CSCI 570 - Fall 2021 - HW 11

Graded Problems

Problem 1

[**15 points**] Given an undirected graph with positive edge weights, the BIG-HAM- CYCLE problem is to decide if it contains a Hamiltonian cycle C such that the sum of weights of edges in C is at least half of the total sum of weights of edges in the graph. Show that finding BIG-HAM-CYCLE in a graph is NP-Complete.

Solution: The certifier takes an undirected graph (the BHC instance) and a sequence of edges as input. It verifies that the sequence of edges form a Hamiltonian cycle and that the total weight of the cycle is at least half of the total weight of the edges in the graph. Thus BIG-HAM-CYCLE is in NP.
We claim that Hamiltonian Cycle is polynomial time reducible to BIG-HAM-CYCLE. To see this, given an instance of problem Hamiltonian cycle in an undirected graph $G = (V,\ E)$, pick an edge e and set its weight to $|E|$ and assign the rest of the edges weight of 1. When this weighted graph is fed into the BIG-HAM-CYCLE decider blackbox, it returns "yes" if and only if G has a Hamiltonian cycle containing the edge e. By repeating the above once for every edge e in the graph G, we can decide if the graph has a Hamiltonian cycle.

Rubric:

- 5 points for proving the problem is in NP.
- 3 points for reducing from Hamiltonian Cycle.
- 7 points for running BIG-HAM-CYCLE on new weighted graph.

Problem 2

[**15 points**] Show that vertex cover remains NP-Complete even if the instances are restricted to graphs with only even degree vertices.

Solution: The certifier takes an undirected graph (the VC instance) and a set of vertices as input. It verifies the union of all edges belong to the each vertex cover all the original graph's edges. Thus problem is in NP.(the same certifier to the original vertex cover problem)

We claim that vertex cover with only even degree vertices remains NP-Complete by showing it is polynomial time reducible to the original vertex cover problem. Let $(G = (V, E), k)$ to be an input instance of Vertex Cover. Because each edge in E contributes a count of 1 to the degree of each of the vertices with which it connects, the sum of the degrees of the vertices is exactly $2|E|$, an even number. Hence, there is an even number of vertices in G that have odd degrees.

Let U be the subset of vertices with odd degrees in G.

Construct a new instance $(\overline{G} = (V_0,\ E_0),\ k + 2)$ of Vertex Cover, where $V_0 = V \cup \{x, y, z\}$ and $E_0 = E \cup \{(x, y), (y, z), (z, x)\} \cup \{(x, v)|v \subset U\}$. That is, we make a triangle with three new vertices, and then connect one of them (say x) to all the vertices in U. The degree of every vertex in V_0 is even. Since a vertex cover for a triangle is of (minimum) size 2, it is clear that V_0 has a vertex cover of size k + 2 if and only if G has a vertex cover of size k.

Hence, vertex cover with only even degree vertices is NP Complete.

Rubric:

- 5 points for proving the problem is in NP.
- 2 points for reducing from vertex cover.
- 8 points for correctly reconstructing the new graph.

Problem 3

[**15 points**] Given an undirected connected graph G= (V, E) in which a certain number of tokens $t(v) \geqslant 1$ placed on each vertex v. You will now play the following game. You pick a vertex u that contains at least two tokens, remove two tokens from u and add one token to any one of adjacent vertices. The objective of the game is to perform a sequence of moves such that you are left with exactly one token in the whole graph. You are not allowed to pick a vertex with 0 or 1 token. Prove that the problem of finding such a sequence of moves is NP-complete by reduction from Hamiltonian Path.

Solution:

Construction: given a HP in G, we construct G' as follows. Traverse a HP in G and placed 2 tokens on the starting vertex and one token on each other vertex in the path.

Claim: G has a HP iff G' has a winning sequence.

- by construction before the last move we will end up with a single vertex having two tokens on it. Making the last move, we will have exactly one token on the board.

-since there is only one vertex with 2 tokens, we will start right there playing the game. Each next move is forced. When we finish the game, we get a sequence moves which represents a HP.

Rubric:

- Didn't prove it is in NP: -5
- Didn't prove it is NP-hard: -10
- Assigned two tokens on one random vertex instead of the starting vertex of Hamiltonian path: -2 (Since it is a reduction from Hamiltonian path, not from Hamiltonian cycle, 2 tokens should be assigned on the starting vertex)
- Assigned wrong number of tokens on one vertex: -3
- Assigned wrong number of tokens on two vertices: -6
- Assigned wrong number of tokens on three or more vertices(considered as not a valid reduction from Hamiltonian path): -7
- Not a valid reduction from Hamiltonian path: -7

Ungraded Problems

Problem 1

You are given a directed graph G = (V, E) with weights we on its edges. The weights can be negative or positive. The Zero-Weight-Cycle Problem is to decide if there is a simple cycle in G so that the sum of the edge weights on this cycle is exactly 0. Prove that this problem is NP-complete.

Solution:
see `https://en.wikipedia.org/wiki/Zero-weight_cycle_problem`

Problem 2

The graph five-coloring problem is stated as follows: Determine if the vertices of G can be colored using 5 colors such that no two adjacent vertices share the same color.
Prove that the five-coloring problem is NP-complete.
Hint: You can assume that graph 3-coloring is NP-complete.

Solution:
Show that 5-coloring is in NP:
Certificate: a color solution for the network, i.e., each node has a color.
Certifier:

i. Check for each edge (u,v), the color of node u is different from the color of node v.
ii. Check at most 5 colors are used.

Show that 5-coloring is NP-hard: prove that 3-coloring $\leqslant$p 5-coloring.
Graph construction:
Given an arbitrary graph G. Construct G' by adding 2 new nodes u and v to G. Connect u and v to all nodes that existed in G, and to each other.
G' can be colored with 5 colors iff G can be colored with 3 colors.

i. If there is valid 3-color solution for G, say using colors 1,2,3, we want to show there is a valid 5-coloring solution to G'. We can color G' using five colors by assigning colors to G according to the 3-color solution, and then color node u and v by additional two different colors. In this case, node u and v have different colors from all the other nodes in G', and together with the 3-coloring solution in G, we use at most 5 colors to color G'.

ii. If there is a valid 5-coloring solution for G', we want to show there is a valid 3-coloring solution in G. In G', since node u and v connect to all the other nodes in G and to each other, the 5-coloring solution must assign two different colors to node u and v, say colors 4 and 5. Then the remaining three colors 1,2,3 are used to color the remaining graph G and form a valid 3-color solution.

Solving 5-coloring to G' is as hard as solving 3-color to G, then 5-coloring problem is at least as hard as 3-coloring, i.e., 3-coloring $\leqslant$p 5-coloring.