Math 307: Problems for section 2.3

October 16, 2012

1. **Let D be the incidence matrix in the example done in the course notes.**

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

Using MATLAB/Octave (or otherwise) compute rref(D) and find the bases for $N(D)$, $R(D)$ and $R(D^T)$. Find a basis for $N(D^T)$ by computing rref(D^T). Verify that every loop vector is a linear combination of vectors in this basis.

We compute

```
> D = [-1 1 0 0; 0 -1 1 0; 0 0 -1 1; 0 -1 0 1;1 0 0 -1];
>rref(D)

ans =

   1   0   0  -1
   0   1   0  -1
   0   0   1  -1
   0   0   0   0
   0   0   0   0
```

This gives the following bases:

$$N(D) : \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{1}$$

$$R(D) : \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$R(D^T) : \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \tag{3}$$

Then

```
>rref(D')

ans =

   1  -0  -0  -0  -1
   0   1  -0   1  -1
   0   0   1   1  -1
   0   0   0   0   0
```

gives the basis

$$N(D^T): \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The loop vectors are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

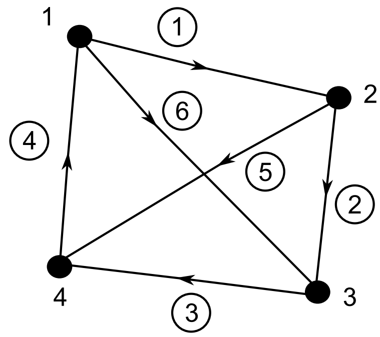
$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = - \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{6}$$

$$\tag{7}$$

2. **Draw the graph corresponding to the incidence matrix**

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$



3. **How many connected components does the graph whose incidence matrix is in the file `hmkgraph.m` have (provided on the website)? Explain how you get your answer.**

   To answer this question we must compute the dimension of $N(D)$ for the matrix in the file. We can define `D` to be the the matrix in the file by typing `hmkgraph`. We could then do `rref(D)` and count the number of free variables. It's a bit easier to do instead

```
>size(D)

ans =

   20   20

>rank(D)

ans =  17
```

   This tells us that D is a 20×20 matrix with rank 17 Thus $\dim(N(D)) = 20 - 17 = 3$.

4. **Let $L = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$ be the Laplacian matrix (with $L^T = L$, C invertible and $N(L)$ equal to all vectors with constant entries).**

   (a) Show that the voltage to current matrix $A - B^T C^{-1} B$ is also symmetric (equal to its transpose).

   (b) Show that the range of $(A - B^T C^{-1} B)$ is equal to $\text{span}\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$. (Hint: Show that if $\mathbf{J} = (A - B^T C^{-1} B)\mathbf{b}$ then $\begin{bmatrix} \mathbf{J} \\ \mathbf{0} \end{bmatrix} = L \begin{bmatrix} \mathbf{b} \\ -C^{-1} B \mathbf{b} \end{bmatrix}$)

   (c) Show that (a) and (b) imply that $A - B^T C^{-1} B$ is a multiple of $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

   (a) Both the matrices A and C are symmetric, so $A^T = A$ and $C^T = C$, (this must be true if L is symmetric). Taking the transpose of $A - B^T C^{-1} B$ we get

   $$(A - B^T C^{-1} B)^T = A^T - (B^T C^{-1} B)^T = A^T - B^T (C^{-1})^T (B^T)^T = A^T - B^T (C^T)^{-1} B = A - B^T C^{-1} B$$

   Thus $A - B^T C^{-1} B$ is symmetric.

