Scaled version of Ford-Fulkerson

Initially $f(e) = 0$ for all e in G

Set Δ to be the largest power of 2 that is no larger than the Max. cap. out of s.

While $\Delta \geq 1$

  While there is an s-t path in $G_f(\Delta)$ — $O(m)$

    Let P be a simple s-t path in $G_f(\Delta)$ — $O(m)$

    $f' = \text{augment}(f, P)$

    $f = f'$

    update $G_f(\Delta)$

  endwhile

  $\Delta = \Delta / 2$ — $O(\log_2 C)$

endwhile

Return f

overall complexity = $O(m^2 \log_2 C)$

weakly polynomial Time Sol.

Strongly versus weakly polynomial
(relevant if input consists of integers)

An algorithm runs in strongly polynomial time if the no. of operations is bounded by a polynomial in the number of integers in the input.

An algorithm runs in weakly polynomial time if the no. of operations is bounded by a polynomial in the number of bits in the input, but not in the number of integers in the input.

Edmonds-Karp

Same as Ford-Fulkerson, except that each augmenting path must be a shortest path with available capacity.

Can be shown to have running time $O(nm^2)$

Ford-Fulkerson	$O(Cm)$ pseudo-polynomial
Scaled version of FF	$O(m^2 \lg C)$ weakly polynomial
Edmonds-Karp	$O(nm^2)$ strongly polynomial
Orlin + KTR	$O(nm)$ " "

Recently developed methods solve max flow in close to linear time WRT m.

approximation

Discussion 8

1. You have successfully computed a maximum s-t flow f for a network G = (V; E) with integer edge capacities. Your boss now gives you another network G' that is identical to G except that the capacity of exactly one edge is decreased by one. You are also explicitly given the edge whose capacity was changed. Describe how you can compute a maximum flow for G' in O(|V| + |E|) time.

2. You need to transport iron-ore from the mine to the factory. We would like to determine how long it takes to transport. For this problem, you are given a graph representing the road network of cities, with a list of k of its vertices ($t_1, t_2, \ldots, t_k$) which are designated as factories, and one vertex S (the iron-ore mine) where all the ore is present.
We are also given the following:

- Road Capacities (amount of iron that can be transported per minute) for each road (edges) between the cities (vertices).
- Factory Capacities (amount of iron that can be received per minute) for each factory (at $t_1, t_2, \ldots, t_k$)
- The amount of ore to be transported from the mine, C

Give a polynomial-time algorithm to determine the minimum amount of time necessary to transport and receive all the iron-ore at factories.

3. In a daring burglary, someone attempted to steal all the candy bars from the CS department. Luckily, he was quickly detected, and now, the course staff and students will have to keep him from escaping from campus. In order to do so, they can be deployed to monitor strategic routes.

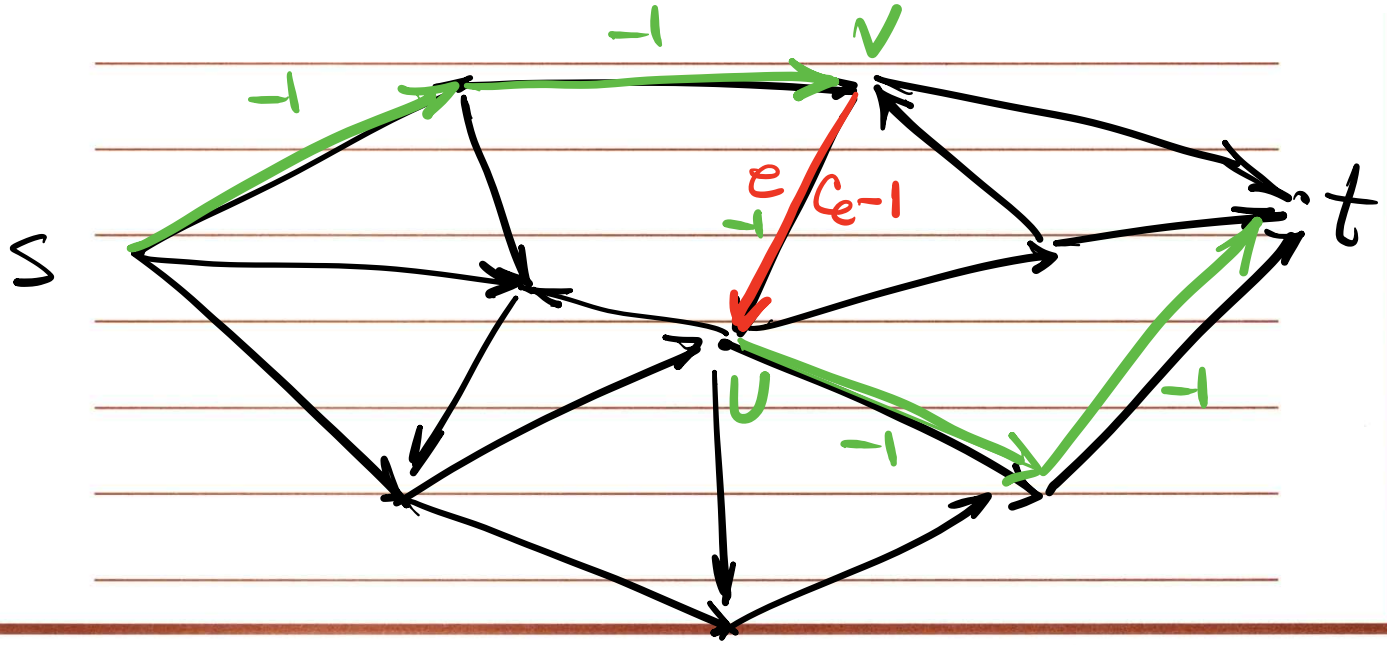
More formally, we can think of the USC campus as a graph, in which the nodes are locations, and edges are pathways or corridors. One of the nodes (the instructor's office) is the burglar's starting point, and several nodes (the USC gates) are the escape points — if the burglar reaches any one of those, the candy bars will be gone forever. Students and staff can be placed to monitor the edges. As it is hard to hide that many candy bars, the burglar cannot pass by a monitored edge undetected.

