ADA Week 13 Notes

(A) Travelling Salesman problem and Hamiltonian cycle.

= Decision version of TSP
Given a set of distances on n cities and a bound D, is there a tour of length / cost atmost D?

Hamiltonian cycle

Hamiltonian cycle, if it visits each vertex exactly once.

Problem statement, Given an undirected graph G, is there a Hamiltonian cycle.

<Q> Show that Hamiltonian cycle problem is NP-complete.

Step1 : Showing Hamiltonian cycle is NP

a. Certificate: ordered list of nodes on the Hamiltonian cycle.

b. Certifier: (i) check to make sure there is an edge between each pair of adjacent nodes in the list

(ii) All nodes are visited

(iii) No repeated nodes

(iv) edge between last and first nodes in the list.

Step2 : Choose a problem already known to be NP - complete

Vertex Cover

Step3: Prove vertex cover $\le_p$ Hamiltonian cycle.

vertex cover $\le_p$ H C

$G=(V,E)$ undirected graph

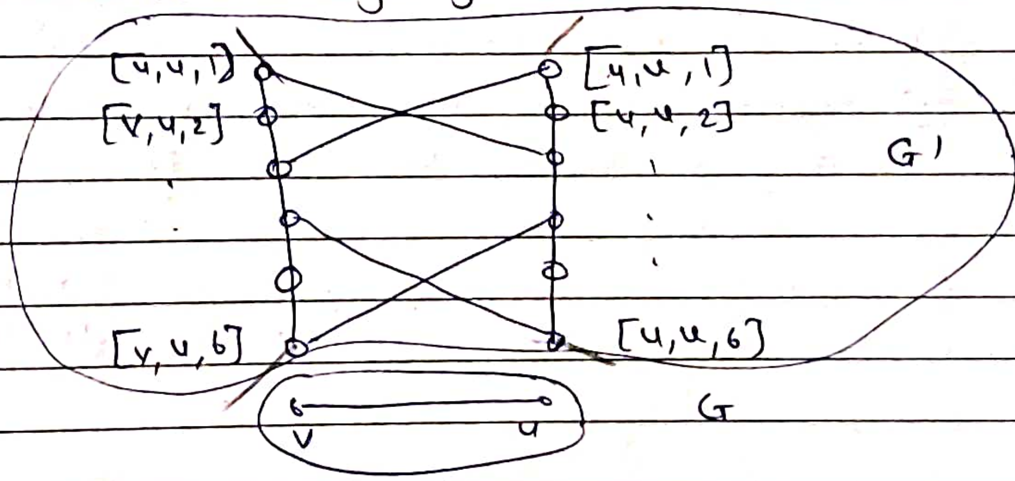
HC needs directed graph.

G'

Plan: Given an undirected graph $G=(V,E)$ and an integer k, we construct $G'=(V',E')$ that has a Hamiltonian cycle if G has a vertex cover of size atmost k

Construction of G'

For each edge (v,u) in G
G' will have one gadget W_{vu}.

