# Motivating Examples

# Plan for the Week

Two sections:

1. Defining random variables
2. Joint distributions

# Plan for the Week (cont.)

At the end of this week, you will be able to:

- Understand how random variables fit as a part of statistical modeling
- Apply the mathematical tools we have for characterizing random variables
- Solve problems by applying properties of expectation and variance

# Introducing Random Variables

# What Is a Random Variable?

Intuition:

- A model for where the numbers in data come from.
- Like a regular variable but with uncertainty about its value.

**Random Variable**

Given probability space $(\Omega, S, P)$, a *random variable* is a measurable function from $\Omega$ to $\mathbb{R}$.

# What Is a Random Variable (Formally)?

160

# What Is a Random Variable (Formally)?

$\longrightarrow$

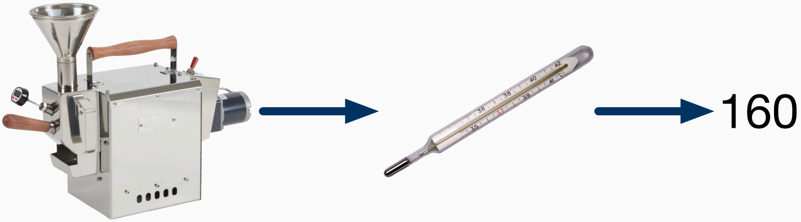


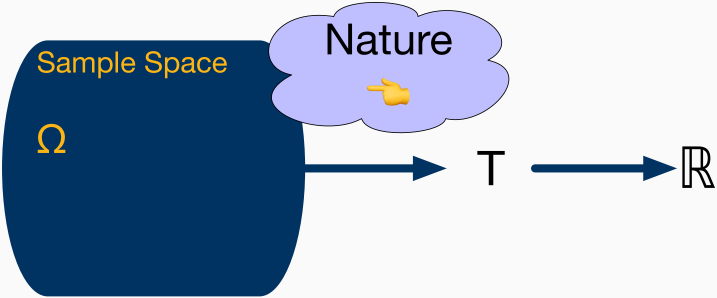
$\longrightarrow$ 160

Sample Space

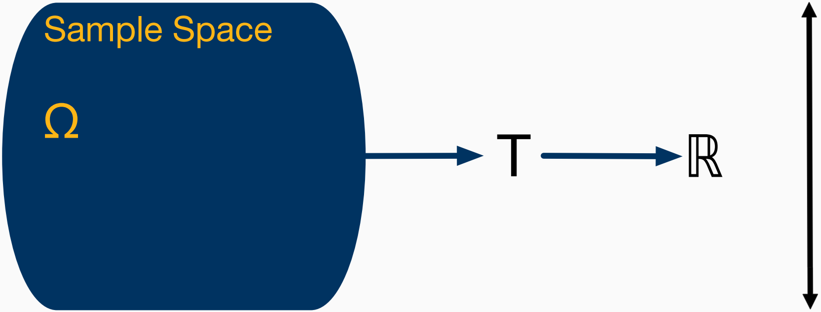
$\Omega$ $\longrightarrow$ $T$ $\longrightarrow$ $\mathbb{R}$

# What Is a Random Variable (Formally)?



# Probabilities of Real Numbers



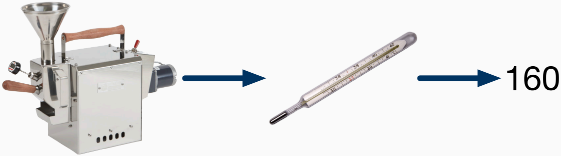
# Reading: Random Variables

# Reading Assignment

Read *Foundations of Agnostic Statistics*, section 1.2.0–1.2.1.

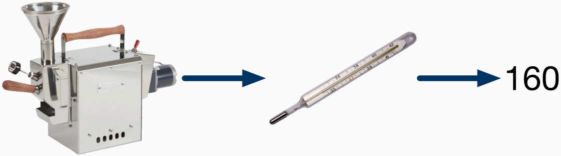
# Functions, Events, and Operators

# Functions of a Random Variable



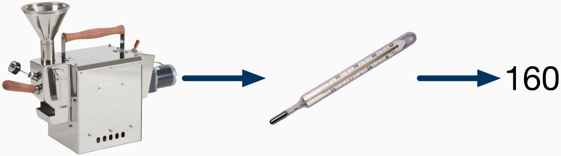

- Random variable $T$ represents temperature in Fahrenheit, but you need Celsius.