Give an algorithm to compute the minimum number of students/staff needed to ensure that the burglar cannot reach any escape points undetected (you don't need to output the corresponding assignment for students — the number is enough). As input, the algorithm takes the graph G = (V,E) representing the USC campus, the starting point s, and a set of escape points $P \subseteq V$. Prove that your algorithm is correct and runs in polynomial time.

4. We define a most vital edge of a network as an edge whose deletion causes the largest decrease in the maximum s-t-flow value. Let f be an arbitrary maximum s-t-flow. Either prove the following claims or show through counterexamples that they are false:

(a) A most vital edge is an edge e with the maximum value of c(e).
(b) A most vital edge is an edge e with the maximum value of f (e).
(c) A most vital edge is an edge e with the maximum value of f (e) among edges belonging to some minimum cut.
(d) An edge that does not belong to any minimum cut cannot be a most vital edge.
(e) A network can contain only one most vital edge.

1. You have successfully computed a maximum s-t flow f for a network G = (V; E) with integer edge capacities. Your boss now gives you another network G' that is identical to G except that the capacity of exactly one edge is decreased by one. You are also explicitly given the edge whose capacity was changed. Describe how you can compute a maximum flow for G' in O(|V| + |E|) time.



Case 1 → $f(e) < c_e$

→ Max Flow stays as it was.

Case 2 → $f(e) = c_e$

O(m) Find a path from s to v on edges carrying flow

O(m) Find . . . . u to t

O(m) Subtract 1 unit of flow on path

$s \to v \to u \to t$

$\Rightarrow$ flow f'

$O(m)$ Construct $G_{f'}$

$O(m)$ find a path from s to t in $G_{f'}$

if no path $\rightarrow$ f' is max flow

$O(m)$ else augment one more unit of flow to f'

$\Rightarrow f''$

2. You need to transport iron-ore from the mine to the factory. We would like to determine how long it takes to transport. For this problem, you are given a graph representing the road network of cities, with a list of k of its vertices ($t_1, t_2, \ldots, t_k$) which are designated as factories, and one vertex S (the iron-ore mine) where all the ore is present.

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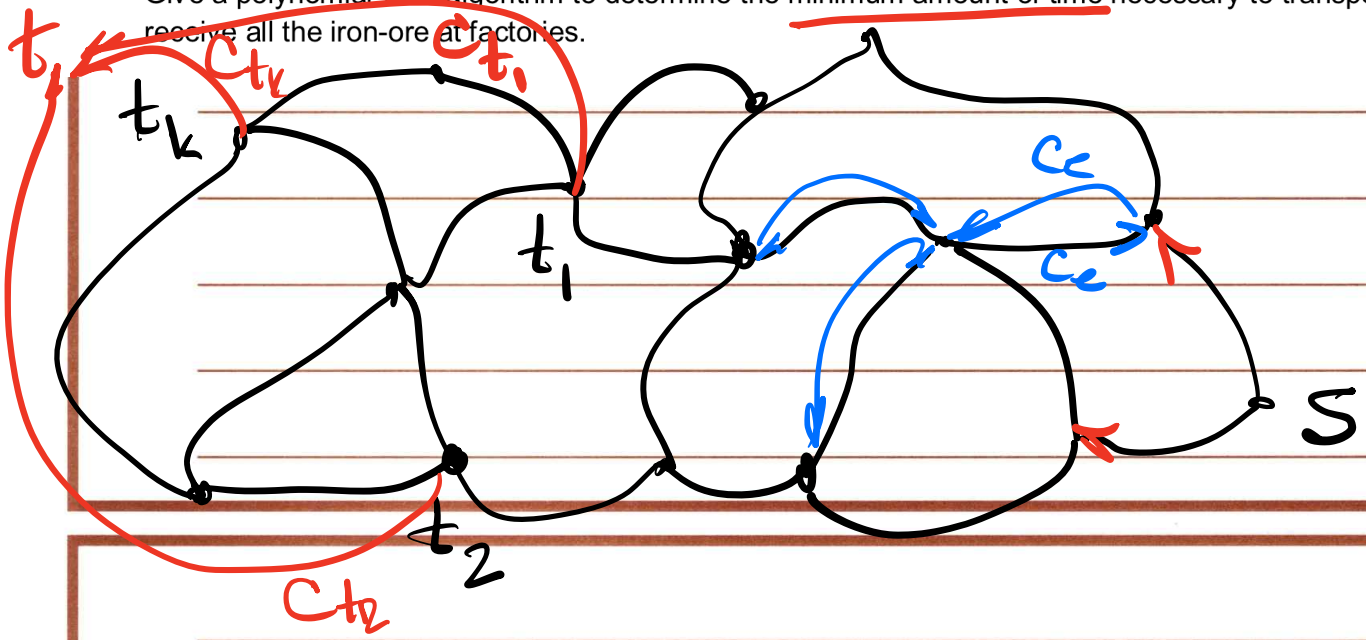
- Road Capacities (amount of iron that can be transported per minute) for each road (edges) between the cities (vertices).
- Factory Capacities (amount of iron that can be received per minute) for each factory (at $t_1, t_2, \ldots, t_k$)
- The amount of ore to be transported from the mine, C → tons