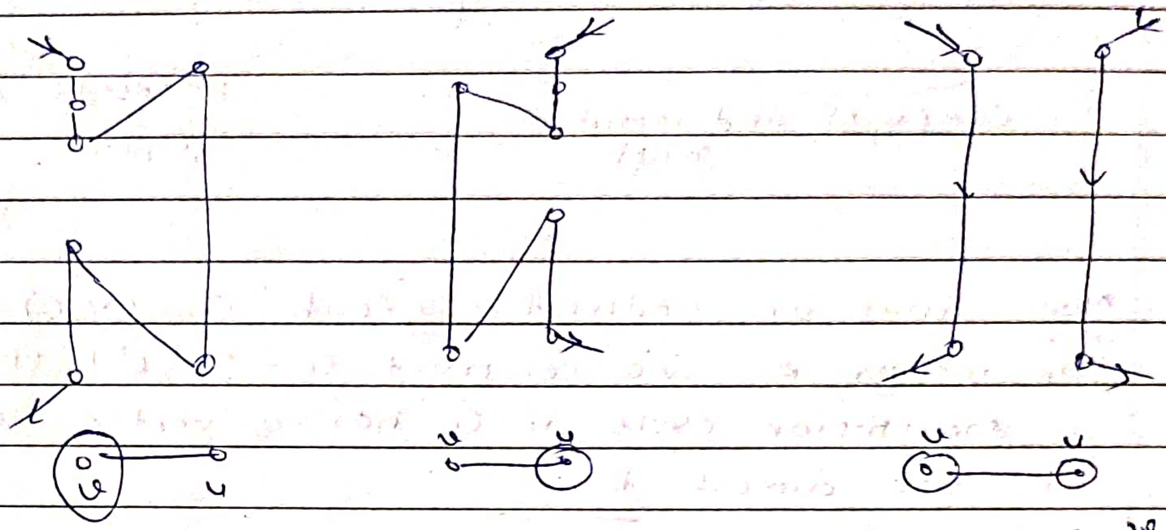


This gadget can be connected to other gadgets only through 4 corner nodes
[v,u,1] [v,u,6] [u,v,1] [u,v,6]

Note $Y \le_p X$

convert Y to prob type X → $\boxed{\text{Blackbox solving } X}$ → Get Ans

There are 3 ways for HC to visit all nodes of this edge.



HC enters the gadget from v.

HC enters the gadget from u

HC covers v and covers some other vertices then cover u.

There are some other vertices in G'

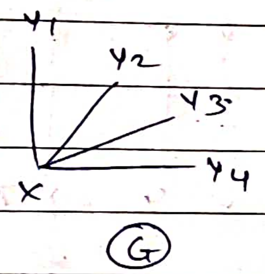
(*) selector vertices: There are K selector vertices in G' $s_1, s_2 \dots s_K$

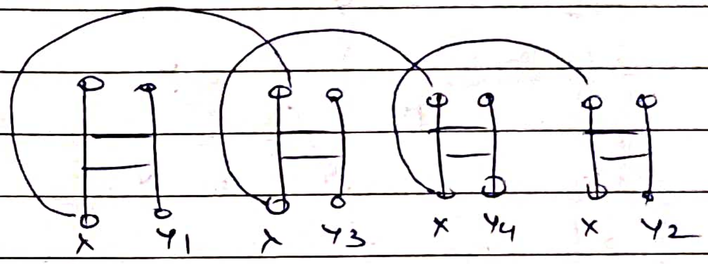
This K comes from vertex cover problem statement.

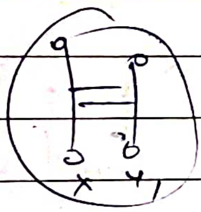
Explanation of how G' is formed.

1. For each vertex $u \in V$ we add edges to join pairs of gadgets in order to form a path going through all the gadgets corresponding to edges incident on u in G.

eg:

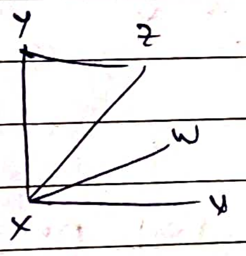


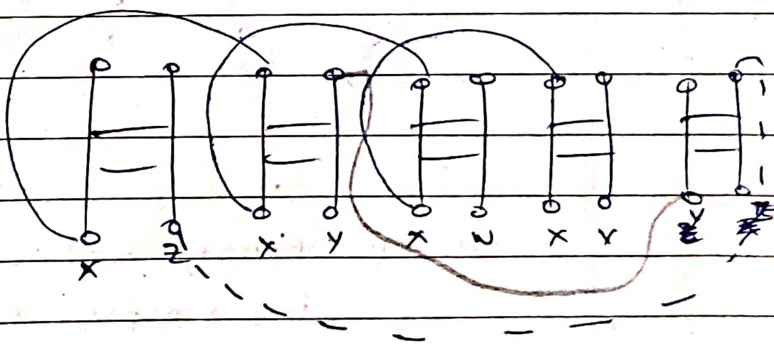




This represents a gadget for edge xy_1

eg:





