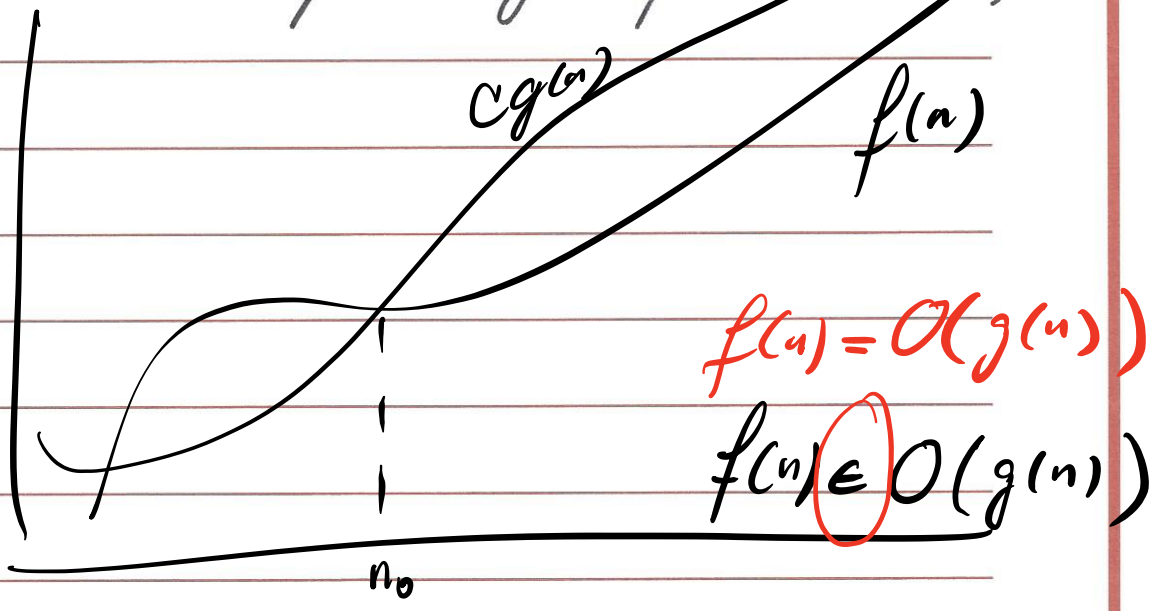
# Review of the Asymptotic Notations

Formally, $O(g(n)) = \{ f(n) \mid$ there exist positive constants $C$ and $n_0$ such that $0 \leq f(n) \leq C\,g(n)$ for all $n \geq n_0 \}$




$f(n) = O(g(n))$

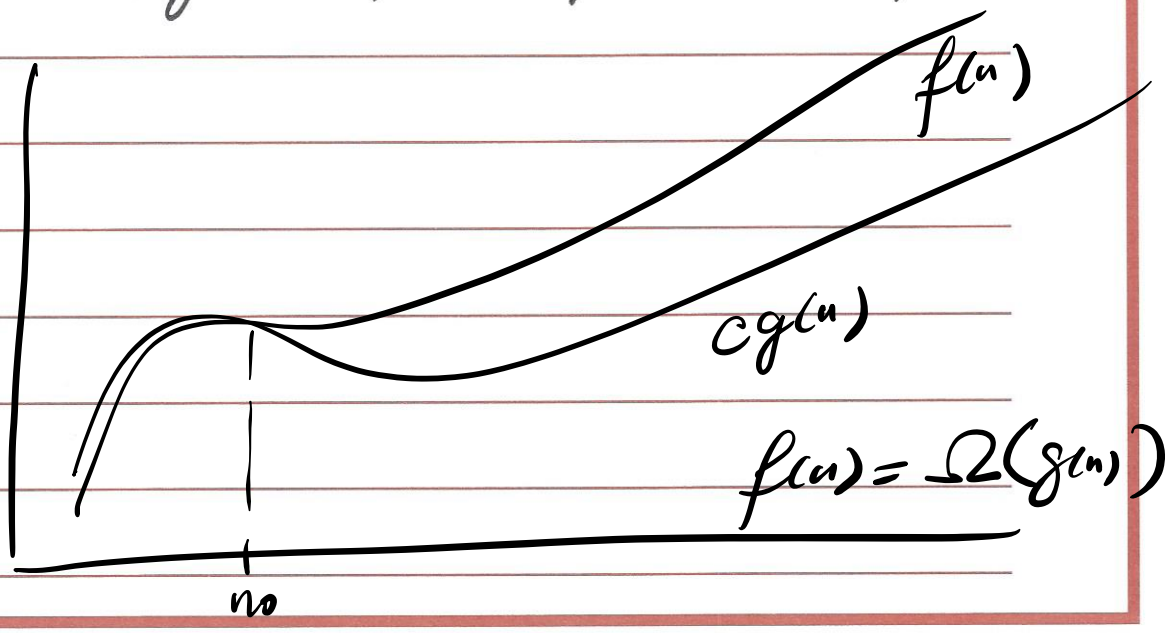
$f(n) \in O(g(n))$

T Any quadratic function is $O(n^2)$

T Any linear " " "

F Any cubic " "

$\Omega(g(n)) = \{ f(n) \mid$ there exist positive constants $c$ and $n_0$ such that $0 \leq cg(n) \leq f(n)$ for $n \geq n_0 \}$

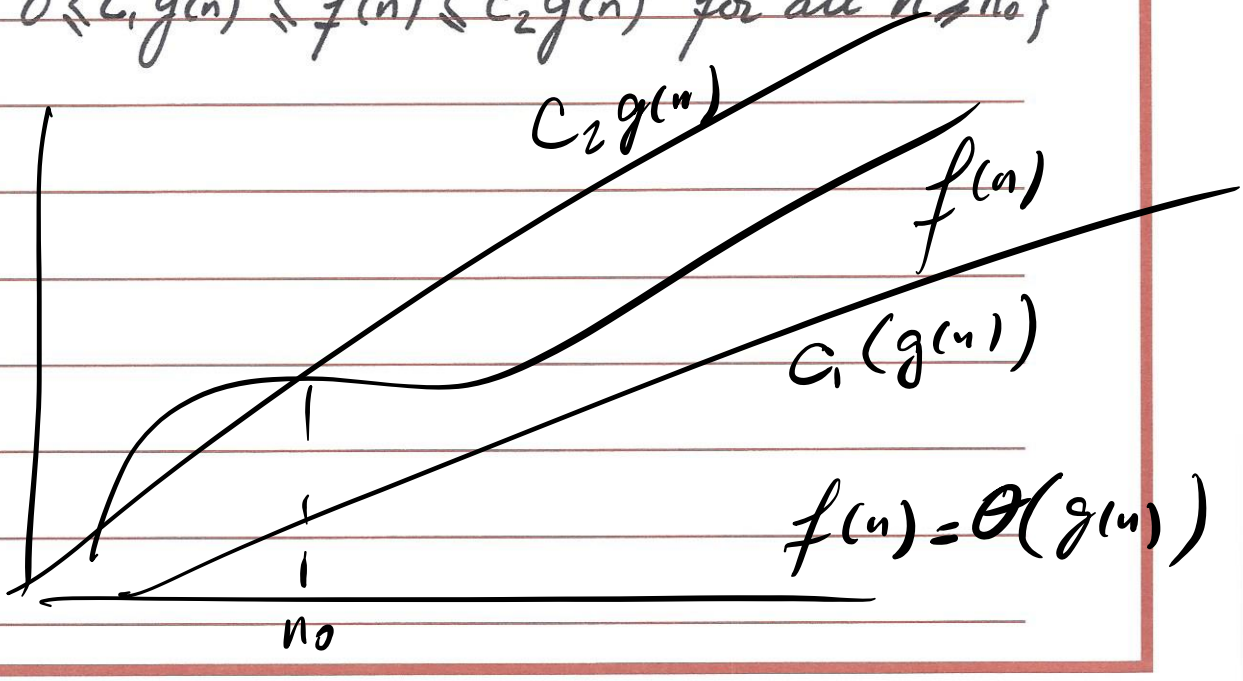



T Any quadratic func. is $\Omega(n^2)$

F Any linear " "

T Any cubic " "

$\Theta(g(n)) = \{ f(n) \mid$ there exist positive constants $c_1, c_2$, and $n_0$ such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0 \}$




T Any quadratic func. is $\Theta(n^2)$

F Any linear " " "

