Math 307: Problems for section 1.3

1. **Write down the vector approximating $f''(x)$ at interior points, the vector approximating $xf(x)$ at interior points, and the finite difference matrix equation for the finite difference approximation with $N = 4$ for the differential equation**

$$f''(x) + xf(x) = 0$$

for $1 \le x \le 3$ subject to

$$f(1) = 1, \quad f(3) = -1.$$

We look for approximations f_0, f_1, f_2, f_3, f_4 to the solution of the differential equation at the points $x_0 = 1$, $x_1 = 3/2$, $x_2 = 2$, $x_3 = 5/2$, $x_4 = 3$. We have $\Delta x = (3-1)/n = (3-1)/4 = 1/2$.

If $\mathbf{F} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$ is the vector containing the approximations to $f(x)$ that we wish to find, then the vector containing the approximations to $f''(x)$ at the interior points is given by

$$\mathbf{F}'' = (\Delta x)^{-2} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \mathbf{F}$$

while the vector containing approximations to $xf(x)$ is

$$\begin{bmatrix} 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_3 & 0 \end{bmatrix} \mathbf{F} = \begin{bmatrix} 0 & 3/2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5/2 & 0 \end{bmatrix} \mathbf{F}$$

Thus the matrix equation that models $f''(x) + xf(x) = 0$ (at the interior points) is

$$\begin{bmatrix} 1 & -2+3/8 & 1 & 0 & 0 \\ 0 & 1 & -2+1/2 & 1 & 0 \\ 0 & 0 & 1 & -2+5/8 & 1 \end{bmatrix} \mathbf{F} = \mathbf{0}$$

(Here we multiplied the equation by $(\Delta x)^2 = 1/4$). Now we adjoin the two boundary value equations $f_0 = 1$ and $f_4 = -1$. This gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -13/8 & 1 & 0 & 0 \\ 0 & 1 & -3/2 & 1 & 0 \\ 0 & 0 & 1 & -11/8 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{F} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

2. **Write down the matrix equation to solve in order to find the finite difference approximation with $N = 4$ for the same differential equation**

$$f''(x) + xf(x) = 0$$

for $1 \leq x \leq 3$ but now subject to

$$f'(1) = 1, \quad f(3) = -1$$

Thus the matrix equation that models $f''(x) + xf(x) = 0$ (at the interior points) is the same as before

$$\begin{bmatrix} 1 & -2+3/8 & 1 & 0 & 0 \\ 0 & 1 & -2+1/2 & 1 & 0 \\ 0 & 0 & 1 & -2+5/8 & 1 \end{bmatrix} \mathbf{F} = \mathbf{0}$$

(Again, we multiplied the equation by $(\Delta x)^2 = 1/4$).

The boundary condition $f'(1) = 1$ can be modeled by $(f_1 - f_0)/(\Delta x) = 1$ or $f_1 - f_0 = 1/2$ while the boundary condition $f(3) = -1$ corresponds to and $f_4 = -1$. This gives

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -13/8 & 1 & 0 & 0 \\ 0 & 1 & -3/2 & 1 & 0 \\ 0 & 0 & 1 & -11/8 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{F} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

3. **Use MATLAB/Octave to solve the matrix equations you derived in the last two problems for the vector F that approximates the solution (i.e., with $N = 4$). Then redo the calculation with $N = 50$ and plot the resulting functions.**

For define N and Δx

```
N=4;
DX=2/N;
```

Define the matrix L where $L\mathbf{F}$ corresponds to $f''(x)$

```
L=diag(-2*ones(1,N+1))+diag(ones(1,N),1)+diag(ones(1,N),-1);
L(1,1)=1;
L(1,2)=0;
L(N+1,N)=0;
L(N+1,N+1)=1;
```

Define the matrix Q where $Q\mathbf{F}$ corresponds to $xf(x)$

```
X=linspace(1,3,N+1);
Q=DX^2*diag(X);
Q(1,1)=0;
Q(N+1,N+1)=0;
```

