In [1]:
```
import warnings
warnings.filterwarnings('ignore')
```

In [18]:
```
from plotting_utils import *
```

A model for two mice in a small cage

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Problem statement: Two mice in a cage

The problem given to us was:

P1. We have two mice in a cage (in a 2-D or 1-D field).

P2. They don't like each other.

P3. Food is present in one given spot in the cage.

P4. There is only enough space for one mouse to eat at a time.

P5. They need to reproduce.

P6. They don't reproduce where they eat.

P7. After being full, mice don't find the feeding spot that attractive anymore.

We are interested in possible trajectories of the two mice.

Assumptions

The problem description brings many details that can be modeled. A model taking every detail into consideration can become quickly very complex. Moreover, the profundity of biological details can become very quickly overwhelming.

For exemplification, however, we want to have a simplistic model that focuses on a couple of key aspects of the problem. Therefore, I will ignore some aspects specified in the problem description and will generalize over some biological details.

A1. The cage

I will assume the mice move inside idealized cages with infinitely high walls such that the mice are not influenced in any way by the outside world. The system is, thus, closed. Therefore, it's state dynamics depends only on variables that can be found inside the cage and we have access to during the modelling task.

I will also assume one can observe the mice from above in an idealized way. This means we always have access to their exact position and this reading is done noise free.

A2. Biological variability of the mice physiology

I will assume that the actual physical biology of the mice is irrelevant for the problem at hand. One could look for biologically plausible movement model of the mouse, in which arms, legs, joins, body and head geometry are considered. However, I do not expect the quality of the desired outputted trajectories to improve dramatically.

As a consequence, in the attempt to keep complexity low, I will idealize the two mice inside a cage to point particles that obey a system of coupled and nonlinear stochastic differential equations.

A3. Movement of the mice

The problem does not express it explicitly, but I will assume that mice usually like exploring their environments and are quite curious. This curiosity should be quantifiable and each mouse should accept its own parametrization.

A4. Mating

Similarly to previous decisions, I will ignore many biologically plausible details, with 2 exceptions:

A4.1. I will assume there is a periodicity in the mating schedule, but keep this very basic.

A4.2. I will assume that proximity to the other mouse produces arousal i.e. even more desire to mate.

A5. Feeding spot

The problem description does not come with many details about feeding. Therefore, I will try to keep feeding as simple as possible.

I will conveniently assume that:

1. food is present in abundance for the mice.
2. there is no feeding schedule or any fixed time window in which feeding can be done.
3. mice are able to feed if they are close enough (a set radius distance apart) from the feeding spot.

As a consequence, infinite amount of food will be **always** present in the cage. This means that the mice can feed as much and as often as they want.

Details ignored from the problem description

1. There is only enough space for one mouse to eat at a time

I will ignore *(P4)* from the problem description. The reasoning is:

1. to the limit $dt \to 0$ two simultaneous feeding events are mathematically impossible.
2. this implies we can always find a small enough discretization s.t. two simultaneous are mathematically excluded.

Moreover, I ignore the fact the mice **might** get hungrier if they are close to the food spot because the problem does not state this. Rather, the assumption is that they know how to navigate their environment and they, somehow, **remember** the spot of the food.

2. They don't reproduce where they eat

After implementing *(P1-P3), (P5), (P7)* I decided to ignore this detail to keep complexity low. In the discussions section I will suggest how to add this aspect.

Methods

The target of the modelling task is to obtain a reasonable system of couples differential equations to model.

Closed form or simulation ?

The natural question arises, if one should and could explicitly solve this system of differential equations.

However, mice take many decisions at each time step randomly, what for the outside observer might appear as totally random. As a simplification, we can bundle all these random decisions in one stochastic process.

As a consequence, our system of differential equations becomes a system of **stochastic differential equations**.

Therefore, finding a closed form is very difficult, if not impossible.

Moreover, closed form solutions are not very descriptive about the task at hand. There are cases when alternative solving methods (phase plane analysis, simulation, etc) could be preferred Strogatz, Nonlinear Dynamics and Chaos, Chapter 2.1 A geometric way of thinking (http://users.uoa.gr/~pjioannou/nonlin/Strogatz,%20S.%20H.%20-%20Nonlinear%20Dynamics%20And%20Chaos.pdf).

I will concentrate in this report on simulating the model on a computer and not think about a closed form solution of the model.

Euler-Maruyama

Dynamical systems can be elegantly implemented using the **Forward Euler** method. **Forward Euler** works by considering the first order Taylor expansion of the dynamical equation.

Due to the stochasticity mentioned above, simple **Forward Euler** does not work. The modelled stochastic process requires a variance that is adapted to the chosen discretization step, usually denoted Δ_t. This method is called **Euler-Maruyama** and is an extension of the **Forward Euler** method [see more details].