F Any Cubic " "

| | Worst Case | Best Case |
|---|---|---|
| Linear Search | $O(n), \Theta(n), \Omega(n)$ | $O(1), \Theta(1), \Omega(1)$ |
| Binary " | $O(\lg n), \Theta(\lg n), \Omega(\lg n)$ | $O(1), \Theta(1), \Omega(1)$ |
| Insertion Sort | $O(n^2), \Theta(n^2), \Omega(n^2)$ | $O(n), \Theta(n), \Omega(n)$ |
| Merge Sort | $O(n \lg n), \Theta(n \lg n), \Omega(n \lg n)$ | $O(n \lg n), \Theta(n \lg n), \Omega(n \lg n)$ |
| ⋮ | | |

Worst case performances:

Algorithm A: $\Theta(4^n n^3 \lg n)$

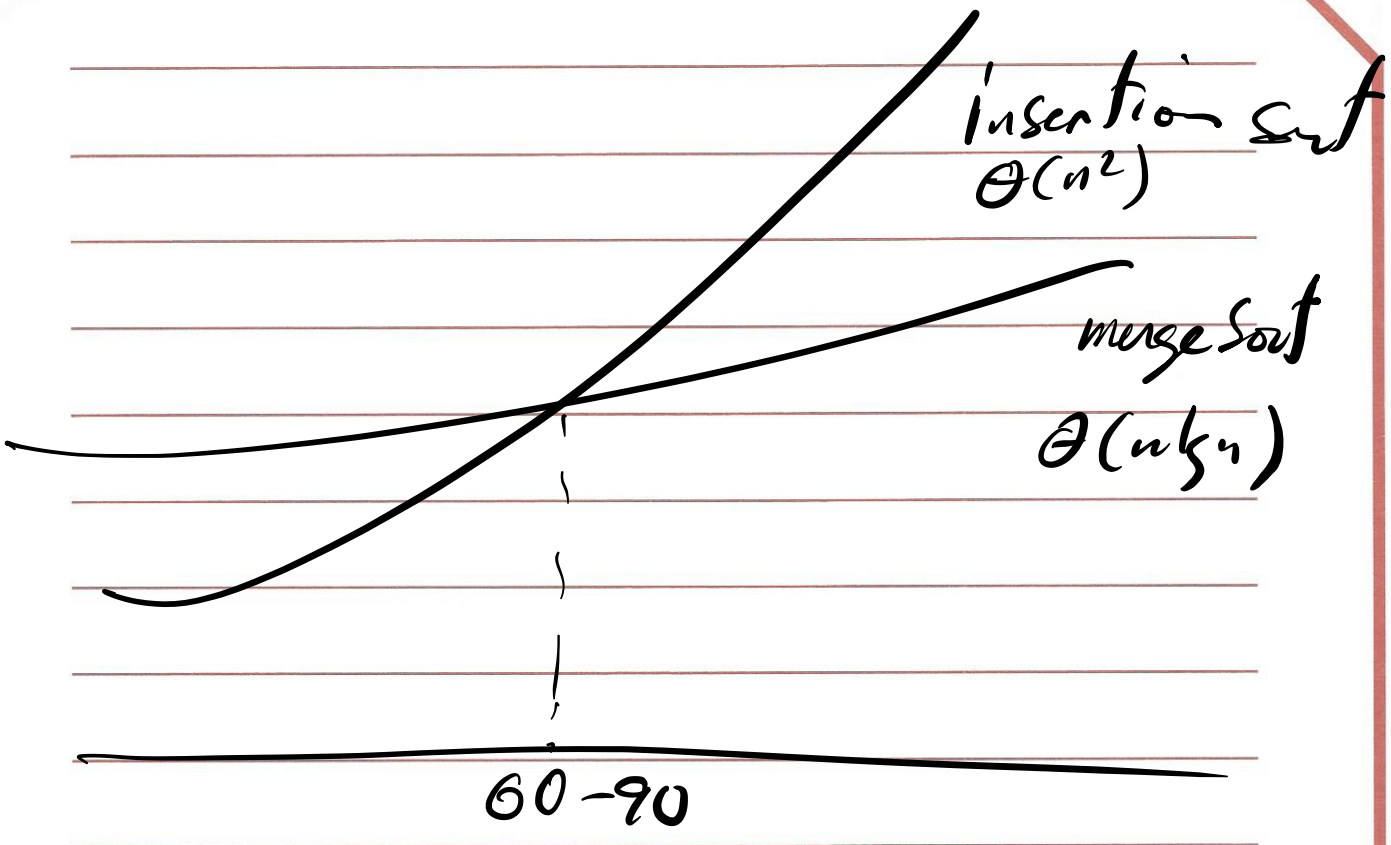
Algorithm B: $\Theta(3^n n^8 (\lg n)^2)$

- Exponential component
- Polynomial "
- Logarithmic "

fastest growing

4

slowest growing


insertion sort
$\Theta(n^2)$
merge sort
$\Theta(n \lg n)$
60-90

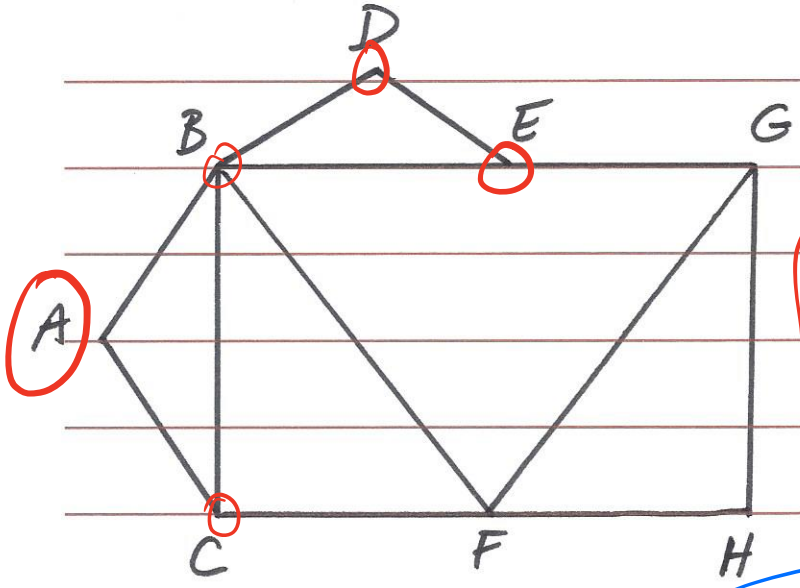
# Review of BFS & DFS

Q: What are we searching for?

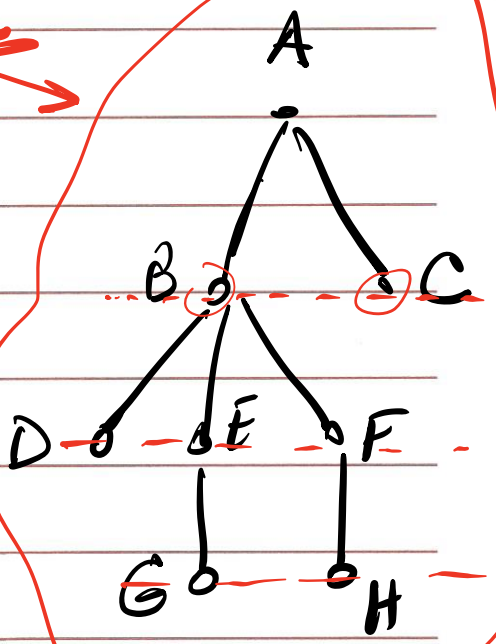
- Find out if there is a path from A to B.

- Find all nodes that can be reached from A.

