Week 6

Hypothesis Testing

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6.1 Historical Development of Frequentist Statistics

Introduction

- Hypothesis, H, is a model for how the world might work
- In practice, evidence is rarely conclusive
- We want $Pr(H|D)$, or the probability of the event that our hypothesis is true

The Dilemma

Whether our hypothesis is true is something we can never know

- The world is not a perfectly controlled lab
- Evidence collected contains information but does not begin to identify a unique model out of all the possible models
- We do not know how to weight all the possibilities out there

Example 1

You flip a coin once and it lands on heads

What is the probability that it is a double-headed coin?

Is there enough information to answer this?

- How did the coin get there?
- The context is missing:
    - How do we choose between the different models?
    - How do we weight all the alternatives?
- Even with more information, you can never know the context completely

Example 2

Isaac Newton

Both motions are consistent with a gravitation attraction that is proportional to the square of the distance between the objects

- What is the probability that Newton's theory of gravity is correct?
- **Problem:** Newton's second theory seemed to work up to the precision of 17th-century instruments
- Only later were instruments developed that were precise enough to show Newton's laws were incorrect

Example 2 (cont.)

- How could Newton decide how probable his model was compared to general relativity (not imagined yet)?
- If he could have imagined another theory, would we be equipped to compare two very different ideas?
- We could never write down the infinite number of models consistent with observations of the planet in order to assign each a probability
- It may not even make sense to assign a probability to Newton's gravity (eg. true state of world or not)

Example 3

You discover three specimens of a new species of squid that measure 3.2, 3.3, and 4.0 feet long

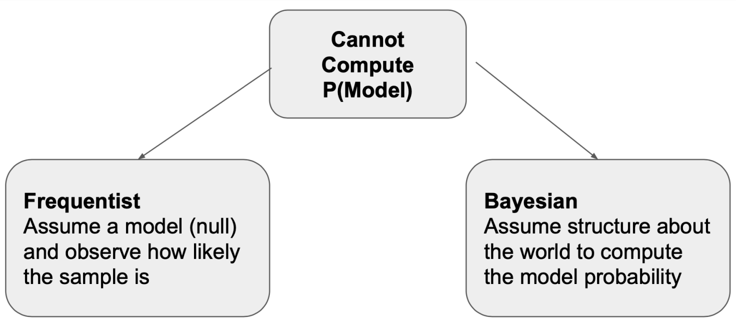
- What is the probability that the average length amount the entire species is 3.5 feet?
    - Probability zero for a single number (point estimate) probability that the average length is between 3 and 4 feet?
    - Positive number for length
    - No new numbers that have not been imagined
    - A better grasp of the possibilities

Example 3 (cont.)

- However, we don’t know how representative the specimens are
    - We still don’t know all the relevant information
    - Examples: deep water pressure, amount of light

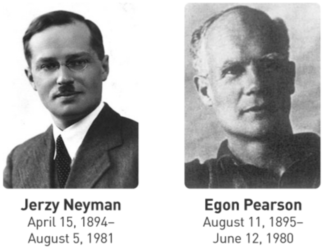
We cannot deduce the probability of our model because we do not know enough about the structure of the world

Two Branches of Statistics



6.4 The Frequentist Approach

The Birth of Modern Statistics



- Before the 1930's, there were lots of statistical procedures but no coherent account of how to choose the right one
- Neyman and Pearson published articles that added a rigorous mathematical treatment, forming the basis of frequentist statistics

The Central Dilemma

- We observe data, D
- Given that this data occurs, we want the probability that our hypothesis $P(H|D)$ is **true**

To a strict frequentist:

- Not just impossible to compute
- Does not even make sense to assign a probability to a hypothesis

Objective Probability

A frequentist defines probability as a matter of long-run frequencies

- We need to specify a collective of elements
  - Eg. throws of a dice
  - **Collective:** a frame of observations that can happen over and over
- As number of observations approaches infinity, proportion of throws of the die that show a 3 is 1/6
- The probability is the long-run frequency of the event relative to all observations (or of 3's relative to all throws)
  - Called **objective probability**

Objective Probability and Hypothesis

- If you view probability as objective, you cannot talk about the probability of a hypothesis
- it is just true or false
- **Subjective Probability:** The probability of a hypothesis
    - Allows for disagreement about what it is
    - Reflects our lack of information

So, what probabilities can we study?

- We need a long-run collective

$PR(D|H)$

- H = hypothesis
- D = data

Assume H is true and call it the null hypothesis

- Has to be quite specific (the only extra assumption we're making)
- Is the basis for predictions we need to make about what data should come out of the experiment
- Governs how the experiment behaves as we run it over and over

$PR(D|H)$ (cont.)

Now we have a meaningful collective

- Can look at the relative frequencies of different outcomes
- Can specifically look at the number of hypothetical experiments in which we would get data at least as extreme as D
  - Captured by *p*-value
  - ***p*-value:** The probability of getting data as extreme as our observations, assuming the null hypothesis is true

Components of a Hypothesis Test

Notes while Screensharing

$$g(x_1, X_2, \ldots, X_n) \to S \in \mathbb{R}$$

$$H_0 = 0$$

Hypothesis Test Example

Mad data science

Suppose that your lab has synthesized a new compound, *Vitamin W*.

Let random variable B represent the change in blood pressure that results from taking *Vitamin W*.

Let $\mu = \mathrm{E}[B]$.

You need to make a decision, to invest resources in Vitamin W or not.

Two Possible States of the World

Goal: Begin with a reasonable default supposition; leave this supposition behind if data provides compelling evidence

Null hypothesis

- Default assumption, status quo, statement that data might overturn
- $H_\varnothing$: Usually $\mu = 0$
- No effect

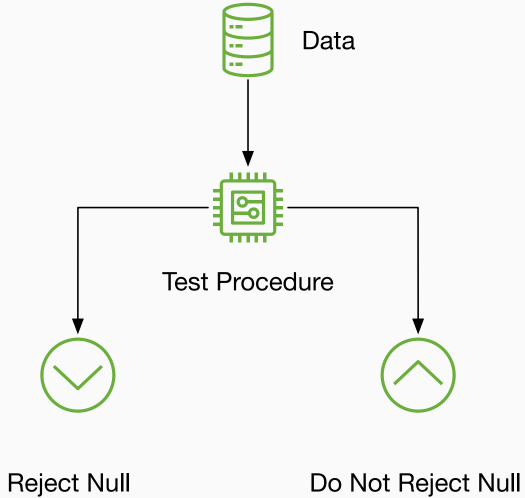
Alternative hypothesis

- Idea or alternative to status quo
- H_a : Usually $\mu \neq 0$
- Some effect exists

With compelling evidence, we leave the specific null hypothesis ($H_\varnothing$) for the alternative (H_a)

A Hypothesis Test

A *hypothesis test* is a procedure.