# Creating Events from a Random Variable




You want to know whether the temperature is below 150.

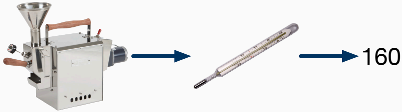
# Operators Applied to a Random Variable




You want to know the minimum possible temperature.

# Learnosity: Roaster Temperature

# Learnosity: Roaster Temperature




Suppose $T$ is a random variable representing the temperature of a roaster.

What kind of object is each of the following?

1. $g(T)$ where $g(t) = \begin{cases} 1, & t < 150 \\ 0, & \text{otherwise} \end{cases}$. Answer: random variable
2. $R$ such that $R[T] = P(T > 150)$. Answer: operator
3. $T < (5/9)(T - 32)$. Answer: event

# Reading Call

# Reading Assignment

Read *Foundations of Agnostic Statistics*, section 1.2.2.

# Discrete Random Variables

# Introducing Discrete Random Variables

A random variable $X$ is *discrete* if its range, $X(\Omega) \in \mathbb{R}$, is a countable set.

- $X(\Omega)$ can be finite.
  - A roll of a die: $\{1, 2, 3, 4, 5, 6\}$
- $X(\Omega)$ can be countably infinite.
  - Edits to an article: $\{0, 1, 2, 3...\}$

# The Bernoulli Distribution

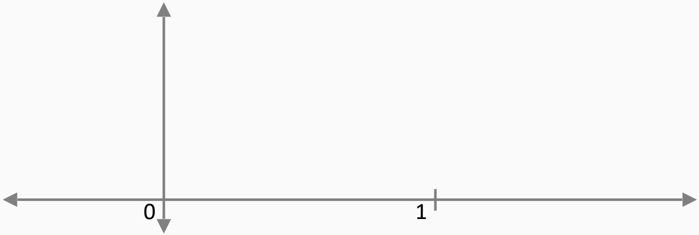
A Bernoulli trial is a simple (but important) discrete random variable.

- It can represent:
  - a coin flip
  - a decision to buy or not to buy
  - success or failure of experiment
- Convention: 1 represents success, 0 failure

# The Probability Mass Function (PMF)

A function that gives the probability that a discrete random variable equals each number in $\mathbb{R}$

- $f_X(x) = P(X = x)$



# Key Properties of the PMF

**Theorem**

For a discrete random variable $X$ with pmf $f$,

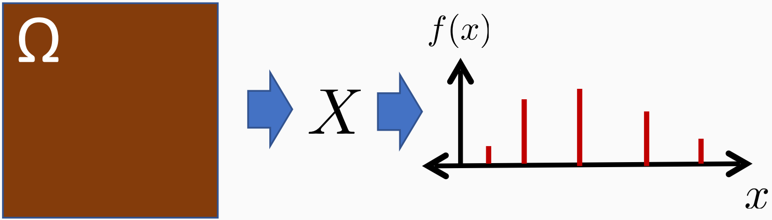
1. For all $x \in \mathbb{R}$, $f(x) \geq 0$.
2. $\sum_{x \in X(\Omega)} f(x) = 1$.

# Computing Other Event Probabilities from the PMF

**Theorem: The PMF fully characterizes a discrete RV**

For a discrete random variable $X$ with pmf $f$, if $D \subseteq \mathbb{R}$,

$$P(X \in D) = \sum_{x \in X(\Omega) \cap D} f(x).$$




Take-away: The pmf contains *all* the information about the probability distribution of $X$.

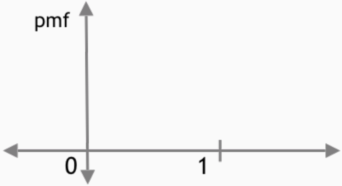
# Reading: Cumulative Distribution Functions

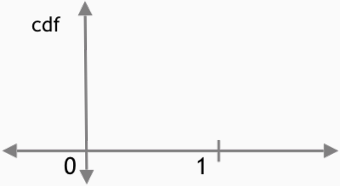
# Reading Assignment

Read *Foundations of Agnostic Statistics*, section 1.2.3.

