Math 307: Problems for section 2.3

October 16, 2012

1. **Let D be the incidence matrix in the example done in the course notes.**

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

   Using MATLAB/Octave (or otherwise) compute rref(D) and find the bases for $N(D)$, $R(D)$ and $R(D^T)$. Find a basis for $N(D^T)$ by computing rref(D^T). Verify that every loop vector is a linear combination of vectors in this basis.

2. **Draw the graph corresponding to the incidence matrix**

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

3. **How many connected components does the graph whose incidence matrix is in the file `hmkgraph.m` have (provided on the website)? Explain how you get your answer.**

4. **Let $L = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$ be the Laplacian matrix (with $L^T = L$, C invertible and $N(L)$ equal to all vectors with constant entries).**

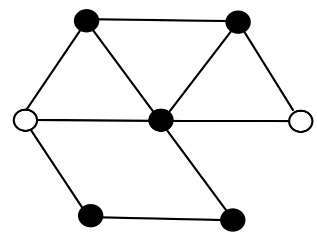
   (a) **Show that the voltage to current matrix $A - B^T C^{-1} B$ is also symmetric (equal to its transpose).**

   (b) **Show that the range of $(A - B^T C^{-1} B)$ is equal to $\text{span}\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$. (Hint: Show that if $\mathbf{J} = (A - B^T C^{-1} B)\mathbf{b}$ then $\begin{bmatrix} \mathbf{J} \\ \mathbf{0} \end{bmatrix} = L \begin{bmatrix} \mathbf{b} \\ -C^{-1} B \mathbf{b} \end{bmatrix}$)**

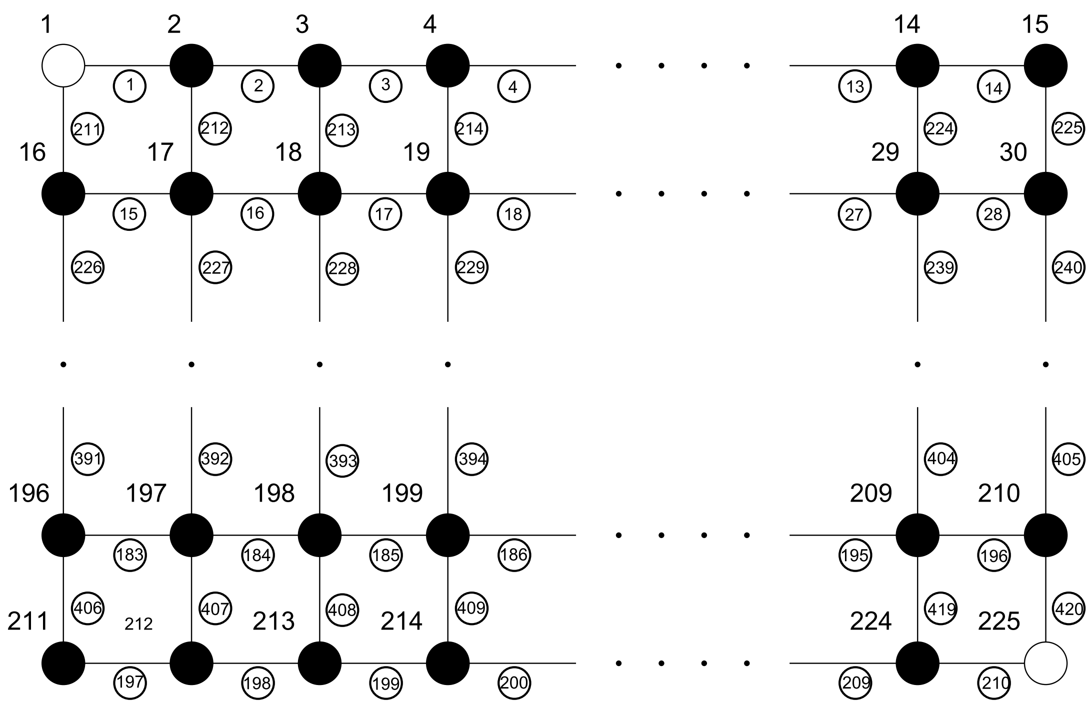
   (c) **Show that (a) and (b) imply that $A - B^T C^{-1} B$ is a multiple of $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$**

5. **Write a MATLAB/Octave function `resistance(L,n,m)` that takes as input the Laplacian matrix L for a circuit and the position of two nodes n and m and returns the effective resistance between those two nodes. Here is a template for the function (provided on the website in `resistance.m`). Edit this file, replacing the stars `*****` with your code. Hand in your code with a list of the changes that you made.**

6. Using the function you wrote in the last question, compute the effective resistance of the following network between the indicated nodes (the hollow nodes). Assume that all resistances have value $R_i = 1$. Provide the MATLAB/Octave commands that you used.