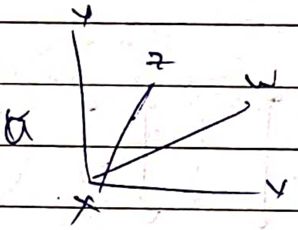
Tips:
- Connection is always Top & Bottom or Bottom & Top.
- For every vertex having more than 1 edge such connection is done

2. Final set of edges in G' join the first vertex [x, y, 1] and last vertex [x, y(deg(x)), 6] of each of these paths to each of the selector vertex.

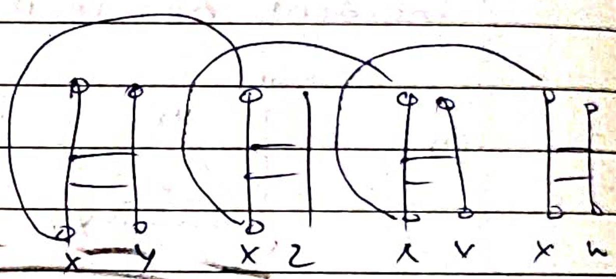
Here only 1 selector vertex as vertex cover k=1 for graph G

Eg(1)

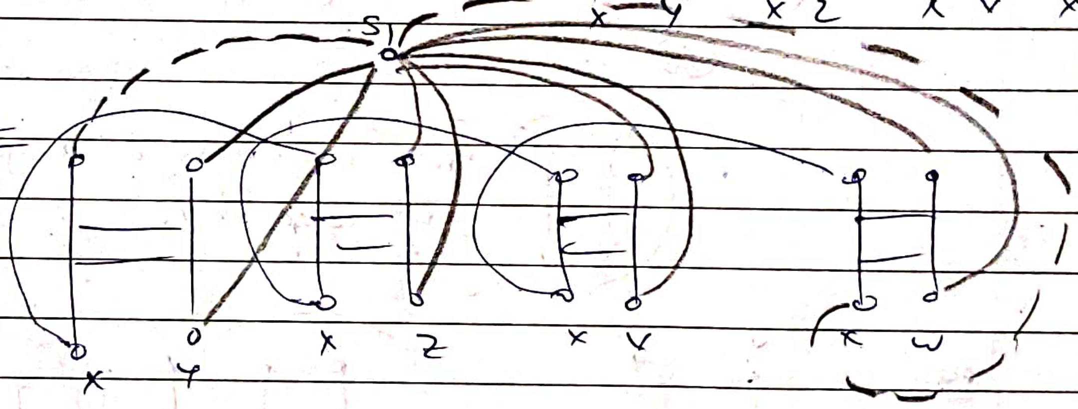
G



Step1



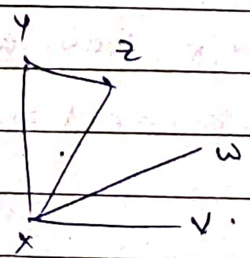
Step2



Pro Tip: Every node that has no connectivity is simply connected to the selector vertex

Eg(2):