   (b) If we let $\mathbf{b} \in \mathbb{R}^2$, then

   $$L \begin{bmatrix} \mathbf{b} \\ -C^{-1} B \mathbf{b} \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ -C^{-1} B \mathbf{b} \end{bmatrix} = \begin{bmatrix} A\mathbf{b} - B^T C^{-1} B \mathbf{b} \\ B\mathbf{b} - C C^{-1} B \mathbf{b} \end{bmatrix} = \begin{bmatrix} (A - B^T C^{-1} B)\mathbf{b} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{J} \\ 0 \end{bmatrix},$$

where $\mathbf{J} = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$. Now $\begin{bmatrix} \mathbf{J} \\ 0 \end{bmatrix}$ is in $R(L)$ and we know that $R(L) = N(L)^\perp$. So $\begin{bmatrix} \mathbf{J} \\ 0 \end{bmatrix}$ must be orthogonal to the vectors in the null space of L, and the null space of L is span$\{\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T\}$. So

$$\langle \begin{bmatrix} \mathbf{J} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \rangle = J_1 + J_2 = 0.$$

Therefore the $\mathbf{J}$ must be of the form $\begin{bmatrix} J \\ -J \end{bmatrix}$ for some $J \in \mathbb{R}$. We also had that $\mathbf{J} = (A - B^T C^{-1} B)\mathbf{b}$ for arbitrary $\mathbf{b} \in \mathbb{R}^2$. Thus all possible vectors $\mathbf{J}$ form the range of $A - B^T C^{-1} B$. So the range of $A - B^T C^{-1} B$ equals span$\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$.

(c) Since $R(A - B^T C^{-1} B) = \text{span}\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$ each column must be a scalar multiple of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So $A - B^T C^{-1} B$ must look like $\begin{bmatrix} c_1 & c_2 \\ -c_1 & -c_2 \end{bmatrix}$. But $A - B^T C^{-1} B$ is also symmetric. This requires that $c_1 = -c_2$ so $A - B^T C^{-1} B = \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

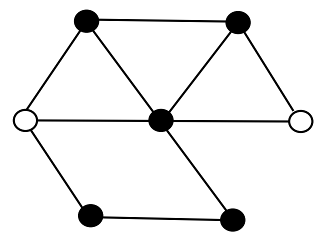
5. **Write a MATLAB/Octave function `resistance(L,n,m)` that takes as input the Laplacian matrix L for a circuit and the position of two nodes n and m and returns the effective resistance between those two nodes. Here is a template for the function (provided on the website in `resistance.m`). Edit this file, replacing the stars ***** with your code. Hand in your code with a list of the changes that you made.**

Here is a version of `resistance.m`

```
function r=resistance(L,n,m)

% if n=m then return with r=0;
if(n==m)
r=0;
return;
end

% if m < n the swap n and m
if(m < n)
temp=m; m=n; n=temp;
end

%find the size of the matrix L
N=length(L);

%swap the nth and mth nodes to positions 1 and 2
L = L([n,m,1:n-1,n+1:m-1,m+1:N],[n,m,1:n-1,n+1:m-1,m+1:N]);

%compute the voltage-to current map DN
A = L(1:2,1:2);
B = L(3:N,1:2);
C = L(3:N,3:N);
DN = A - B'*C^(-1)*B;
```

```
%the effective resistance is the reciprocal of the 1 1 entry of DN
r = 1/DN(1,1);
end
```

6. **Using the function you wrote in the last question, compute the effective resistance of the following network between the indicated nodes (the hollow nodes). Assume that all resistances have value $R_i = 1$. Provide the MATLAB/Octave commands that you used.**



The Laplacian matrix for the graph (with vertices numbered from left to right, and top to bottom) is