# Lightboard: The Cumulative Distribution Function

# The Cumulative Distribution Function



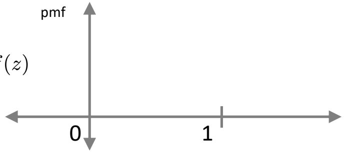



# The Cumulative Distribution Function (Solution)

For R.V. X, $F(x) = P(X \leq x)$ for all $x \in \mathbb{R}$

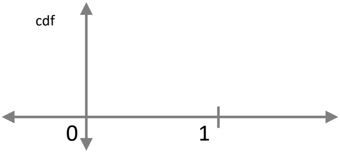
For discrete X,

$$P(X \leq x) = P\big(X \in (-\infty, x]\big) = \sum_{z \in X(\Omega), z \leq x} f(z)$$




Ex: X ~ Bernoulli (p)

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - p, & 0 \leq x < 1 \\ 1, & 1 \leq x \end{cases}$$

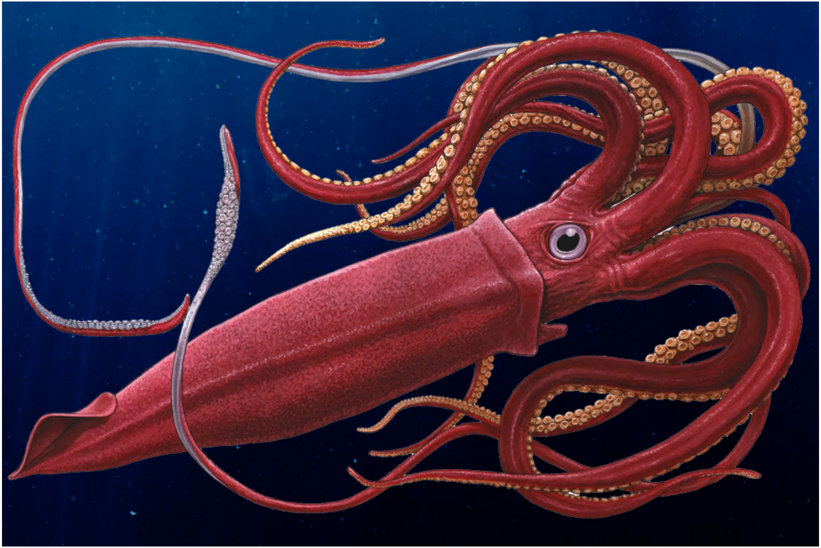



Key properties

1. F is Nondecreasing
2. $lim_{x \to -\infty} F(x) = 0$
3. $lim_{x \to \infty} F(x) = 1$

# Continuous Random Variables

# Motivation for Continuous Random Variables



# Motivation for Continuous Random Variables

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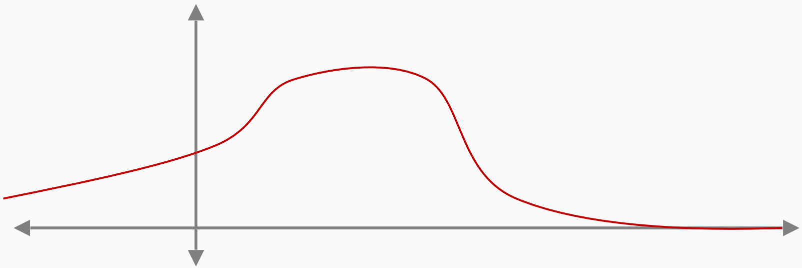
# The Probability Density Function



# Reading: Continuous Random Variables

# Reading: Continuous Random Variables

Read *Foundations of Agnostic Statistics*, section 1.2.4.