G



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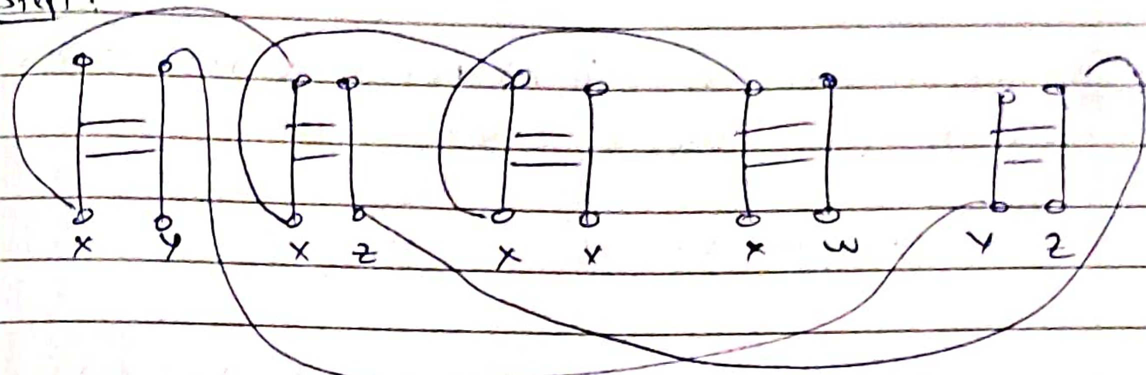
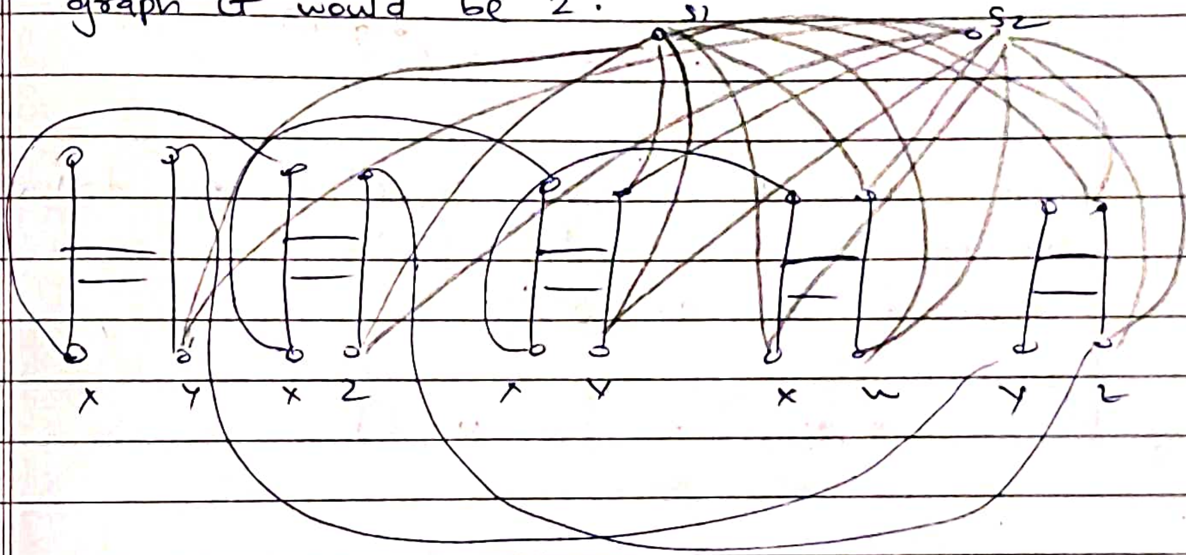
Step1 :



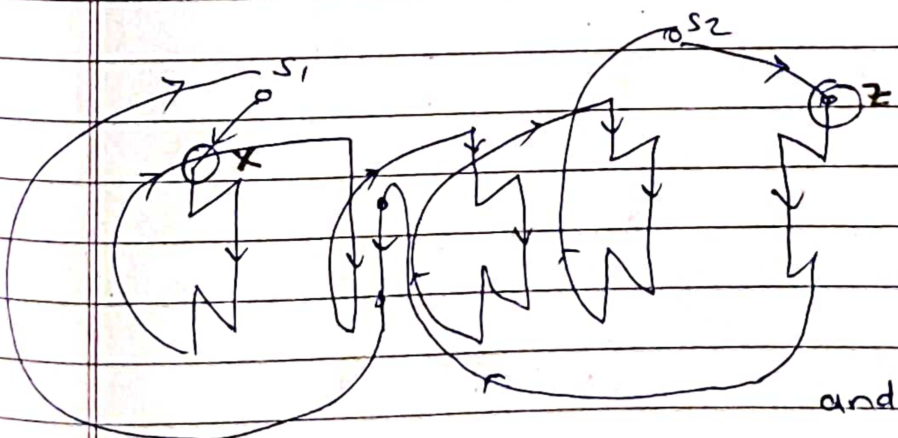
Step2 : Here $K=2$ ~~are~~ as vertex cover in graph G would be 2.