Define the right side of the equation for the boundary conditions of the first problem.

```
b=zeros(N+1,1);
b(1)=1;
b(N+1)=-1;
```

Solve for **F**

```
F=(L+Q)\b
```

The result is

```
F =
   1.00000
   0.17778
  -0.71111
  -1.24444
  -1.00000
```

To solve the second problem we have to change the terms corresponding to the boundary conditions.

```
L(1,1)=-1;
L(1,2)=1;
b(1)=DX;
F=(L+Q)\b
```

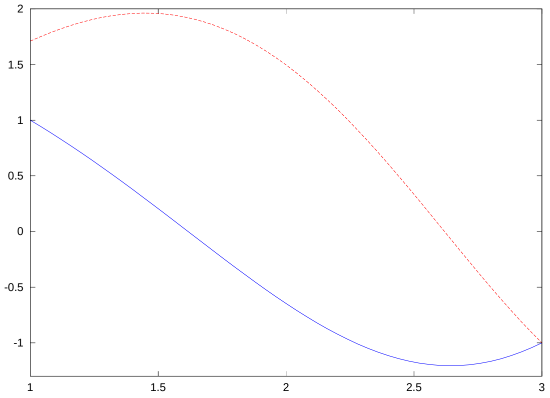
The result is

```
F =
   1.65385
   2.15385
   1.84615
   0.61538
  -1.00000
```

For the second part of the problem we change `N=4` to `N=50` above. Then at the end, we can plot the solution against the correct values of x using

```
plot(X,F);
```

Here are the resulting plots:



Questions 4–6 deal with the steady heat equation in a one-dimensional rod considered in the notes:

$$0 = kT''(x) - HT(x) + S(x),$$

where k and H are constants, subject to the boundary conditions

$$T = T_l \textbf{ at } x = x_l \textbf{ and } T = T_r \textbf{ at } x = x_r.$$

The MATLAB/Octave commands needed to find the finite difference approximation for $T(x)$ in the case $k = 1$, $H = 0$, $S(x) = 1$, $T_l = T_r = 1$, $x_l = 0$ and $x_r = 1$ are provided in `heat.m`.

4. **Modify the commands provided in `heat.m` to calculate the temperature profile in a rod cooled by the air in the case $k = 1$, $H = 1$, $S(x) = 1$, $T_l = 0$, $T_r = 2$, $x_l = -1$ and $x_r = 1$. Describe briefly the modifications made, and hand in a plot of the solution for $n = 50$.**

   The changes to the problem solved using the commands in `heat.m` are

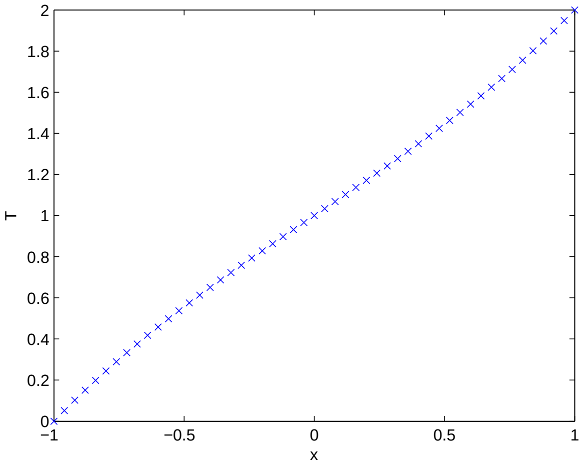
   (a) there is cooling to the air ($H \neq 0$), so we must add a matrix Q that describes the term $-T(x)$ in the differential equation

   (b) the locations of and values at the boundary conditions have changed.