## BFS




BFS Tree →

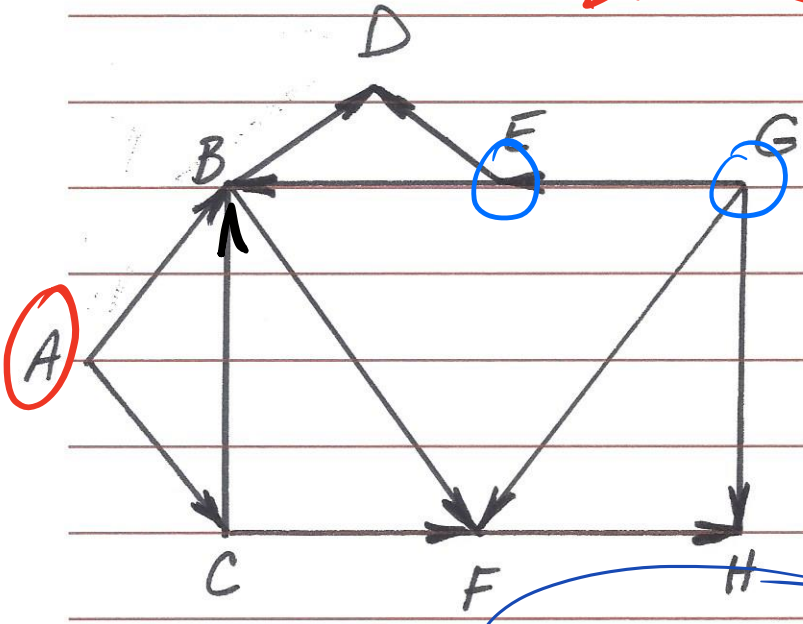



Takes $O(m+n)$

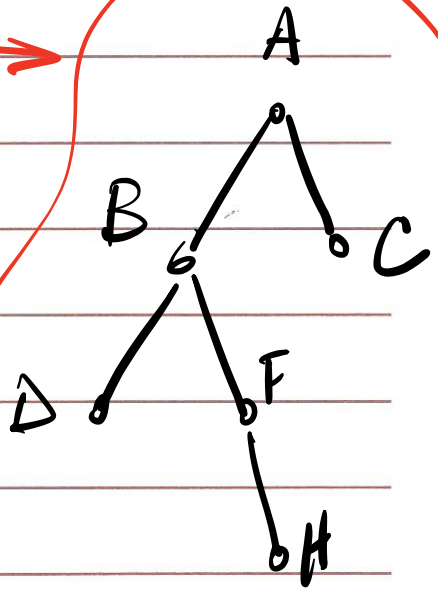
m: no. of edges

n: " " nodes

## DFS



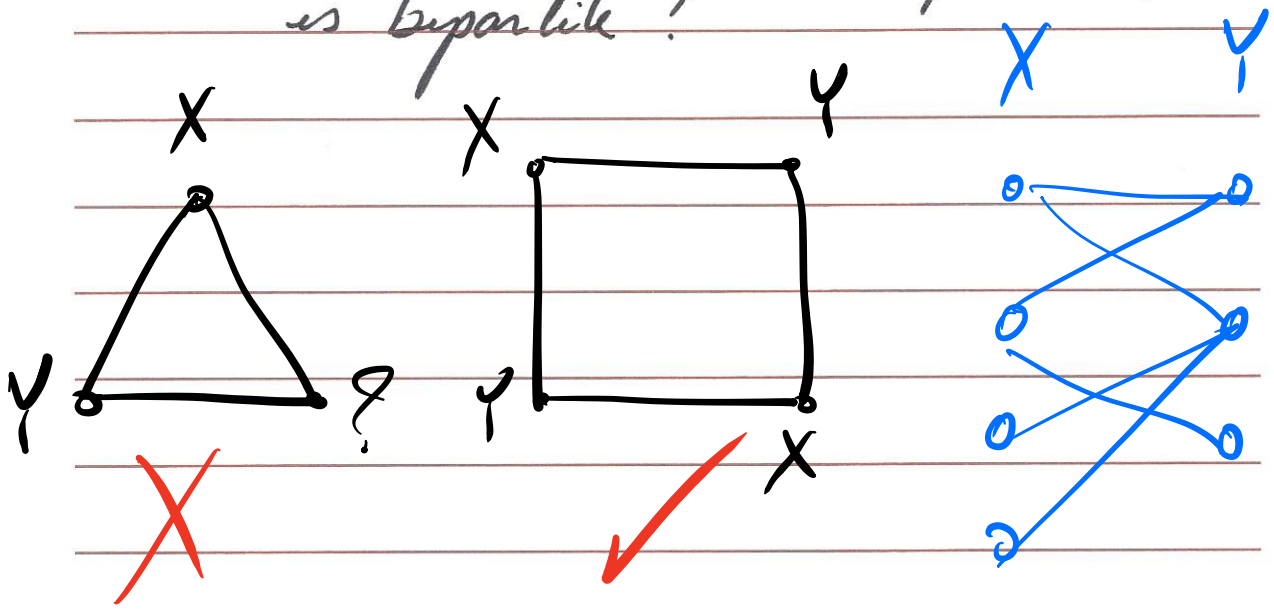

DFS Tree →

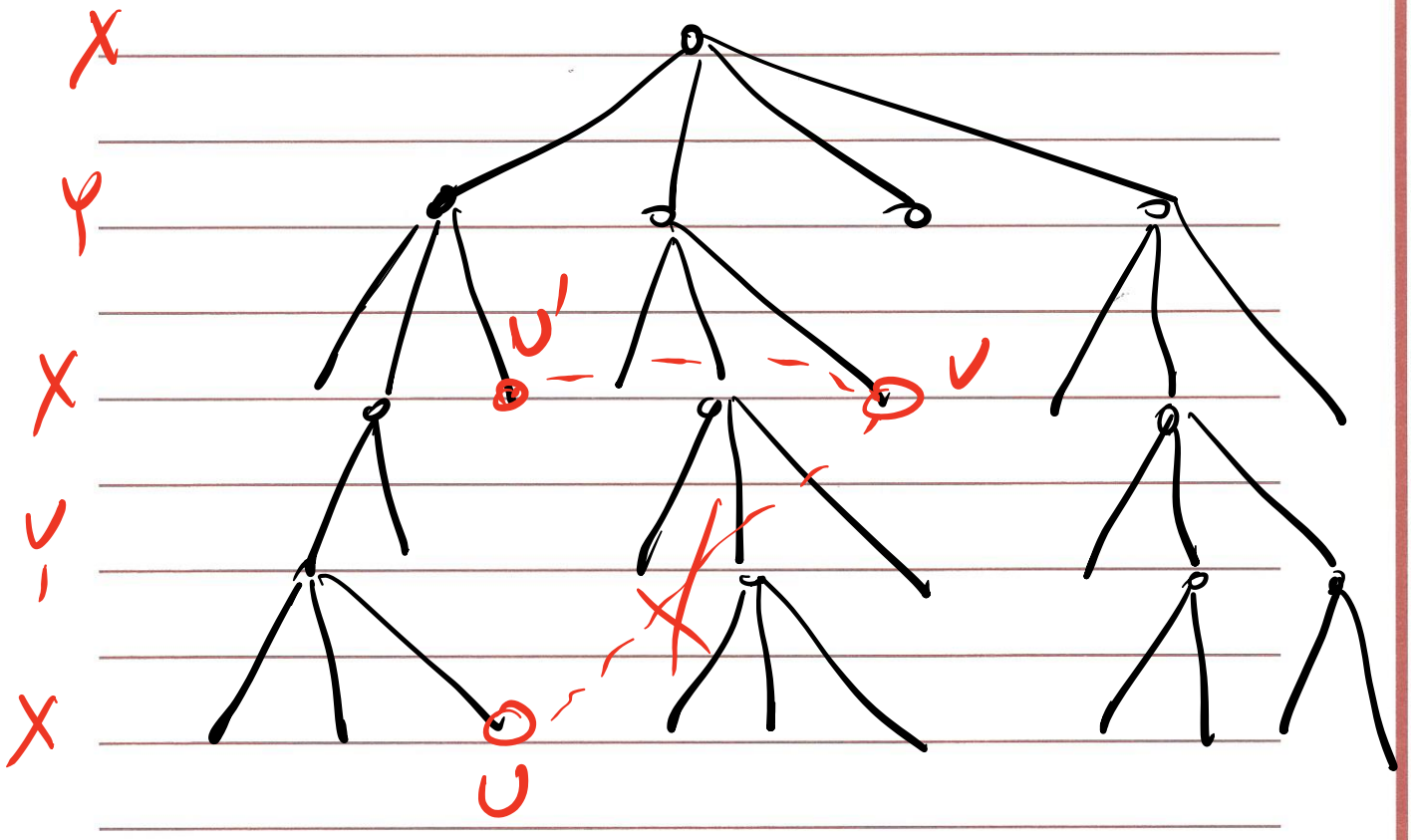


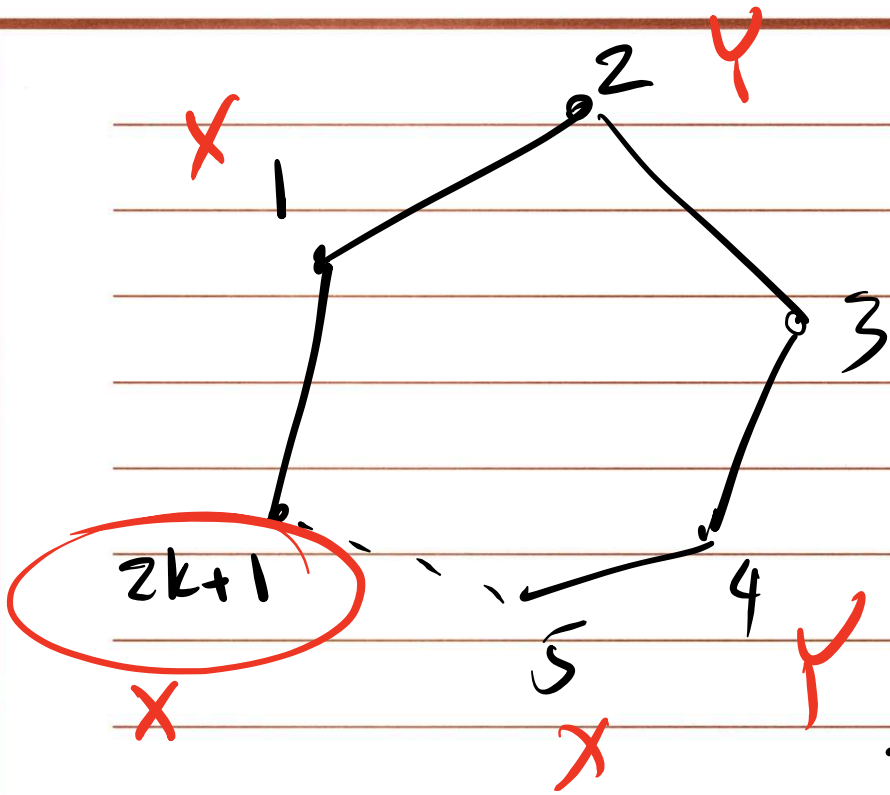

takes $O(m+n)$ $O(m \log n + n)$

$\Theta(m+n)$ ~~$\Theta(m \log n + n)$~~

Q: How do you determine if a graph is bipartite?