Because it is very simple to implement, I will code the **Euler-Maruyama** integration scheme directly and not use any external libraries.

Splitting the problem into smaller ones

A dynamical system of differential equations can be seen as a gradient descent on an energy landscape.

Luckily, gradients are a vector field. Therefore, we can actually split the problem into smaller ones. At the end we will be able to recover full expressivity of the model by linearly adding up these smaller components.

This is particularly advantageous when implementing the model because it allows me to split the problem into smaller ones, that have, independently, manageable complexity.

Therefore, I will subsequently present, in isolation three models:

1. a movement model
2. a mating model
3. a feeding model

Our final model, the Mouse Model will be composed of the mentioned three models see.

Movement Model

Implementing P1 , A2 and A3 (i.e. 2-D movement, point particles, exploring mice) I find appropriate a brownian movement with white noise model. The component of the movement model in the system of SDEs the becomes:

$$\tau_{mouse} \frac{d\vec{m}}{dt} = \sigma_{mouse} \cdot \overrightarrow{\xi(t)} \ \ (1)$$

where $\overrightarrow{\xi(t)} \sim \vec{\mathcal{N}}(0,1)$ is called noise.

The noise model I chose is gaussian white-noise, which is defined by the following two properties:

Noise 1. $< \overrightarrow{\xi_t} >= \vec{0}$

Noise 2. $< \xi_t \cdot \xi_t' >= \delta(t - t')$

where $\delta(t)$ is the **Dirac delta** generalized function. The **Dirac delta** has the very important property that $\int_{-\infty}^{\infty} \delta(t)dt = 1 \ \ (2)$.

By multiplying both sides of eq. (1) with dt we obtain:

$$\tau_{mouse} d\vec{m} = \sigma_{mouse} \cdot dt \cdot \overrightarrow{\xi(t)} = \sigma_{mouse} \cdot d\overrightarrow{W_t} \ \ (3)$$

Eq. (3) is known in the literature as an Ornstein-Uhlenbeck stochastic process, dW_t being the kernel. The vector notation means we are drawing two components, one for each (x,y)-axis.

Because we have to simulate the time evolution on a computer, we have to choose a discretization for our simulation $dt \mapsto \Delta t$. Thus, we have (Noise 2.) in discrete form: $< \xi_{\Delta t} \cdot \xi_{\Delta t} >= \frac{1}{\Delta t}$ to fulfill property (2) of the Dirac delta i.e. integration to exactly 1 value.

Thus: $< \Delta W_t^2 >=< \Delta\xi_t^2 \cdot \Delta t^2 >= \Delta t^2 \cdot < \Delta\xi_t^2 >= \frac{\Delta t^2}{\Delta t} = \Delta t$.