   The modified commands are:

```
n = 50;
X = linspace(-1,1,n+1);
dx = 2/n;
L = ( diag(-2*ones(1,n+1)) + diag(ones(1,n),-1) + diag(ones(1,n),1));
L(1,1) = 1;
L(1,2) = 0;
L(n+1,n+1) = 1;
L(n+1,n) = 0;
Q = -diag(ones(1,n+1))*dx^2;
Q(1,1) = 0;
Q(n+1,n+1) = 0;
r = -ones(n+1,1)*dx^2;
r(1) = 0;
r(n+1) = 2;
T = (L+Q)\r;
```

   The plotting commands remain the same. The plot for $n = 50$ is below.

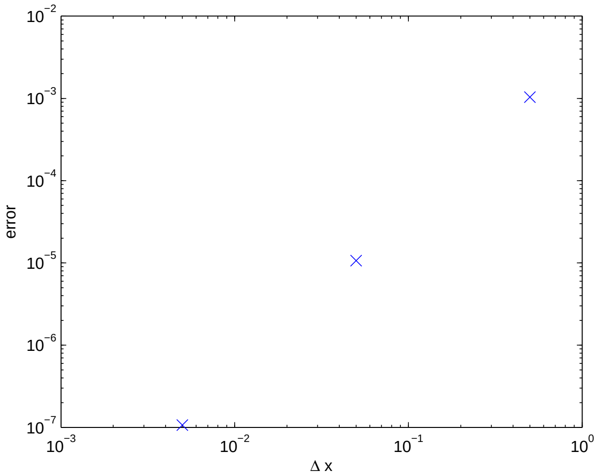


5. **For the case given in Q4, compute the finite difference approximation at $x = -0.5$ for $n = 4$, 40 and 400. The true solution at this point is $1 - \sinh 0.5/\sinh 1$. Make a log-log plot of the magnitude of the error in the finite difference approximation against Δx. What is the approximate slope of this curve?**

   The magnitude of the error for the three values of n is

4	0.0010350
40	1.0669×10^{-5}
400	1.0672×10^{-7}

   A log-log plot of these errors against $\Delta x = 2/n$ is shown below.



   The slope of the curve is approximately 2 in this plot. This suggests that the error in the solution is proportional to Δx^2 (which is in fact the case and can be shown using Taylor series expansions).

6. **The boundary condition $T'(x) = 0$ at $x = x_l$ or $x = x_r$ describes an insulating end to the rod. Write down an approximation for $T'(x_l)$ using T_0 and T_1. Also write down an approximation for $T'(x_r)$ using T_{n-1} and T_n. Find the modification needed to the matrix**

equation if insulating boundary conditions are placed at $x = x_l$ and $x = x_r$ (you should find that two rows of the matrix change and two entries of the vector on the right-hand-side change). Modify the commands provided in `heat.m` to calculate the temperature profile in a heated rod in the case $k = 1$, $H = 0$, $S(x) = 1$, with insulating boundary conditions at $x = 0$ and $x = 1$ (representing a continuously heated rod from which no heat escapes). Try to find the solution for $n = 10$. Is the solution reasonable?

$$T'(x_l) \approx \frac{T_1 - T_0}{\Delta x}, \qquad T'(x_r) \approx \frac{T_n - T_{n-1}}{\Delta x}$$

Insulating boundary conditions at $x = x_l$ and $x = x_r$ imply that $T'(x_l) = T'(x_r) = 0$ and so $T_0 = T_1$ and $T_n = T_{n-1}$ or

$$T_0 - T_1 = 0 \qquad \text{and} \qquad T_n - T_{n-1} = 0.$$

The equations for the interior points do not change. Only the equations for the boundary conditions ($i = 0$ and $i = n$) need to be altered.