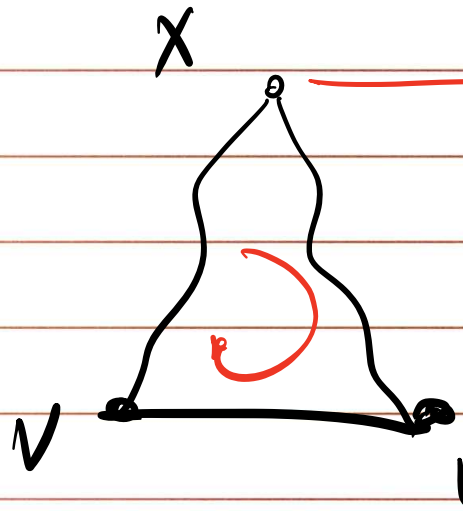





FACT:

If a graph G is bipartite it cannot have an odd cycle.




level j

length of cycle = 2*(j-i) + 1

odd!

Solution:

$O(m+n)$ Run BFS starting from any node, say $S$. Label each node Red or Blue depending on whether they appear at an odd or even level on the BFS tree.

$O(m)$ Then, go through all edges and examine the labels at the two ends of the edge. If all edges have a Red end and a Blue end, then the graph is bipartite.

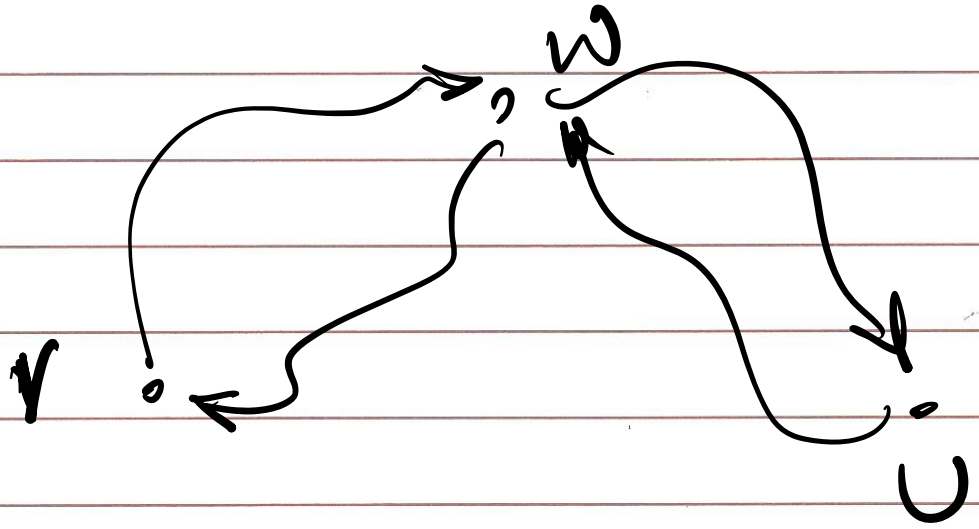
Otherwise, the graph is not bipartite.

overall complexity $= O(m+n)$

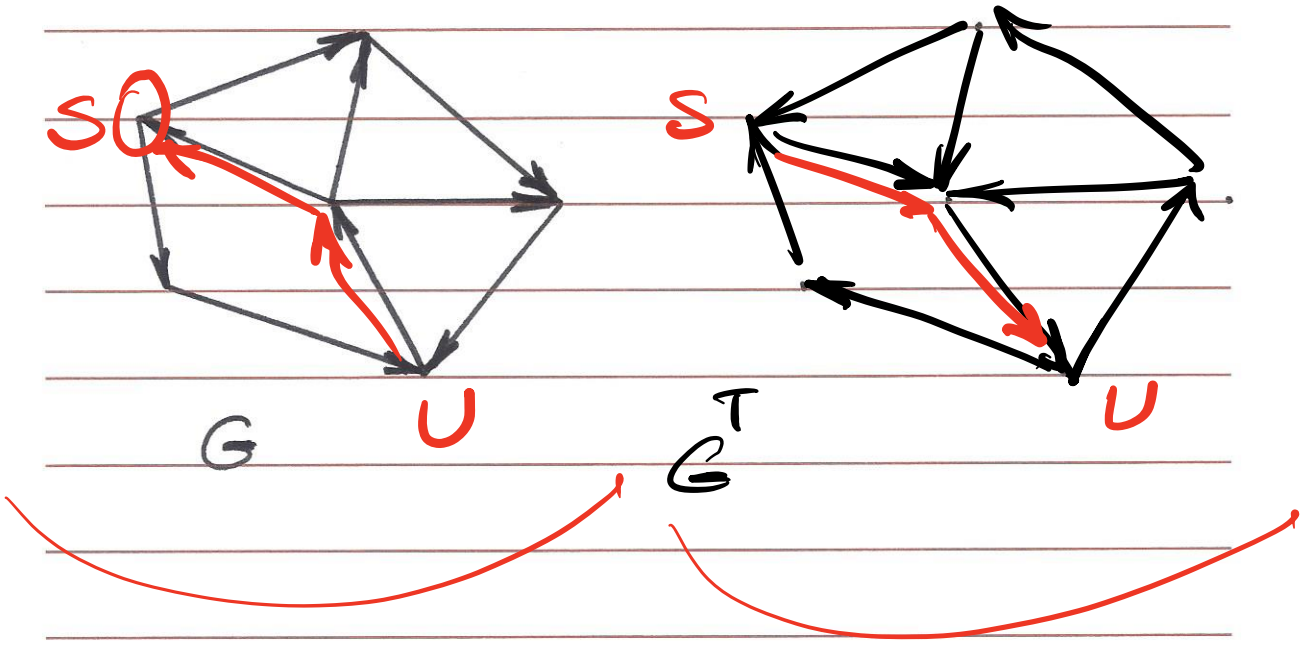
**Def.** A directed graph is strongly connected if there is a path from any point to any other point in the graph.

Q: How do you know if a given directed graph is strongly connected?

Brute Force Sol.: Run BFS/DFS $n$ times, once from each node $\rightarrow O(n^2+mn)$



# Transpose of a directed graph




# Mutually Reachable Nodes

Solution:

1. Use BFS or DFS to find all nodes reachable from s (an arbitrary node) in G. If some nodes are not reachable from s, stop. The graph is not strongly connected.

O(m+n)

Otherwise, continue with step 2.

2. Create $G^T$ (Transpose of G)

O(m+n)

3. Use BFS or DFS to find all nodes reachable from s in $G^T$. If some nodes are not reachable from s, then the graph is not strongly connected.

O(m+n)

Otherwise, the graph is strongly connected.

Overall Complexity = O(m+n)

# Discussion 2

**1.** Arrange the following functions in increasing order of growth rate with g(n) following f(n) in your list if and only if *f(n) = O(g(n))*

$$\log n^n,\ n^2,\ n^{\log n},\ n \log \log n,\ 2^{\log n},\ \log^2 n,\ n^{\sqrt{2}}$$

**2.** Suppose that *f(n)* and *g(n)* are two positive non-decreasing functions such that *f(n) = O(g(n)).* Is it true that $2^{f(n)} = O(2^{g(n)})$?