Give a polynomial-time algorithm to determine the minimum amount of time necessary to transport and receive all the iron-ore at factories.



Run Max Flow

$\xrightarrow{F}$ $v(F) = X$ ton/min

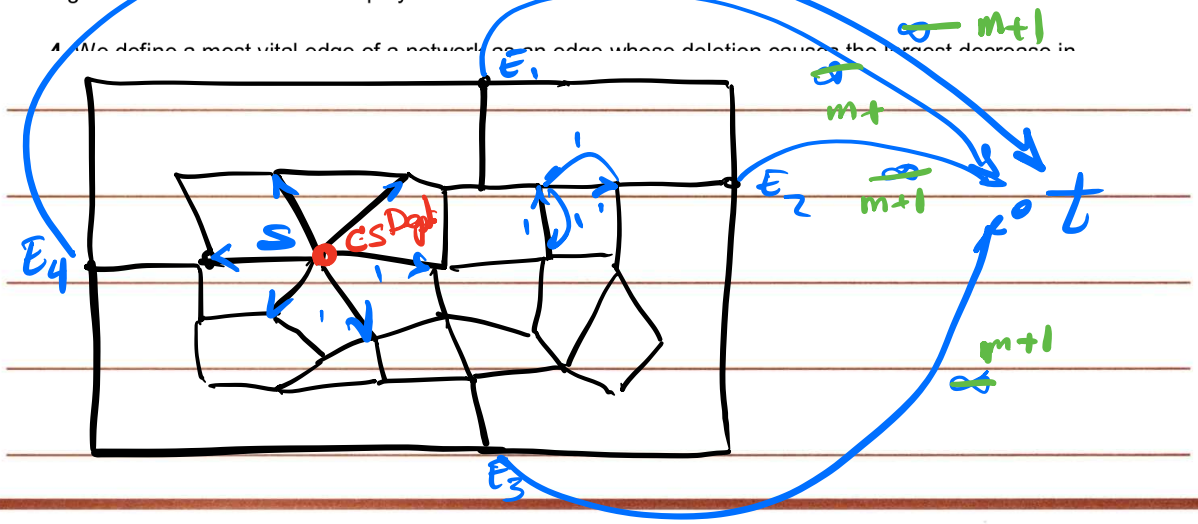
$t = \frac{C}{X}$ mins

3. In a daring burglary, someone attempted to steal all the candy bars from the CS department. Luckily, he was quickly detected, and now, the course staff and students will have to keep him from escaping from campus. In order to do so, they can be deployed to monitor strategic routes.

More formally, we can think of the USC campus as a graph, in which the nodes are locations, and edges are pathways or corridors. One of the nodes (the instructor's office) is the burglar's starting point, and several nodes (the USC gates) are the escape points — if the burglar reaches any one of those, the candy bars will be gone forever. Students and staff can be placed to monitor the edges. As it is hard to hide that many candy bars, the burglar cannot pass by a monitored edge undetected.

Give an algorithm to compute the minimum number of students/staff needed to ensure that the burglar cannot reach any escape points undetected (you don't need to output the corresponding assignment for students — the number is enough). As input, the algorithm takes the graph G = (V,E) representing the USC campus, the starting point s, and a set of escape points P ⊆ V . Prove that your algorithm is correct and runs in polynomial time.

4. We define a most vital edge of a network as an edge whose deletion causes the largest decrease in



Ford-Fulkerson → $O(Cm)$

$O(nm)$ ←

4. We define a most vital edge of a network as an edge whose deletion causes the largest decrease in the maximum s-t-flow value. Let f be an arbitrary maximum s-t-flow. Either prove the following claims or show through counterexamples that they are false:

(a) A most vital edge is an edge e with the maximum value of c(e).

(b) A most vital edge is an edge e with the maximum value of f (e).

(c) A most vital edge is an edge e with the maximum value of f (e) among edges belonging to some minimum cut.

(d) An edge that does not belong to any minimum cut cannot be a most vital edge.

(e) A network can contain only one most vital edge.