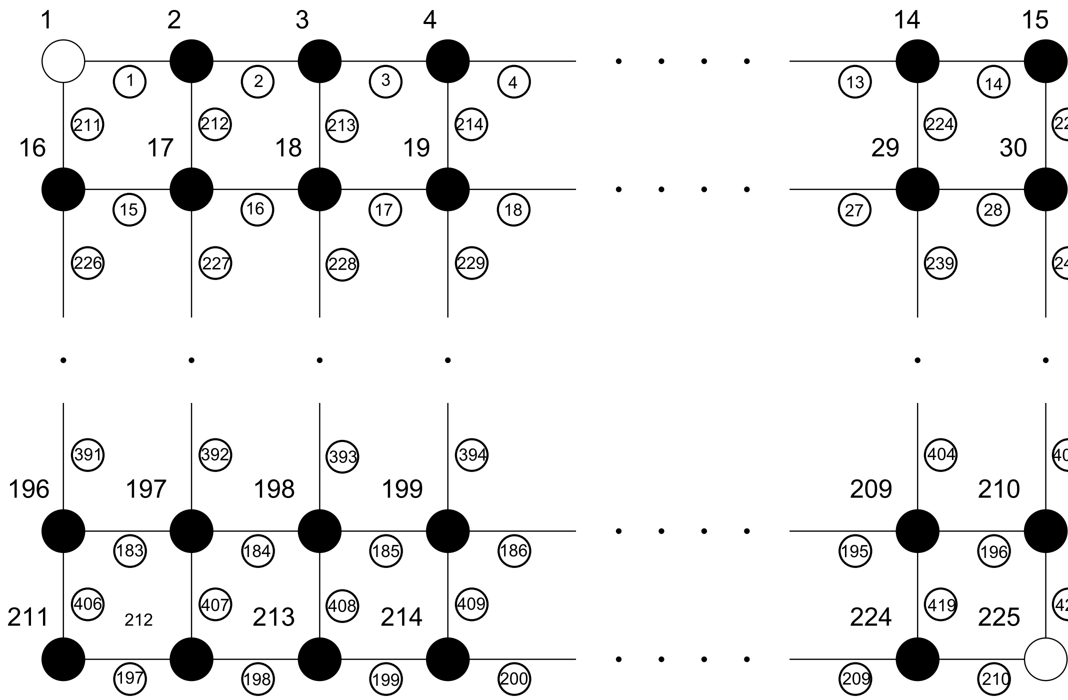
$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 3 & -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 5 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Using the function written in the last question we find

```
>resistance(L,3,5)

ans =   1.0526
```

Using `format rat` we find that the resistance is 20/19.

7. (***Resistor network analogy to lightning***) **Air is generally a very poor electrical conductor. Its ability to conduct depends on such factors as humidity and density, variations of which can be considered as essentially random over the depth of the troposphere. Nevertheless it can conduct, as observed in lightning strikes. This question investigates the nature of lightning, using a simple model of the electrical resistance of the atmosphere as the resistor network shown below, with resistances randomly assigned to each edge.**



Consider resistances of the form $R_{j} = \exp(\alpha p_{j})$, where α is a parameter that we can vary and p_{j} is a (uniformly distributed) random number between 0 and 1 assigned to edge j. For a resistor network of 15×15 nodes, we apply a voltage of 1 at the first node (representing a point high in the atmosphere) and 0 at the last node (representing a point on the ground). How does the path that the resulting current follows change as α is increased from 0 to 30?

To answer this question you may use the MATLAB/Octave `.m` file `lightning.m` provided on the website. This file contains a function `lightning(n,a,p)` that returns a vector of length $2(n-1)n$ containing currents along each edge for an $n \times n$ network of nodes with a parameter value of a and for a vector p (of length $2(n-1)n$) that contains the probability for each edge.

If you like a challenge you can try to write this function yourself. If you use the provided function, write a couple of short sentences describing how the function computes the current through each edge.

The MATLAB/Octave `.m` file `lightningplot.m` provided on the website contains a function `lightningplot(n,I)` that plots the grid with larger currents represented by thicker lines.

Use the functions `lightning(n,a,p)` and `lightningplot(n,I)` to produce a series of plots showing how the paths that the current takes change as α increases.

Hand in the list of MATLAB/Octave commands that you used to solve the problem, and the plots.

For small values of α the directed graph with weighted edges that represents the resistor network is said to have *weak disorder* (essentially the weights assigned to all edges are similar) whereas for large α it is said to have *strong disorder* (essentially there is a wide range of weights). By comparing your plots with lightning, would you say that the resistance of the atmosphere has weak or strong disorder?

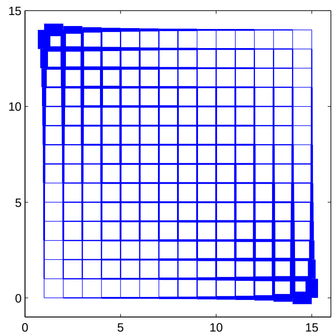
Here is MATLAB/Octave `lightning.m` file is provided on the website.

