Math 307: Problems for section 4.1

1. **For the following matrices find**

   (a) all eigenvalues

   (b) linearly independent eigenvectors for each eigenvalue

   (c) the algebraic and geometric multiplicity for each eigenvalue

   and state whether the matrix is diagonalizable.

$$A = \begin{bmatrix} 3 & 7 \\ 2 & -2 \end{bmatrix} \quad \textbf{(calculate by hand)}$$

$$B = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \quad \textbf{(calculate using Matlab/Octave or otherwise)}$$

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{bmatrix} \quad \textbf{(calculate using Matlab/Octave or otherwise)}$$

For matrix A,

$$\lambda I - A = \begin{bmatrix} \lambda - 3 & -7 \\ -2 & \lambda + 2 \end{bmatrix}.$$

Therefore $\det(\lambda I - A) = (\lambda - 3)(\lambda + 2) - 14 = \lambda^2 - \lambda - 20 = (\lambda - 5)(\lambda + 4)$. Roots of this polynomial are -4 and 5 and so these are the eigenvalues. Each has algebraic multiplicity 1.

For $\lambda = 5$ we have

$$(\lambda I - A)\mathbf{v} = \begin{bmatrix} 2 & -7 \\ -2 & 7 \end{bmatrix} \mathbf{v} = \mathbf{0}$$

and so an eigenvector is $\mathbf{v} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$. The geometric multiplicity is 1.

For $\lambda = -4$ we have

$$(\lambda I - A)\mathbf{v} = \begin{bmatrix} -7 & -7 \\ -2 & -2 \end{bmatrix} \mathbf{v} = \mathbf{0}$$

and so an eigenvector is $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The geometric multiplicity is 1.

Matrix A is diagonalizable (the algebraic and geometric multiplicities match).

For matrix B, we use Octave:

```
>B = [1 -3 3; 3 -5 3; 6 -6 4]
B =

   1  -3   3
```

```
   3  -5   3
   6  -6   4

> [V D] = eig(B)
V =

   0.408248   0.408248  -0.087127
   0.408248  -0.408248  -0.746633
   0.816497  -0.816497  -0.659506

D =

   4.00000   0.00000   0.00000
   0.00000  -2.00000   0.00000
   0.00000   0.00000  -2.00000
```

The eigenvalues are 4 (with algebraic multiplicity 1) and -2 (with algebraic multiplicity 2). For $\lambda = 4$, an eigenvector is $\begin{bmatrix} 0.408248 \\ 0.408248 \\ 0.816497 \end{bmatrix}$, or a cleaner choice $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and the geometric multiplicity is 1. For $\lambda = -2$, two linearly independent (by inspection) eigenvectors are $\begin{bmatrix} 0.408248 \\ -0.408248 \\ -0.816497 \end{bmatrix}$ and $\begin{bmatrix} -0.087127 \\ -0.746633 \\ -0.659506 \end{bmatrix}$, or cleaner choices are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The geometric multiplicity is 2. The matrix is diagonalizable.

For matrix C using Octave again:

```
> C = [1 2 1; 2 0 -2; -1 2 3]
C =

   1   2   1
   2   0  -2
  -1   2   3

> [V D] = eig(C)
V =

   0.57735 + 0.00000i  -0.70711 - 0.00000i  -0.70711 + 0.00000i
  -0.57735 + 0.00000i  -0.00000 - 0.00000i  -0.00000 + 0.00000i
   0.57735 + 0.00000i  -0.70711 + 0.00000i  -0.70711 - 0.00000i

D =

   0.00000 + 0.00000i   0.00000 + 0.00000i   0.00000 + 0.00000i
   0.00000 + 0.00000i   2.00000 + 0.00000i   0.00000 + 0.00000i
   0.00000 + 0.00000i   0.00000 + 0.00000i   2.00000 - 0.00000i
```