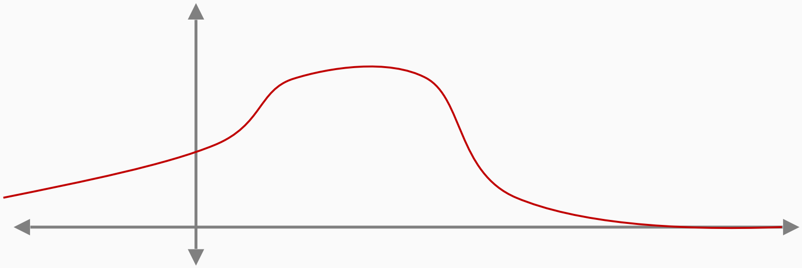
# Properties of Continuous Random Variables

# Continuous Random Variables

**Definition**

A random variable $X$ is *continuous* if there exists a non-negative function $f : \mathbb{R} \to \mathbb{R}$ such that the cdf of $X$ is

$$F(x) = \int_{-\infty}^{x} f(u)du, \text{ for all } x \in \mathbb{R}$$

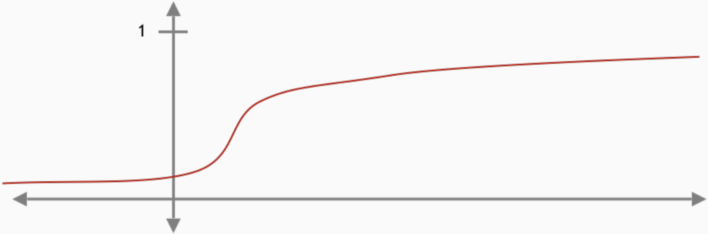


# Characterizing Continuous Random Variables

## Deriving the PDF from the CDF

For a continuous random variable with cdf $F$, the probability density function of $X$ is

$$f(x) = \left.\frac{dF(u)}{du}\right|_{u=x}, \text{ for all } x \in \mathbb{R}$$



**Properties of Probability Density Functions**

For a continuous random variable $X$ with pdf $f$,

1. For all $x \in \mathbb{R}$, $f(x) \geq 0$.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

# Support

**Definition**

For a random variable $X$ with pmf or pdf $f$, the *support* of $X$ is

$$\text{Supp}[X] = \{x \in \mathbb{R} : f(x) > 0\}$$

# Uniform Random Variables

# Uniform Random Variables

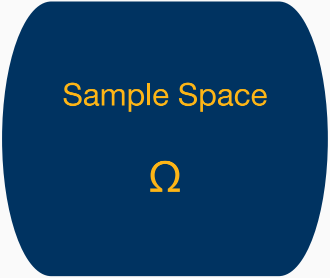
**Note: This is a placeholder for a video that we're going to re-use from:** Use the old video 3.9 Uniform Random Variables - only the start through 6.12, before I start talking about expectation.

# Normal Random Variables

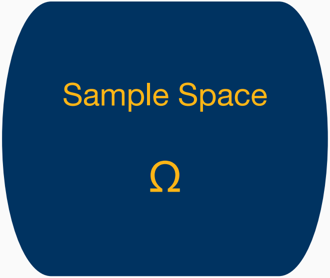
Use the old video 3.10 The Normal Distribution

# Joint Distributions

# Introducing Joint Distributions



# Introducing Joint Distributions



# Reading Assignment

# Reading Assignment

Read *Foundations of Agnostic Statistics*, section 1.3–1.3.1.

# Discrete Bivariate Distributions

# Equality of Random Variables

**Equality of random variables**

Let $X$ and $Y$ be random variables. Then, $X = Y$ if, for all $\omega \in \Omega, X(\omega) = Y(\omega)$.

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Don't assume $X = Y$ just because:

1. $X$ and $Y$ have the same distribution
    - $X = 1$ if Paul's coin lands heads, 0 otherwise
    - $Y = 1$ if Alex's coin lands heads, 0 otherwise
    - $f_X = f_Y$ but $X \neq Y$

# Equality of Random Variables

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   - $Y = 1$ if Alex's coin lands heads, 0 otherwise
   - $f_X = f_Y$ but $X \neq Y$
2. $P(X = Y) = 1$
   - $X = 1$ if Paul climbs Half Dome, 0 otherwise
   - $Y = 1$ if Alex climbs El Capitan, 0 otherwise
   - $P(X = Y) = 1$ but $X \neq Y$