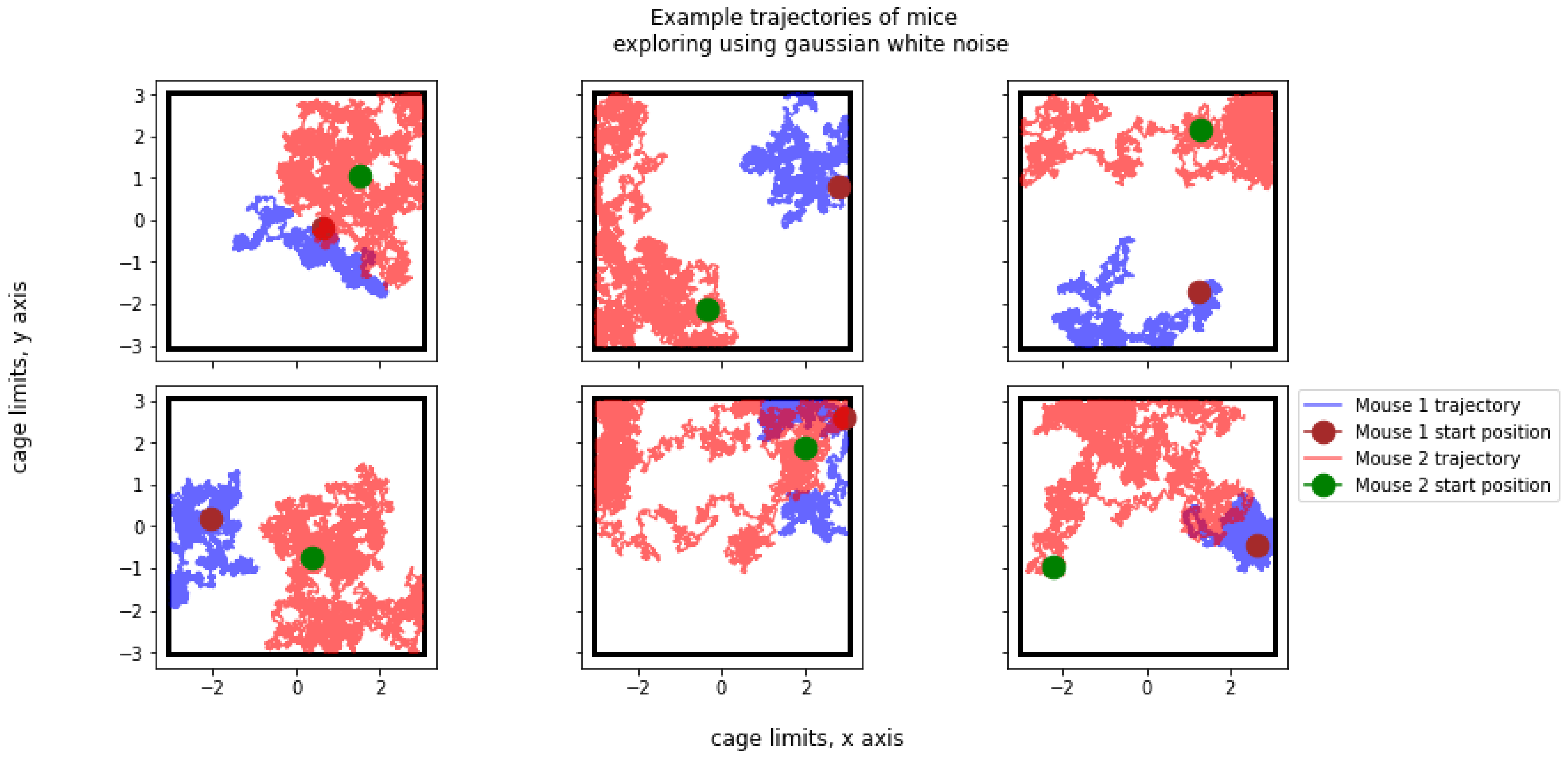
(3) becomes: $\tau_{mouse} \Delta\vec{m} = \sigma_{mouse} \cdot \ \sqrt{\Delta t} \cdot \overrightarrow{\xi_t}$ from which it follows that:

$$\tau_{mouse} \Delta\vec{m}_{movement} \sim \vec{\mathcal{N}}(0,\ \sigma^2_{mouse} \cdot \Delta t \cdot I_2) \quad (4)$$

We can implement relationship (4) directly using the mentioned **Euler-Maruyama** scheme.

Typical trajectories

```
In [3]: plot_example_exploration_trajectories()
```



Mating Model

Considering P5 (mice need to reproduce) and A5 (periodicity of mating period), I chose as an appropriate model a model with 2 components.

Periodicity of mating desire

The first one takes into consideration the fact that mating desire is periodical. For ease of implementation, I took $\sin(\omega_{mating} \cdot t)$ function. The advantage of the sinusoidal function is that there is one parameter controlling the periodicity and one for the start of the phase at the beginning os the simulation.

Sinusoidals also come with the advantage that they vary periodically in the interval [-1, 1]. In our case, the value 1 represents the fact that the two mice are attracting (i.e. condition P5) each other and -1 the fact that the mice are currently avoiding each other (i.e. condition P2).

Mating desire

The model takes into consideration A4.2, the fact that mice are aroused with increased intensity as the other mouse is in the vicinity.