**3.** Find an upper bound (Big O) on the worst case run time of the following code segment.

```
void bigOh1(int[] L, int n)
   while (n > 0)
      find_max(L, n); //finds the max in L[0…n-1]
      n = n/4;
```

Carefully examine to see if this is a tight upper bound (Big θ)

**4.** Find a lower bound (Big Ω) on the best case run time of the following code segment.

```
string bigOh2(int n)
   if(n == 0) return "a";
   string str = bigOh2(n-1);
   return str + str;
```

Carefully examine to see if this is a tight lower bound (Big θ)

**5.** What Mathematicians often keep track of a statistic called their Erdős Number, after the great 20th century mathematician. Paul Erdős himself has a number of zero. Anyone who wrote a mathematical paper with him has a number of one, anyone who wrote a paper with someone who wrote a paper with him has a number of two, and so forth and so on. Supposing that we have a database of all mathematical papers ever written along with their authors:

a. Explain how to represent this data as a graph.
b. Explain how we would compute the Erdős number for a particular researcher.
c. Explain how we would determine all researchers with Erdős number at most two.

**6.** In class, we discussed finding the shortest path between two vertices in a graph. Suppose instead we are interested in finding the *longest* simple path in a directed acyclic graph. In particular, I am interested in finding a path (if there is one) that visits all vertices.
Given a DAG, give a linear-time algorithm to determine if there is a simple path that visits all vertices.

**1.** Arrange the following functions in increasing order of growth rate with g(n) following f(n) in your list if and only if *f(n) = O(g(n))*

$\log n^n$, $n^2$, $n^{\log n}$, $n \log \log n$, $2^{\log n}$, $\log^2 n$, $n^{\sqrt{2}}$

$2^{\log n} = n^1$

$\log n^n = n \log n$

$\log^2 n,\ n^1,\ n \log\log n,\ n \log n,\ n^{\sqrt{2}},\ n^2,\ n^{\log n}$

$n^2 > n \log n$

$n^{1.001} > n \log n$

**2.** Suppose that *f(n)* and *g(n)* are two positive non-decreasing functions such that *f(n)* = *O(g(n))*. Is it true that $2^{f(n)} = O(2^{g(n)})$?

$f(n) = 2n \qquad g(n) = n$

$2^{2n} \neq O(2^n)$

**3.** Find an upper bound (Big O) on the worst case run time of the following code segment.

```
void bigOh1(int[] L, int n)
  while (n > 0)
    find_max(L, n); //finds the max in L[0…n-1]
    n = n/4;
```

O(n) ← Find_Max runs in linear time WRT n

$\log_4 n$ (

n = 3/4 → 0

Carefully examine to see if this is a tight upper bound (Big θ)

Runs in $O(n \log n)$ ✓

more careful examination:

$cn + cn/4 + cn/16 + \cdots$

$< 2cn = O(n)$

$n + n/2 + n/4 + n/8 + \cdots$

$2n$

**4.** Find a lower bound (Big Ω) on the best case run time of the following code segment.

```
string bigOh2(int n)
   if(n == 0) return "a";
   string str = bigOh2(n-1);
   return str + str;
```

Carefully examine to see if this is a tight lower bound (Big θ)

$\Omega(1)$ ✓ $\Omega(n)$ ✓

| n | output |
|---|---|
| 0 | a |
| 1 | aa |
| 2 | aaaa |
| ⋮ | |
| n | aa...aa ($2^n$) |

$\Theta(2^n)$

Carefully examine to see if this is a tight lower bound (Big θ)

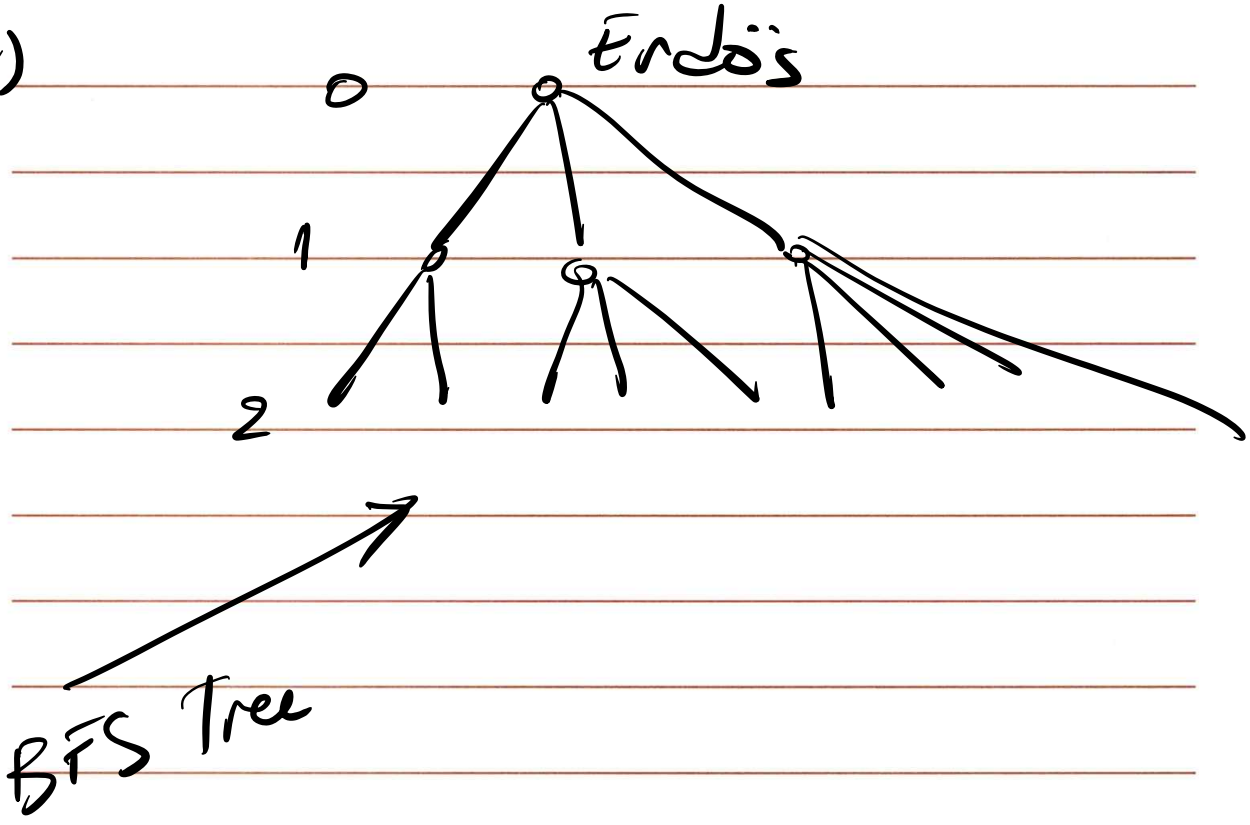
**5.** What Mathematicians often keep track of a statistic called their Erdős Number, after the great 20th century mathematician. Paul Erdős himself has a number of zero. Anyone who wrote a mathematical paper with him has a number of one, anyone who wrote a paper with someone who wrote a paper with him has a number of two, and so forth and so on. Supposing that we have a database of all mathematical papers ever written along with their authors:

a. Explain how to represent this data as a graph.
b. Explain how we would compute the Erdős number for a particular researcher.
c. Explain how we would determine all researchers with Erdős number at most two.

a) use an undirected graph where
- nodes represent Mathematicians
- edges " co-authorship

b) Run BFS from Erdös (or the math') and find the other node

c)



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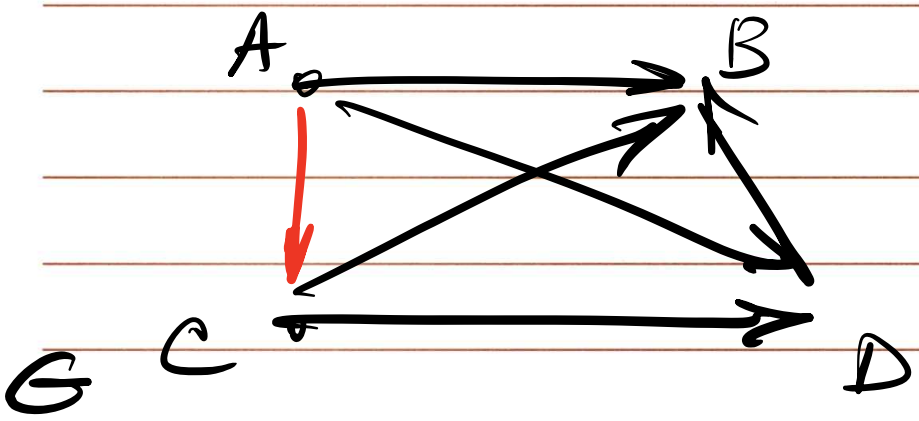
Given a DAG, give a linear-time algorithm to determine if there is a simple path that visits all vertices.