The eigenvalues are 0 (with algebraic multiplicity 1) and 2 (with algebraic multiplicity 2). For $\lambda = 0$ an eigenvector is $\begin{bmatrix} 0.57735 \\ -0.57735 \\ 0.57735 \end{bmatrix}$, or a cleaner choice $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. The geometric multiplicity is 1. For $\lambda = 2$, Octave only provides one linearly independent eigenvector: $\begin{bmatrix} -0.70711 \\ 0 \\ -0.70711 \end{bmatrix}$, or a cleaner choice $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. We can check that no other vectors can be found using

```
6> rref(2*eye(3)-C)
ans =

   1   0  -1
   0   1  -0
   0   0   0
```

Only one free parameter exists and so the nullity of $2I - C$ is one and no other linearly independent eigenvectors exist for $\lambda = 2$. Thus the geometric multiplicity is 1. The matrix is not diagonalizable because the algebraic and geometric multiplicities do not match for this eigenvalue.

2. **Find a 3×3 real, non-zero (*i.e.* not all entries zero) matrix which has all three eigenvalues zero.**

   A possible matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

3. **(a) By hand find a matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ and corresponding eigenvectors**

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   (b) Using Matlab/Octave or otherwise, find a matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$ and corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 9 \\ 4 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

   (a)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 7 & -2 \\ 2 & 2 \end{bmatrix}$$

   (b) Using Octave:

```
> S = [2 9 2; 1 4 1; 0 4 2]
S =

   2   9   2
   1   4   1
   0   4   2

> D = diag([1 2 3])
D =

   1   0   0
   0   2   0
   0   0   3
```

```
 A = S*D*S^(-1)
A =

   2  -2   2
   0   1   1
  -4   8   3
```

4. **Show that if A is an $n \times n$ square matrix and each column sums to c, then c is an eigenvalue of A. *Hint: if you cannot show this in a few lines, try another approach.***

   Consider $\det(cI - A)$. The determinant is unchanged by replacing the last row of the matrix $cI - A$ by the sum of all the rows. But then the last entry in each column of the matrix is c minus the sum of all the entries in the corresponding column of A, which was given to be c. Therefore all the entries in the last row of $cI - A$ are zero and so $\det(cI - A) = 0$.

   Here is a another way to do this problem. First notice that the eigenvalues of A and A^T are the same. To see this recall that $\det(B) = \det(B^T)$ for any matrix B. Thus $\det(\lambda I - A) = \det((\lambda I - A)^T) = \det(\lambda I - A^T)$. This shows that A and A^T have the same characteristic polynomial, and hence the same eigenvalues. For the matrix A in the problem A^T has *rows* that sum to c. For this matrix, it is easy to see that $\mathbf{v} = [1, 1, \ldots, 1]^T$ is an eigenvector with eigenvalue c.

5. **If $p(\lambda)$ is the characteristic polynomial of an $n \times n$ invertible matrix A, find an expression for the characteristic polynomial of A^{-1} in terms of the characteristic polynomial of A.**

   Let the characteristic polynomial of A^{-1} be $q(\mu)$, then

$$
\begin{aligned}
q(\mu) &= \det(A - \mu I) = \det\left(A^{-1}(I - \lambda A)\right) \\
&= \det A^{-1}. \det(I - \lambda A) \quad (\text{because } \det(AB) = \det A. \det B) \\
&= \det A^{-1}. \det\left(-\lambda\left(A - \frac{1}{\lambda}I\right)\right) \\
&= \det A^{-1}(-\lambda)^n \det\left(A - \frac{1}{\lambda}I\right) \quad (\text{linearity of the determinant}) \\
&= \det A^{-1}(-\lambda)^n p\left(\frac{1}{\lambda}\right).
\end{aligned}
$$

6. **Find the Jordan canonical form of the matrix**

$$
A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

   The characteristic polynomial of A is $(\lambda - 1)^4$. So A has only one distinct eigenvalue with value 1.

$$
I - A = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

   So $(I - A)\mathbf{v} = \mathbf{0}$ only has one linearly independent solution, the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Therefore the Jordan canonical form for A has only one Jordan block and so must be

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$