False Positive and False Negative Errors

	True state of the world	
	The null is true	*The null is false*
Reject the null	False Positive (Type I Error)	
Do not reject the null		False Negative (Type II Error)

False Positive and False Negative Errors (cont.)

False Positive Errors

- Typically the most destructive
- Error rate, denoted α, is the probability of rejecting the null hypothesis when we should not; $P(\text{Reject } H_\varnothing | H_\varnothing)$
- Starting with Ronald Fisher: set $\alpha = 0.05$

A hypothesis test is a procedure for rejecting or not rejecting a null, such that the false positive error rate is controlled ($\alpha = 0.05$).

Breaking Down a Test Procedure

A test statistic

- A function of our sample
- Measures deviations from the null hypothesis
- Distribution must be completely determined by the null

A rejection region

- A set of values for which we will reject the null
- Chosen to be contrary to the null
- Total probability must be $\alpha = 0.05$

What a Hypothesis Test Doesn't Do

A hypothesis test does not prove the null hypothesis.

- We control Type 1 error rates
- We cannot control Type 2 error rates
- How can you be sure the real B is not 0.01? Or 0.00001?

Never accept the null hypothesis.

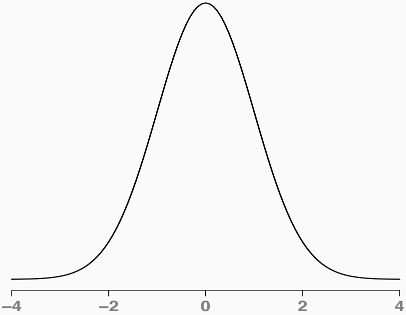
- The valid decisions are reject and fail to reject.

The One-Sample z-Test

Vitamin W Example

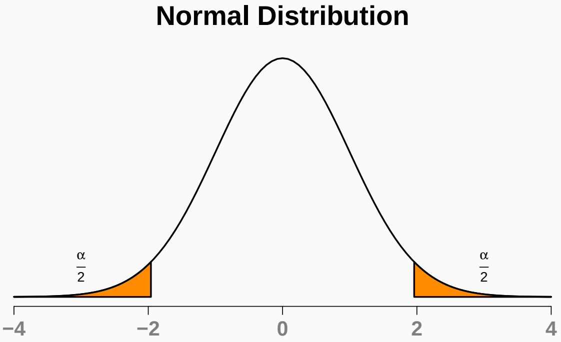
Suppose $(B_1, .., B_{100})$ are i.i.d. random variables with mean $\mu = \mathrm{E}[B]$, representing changes in blood pressure.

Assume $B \sim N(\mu, \sigma)$. Assume we know $\sigma[B] = 20$.



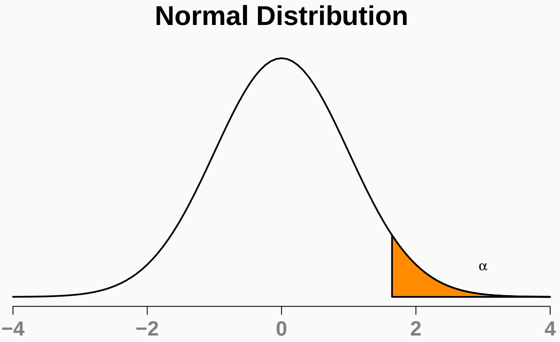
One- and Two-Tailed Tests

The Two-Tailed z-Test



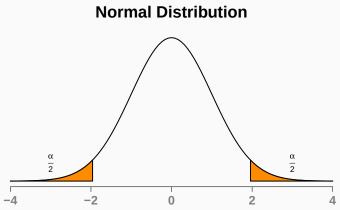
- **Null hypothesis**: $\mu = 0$
- **Alternative hypothesis**: $\mu \neq 0$

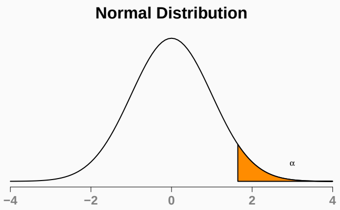
The One-Tailed z-Test



- **Null hypothesis**: $\mu = 0$
- **Alternative hypothesis 1**: $\mu > 0$
- **Alternative hypothesis 2**: $\mu < 0$

Choosing One or Two Tails





Switching your test after you see the statistic is cheating.

One-Tailed Test: Things to Consider

Before using a one-tailed test, ask yourself these questions:

1. Will the audience believe that I started with one tail before I saw the data?
2. Will the audience share my opinion of which tail is interesting?
3. Am I really 100% committed to only this tail?
    - What if the effect turns out to be huge, but in the other direction?
    - Would I be willing to call that a negative result?
    - Can I convince my audience I have this much commitment?

T-Test Assumptions

T-Test Assumptions, Part I

Assumptions of t-test

The textbook assumptions

- X is a metric variable.
- $\{X_1, X_2, \ldots, X_n\}$ is a random sample.
- X has a normal distribution.

Variables are almost never normal.

T-Test Assumptions, Part II

But, in the large sample case, this is more plausible.

Large sample t-test assumptions

If:

- X is a metric variable
- $\{X_1, X_2, ..., X_n\}$ is a random sample
- n is large enough that the CLT implies a normal distribution of mean

Then: The t-test is asymptotically valid

T-Test Assumptions, Part III

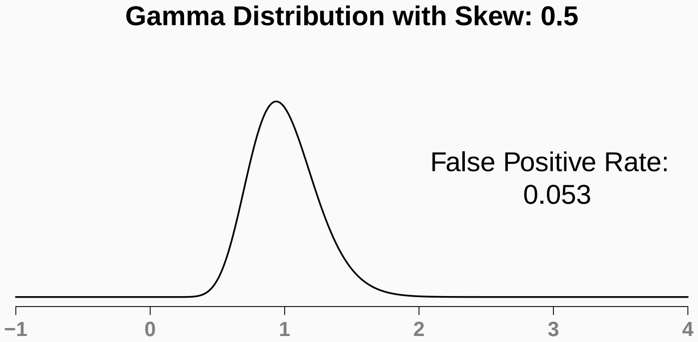
T-Test Assumptions, Part IV

The t-test is considered "reasonably robust," even when $n < 30$, as long as deviations from normality are moderate.

However, watch out for strong skewness, especially when $n < 30$.