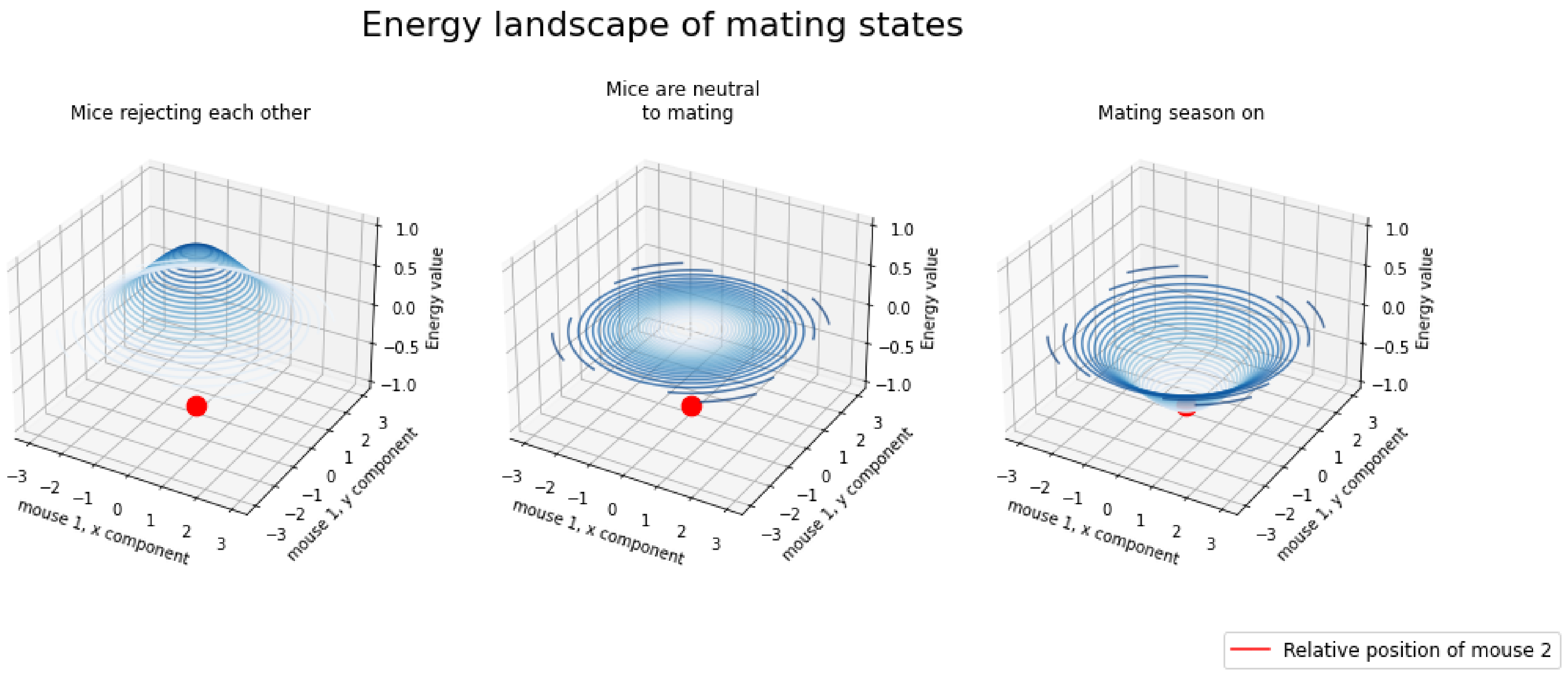
As a consequence, I chose as an appropriate description a potential well of Gaussian shape. When mating, from the perspective of the first mouse, there is an energy landscape with the lowest energy value pointing to the other mouse. The second mouse is, of course, handled symmetrically.

The mating energy model thus looks like:

$$E_{mating} = -\sin(\omega_{mating} t) \cdot \exp(-\frac{\left\| \vec{m_1} - \vec{m_2} \right\|^2}{\sigma^2_{mating}}) \quad (5)$$

This produces, from the perspective of mouse 1, the following energy landscape:

```
In [4]: plot_mating_energy_landscape(figsize=(16,6))
```



As mentioned before, the dynamics of the system is to do gradient descent on this energy landscape.

The gradient of mouse 1 follows from (5):

$$\frac{\partial E_{mating}}{\partial \overrightarrow{m}1} = \sin(\omega_{mating}t) \cdot \frac{2}{\sigma^2_{mating}} \cdot (\overrightarrow{m}_1 - \overrightarrow{m}_2) \cdot \exp(-\frac{\left\|\overrightarrow{m}_1 - \overrightarrow{m}_2\right\|^2}{\sigma^2_{mating}}) \quad (6)$$