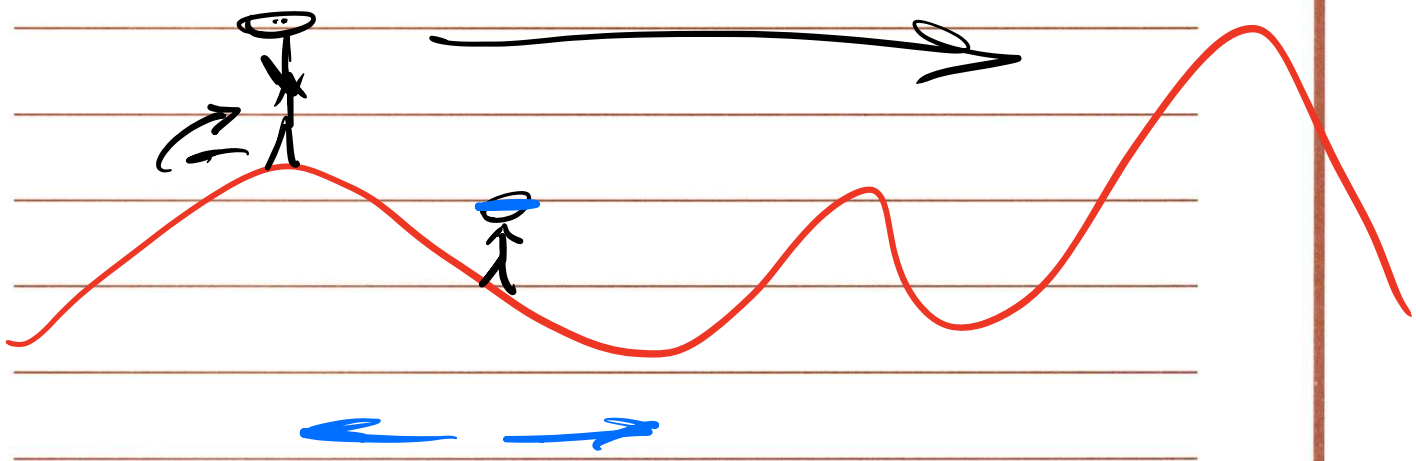



topological orderings of G:

A C D B

~~C A D B~~

takes O(m+n)

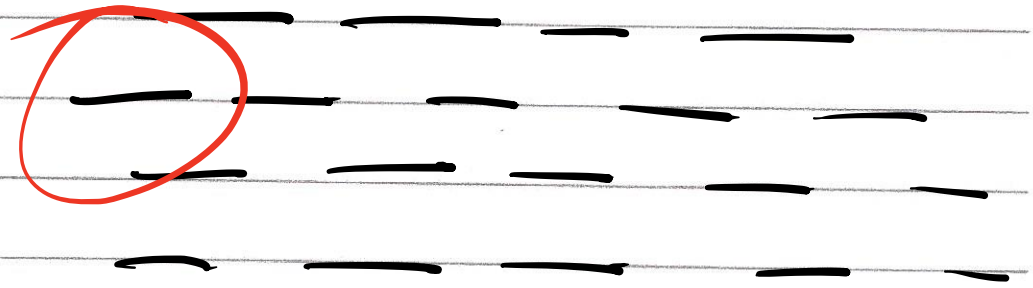


# Interval scheduling Problem

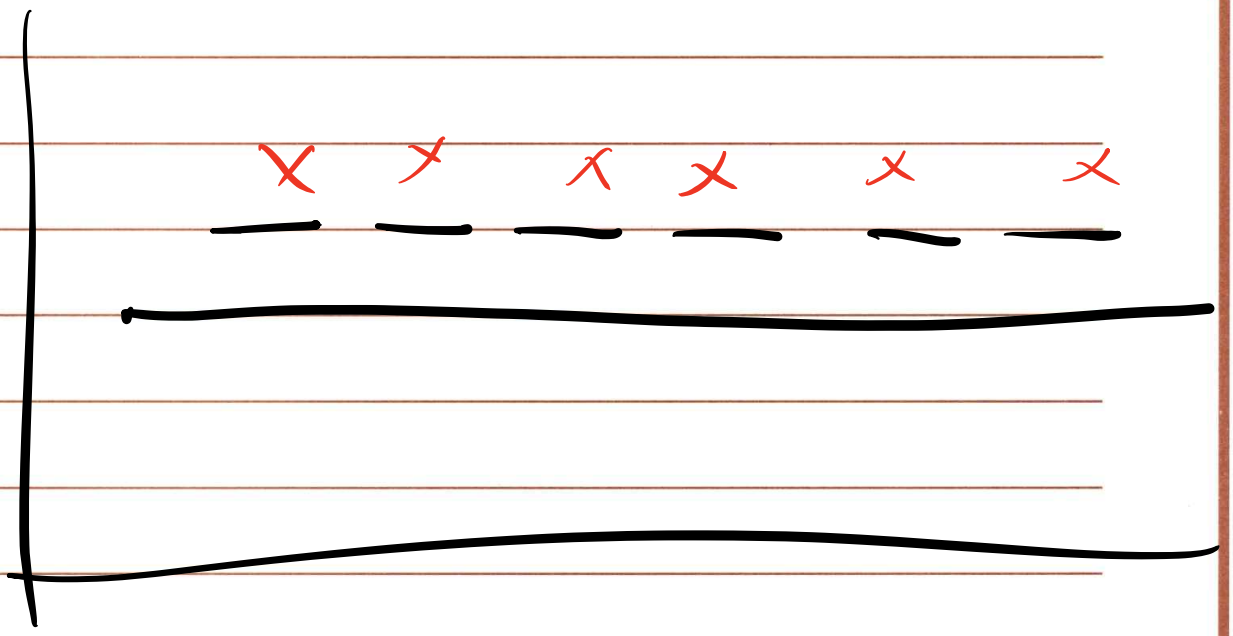
Input: Set of requests {1..n}

$i^{th}$ request starts at s(i) and ends at f(i)

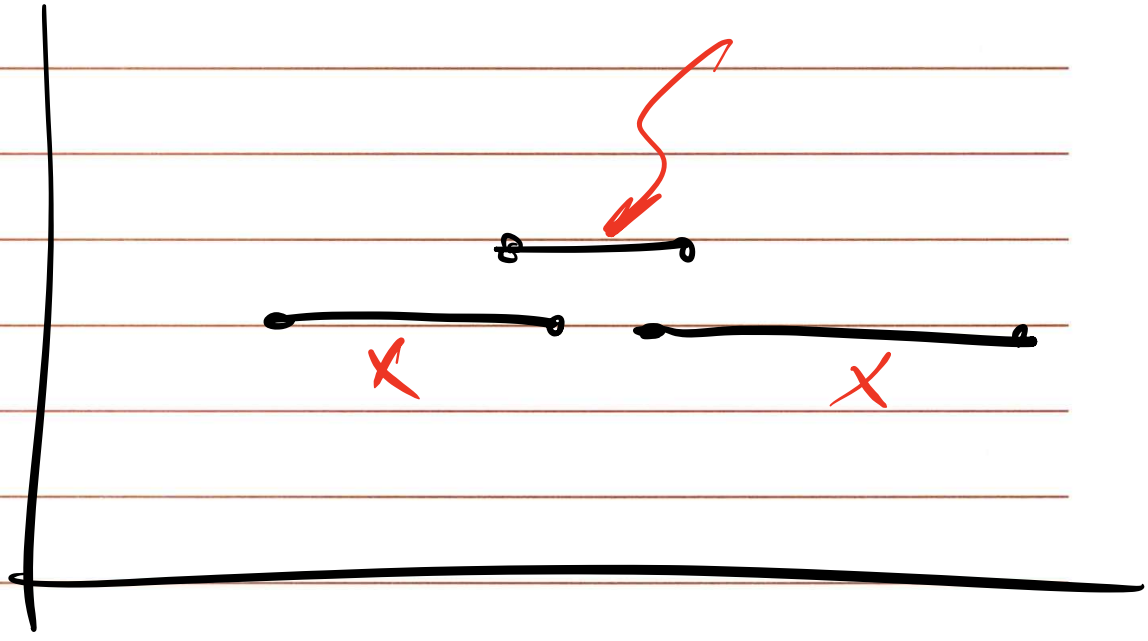
Objective: To find the largest compatible subset of these requests