Gamma With Increasing Skew

Twenty draws from gamma distributions



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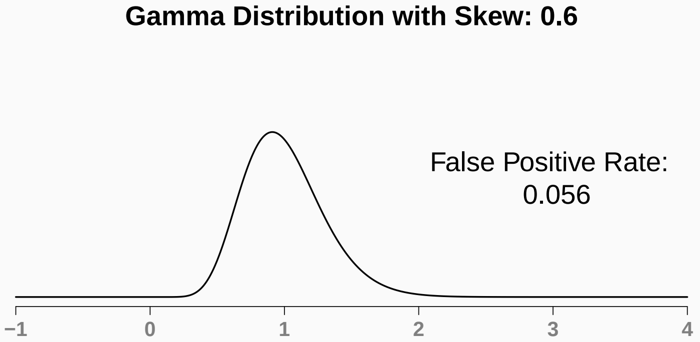

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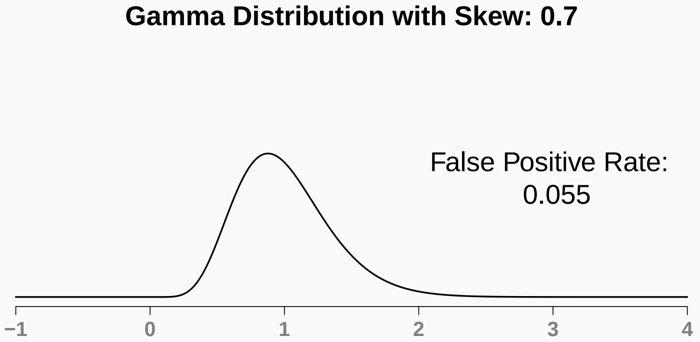

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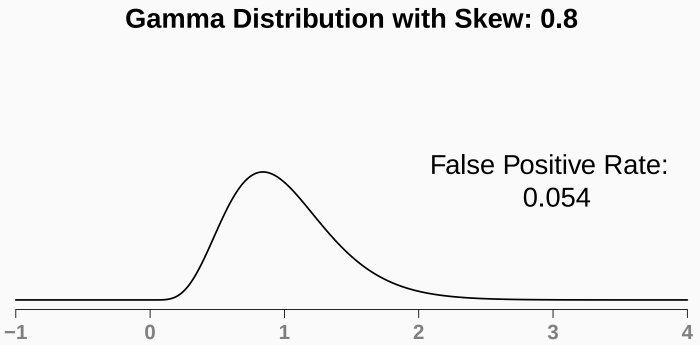

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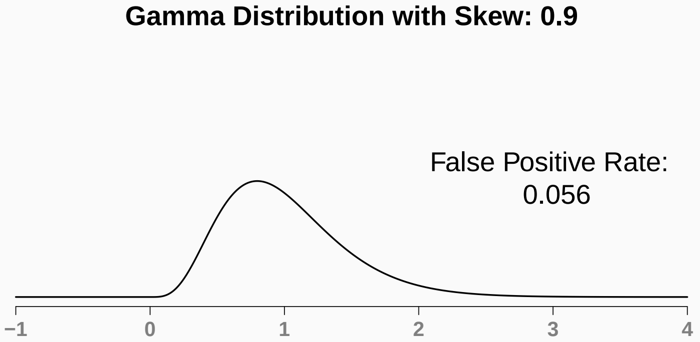

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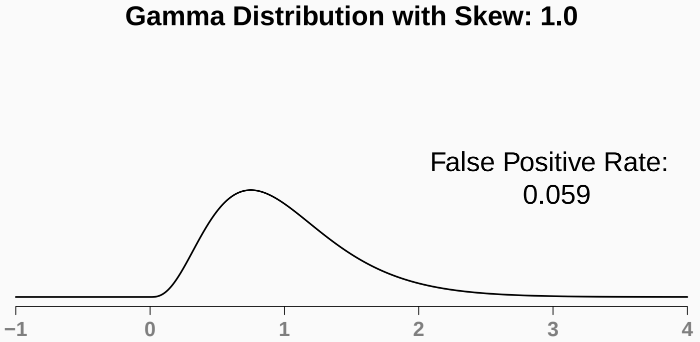

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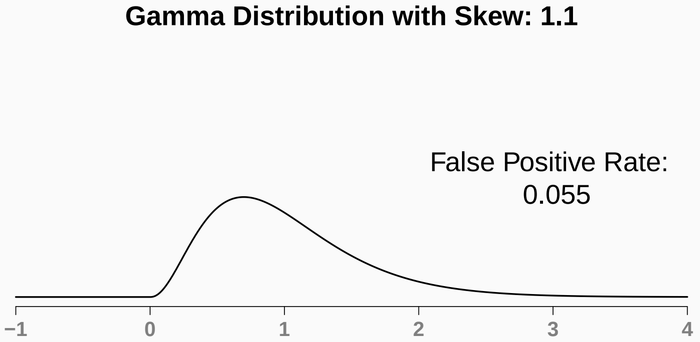

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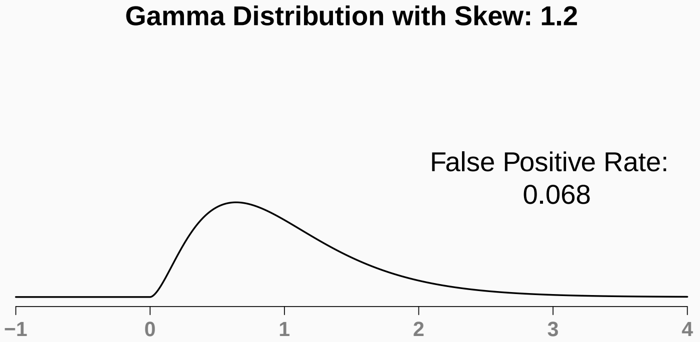

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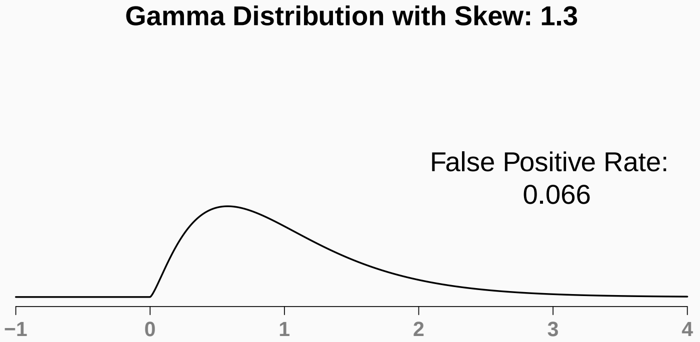