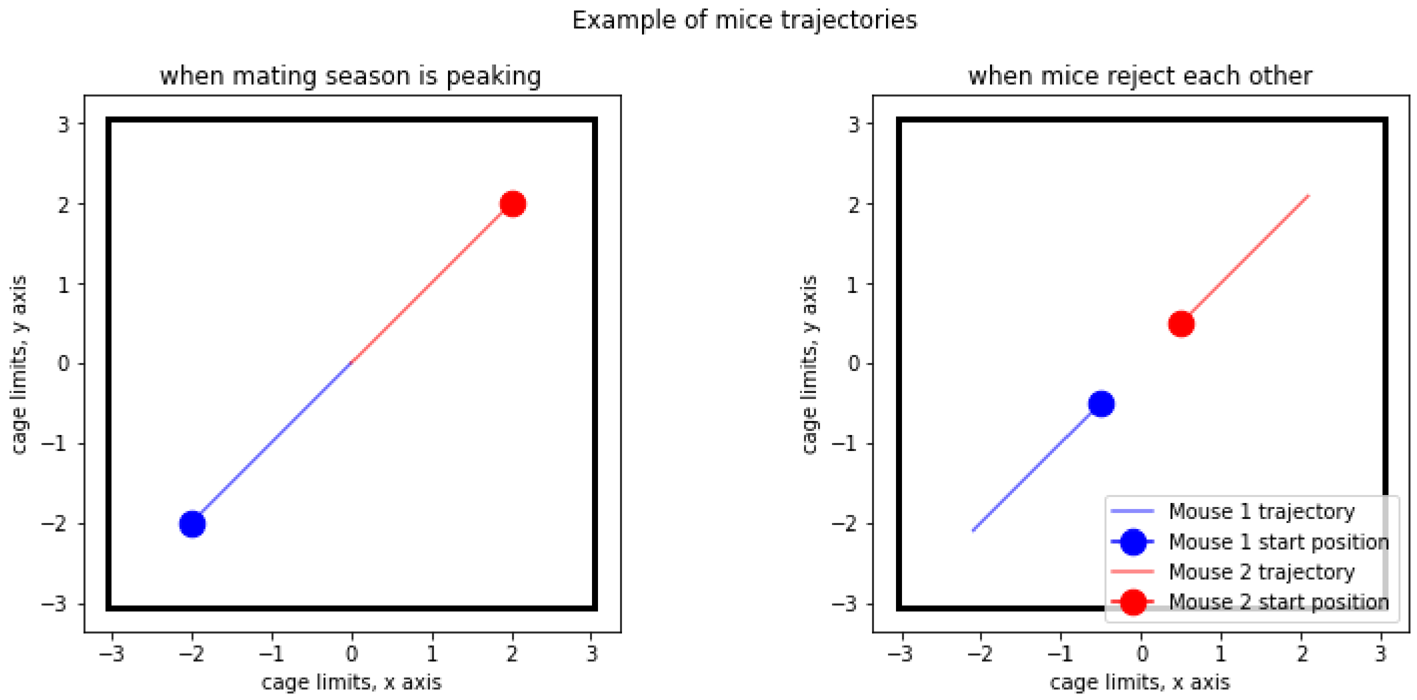
Here it is important that $\frac{\partial E_{mating}}{\partial \overrightarrow{m}1} \propto (\overrightarrow{m}_1 - \overrightarrow{m}_2)$.

Due to gradient **descent**, depending on the value of $\sin(\omega_{mating}t)$ the system will either try to:

1. reduce the distance between $\overrightarrow{m_1}$ and $\overrightarrow{m_2}$ in case $\sin(\omega_{mating} t) = 1$
2. increase the distance between $\overrightarrow{m_1}$ and $\overrightarrow{m_2}$ in case $\sin(\omega_{mating} t) = -1$

thus implement either mating desire (P5) or rejection (P6)

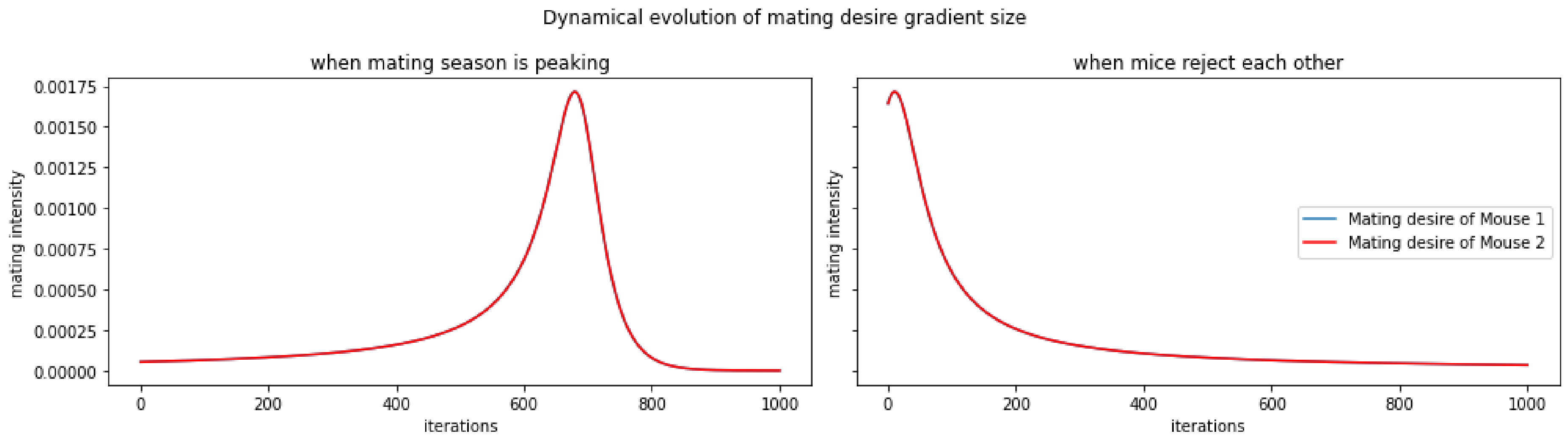
In [17]:
```
always_mating, always_rejecting = plot_mice_trajetories_for_mating_season_on_and_off(figsize=(14, 5))
```



The interpretation of these two graphs is rather easy. The mice move towards each other when desiring to mate and opposite to each other when they reject each other. This is how we would want out mating model to work.