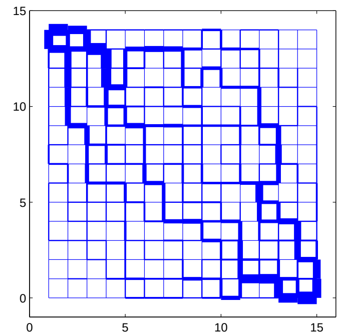
```
function I = lightning(n,a,p)

% This function calculates the currents along each edge
% for an n  n network of nodes with a parameter value of a
% and for a vector p (of length 2(n  1)n)
% that contains the probability for each edge.

  % create the incidence matrix D
  D = zeros(2*n*(n-1),n^2);

  % first deal with the horizontal edges
  for i=1:n
    for j=1:n-1
      D((n-1)*(i-1)+j,n*(i-1)+j)   = -1;
      D((n-1)*(i-1)+j,n*(i-1)+j+1) =  1;
    end
  end

  % now deal with the vertical edges
  for i=1:n-1
    for j=1:n
      D((n-1)*n+n*(i-1)+j,n*(i-1)+j) = -1;
      D((n-1)*n+n*(i-1)+j,n*i+j)     =  1;
    end
  end

  % find the resistance matrix according to the rule given in the
  % question
  R = diag(exp(a*p));

  % find the Laplacian matrix
  L = D'*R^(-1)*D;

  % reorder the nodes so that the first two have the prescribed
  % voltage
  L = L([1 n^2 2:n^2-1],[1 n^2 2:n^2-1]);

  % find the submatrices A, B and C
  A = L(1:2,1:2);

  B = L(3:n^2,1:2);

  C = L(3:n^2,3:n^2);

  % calculate the voltage at all the remaining nodes
  v = -C\B*[1; 0];

  % add in the prescribed voltages at the first and last nodes
  v = [1; v; 0];

  % calculate the current through each edge
```

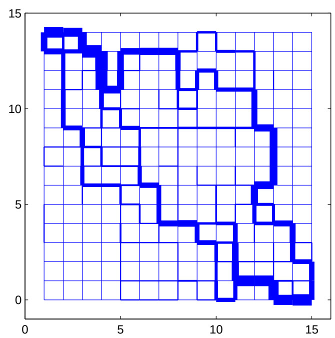
```
  I = R^(-1)*D*v;
```

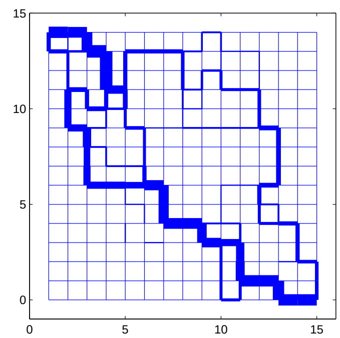
Sample plots for $a = 0$, $a = 5$, $a = 10$, $a = 20$ and $a = 30$ are shown below. They were created using

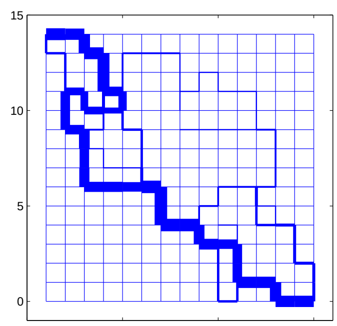
```
p = rand(2*14*15,1);
I=lightning(15,0,p);
lightningplot(15,I);
I=lightning(15,5,p);
lightningplot(15,I);
I=lightning(15,10,p);
lightningplot(15,I);
I=lightning(15,20,p);
lightningplot(15,I);
I=lightning(15,30,p);
lightningplot(15,I);
```







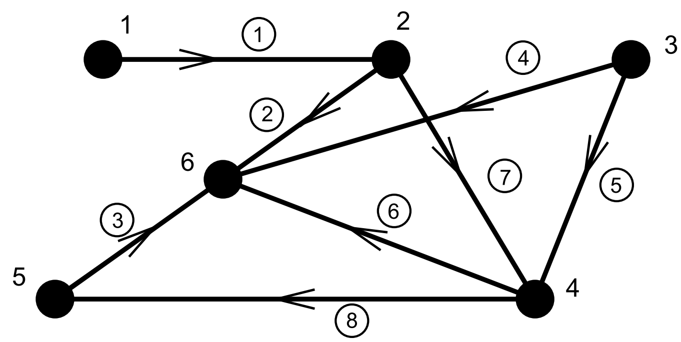




The fact that the current passes predominately along a single path for $a = 30$ in the plot above (with a few small side-shoots) is reminiscent of lightning paths, and suggests that the resistance of the atmosphere has strong disorder.