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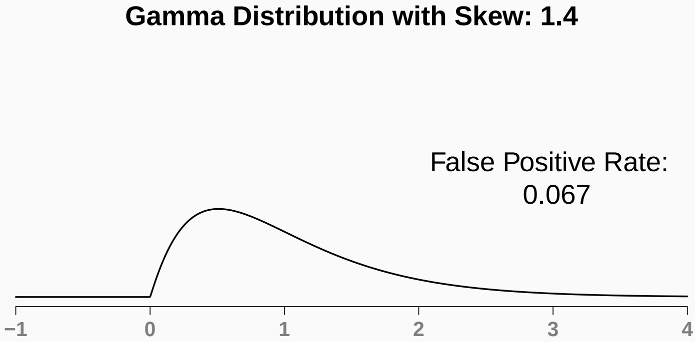

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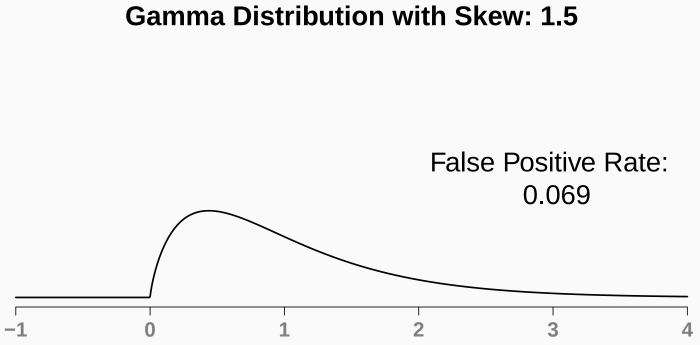

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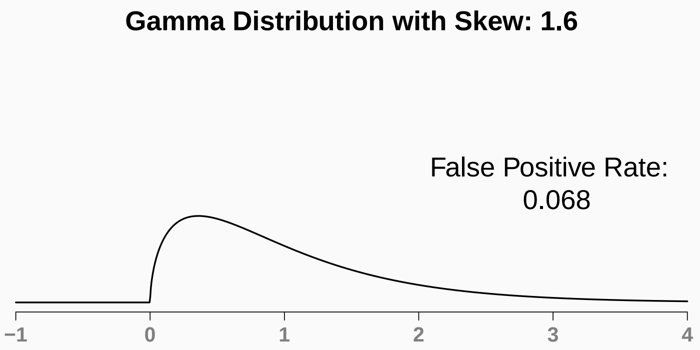

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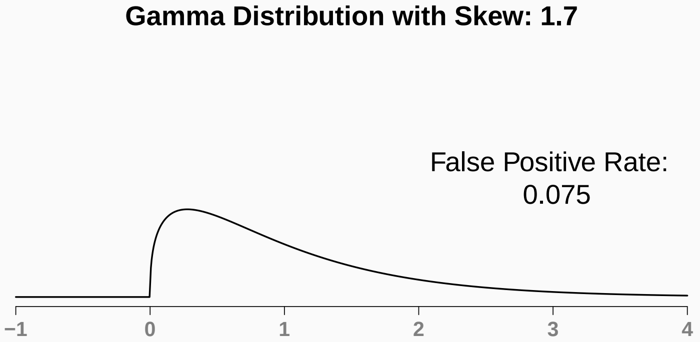

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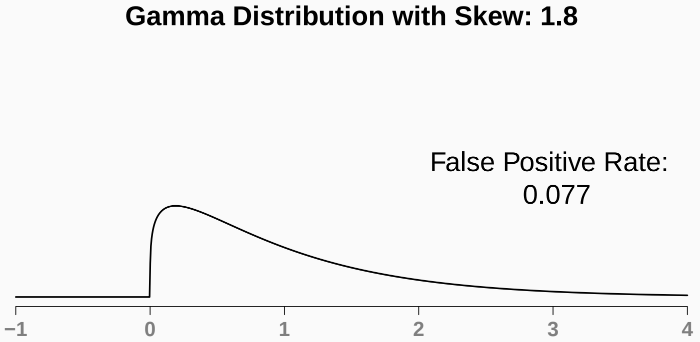

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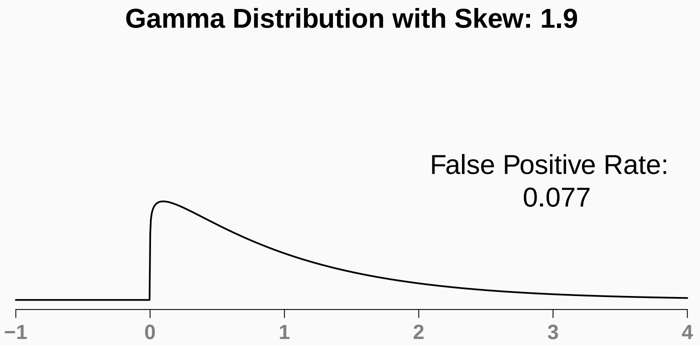

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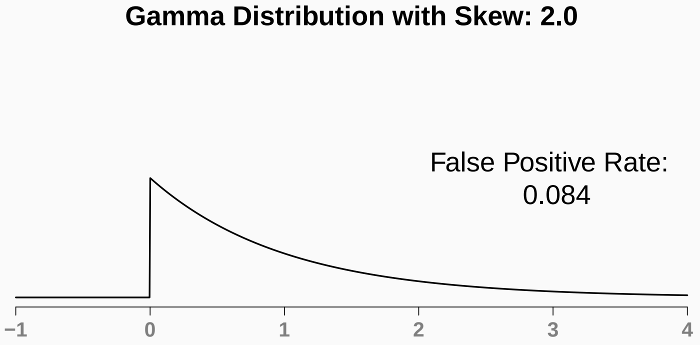

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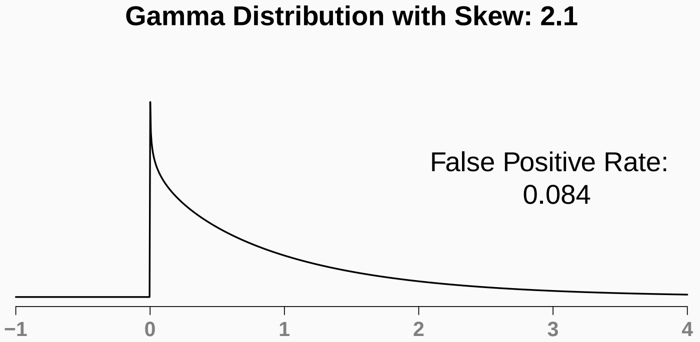