One can also observe the evolution of the mating desire gradient length when mice are closing in on each other.

```
In [16]: plot_mating_desire_gradients_for_mating_on_and_off(always_mating, always_rejecting, figsize=(14, 4))
```



As wanting to model:

1. during mating season, the closer they get to each other, the higher the attraction
2. during rejection season, the closer they get to each other the stronger they reject each other

Feeding Model

The feeding model is implemented with the same idea of an energy landscape as the **mating model** is.

Basically, I have modelled

$$E_{feeding}(t, \vec{m}) = hunger_{max} \cdot \sigma(t) \cdot ||\vec{m} - \overrightarrow{foodspot}||^2 \quad (7)$$

where

$$\sigma(t) = \frac{1}{1 + \exp(-\frac{t-(t_{last feeding}+t_{hunger})}{slope})}$$

$\vec{m}$ can be either $\vec{m_1}$ or $\vec{m_2}$ and $\overrightarrow{foodspot}$ represents the vector position of the center of the food spot considered from the centered of our reference frame, the point $\overrightarrow{(0,0)}$.

From (7) is follows that:

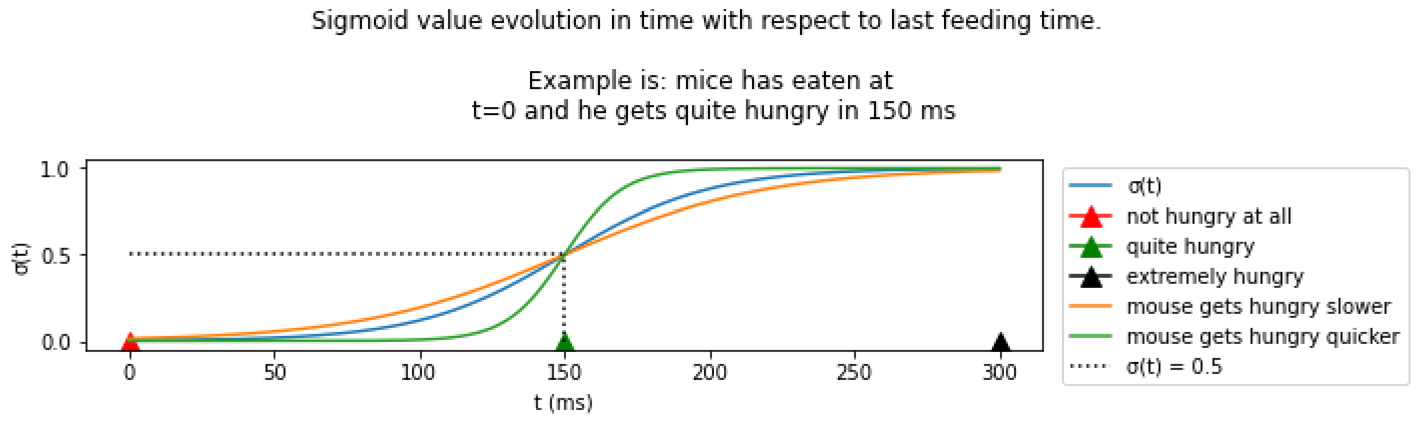
$$\frac{\partial E_{feeding}}{\partial \vec{m}_1} = 2 \cdot hunger_{max} \cdot \sigma(t) \cdot (\vec{m} - \overrightarrow{foodspot}) \quad (8)$$

The strength of the hunger feeling

$\sigma(t)$ conveniently models how hungry a mouse currently is. I have chosen the sigmoidal function because it can easily:

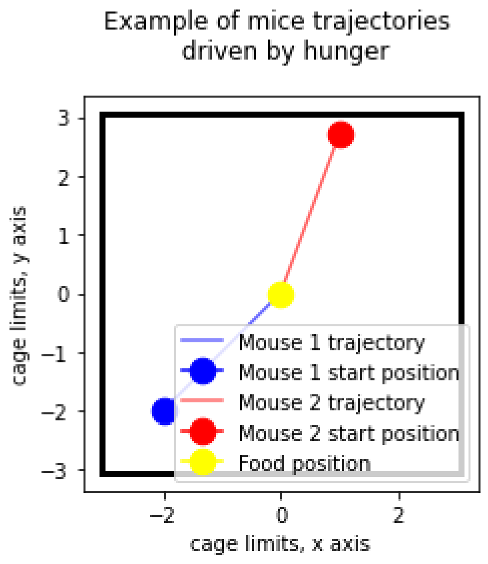
[illegible]

In [7]:
```
plot_sigmoid_function_output()
```



This produces, in isolation, the following example trajectory:

```
In [19]: plot_feeding_example_trajectory(figsize=(7, 4))
```



The observation to be made here is that because there is nothing else to stop/interest them, mice will always head towards the feeding spot.