Combining these equations with the equations derived in class for the interior points, we obtain the matrix equation

$$\begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{n-3} \\ T_{n-2} \\ T_{n-1} \\ T_n \end{bmatrix} = \Delta x^2 \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \\ \vdots \\ -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

The modified commands to solve this equation using MATLAB/Octave are:

```
n = 10;
X = linspace(0,1,n+1);
dx = 1/n;
L = ( diag(-2*ones(1,n+1)) + diag(ones(1,n),-1) + diag(ones(1,n),1));
L(1,1) = 1;
L(1,2) = -1;
L(n+1,n+1) = -1;
L(n+1,n) = 1;
r = -ones(n+1,1)*dx^2;
r(1) = 0;
r(n+1) = 0;
T = L\r;
```

Trying these commands with MATLAB/Octave we find that the matrix P is singular to machine precision. This suggests that there may be something wrong with the physics of the problem. If check the reduced row echelon form of the augmented matrix (`rref([L r])`) we get

$$\begin{bmatrix} 1 & 0 & \dots & 0 & -1 & 0 \\ 0 & 1 & \dots & 0 & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

From the last line we can see that the system of equations has no solution in this case. This is because, physically, we cannot have a steady (time-independent) temperature profile in the rod if we are continually supplying heat but not allowing any to escape.

7. **In this problem we will use finite differences to solve Laplace's equation on a square. We want to approximate the solution $f(x, y)$ to the partial differential equation (Laplace's equation)**

$$f_{xx}(x, y) + f_{yy}(x, y) = 0 \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

subject to the boundary conditions

$$\begin{aligned} f(x, 0) &= a_1(x) \quad 0 \le x \le 1 \\ f(0, y) &= a_2(y) \quad 0 \le y \le 1 \\ f(x, 1) &= a_3(x) \quad 0 \le x \le 1 \\ f(1, y) &= a_4(y) \quad 0 \le y \le 1 \end{aligned}$$

You can think of $f(x, y)$ as the shape (i.e., the height) of a streched rubber membrane attached along the edges of a square to a wire described by the four known functions $a_1, \ldots, a_4$.

Pick N equally spaced points $x_k = k/N$ and $y_k = k/N$, $k = 0, \ldots N$ along the x and y axes with spacing $\Delta x = \Delta y = (1/N)$. Then, in our equations the unknown function $f(x, y)$ will be replaced by a grid of unknown values $f_{i,j}$ with $i = 0, \ldots N$ and $j = 0, \ldots N$ with the idea that $f(x_i, y_j) \sim f_{i,j}$.

For interior points (x_i, y_j) (i.e., $1 \le i \le N - 1$ and $1 \le j \le N - 1$) we can write down approximations to the second partial derivatives using the formula we derived for single variable B.V.P.:

$$\begin{aligned} f_{x,x}(x_i, y_j) &\sim (\Delta x)^{-2} (f_{i+1,j} - 2f_{i,j} + f_{i-1,j}) \\ f_{y,y}(x_i, y_j) &\sim (\Delta y)^{-2} (f_{i,j+1} - 2f_{i,j} + f_{i,j-1}) \end{aligned}$$

Adding these together and setting the result to zero is the discrete analogue of Laplace's equation. This gives a linear equation in the unknowns $f_{i,j}$ for each interior point:

$$f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j} = 0. \tag{1}$$

Here we have cancelled a factor of $N^2 = (\Delta x)^{-2} = (\Delta y)^{-2}$ from each side of the equation.

Next we set the boundary values. This is done with the equations

$$\begin{aligned} f_{i,0} &= a_1(x_i), \quad i = 0, \ldots N & (2) \\ f_{0,j} &= a_2(y_j), \quad j = 0, \ldots N & (3) \\ f_{i,N} &= a_3(x_i), \quad i = 0, \ldots N & (4) \\ f_{N,j} &= a_4(y_j), \quad j = 0, \ldots N & (5) \\ & & (6) \end{aligned}$$

In total this gives $(N + 1)^2$ equations in $(N + 1)^2$ unknowns $f_{i,j}$.