G'



S_1 and S_2 have same connectivity

is there a Hamiltonian cycle in G' ? (If yes, then we will get vertex cover in G)



Yes there is Hamiltonian cycle and

and $\{x, z\}$ is vertex cover of graph G.

S_1 selected x to be in vertex cover
S_2 selected z to be in vertex cover

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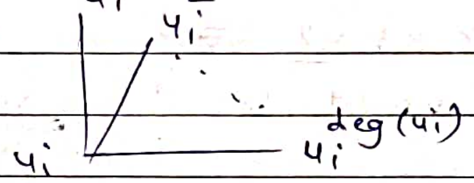
Step 4: Proof

(A) Suppose that $G = (V, E)$ has a vertex cover of size $\underline{\underline{k}}$, let vertex cover set be

$S = \{u_1, u_2 \dots u_k\}$

we will identify neighbors of u_i as shown here



Form a Hamiltonian cycle in G' by ~~the~~ following the nodes in G in this order.

start at s_1 and go to

$s_1 \xrightarrow{\text{selects } u_1} [u_1, u_1^1, 1] \dots [u_1, u_1^1, 6]$

neigh of u_1

$[u_1, u_1^2, 1] \dots [u_1, u_1^2, 6]$

$\vdots$

$[u_1, u_1^{deg u_1}, 1] \dots [u_1, u_1^{deg u_1}, 6]$

all neighbours of u_1 visited

$s_2 \xrightarrow{\text{select } u_2} [u_2, u_2^1, 1] \dots [u_2, u_2^1, 6]$

$[u_2, u_2^{deg u_2}, 1] \dots [u_2, u_2^{deg u_2}, 6]$

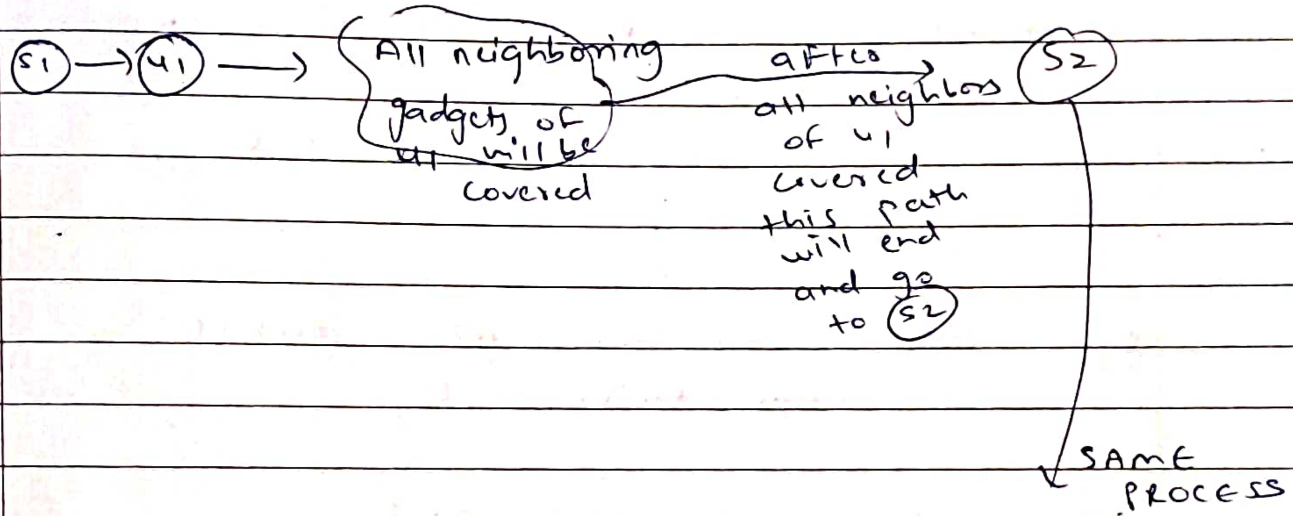
s_3 $\dots$

$[u_k, u_k^{deg u_k}, 6]$

Explanation:

as u_1 is in vertex cover set in G

In G' the selected vertex (s_1) will select u_1 to be in vertex cover set.



(B) Suppose G' has a Hamiltonian cycle C, then the set

$$S = \{ u_j \in V : (s_j , [u_j , u_j' , 1]) \in C \quad \text{For some } 1 \le j \le k \}$$

will be a vertex cover set in G

∴ Proved Hamiltonian cycle is NP-complete.

<Q> Now prove TSP is NP-complete

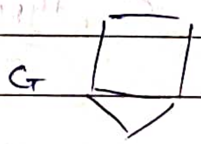
Step1: Prove TSP in NP

a. certificate: tour of cost no more than C

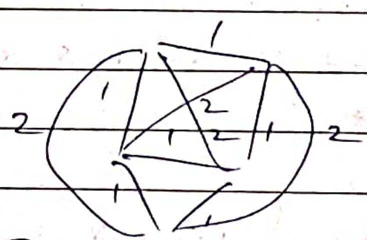
b. certifier: Everything in Hamiltonian cycle
+
check total cost $\leq$ C

Step2: choose an NP complete problem
(Hamiltonian cycle)

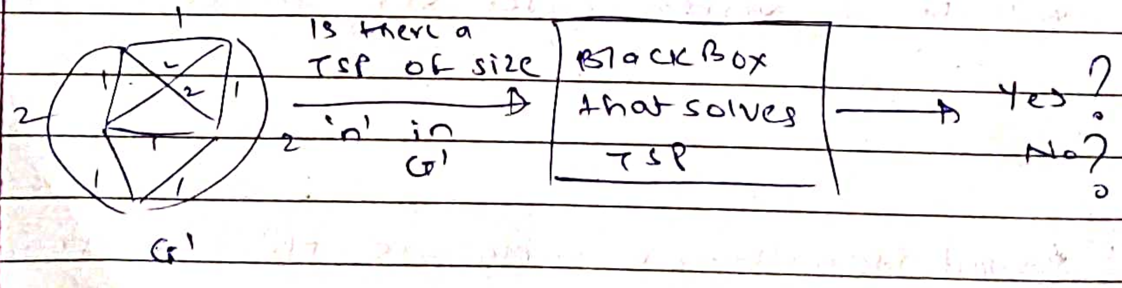
Step3: Prove HC $\leq_p$ TSP



Instance of
HC



G'
- weighted
- directed
- fully connected



There is a tour in G' of size n if and only if there is Hamiltonian cycle in G.

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Step 4 : Proof of reduction

(A) If there is a HC in G then there is TSP in G' of cost 'n'

If there is HC in G
and in G' we put all that edges with cost = 1

∴ There will be a TSP of size 'n' in G'

(B) If there is a TSP of size n in G' then there is HC in G

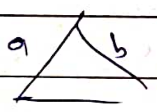
If TSP cost = n and there are 'n' edges selected
∴ cost of each edge = 1

∴ all cost 1 edges selected
∴ HC in G.