# The Joint PMF and Car Dealers

$$\begin{aligned} B: &\quad \Omega \to \{0, 1\}, && \text{Does the customer buy?} \\ D: &\quad \Omega \to \{0, 1, 2\}, && \text{Number of test drives} \end{aligned}$$

$$f(b, d) = P(B = b, D = d)$$

# The Joint PMF and Car Dealers (cont.)

| | $D = 0$ | $D = 1$ | $D = 2$ |
|---|---|---|---|
| $B = 0$ | 0.1 | 0.4 | 0.1 |
| $B = 1$ | 0.1 | 0.1 | 0.2 |

# The Joint PMF and Car Dealers (cont.)

| | $D = 0$ | $D = 1$ | $D = 2$ |
|---|---|---|---|
| $B = 0$ | 0.1 | 0.4 | 0.1 |
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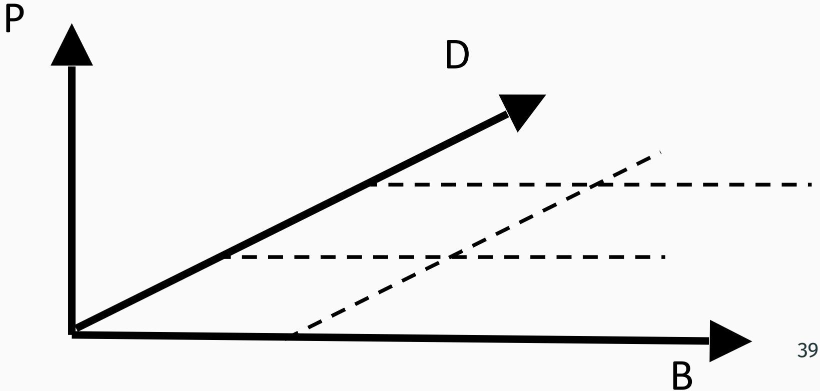


# The Joint Cumulative Probability Function

$$F(b, d) = P(B \leq b, D \leq d)$$

- Recall that $F_X(x) = \int_{-\infty}^{x} f(x)$ relates the CDF and PDF for the single variable case.
- In a similar way:

$$F_{Y,X} = \int_{-\infty}^{x} \int_{-\infty}^{y} f(y, x) dydx$$

- This can be hard to work with but can describe *any* random variables.

# Learnosity: The Probability Mass Function

# Learnosity: The Probability Mass Function

Suppose that $X_1$ is a Bernoulli random variable representing a fair coin. Suppose that $X_2$ is a Bernoulli random variable representing a second fair coin. Let $Y = X_1 + X_2$.

Fill in the following table for the PMF of $X_1$ and $Y$.

| | $X_1 = 0$ | $X_1 = 1$ |
|---|---|---|
| $Y = 0$ | | |
| $Y = 1$ | | |
| $Y = 2$ | | |

# Discrete Marginal and Conditional Distributions

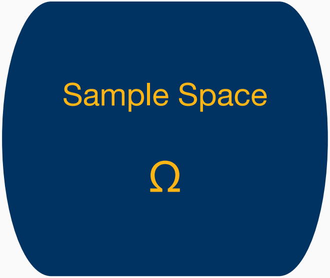
# Reading Assignment: Discrete Marginal and Conditional Distributions

# Reading Assignment

Read *Foundations of Agnostic Statistics*, section 1.3.2.

# Discrete Marginal Distributions

# Marginal vs. Joint Distributions



# From Joint to Marginal Distributions

$$f_B(b) = P(B = b) = \sum_{d \in \text{Supp}[D]} f(b, d) \text{ for all } b \in \mathbb{R}$$

| | $B = 0$ | $B = 1$ |
|---|---|---|
| $D = 0$ | .1 | .1 |
| $D = 1$ | .4 | .1 |
| $D = 2$ | .1 | .2 |

# From Marginal to Joint Distributions?

Is it possible to compute the joint distribution, $f$, given the marginals $f_B$ and $f_D$?

| | $B = 0$ | $B = 1$ | $F_D$ |
|---|---|---|---|
| $D = 0$ | | | 0.2 |
| $D = 1$ | | | 0.5 |
| $D = 2$ | | | 0.3 |
| $F_B$ | 0.6 | 0.4 | |

# From Marginal to Joint Distributions? (cont.)

Two joint distributions with the same marginals:

| | $B = 0$ | $B = 1$ | $F_D$ |
|---|---|---|---|
| $D = 0$ | 0.1 | 0.1 | 0.2 |
| $D = 1$ | 0.4 | 0.1 | 0.5 |
| $D = 2$ | 0.1 | 0.2 | 0.3 |
| $F_B$ | 0.6 | 0.4 | |

| | $B = 0$ | $B = 1$ | $F_D$ |
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| $D = 0$ | | | 0.2 |
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| $D = 2$ | | | 0.3 |
| $F_B$ | 0.6 | 0.4 | |