The only real difficulty in writing this system down as a matrix equation is that the unknowns $f_{i,j}$ are indexed by a double index. To write the matrix equation we need to number the unknowns by a single index. In other words we want a vector $F =$

$[F_1, F_2, \ldots, F_{(N+1)^2}]^T$ that contains the $f_{i,j}$'s in some order. Then, we can write the equation in the form $LF = Y$ for some matrix L and vector Y. If we order the $f_{i,j}$'s by starting on the bottom row and working our way up, i.e.,

$$\begin{bmatrix} f_{0,N} & f_{1,N} & f_{2,N} & \cdots & f_{N,N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f_{0,2} & f_{1,2} & f_{2,2} & \cdots & f_{N,2} \\ f_{0,1} & f_{1,1} & f_{2,1} & \cdots & f_{N,1} \\ f_{0,0} & f_{1,0} & f_{2,0} & \cdots & f_{N,0} \end{bmatrix} = \begin{bmatrix} F_{N(N+1)+1} & F_{N(N+1)+2} & F_{N(N+1)+3} & \cdots & F_{(N+1)^2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ F_{2(N+1)+1} & F_{2(N+1)+2} & F_{2(N+1)+3} & \cdots & F_{3(N+1)} \\ F_{(N+1)+1} & F_{(N+1)+2} & F_{(N+1)+3} & \cdots & F_{2(N+1)} \\ F_1 & F_2 & F_3 & \cdots & F_{N+1} \end{bmatrix}$$

then we see that

$$f_{i,j} = F_{\kappa(i,j)}$$

where $\kappa(i,j) = j(N+1) + i + 1$ is a re-indexing function. Now the equations 1 for each interior point (i,j) correspond to the $\kappa(i,j)$th row in our equation $LF = Y$ and can be written

$$F_{\kappa(i+1,j)} + F_{\kappa(i-1,j)} + F_{\kappa(i,j+1)} + F_{\kappa(i,j-1)} - 4F_{\kappa(i,j)} = 0$$

In other words, the $\kappa(i,j)$th row in the matrix L has a 1 in the $\kappa(i+1,j)$, $\kappa(i-1,j)$, $\kappa(i,j+1)$ and $\kappa(i,j-1)$ spots, and a -4 in the $\kappa(i,j)$ spot. The vector Y has a 0 in the $\kappa(i,j)$ spot.

Similarly, the equations 2 for each boundary point also correspond to a row in our equation $LF = b$. These equations can be written

$$\begin{aligned} F_{\kappa(i,0)} &= a_1(x_i), \quad & i &= 0, \ldots N \\ F_{\kappa(0,j)} &= a_2(y_j), \quad & j &= 0, \ldots N \\ F_{\kappa(i,N)} &= a_3(x_i), \quad & i &= 0, \ldots N \\ F_{\kappa(N,j)} &= a_4(y_j), \quad & j &= 0, \ldots N \end{aligned}$$

from which the entries in L and Y can be deduced.

Here are MATLAB/Octave commands to implement this procedure when

$$\begin{aligned} a_1(x) &= \sin(\pi x) \\ a_2(y) &= 0 \\ a_3(x) &= 0 \\ a_4(y) &= 0. \end{aligned}$$

These commands can be found in the file `laplaceeqn.m`

First we choose `N`, initialize the matrix `L` and the vector `Y`, and set `X` to the vector of $N+1$ equally spaced points between 0 and 1.

```
N=30
L=zeros((N+1)^2,(N+1)^2);
Y=zeros((N+1)^2,1);
X=linspace(0,1,(N+1));
```

Next we define the re-indexing function that converts the double index (i,j) into the single index $\kappa(i,j) = i(N+1) + j + 1$. In Octave, as long as a function is not the first thing in a .m file, the filename does not have to match the function name.