The Mice Model: combining all our simple models

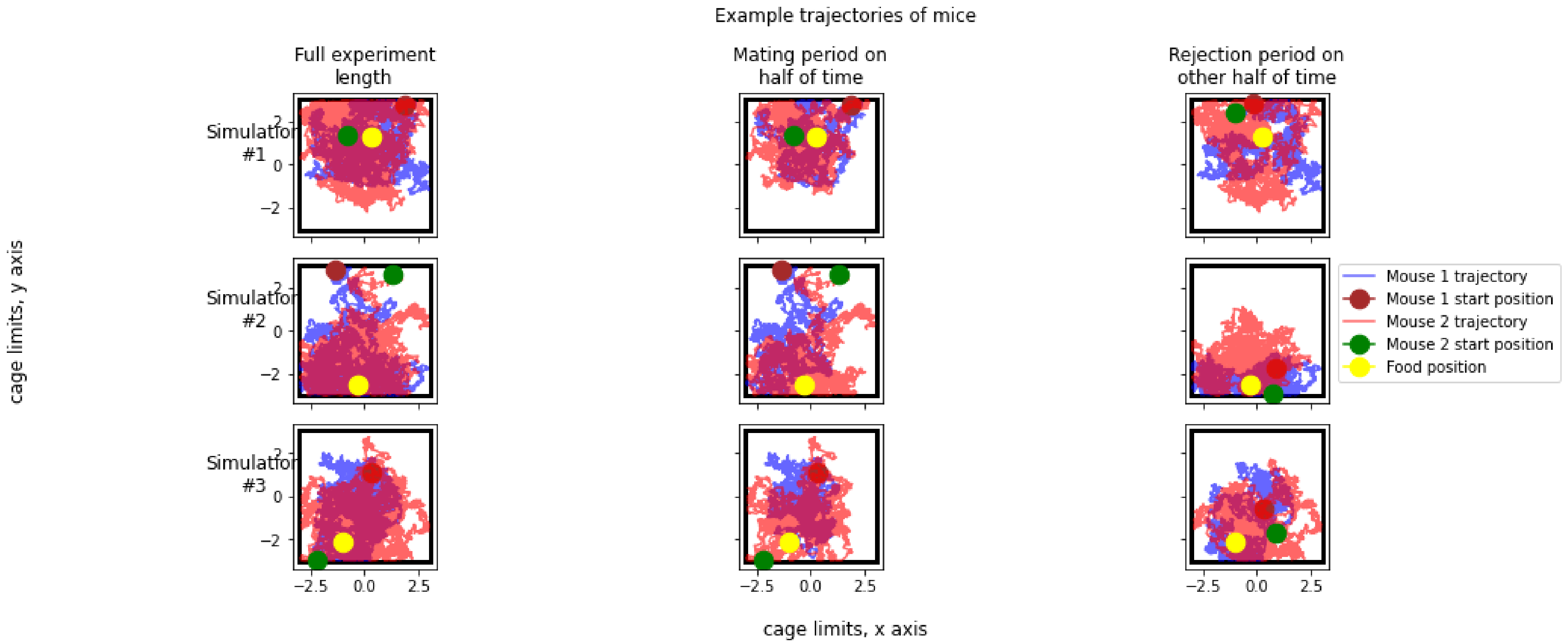
$$\tau_1 \dot{\vec{m_1}} = \tau_1 \frac{\vec{m_1}}{dt} = \sigma_{mouse} \cdot \xi(t) - \frac{\partial E_{mating}}{\partial \vec{m}1} - \frac{\partial E_{feeding}}{\partial \vec{m}1}$$

$$= \overbrace{\sigma_{mouse} \cdot \xi(t)}^{movement} - \underbrace{\sin(w_{mating} t) \cdot \frac{2}{\sigma_{mating}^2} \cdot (\vec{m_1} - \vec{m_2}) \cdot \exp(-\frac{\left\|\vec{m_1} - \vec{m_2}\right\|^2}{\sigma_{mating}^2})}_{mating} - \overbrace{hunger_{max} \cdot \sigma(t) \cdot ||\vec{m_1} - \vec{foodspot}||}^{feeding}$$

$$\tau_2 \dot{\vec{m_2}} = \sigma_2 \cdot \xi(t) - \sin(w_{mating} t) \cdot \frac{2}{\sigma_{mating}^2} \cdot (\vec{m_2} - \vec{m_1}) \cdot \exp(-\frac{\left\|\vec{m_2} - \vec{m_1}\right\|^2}{\sigma_{mating}^2}) - hunger_{max} \cdot \sigma(t) \cdot ||\vec{m_2} - \vec{foodspot}||$$

Results

```
In [22]: plot_example_full_trajectories(figsize=(16, 6))
```



It was quite difficult to place so much information on the graphs and they look quite heavy. Therefore, I chose split the full trajectories from column 1 into two half trajectories, one during mating time, the other during rejection.