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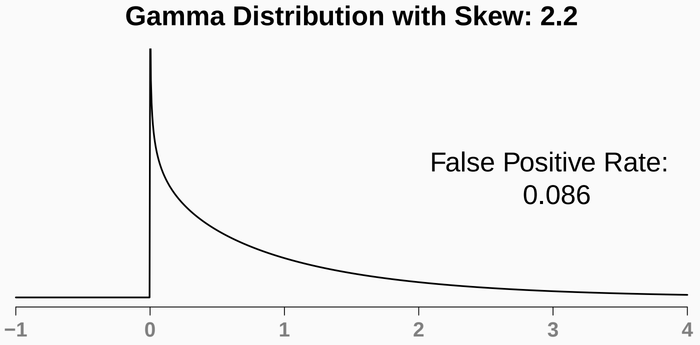

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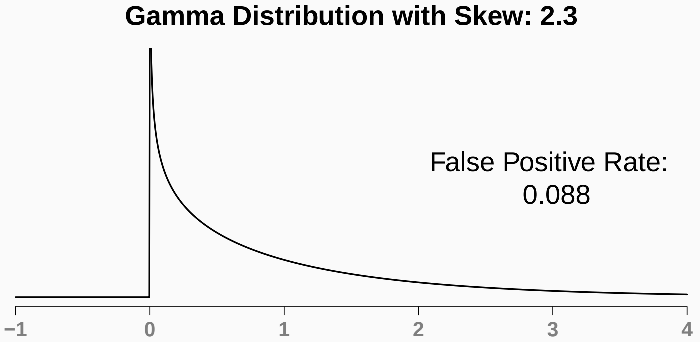

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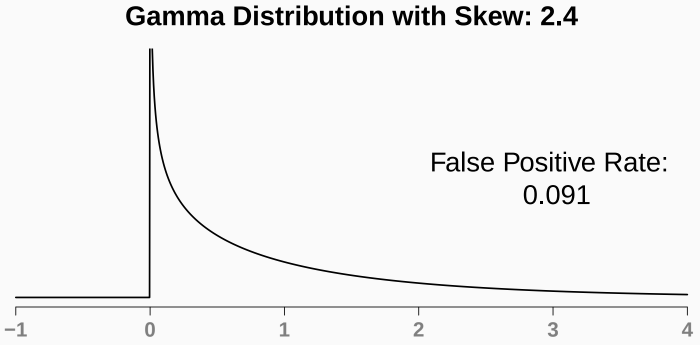

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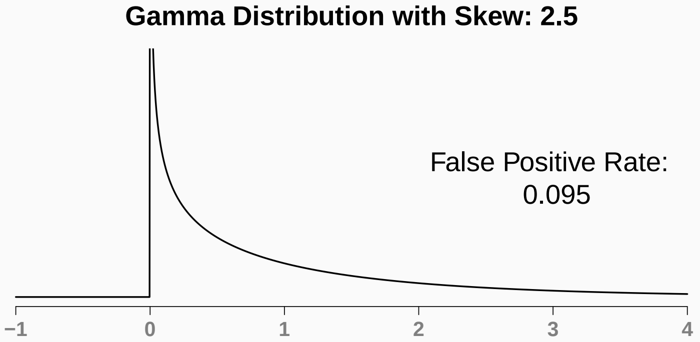

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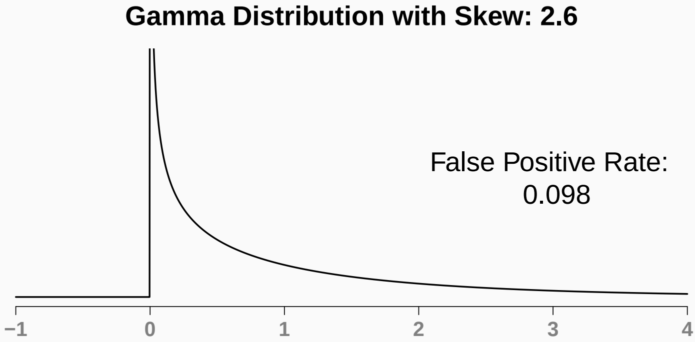

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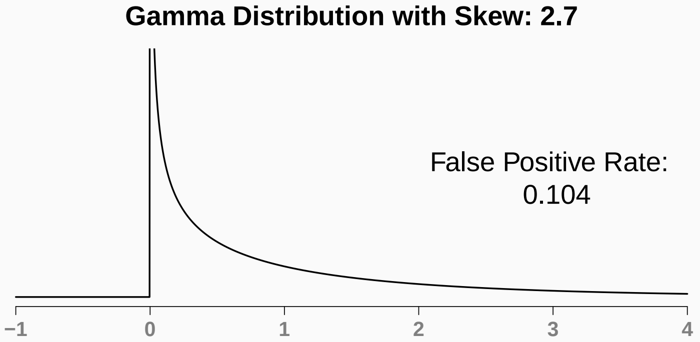

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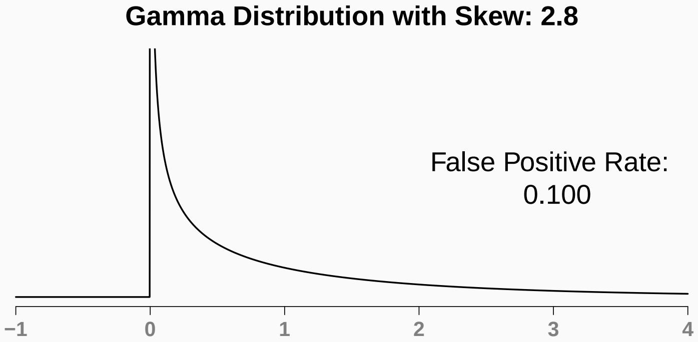

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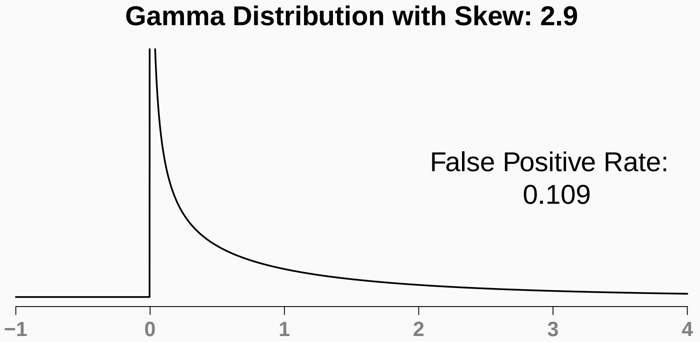

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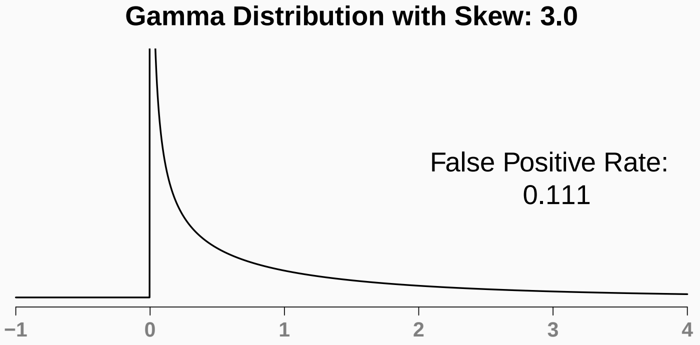