In MATLAB you have to take the following three lines and put them in a separate file called `k.m`

```
function k=k(i,j,N)
    k=j*(N+1)+i+1;
end
```

Now we define the parts of the matrix L and vector Y that correspond to the boundary conditions along the sides of the square.

```
for n=0:N
    L(k(0,n,N),k(0,n,N))=1;
    Y(k(0,n,N))=0;
    L(k(N,n,N),k(N,n,N))=1;
    Y(k(N,n,N))=0;
end
```

Next we define the parts of the matrix L and vector Y that correspond to the boundary conditions along the bottom and top of the square.

```
for n=1:N-1
    L(k(n,0,N),k(n,0,N))=1;
    Y(k(n,0,N))=sin(pi*X(n+1));
    L(k(n,N,N),k(n,N,N))=1;
    Y(k(n,N,N))=0;
end;
```

Finally we define the parts of the matrix L and vector Y that correspond to the equations for the interior points.

```
for i=1:N-1
    for j=1:N-1
        L(k(i,j,N),k(i,j,N))=-4;
        L(k(i,j,N),k(i+1,j,N))=1;
        L(k(i,j,N),k(i-1,j,N))=1;
        L(k(i,j,N),k(i,j+1,N))=1;
        L(k(i,j,N),k(i,j-1,N))=1;
    end
end
```

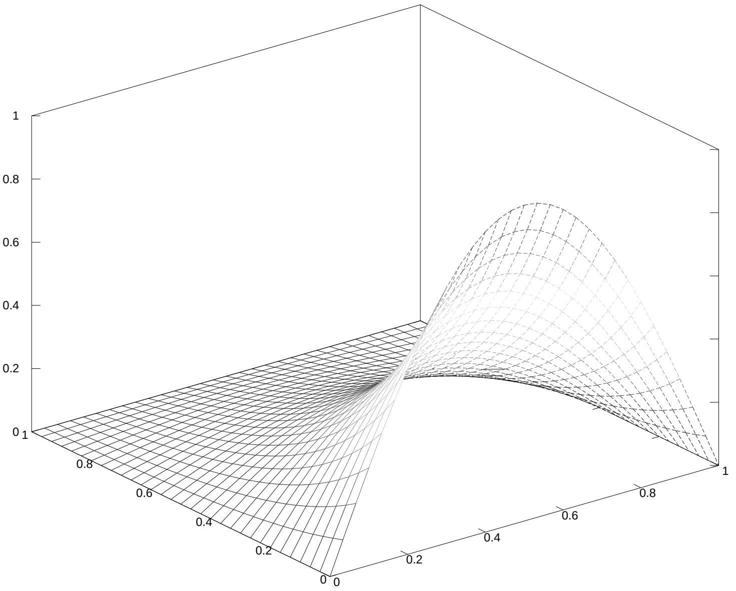
Now we can solve the equation for F and plot the result. To do this we have to put the `F` values in a two dimensional grid `FF` and use the `mesh` command to do the 3-d plot. If `X` and `Y` are vectors of length n and `Z` is an nxn matrix then `mesh(X,Y,Z)` plots the points `[X(j),Y(i),Z(i,j)]`.

```
F=L\Y;

FF=zeros(N+1,N+1);
for i=0:N
    for j=0:N
        FF(j+1,i+1)=F(k(i,j,N));
    end
end

mesh(X,X,FF);
```

We can print out the resulting graph using `print laplace1.pdf` (or `print laplace1.jpg` or `print laplace1.eps`). This will produce a pdf file (or jpg or eps file) containing the graph. Here is the result:



Run the file `laplaceeqn.m` to produce this picture. Then modify the code to solve Laplace's equation with boundary conditions:

$$\begin{aligned} f(x,0) &= \sin(\pi x) \quad 0 \le x \le 1 \\ f(0,y) &= 0 \quad 0 \le y \le 1 \\ f(x,1) &= \sin(\pi x) \quad 0 \le x \le 1 \\ f(1,y) &= 0 \quad 0 \le y \le 1 \end{aligned}$$

Say what code you modified, and hand in the resulting picture. Finally modify the code to solve Laplace's equation with boundary conditions:

