Linear Algebra: wk4 Four Subspaces

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```
library(far)
```

```
## Loading required package: nlme
```

```
## far library : Modelization for Functional AutoRegressive processes
```

```
## version 0.6-4 (2014-12-07)
```

```
library(MASS)
library(pracma)
```

Fundamental four subspaces of Matrix

- $R(\mathbb{A}), N(\mathbb{A}), C(\mathbb{A}), N(\mathbb{A}^T)$

Exercise.

- Get 5 matrices from class
- Given a matrix and a vector,

(1) find out where the vector lives

- Space and subspace

(2) basis of the subspace
(3) Provide a vector that is not in the span of these two subspaces

Finding a basis of a subspace

Column space basis

use rref($\mathbb{A}$)

```
A = matrix(c(-3,6,-1,1,-7,1,-2,2,3,-1,2,-4,5,8,-4),nrow=3,ncol=5,byrow=TRUE)
print(A)
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]   -3    6   -1    1   -7
## [2,]    1   -2    2    3   -1
## [3,]    2   -4    5    8   -4
```

```
rref(A)
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    1   -2    0   -1    3
## [2,]    0    0    1    2   -2
## [3,]    0    0    0    0    0
```

use orth()

```
C_A = orth(A)
C_A
```

```
##             [,1]        [,2]
## [1,]  0.03354216  0.99686846
## [2,] -0.36102371 -0.05472854
## [3,] -0.93195322  0.05707950
```

Left Nullspace($\mathbb{A}^T$)

```
N_AT <- null(t(A))
N_AT
```

```
##             [,1]
## [1,] -0.07161149
## [2,] -0.93094934
## [3,]  0.35805744
```

Basis for output space

```
OUT <- cbind(C_A, N_AT)
rref(OUT)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

Row space basis

Using transformation

using orth()

```
C_AT <- orth(t(A))
C_AT
```

```
##            [,1]        [,2]
## [1,] -0.1940942  0.29847614
## [2,]  0.3881884 -0.59695228
## [3,] -0.4519731  0.08359359
## [4,] -0.7098521 -0.13128896
## [5,]  0.3216637  0.72824124
```

Nullspace basis

- Suppose $T(\vec{x}) = A\vec{x}$, then the `kernel` or null space of such T can be found as below.

using nullspace()

```
N_A <- nullspace(A)
N_A
```

```
##             [,1]       [,2]        [,3]
## [1,]  0.04416189  0.3476413 -0.86627634
## [2,]  0.41326313  0.5658536 -0.04450838
## [3,]  0.85611211 -0.2202874  0.08531064
## [4,] -0.25090190  0.5572485  0.32464681
## [5,]  0.17715416  0.4471048  0.36730213
```

Basis spanning the input space

```
IN <- cbind(C_AT, N_A)
IN
```

```
##            [,1]        [,2]        [,3]       [,4]        [,5]
## [1,] -0.1940942  0.29847614  0.04416189  0.3476413 -0.86627634
## [2,]  0.3881884 -0.59695228  0.41326313  0.5658536 -0.04450838
## [3,] -0.4519731  0.08359359  0.85611211 -0.2202874  0.08531064
## [4,] -0.7098521 -0.13128896 -0.25090190  0.5572485  0.32464681
## [5,]  0.3216637  0.72824124  0.17715416  0.4471048  0.36730213
```

- Suppose we have $\vec{H} = [a - 3b, b - a, a, b]^T$, this can be written as linear combination of two vectors $a\vec{v}_1$ and $b\vec{v}_2$ where $\vec{v}_1 = [1, -1, 1, 0]$ and $\vec{v}_2 = [-3, 1, 0, 1]$.
- This is very useful technique of expressing a subspace of $\vec{H}$ as the linear combination of some small collectoin of vectors.
- Subspace of $\vec{H} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$

How to find the basis of null space

- Step 1: Given $\mathbb{A}$, find its `rref`
- Step 2: Solve for $\vec{x}$ in $\mathbb{A}\vec{x} = \vec{0}$
- Step 3: express $\vec{x}$ as linear combination of smaller vectors.
- Step 4: identify basis spanning the null space

```r
r1 <- c(-3,6,-1,1,-7)
r2 <- c(1,-2,2,3,-1)
r3 <- c(2,-4,5,8,-4)

A <- rbind(r1,r2,r3)

rref(A)
```

```
##    [,1] [,2] [,3] [,4] [,5]
## r1    1   -2    0   -1    3
## r2    0    0    1    2   -2
## r3    0    0    0    0    0
```

```r
n1 <- c(2,1,0,0,0)
n2 <- c(1,0,-2,1,0)
n3 <- c(-3,0,2,0,1)

#########################################
# Any vector in the null space with A
#########################################

print(A%*%n1)
```

```
##    [,1]
## r1    0
## r2    0
## r3    0
```

```r
print(A%*%(n1+n2+n3))
```

```
##    [,1]
## r1    0
## r2    0
## r3    0
```

```r
print(round(A%*%(100*n1+0.1*n2-305*n3),3))
```

```
##    [,1]
## r1    0
## r2    0
## r3    0
```

Concept check questions

- What is the relationship between $C(A)$ and $N(A^T)$?
- What is the relationship between $R(A)$ and $N(A)$?
- What is the relationship between $C(A)$ and $R(A)$?
- What is the relationship between $N(A)$ and $N(A^T)$?
- If the basis spanning $C(A)$ are given, can you find out the basis spanning $N(A^T)$?
- If the basis spanning $R(A)$ are given, can you find out the basis spanning $N(A^T)$?