# Discrete Conditional Distributions

# The Conditional PMF

- Given two test drives ($D = 2$), what does the model say about buying behavior ($B$)?
- What is the conditional pmf, $f_{B|D}$?

| | $B = 0$ | $B = 1$ | $F_D$ |
|---|---|---|---|
| $D = 0$ | 0.1 | 0.1 | 0.2 |
| $D = 1$ | 0.4 | 0.1 | 0.5 |
| $D = 2$ | 0.1 | 0.2 | 0.3 |
| $F_B$ | 0.6 | 0.4 | |

# The Conditional PMF (cont.)

## Conditional expectation, car dealers

$$
\begin{aligned}
f_{B|D}(b|d) &= P(B = b|D = d) \\
&= \frac{P(B = b, D = d)}{P(D = d)} = \frac{f(b, d)}{f_D(d)}
\end{aligned}
$$

# The Conditional PMF (cont.)

## Conditional expectation, car dealers

$$\begin{aligned} f_{B|D}(b|d) &= P(B = b|D = d) \\ &= \frac{P(B = b, D = d)}{P(D = d)} = \frac{f(b, d)}{f_D(d)} \end{aligned}$$

$$\text{conditional} = \frac{\text{joint}}{\text{marginal}} \Leftrightarrow$$

# The Conditional PMF (cont.)

## Conditional expectation, car dealers

$$f_{B|D}(b|d) = P(B = b|D = d)$$
$$= \frac{P(B = b, D = d)}{P(D = d)} = \frac{f(b, d)}{f_D(d)}$$

$$\text{conditional} = \frac{\text{joint}}{\text{marginal}} \Leftrightarrow \text{joint} = \text{marginal} \cdot \text{conditional}$$

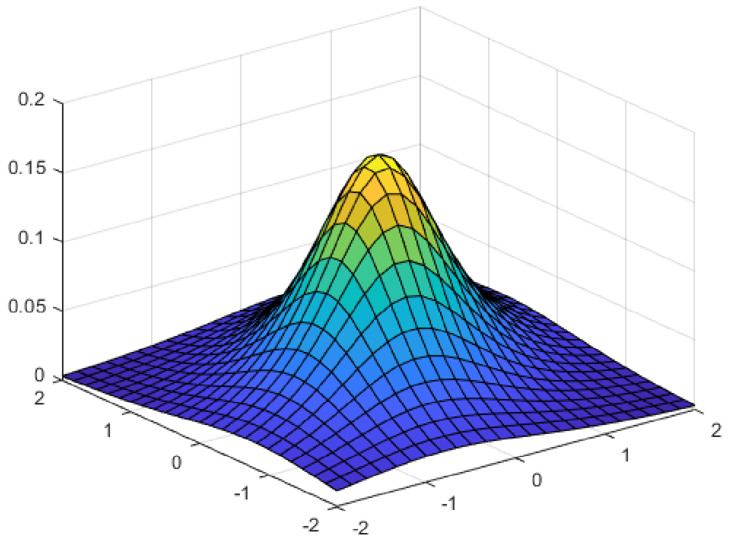
# Reading: Jointly Continuous Random Variables

# Reading Assignment

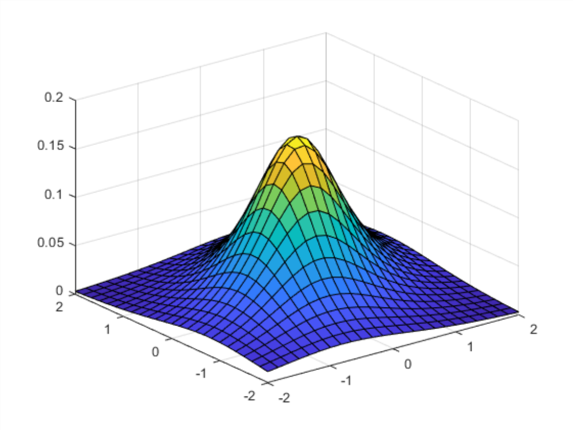
Read *Foundations of Agnostic Statistics*, section 1.3.3.