A reference for the concepts of weak and strong disorder in random resistor networks is *Current flow in random resistor networks: The role of percolation in weak and strong disorder*, Physical Review E, volume 71, 045101 (2005).

8. **The ideas we have used for resistor networks can also be applied to the flow of fluid through networks, for example air flow through connected tunnels or the flow of oil through sandstone (the paths between pores in the sandstone correspond to edges and pores correspond to nodes). Consider the following network of tubes:**



Let $Q_{(j)}$ be the volume flux of fluid through tube (j) and q_j be the rate at which fluid is injected/extracted from the network of tubes at node j. Let p_j be the pressure applied at each node and $P_{(j)}$ be the pressure difference between the ends of tube (j). Let $R_{(j)}$ be the resistance of the tube (it depends on the geometry of the tube).

The physical laws governing (slow) flow of an incompressible fluid through this network of tubes are similar to the electrical circuit rules of a resistor network:

(a) **The Hagen–Poiseuille law states that** $R_{(j)} = \dfrac{P_{(j)}}{Q_{(j)}}$

(b) **There can be no pressure difference around a closed loop.**

(c) **The total volume flux coming in to a node must equal the total volume flux coming out.**

For the network of tubes sketched above, find the incidence matrix D. Find bases for and give the meaning of $R(D)$, $N(D)$, $R(D^T)$, $N(D^T)$

The incidence matrix:

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

The nullspace of D is the set of all vectors of pressures applied at the nodes such that there is no pressure difference along any of the tubes. Therefore the applied pressures at the nodes must all be equal and a basis for $N(D)$ is $\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}$.

The range of D is the set of all vectors of pressure differences along the tubes. The rank of the matrix is $r(D) = 6 - n(D) = 5$. Because the vectors of pressure differences are in $\mathbb{R}^8$, there must be 3 constraints on the possible vectors of pressure differences. We use the fact that the pressure difference around a closed loop must be zero. Three linearly independent constraints are:

$$\begin{aligned}-P_{②} + P_{⑥} + P_{⑦} &= 0\\ -P_{④} + P_{⑤} + P_{⑥} &= 0\\ P_{③} - P_{⑥} + P_{⑧} &= 0\end{aligned}$$

Setting $P_{⑤}$, $P_{⑥}$, $P_{⑦}$, $P_{⑧}$ as free variables we have the basis

$$\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\0\\0\\0\\0\\1\end{bmatrix},$$

where the first vector represents the fact that the pressure at the first point is not contrained by any loop.

The nullspace of D^T is the set of all vectors of volume fluxes through the tubes such that the rate of fluid injection/extraction from the network at each of the nodes is zero. If the network is isolated then these are the only permissible volume fluxes. We have $n(D^T) = 8 - r(D^T) = 8 - r(D) = 3$. A basis for $N(D^T)$ is given by three linearly independent loop vectors:

$$\begin{bmatrix}0\\-1\\0\\0\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\-1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\-1\\0\\1\end{bmatrix}.$$

The range of D^T is the set of all vectors of rates of fluid injection/extraction at each node. We have $r(D^T) = r(D) = 5$. Because the vectors of rates of fluid injection/extraction are in $\mathbb{R}^6$, there is one constraint. This is that the total rate of injection/extraction from the network must be zero, or equivalently:

$$q_1 + q_2 + q_3 + q_4 + q_5 + q_6 = 0.$$

A basis is thus

$$\begin{bmatrix}-1\\0\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\-1\\1\end{bmatrix}.$$