7. (***Resistor network analogy to lightning***) Air is generally a very poor electrical conductor. Its ability to conduct depends on such factors as humidity and density, variations of which can be considered as essentially random over the depth of the troposphere. Nevertheless it can conduct, as observed in lightning strikes. This question investigates the nature of lightning, using a simple model of the electrical resistance of the atmosphere as the resistor network shown below, with resistances randomly assigned to each edge.



Consider resistances of the form $R_{j} = \exp(\alpha p_{j})$, where α is a parameter that we can vary and p_{j} is a (uniformly distributed) random number between 0 and 1 assigned to edge j. For a resistor network of 15×15 nodes, we apply a voltage of 1 at the first node (representing a point high in the atmosphere) and 0 at the last node (representing a point on the ground). How does the path that the resulting current follows change as α is increased from 0 to 30?

To answer this question you may use the MATLAB/Octave `.m` file `lightning.m` provided on the website. This file contains a function `lightning(n,a,p)` that returns a vector of length $2(n-1)n$ containing currents along each edge for an $n \times n$ network of nodes with a parameter value of a and for a vector p (of length $2(n-1)n$) that contains the probability for each edge.

If you like a challenge you can try to write this function yourself. If you use the provided function, write a couple of short sentences describing how the function computes the current through each edge.

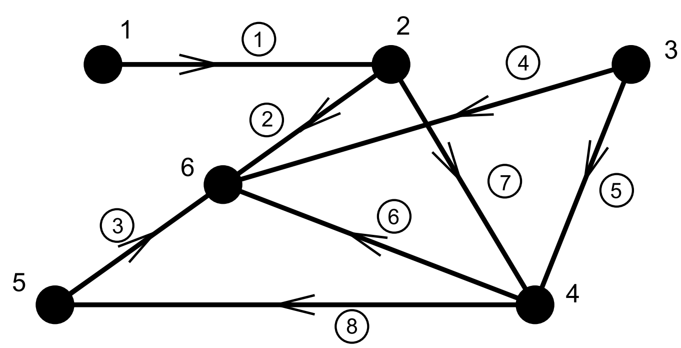
The MATLAB/Octave `.m` file `lightningplot.m` provided on the website contains a function `lightningplot(n,I)` that plots the grid with larger currents represented by thicker lines.

Use the functions `lightning(n,a,p)` and `lightningplot(n,I)` to produce a series of plots showing how the paths that the current takes change as α increases.

Hand in the list of MATLAB/Octave commands that you used to solve the problem, and the plots.

For small values of α the directed graph with weighted edges that represents the resistor network is said to have *weak disorder* (essentially the weights assigned to all edges are similar) whereas for large α it is said to have *strong disorder* (essentially there is a wide range of weights). By comparing your plots with lightning, would you say that the resistance of the atmosphere has weak or strong disorder?

8. The ideas we have used for resistor networks can also be applied to the flow of fluid through networks, for example air flow through connected tunnels or the flow of oil through sandstone (the paths between pores in the sandstone correspond to edges and pores correspond to nodes). Consider the following network of tubes:



Let Q_{j} be the volume flux of fluid through tube j and q_j be the rate at which fluid is injected/extracted from the network of tubes at node j. Let p_j be the pressure applied at each node and P_{j} be the pressure difference between the ends of tube j. Let R_{j} be the resistance of the tube (it depends on the geometry of the tube).

The physical laws governing (slow) flow of an incompressible fluid through this network of tubes are similar to the electrical circuit rules of a resistor network:

(a) The Hagen–Poiseuille law states that $R_{j} = \dfrac{P_{j}}{Q_{j}}$

(b) There can be no pressure difference around a closed loop.

(c) The total volume flux coming in to a node must equal the total volume flux coming out.

For the network of tubes sketched above, find the incidence matrix D. Find bases for and give the meaning of $R(D)$, $N(D)$, $R(D^T)$, $N(D^T)$