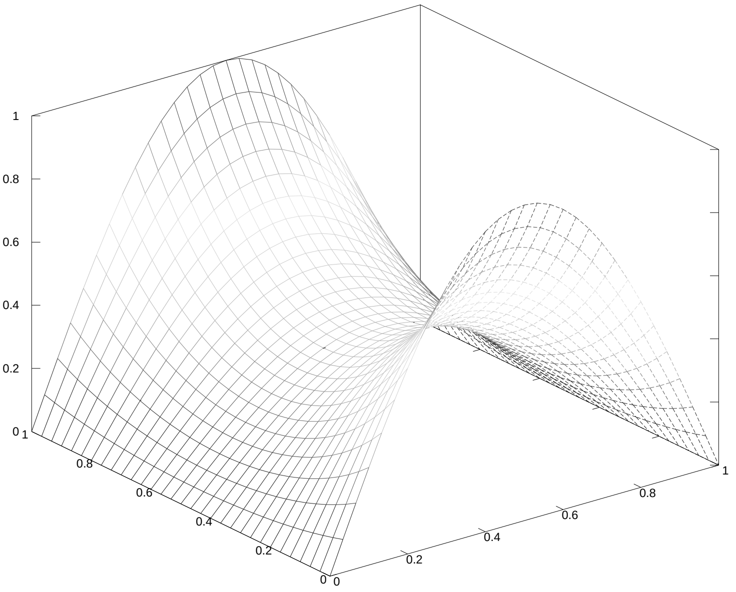
$$\begin{aligned} f(x,0) &= \sin(\pi x) \quad 0 \le x \le 1 \\ f(0,y) &= 0 \quad 0 \le y \le 1 \\ f_y(x,1) &= 0 \quad 0 \le x \le 1 \\ f(1,y) &= 0 \quad 0 \le y \le 1 \end{aligned}$$

The third boundary condition is called a Neumann boundary condition, and corresponds to detaching the rubber membrane from the wire along the top boundary. Again, say what code you modified, and hand in the resulting picture.

The first modification is to change the code defining the top and bottom boundary conditions to

```
for n=1:N-1
L(k(n,0,N),k(n,0,N))=1;
Y(k(n,0,N))=sin(pi*X(n+1));
L(k(n,N,N),k(n,N,N))=1;
Y(k(n,N,N))=sin(pi*X(n+1));
end;
```

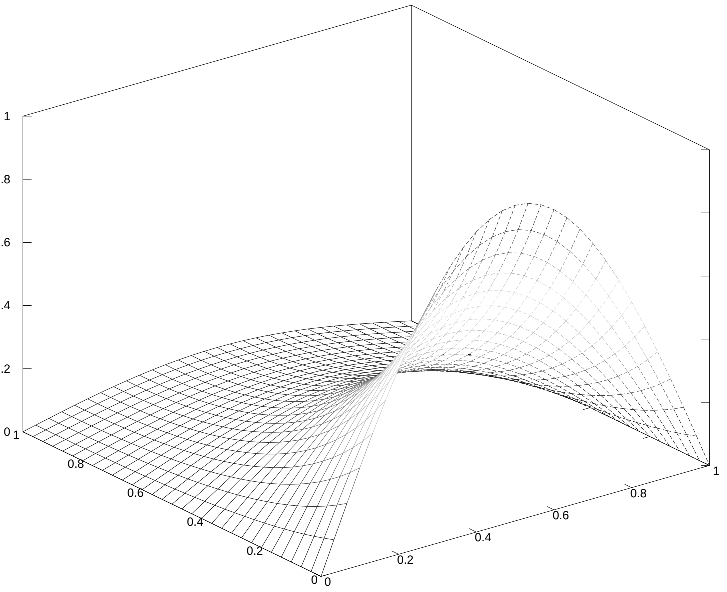
The resulting plot looks like



The second modification is to change the code defining the top and bottom boundary conditions to

```
for n=1:N-1
L(k(n,0,N),k(n,0,N))=1;
Y(k(n,0,N))=sin(pi*X(n+1));
L(k(n,N,N),k(n,N,N))=-1;
L(k(n,N,N),k(n,N-1,N))=1;
Y(k(n,N,N))=0;
end;
```

The resulting plot looks like


1
0.8
0.6
0.4
0.2
0
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1