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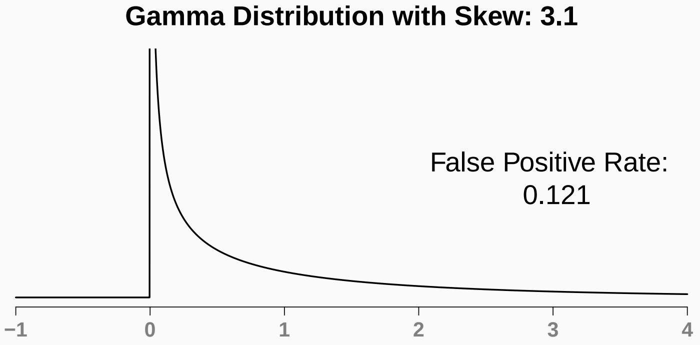

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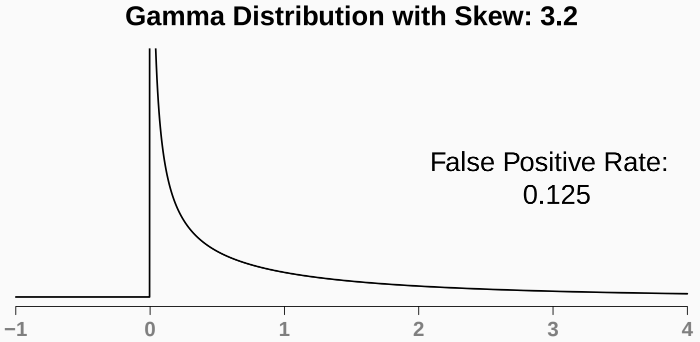

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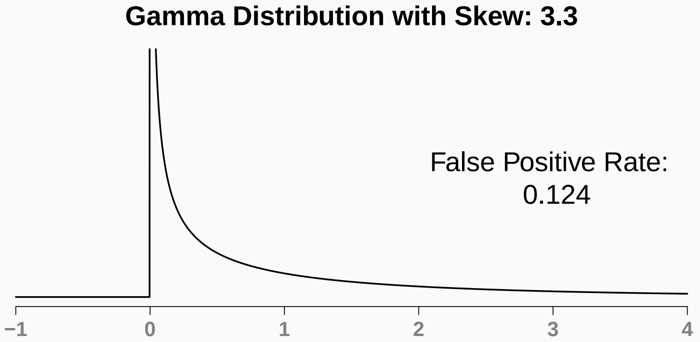

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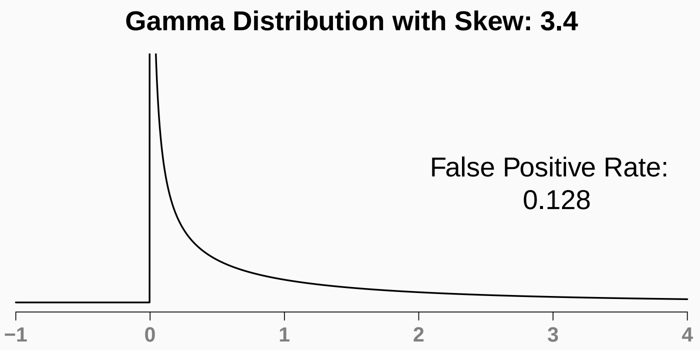

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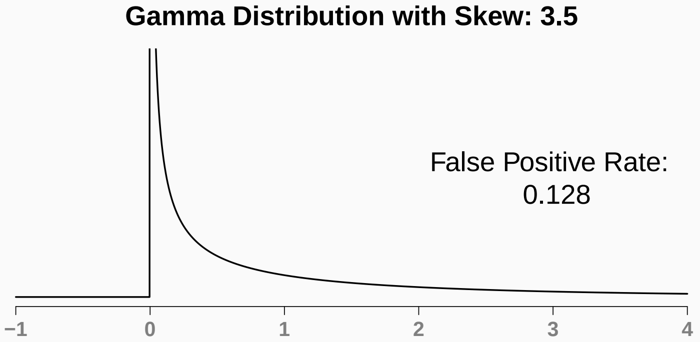

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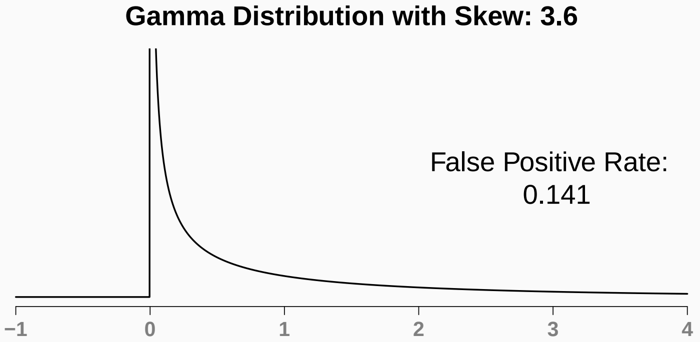

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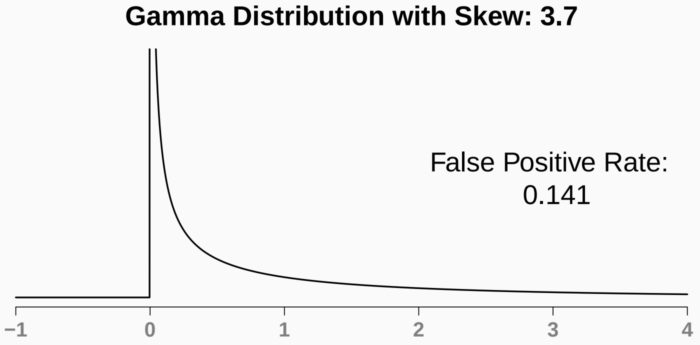

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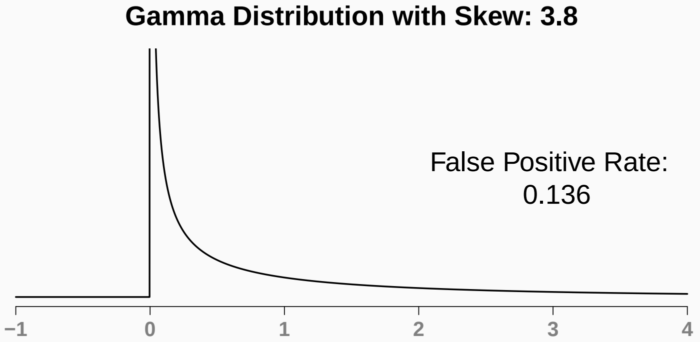