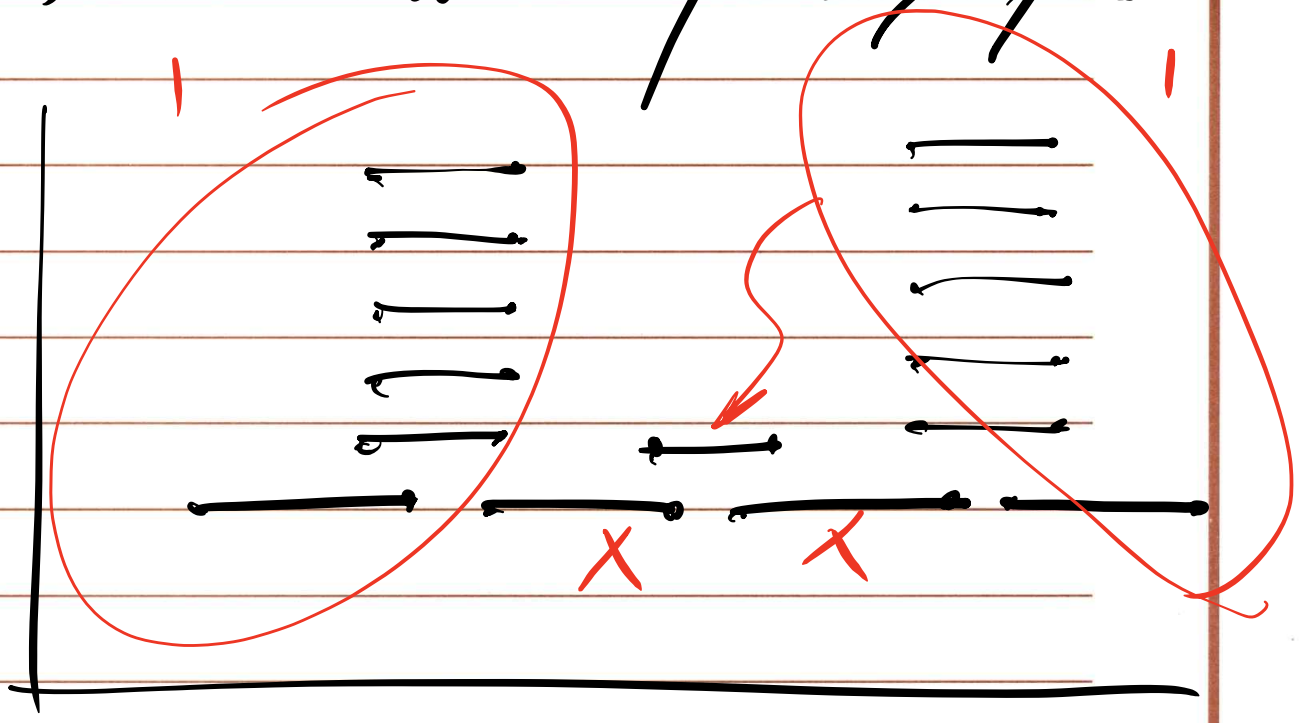
try #1 Earliest start time first.



try #2 Smallest requests first



try #3 Smallest no. of overlaps first.




try #4 Smallest finish time first.

Solution:

Initially R is the complete set of requests
& A is empty

While R is not empty

Choose a request $i \in R$ that has the smallest finish time

Add request i to A

Delete all requests from R that are not compatible w/ i

endwhile

Return A

# Proof of Correctness

(1) Show that A is a compatible set

(2) Show that A is an optimal set

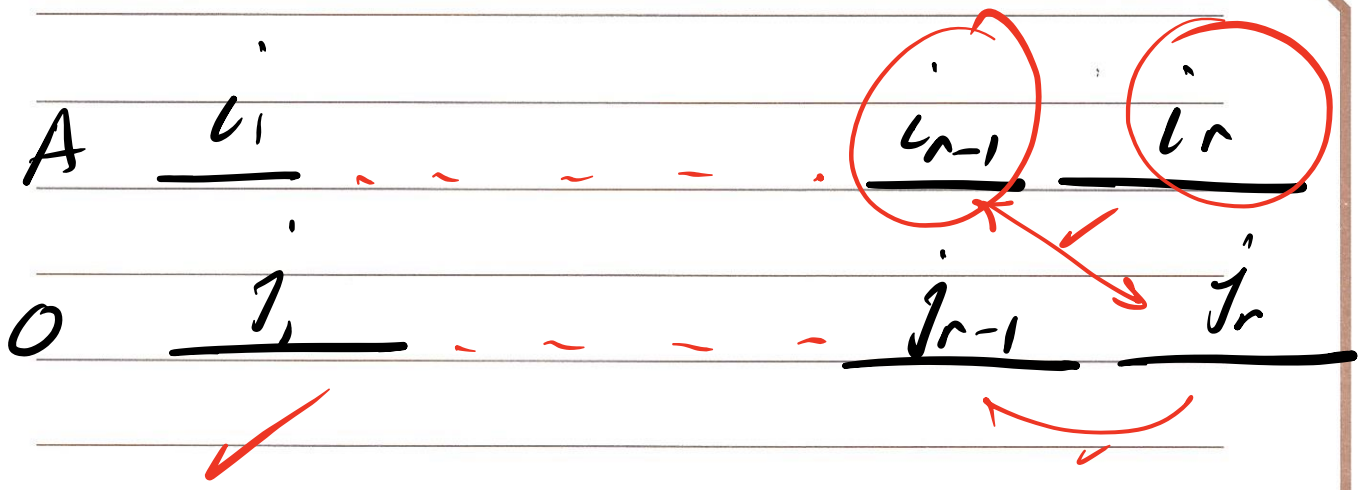
Say A is of size k

Say there is an opt. solution O

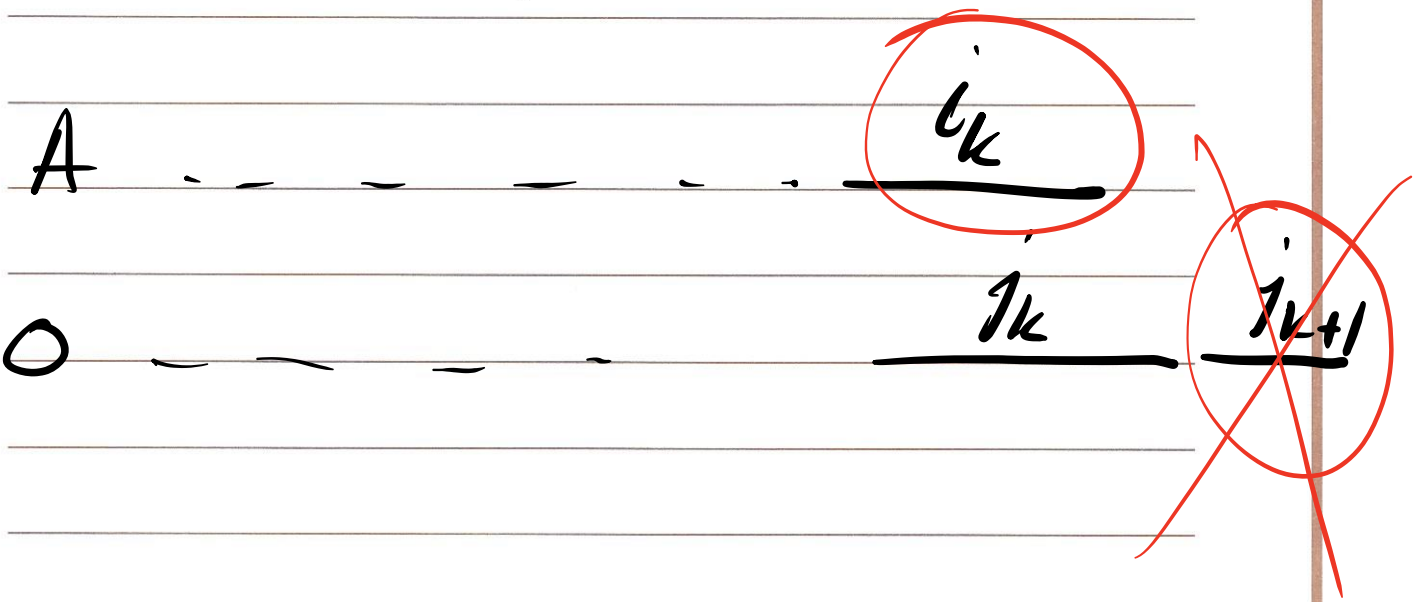
Requests in A: $i_1, \ldots, i_k$

" " O: $j_1, \ldots, j_m$

We will first prove that for all indices $r \leq k$, we have $f(i_r) \leq f(j_r)$