I would draw the attention to the following:

1. The start position at the beginning of the simulation is drawn totally random and is irrelevant
2. The **start** position of the mice at the beginning of the *rejection* part **is quite close**, meaning the mice have been **drawn together** by the mating desire
3. Because I did not implement the *they don't mate where they feed* feature, when hungry and mating the mice will be strongly drawn to the feeding spot

The model is unable to predict a known feature of mice (maybe due to ineffective parameter tuning) the fact that mice prefer to stay in corners when they reject each other).

Discussion

In my opinion, the implementation looks elegant because each model (Movement, Mating, Food) works in isolation from the others.

Although I tried to keep the model simple in the end it has many free variables.

The Movement Model comes with 2 free parameters: $\sigma_{movement,mouse_1}$, $\sigma_{movement,mouse_2}$ that controls the variance of dW_t.

The Mating Model comes with 5 free parameters:

1. one ω_{mating} - angular speed at which mating peak comes and goes
2. two $\sigma_{mating,mouse}$ - how wide can the mice feel aroused by the other
3. two $mating_{max,mouse}$ - how strong each mouse reacts to the mating desire

The Food Model comes with:

1. one feeding radius - how close must the mice be to be able to feed
2. two hunger maximum values
3. two $t_{hunger,mouse}$ - controlling when the mouse gets hungry after he just fed

As a consequence we have 12 parameters we have to tweak and find a sweet spot in which the models are balanced. This would mean that there is no particular model stronger than the others all the time.

This proved quite difficult and time laborious, hinting at the fact that the model is already quite complex.

I have not considered P6, the fact that the mice don't reproduce where they eat. This could be implemented by making the mating model accept two models as parameters:

1. This current mating model implementation
2. a high energy component with the center on the food spot, making the mice unavailable to reach the food spot while mating season is on. This energy landscape is known as a **mexican hat** energy model.

Looking at actual simulated mice trajectories also look hard to interpret, but it is the way the problem was stated. My opinion is, however, that using **Fokker-Planck** equation to see how the diffusion of the mice movement under the effect of the other models would be preferable. The output should be, then, a heatmap showing a 2-D difussive process.

As a final statement, the report was written as a conclusion to the Modelling Seminar led by Prof. Sprekeler in the winter term 2022. I hope I was able to showcase a couple of skills I have gathered during the seminar.

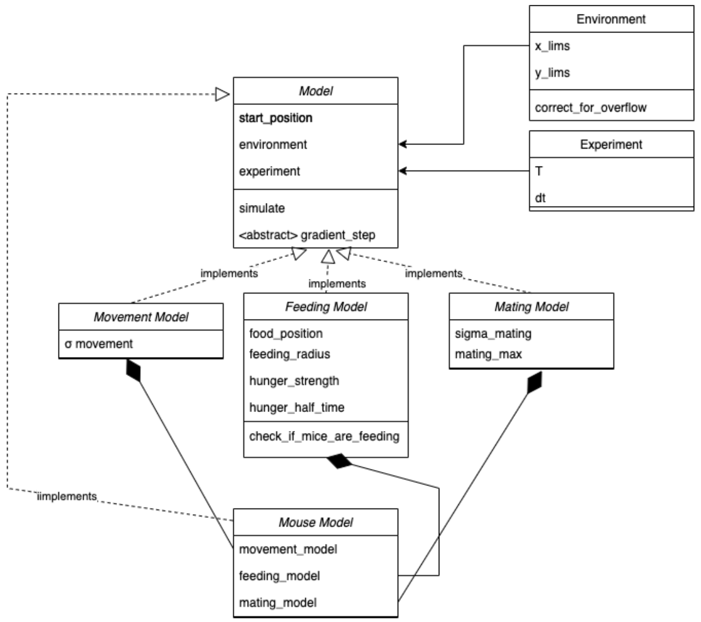
Appendix

Implementation Details

It was an interesting exercise to make the model:

1. testable
2. pluggable i.e. being able to turn on/off parts of the model to verify their contribution.

Thus, I have implemented the following software architecture:



The interesting part is that:

1. Euler-Maruyama is implemented only once, in the abstract class **Model**.
2. One could make a model fully extensible by providing a list of Models.
3. One must implement in each Model subclass the **gradient_step** method. Moreover, because each model has the information if it is deterministic or stocastic, the proper scaling with Δ_t must be implemented here.

Thus, the software is modular and easy to extend, implementing the **Open Closed Principle** .

The whole code base can be found on github (https://github.com/midumitrescu/modelling-course-prof-sprek-winter-2022).