# Jointly Continuous Random Variables

# The Joint Density Function

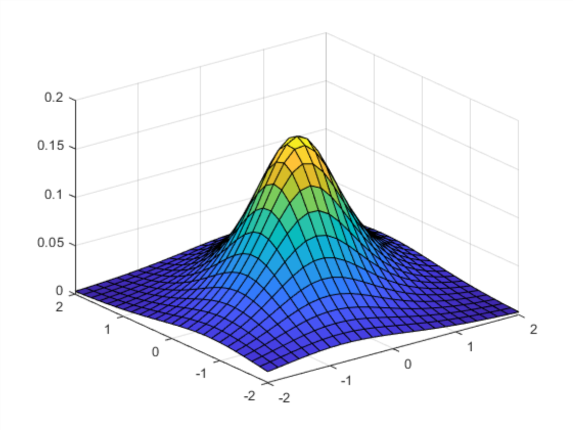


# Probabilities of Events



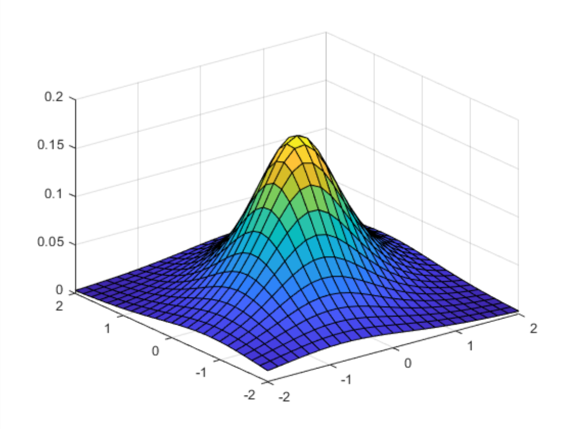

How to compute:
$P((X, Y) \in A)$

# Continuous Marginal Distributions



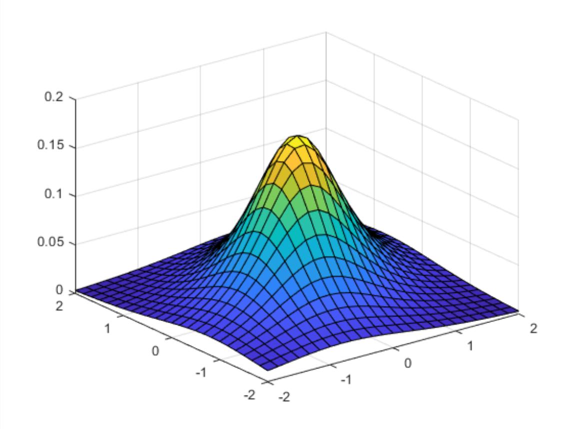

How to compute:
$f_X(x)$

# Continuous Conditional Distributions




How to compute:
$f_{B|D}(b|d)$

# Continuous Conditional Distributions




How to compute:
$f_{B|D}(b|d)$

$$\text{conditional} = \frac{\text{joint}}{\text{marginal}}$$

# Reading: Independent Random Variables

# Reading Assignment

Read *Foundations of Agnostic Statistics*, section 1.3.4.

# Independent Random Variables

# Independence

If $X$ and $Y$ are independent, $X \perp\!\!\!\perp Y$.

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# Independence

If $X$ and $Y$ are independent, $X \perp\!\!\!\perp Y$.

1. The marginals fully describe the distribution:
   $f(x, y) = f_X(x) f_Y(y)$
2. Knowing X tells you nothing about Y:
   $f_{Y|X}(y|x) =$

# Drawing Independent Continuous RV

# Drawing Non-Independent Continuous RV

# Importance of Independence

$B_1$ represents blood pressure of patient 1.

$B_2$ represents blood pressure of patient 2.

$\vdots$

Without simplifying assumptions, the joint distribution of $B_1, B_2, \ldots$ is too complex to estimate with limited data.

Independence $\implies$ only need to estimate the marginals

# Assessing Independence in Practice

Independence is rarely (if ever) perfectly met.

What if:

- Patient 1 and patient 2 are related
- Technicians trade off every 10 patients
- After seeing an unusually high blood pressure, the technician adjusts their mannerisms

These are potential *dependencies*.