We can then easily prove that $|A| = |O|$

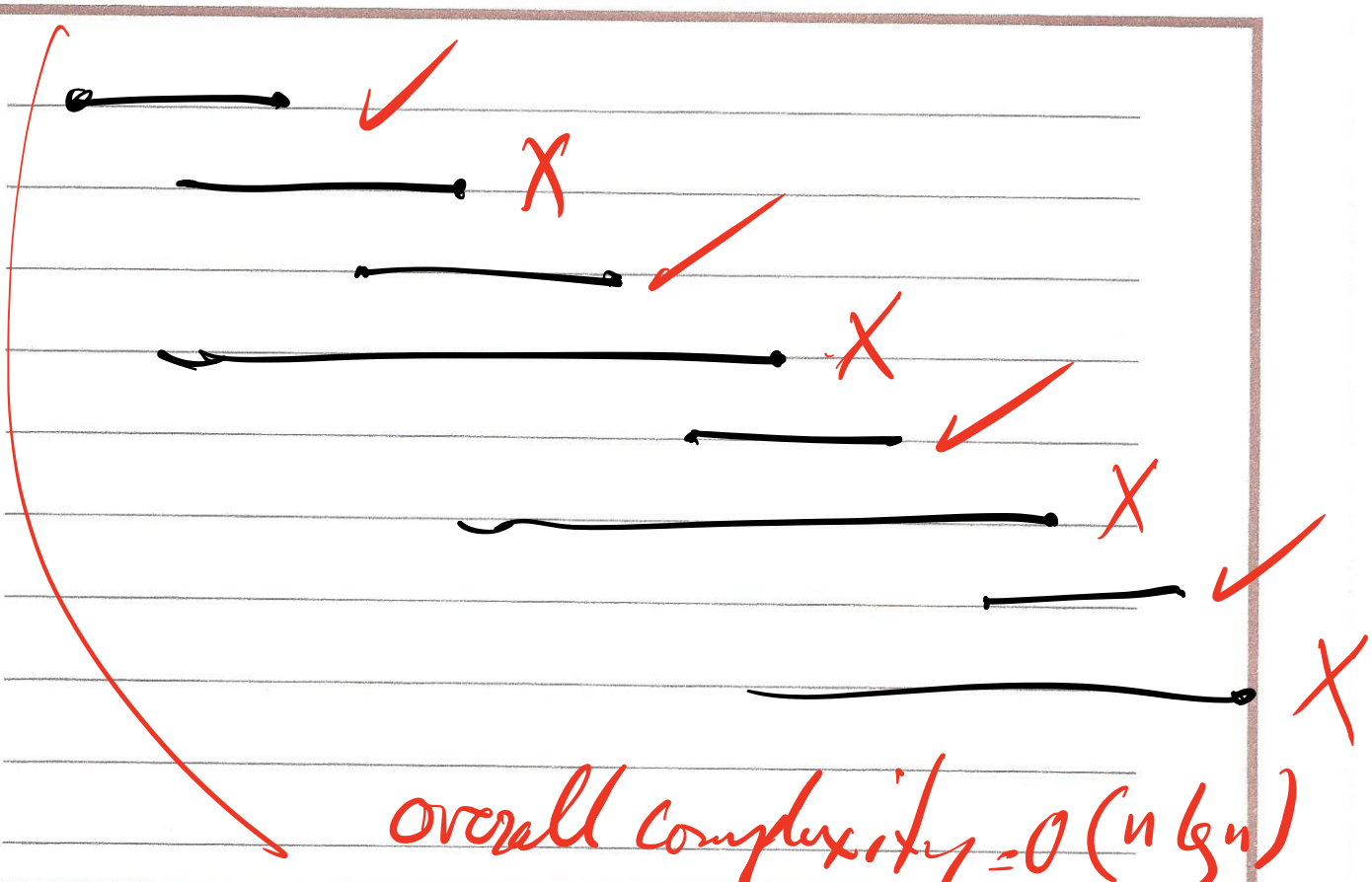


## Implementation

- Sort requests in order of finish time and label in this order: $O(n \log n)$

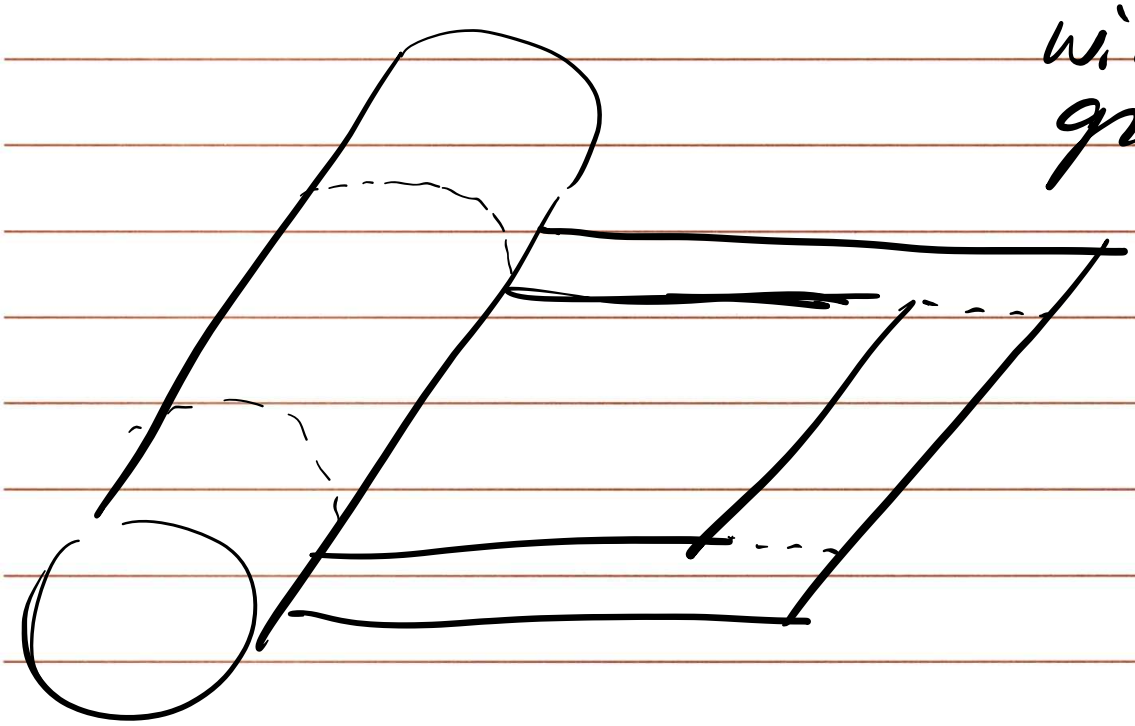
  $f(i) \leq f(j)$ where $i < j$

- Select requests in order of increasing $f(i)$, always selecting the first. Then iterate through the intervals in this order until reaching the first interval for which $s(j) \geq f(i)$ $O(n)$

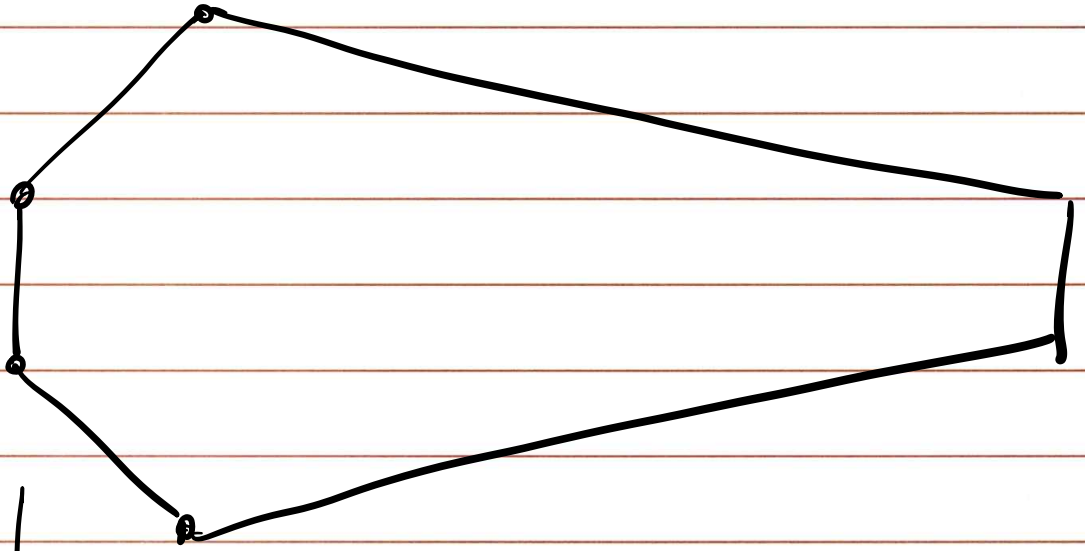


overall complexity $= O(n \log n)$

Orders: Qty
width
grade



Coffin Sched.



warm up.

# Fractional Knapsack

Knapsack has a weight capacity of $W$.

We are given as input, a set of $n$ objects with weight $w_i$ and value $v_i$.

Objective: Fill up the knapsack to its weight capacity such that the value of items in knapsack is maximized.

Ex. knapsack weight cap: 10

| items | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Values | 10 | 20 | 15 | 2 | 8 |
| weights | 4 | 10 | 5 | 1 | 2 |

Come up with a greedy solution to this problem on your own and solve the above example numerically. Then try to prove that your solution is optimal.