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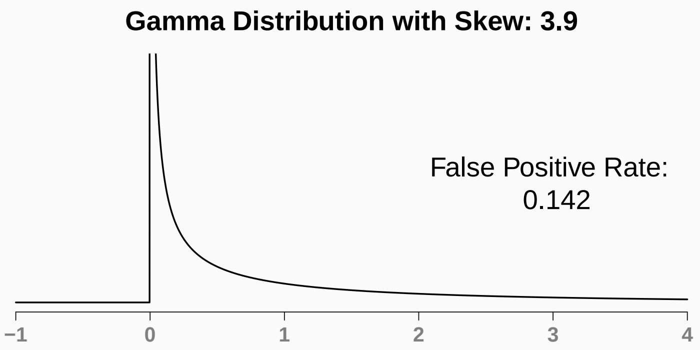

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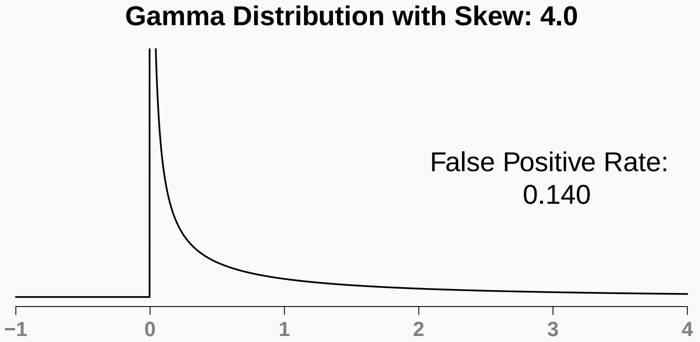


T-Test Assumptions

More practical guidance:

- X is a metric variable.
- $\{X_1, X_2, \ldots, X_n\}$ is a random sample.
- The distribution is not too non-normal, considering n.

When the t-test is not valid, consider using a non-parametric test instead.

Introduction to P-Values

Introducing P-Values

The p-value is the probability, calculated assuming that the null hypothesis is true, of obtaining a value of the test statistic at least as contradictory to H_0 as the value calculated from the available sample.

Jay L. Devore (2015)

Z-Distribution

The P-Value for a Z-Test

Vitamin W

You measure the effects of Vitamin W on blood pressure (measured in *mmHg*) for 100 patients and get $\bar{X} = 3$.

Assume $X \sim N(\mu, 20)$.

- $H_0 : \mu = 0$
- $z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$

The P-Value and Decision Rules, Part I

Neyman-Pearson hypothesis testing: rules to make a decision and usually be right ($\alpha = 0.05$)

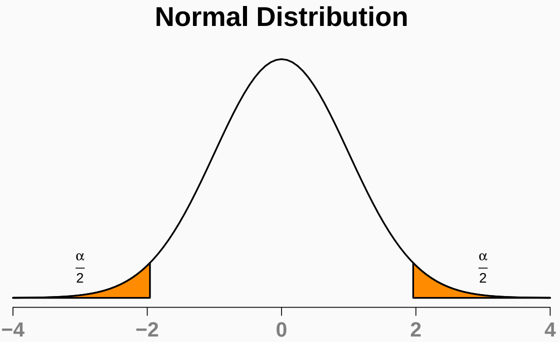
A classic z-test

- z=1 $\rightarrow$ Do not reject null.
- z=2 $\rightarrow$ Reject null.
- z=10 $\rightarrow$ Reject null.

- Strict frequentist with a dichotomous decision rule: treat $z = 2$ and $z = 10$ identically.
- But is there value in knowing *how contrary* the data is to the null?

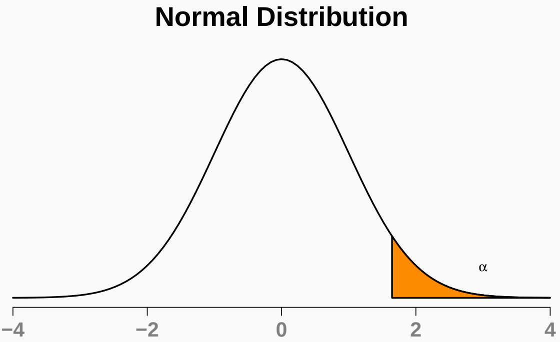
The P-Value and Decision Rules, Part II

$$|z| > \text{critical value} \Rightarrow \text{reject } H_0$$
$$|z| < \text{critical value} \Rightarrow \text{fail to reject } H_0$$



The P-Value and Decision Rules, Part III

$$|z| > \text{critical value} \Rightarrow \text{reject } H_0$$
$$|z| < \text{critical value} \Rightarrow \text{fail to reject } H_0$$



An Equivalent Decision Procedure

Compute p-value.

- If $p < .05 \Rightarrow$ reject H_0
- If $p \geq .05 \Rightarrow$ do not reject H_0

But, can you justify making such a bright-line statement after reducing information so much?

1. Concept
2. Measurement
3. Statistic
4. Assumptions about distribution
5. **p-value**
6. Reject/fail to reject

t-Test and p-Values

P-Value Convention

p-value range	Convention	Symbol
$p > 0.10$	Non-significant	
$0.10 > p > 0.05$	Marginally-significant	.
$p < 0.05$	Significant	*
$p < 0.01$	Highly significant	**
$p < 0.001$	Very highly significant	***

Reporting Test Results

- A t-test for the effect of Vitamin W on blood pressure was highly significant ($t = 3.1$, $p = .008$).
- We found evidence that Vitamin W decreases blood pressure ($t = 2.3$, $p = .04$).
- The effect of Vitamin X on blood pressure was not statistically significant ($t = 1.2$, $p = .23$).

Vitamin W	Vitamin X
2.2 **	1.2
(0.6)	(0.8)

This is half the story; next, you'll need to describe practical significance.

Variable Importance and P-Values

Does a small p-value mean that a variable is "important"?

- Statistical significance
- Practical significance

A Warning

A very common mistake is to assume a p-value is the chance the null hypothesis is true.

Frequentist statistics cannot tell you the probability of a hypothesis!

A Warning (cont.)

Example

I test whether Vitamin X decreases blood pressure: $p = 0.03$.

However, you know that Vitamin X is secretly cornstarch because you created it yourself.

My test will not convince you that there is a 97% chance Vitamin X decreases blood pressure.