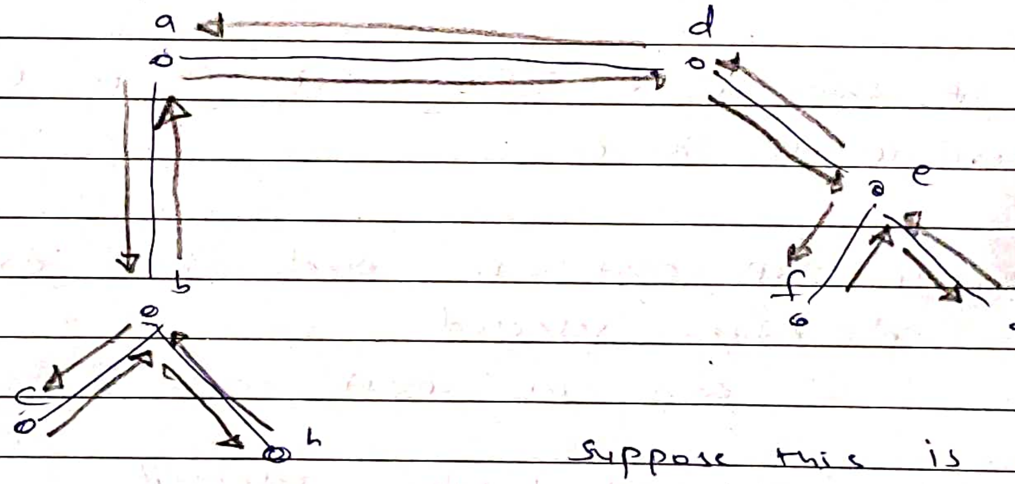
List of NP-complete problems-

* 3-SAT, vertex cover, indp set, set cover, Hamiltonian cycle, TSP
0/1 knapsack, subset sum, graph 3-coloring.

(*) Travelling salesman problem (with △ inequalities)



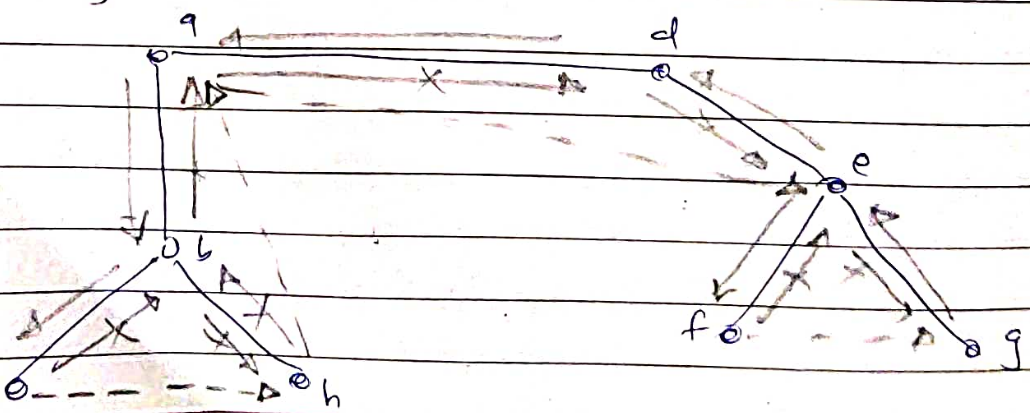
$(a+b) > c$



Suppose this is MST

initial tour cost = 2 * cost of MST

Applying △ inequalities



$(cb) + (bh) > (ch)$

My approx cost now

cost $\leq$ 2 * cost of MST

(cost of optimal Tour) $\geq$ cost of MST (As in optimal soln each path will be taken atleast once)

$\therefore$ cost of our approx soln $\leq$ 2 * cost of optimal solution.

This is a 2-approximation $\rightarrow$ called "2-approximation" as we are in range of factor of 2 to the optimal soln.

Read ppt

Note: As P $\neq$ NP, then for any constant $\rho \geq 1$, there is no polynomial time approximation algorithm with approximation ratio ρ for the general TSP.

explanation: Since NP problems cannot be solved in polynomial Time.
we develop algorithms to give approximate solutions to these problems
But we can never say that approximate algo will give answer in (k^{th} factor) of optimal answer.

If we say so then basically we have proved P=NP.