Statistical Power

False Positive and False Negative Errors

	The null is true	The null is false
Reject the null	False Positive (I)	
Do not reject the null		False Negative (II)

- False Positive (I) errors are jumping without cause
- False Negative (II) errors are failing to jump when you should
    - Failing to detect a real effect
    - Missed opportunity to create a product, publish a paper, or advance knowledge

Statistical Power, Part I

Much Vitamin W

Consider a *specific* alternate hypothesis:

- H_a : Vitamin W decreases blood pressure by 20 mmHg

- False Negative Error Rate: $\beta = P(\text{not rejecting } H_0 | H_a)$
- Statistical power: $1 - \beta$
- Statistical power is the probability of supporting the alternate hypothesis, assuming it is true

Statistical Power, Part II

Statistical Power, Part II

Statistical Power, Part III

How to increase power

- Increase sample size.
- Choose a powerful test (if you can justify its assumptions).

Practical Significance

Practical significance

Statistical significance

- How much does the data support the existence of an effect?

Practical significance

- Is the size of this effect important?
- What is the magnitude of the effect?
- Should we care about this effect?

Example

Productivity supplements

Vitamin W

$$n = 30$$
$$\mu_{treat} = 12.6$$
$$\mu_{control} = 6.1$$
$$p = 0.11$$

"The difference between groups was not statistically significant, ($t = 1.34$, $p = 0.11$)."

Vitamin Q

$$n = 30,000$$
$$\mu_{treat} = 6.25$$
$$\mu_{control} = 6.21$$
$$p = 0.0005$$

"The difference between the two groups was highly significant, ($t = 3.34$, $p < 0.001$)."

Practical Significance: Context

Primary goal: Provide context for your audience to reason about results.

- Who is your audience?
- What action might be taken based on these results?
- How does this result alter how you would run the business?
- What is the cost-benefit for implementing a change based on this result?
- How does this result “stack up” to other effects?

Practical Significance: Model Explainability

- Some tasks require *explainable* models.
- Finance, healthcare, insurance, and other regulated industries stipulate specific model forms .
- Humans reason in linear hypotheses—higher-dimensional and conditional hypotheses are too much to keep in mind.

Practical Significance: Effect Sizes

Effect sizes

- Single-number metrics that characterize the magnitude of an effect
- Population parameters that we estimate—*do not vary based on sample size*

Invalid effect size metrics

- t-stat
- p-value

Valid effect size metrics

- Mean values
- Difference in means between groups

Standard Effect Size Measures

Standardized effect sizes are designed to be flexible and apply in many scenarios:

- Cohen's d
- Correlation ρ
- Cramer's V

General metrics ignore the specific context around your research or business question.

Cohen's d

Sometimes, a mean (or difference in means) is hard to assess because the units are unfamiliar.

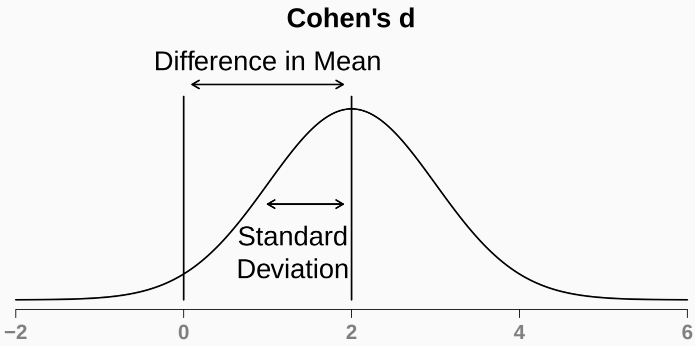
- **Example**: The effect of angled bristles on tooth decay is 5 millicaviparsecs per brushstroke

Cohen's d

Compare effect size relative to the underlying natural variation in the outcome.

$$\text{Cohen's } d = \frac{\text{mean difference}}{\text{standard deviation}}$$

Cohen's d (cont.)



Rules of thumb (according to Cohen)

Small effect	$d = 0.2$
Medium effect	$d = 0.5$
Large effect	$d = 0.8$

- Applicable across a huge number of contexts
- Ignores any important differences between context
- Saving dollars or saving lives are the same to Cohen's d

Takeaways

- After a statistical test, it's important to assess both statistical significance and practical significance.
- Standard effect size measures can help in a wide variety of situations.
- But don't get carried away and reach for them automatically.
- The main objective is to clearly explain how important the magnitude of the effect is.

Guidelines For Statistical Reporting

Guidelines for Statistical Reporting

- Communicating your results is a key part of statistical analysis
- In this class (and other classes in the program) we'll ask you to submit your analysis as a written report
- Next are some guidelines to keep in mind when writing a report

In this case, the guidelines are specific to exploratory analysis

Guideline One

A statistical analysis is a written argument

- A good writing style is key
- This is technical writing: aim for clarity and exposition
- All rules of good writing apply
    - Organize your argument clearly
    - Guide reader through the evidence in the data
    - Proofread

Guideline Two

If you don't have something nice to say (about your output), don't display it at all

- There should be no output dumps
- Every graph should be mentioned in your writing and should have some purpose
- Explain what the graphs and numbers mean

Guideline Three

You should document decisions

- If you decide that observations should be removed, state which ones
- If values are suspicious, but you leave them in, state that too
- If you transform a variable, for example, by taking the logarithm, state that
- Your justification can often be very brief (just a sentence), but make sure that the reader can follow your logic

Guideline Four

Identify features that should be reflected in statistical models

- This will make more sense once you have experience building models
- Keep in mind the purpose of the analysis
- Eg. if you're interested in explaining the price of a house, look to see what kind of relationship that variable has with the explanatory variables
    - Is it linear?
    - Is it exponential?
    - Are there values that don't seem to fit with the overall trend?

Guideline Five

Remember the difference between sample and population

- At this point, we don't know how to model a population This means that you must confine your conclusions to the sample
  - You can talk about sample means, sample covariances
- You can't say anything about the population that generated your sample

Guideline Five (cont.)

Remember the difference between sample and population

- Be wary of technical words–in particular the word *significant*
  - People might casually say one value is significantly bigger than another
  - But this has a technical meaning, and it implies that we've built a model and performed a statistical test

Guideline Six

Show us the code (a guideline for this class)

- We really want to see the code that generates your output so we can follow your analysis in detail, step by step
  - Typically, your software will have a setting to suppress the code when generating a pdf report, but don't do it
- This is probably the biggest difference between writing analyses in school and in a professional context

Guideline Six (cont.)

Show us the code (a guideline for this class)

- In most situations, you have to think about different levels of detail for different audiences
  - It's usually a good idea to provide an executive summary
  - Not everyone can read 50 pages of output
  - Often, you'll want to move details like your script to an appendix
- In this class, however, we'd like to see your code in the body of your report so we can evaluate it effectively