Math 307: Problems for section 4.3

1. **The matrix**
$$A = \begin{bmatrix} 6 & 1 & 2 & 2 & 1 \\ 1 & 5 & 2 & 1 & 2 \\ 2 & 2 & 3 & 1 & 2 \\ 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 2 & 3 \end{bmatrix}$$
has positive eigenvalues. Use the power method to find the largest and the smallest ones, and the corresponding eigenvectors. Check whether the two eigenvectors you have computed are orthogonal.

We just iterate a fixed number of times. A more sophisticated program would check for convergence, say by computing the norm of the distance of each iterate from the previous one, and stopping when this is less than some predetermined error. We can use `X'*A*X/norm(X)^2` to compute the eigenvalue corresponding to the eigenvector `X` at the end of the calculation. Notice that this computes eigenvalues of `A`, even if we iterated `A^(-1)` to find `X`.

```
A=[6  1  2  2  1; 1  5  2  1  2; 2  2  3  1  2; 2  1  1  3  2; 1  2  2  2  3]
X = rand(5,1);
for k = 1:100
  Y=A*X;
  X=Y/norm(Y);
end
XTOP=X

XTOP =

   0.54452
   0.46678
   0.42598
   0.37128
   0.40781

LAMBDATOP = X'*A*X/norm(X)^2

LAMBDATOP =  10.534
```

Now we iterate `A^(-1)` to find the smallest eigenvector (that is, closest to zero). As discussed in the notes, we shouldn't actually compute `A^(-1)`.

```
X = rand(5,1);
for k = 1:100
  Y=A\X;
  X=Y/norm(Y);
```

```
end
XBOT=X

XBOT =

  -0.233357
   0.027144
   0.483210
   0.499856
  -0.679308

LAMBDABOT = X'*A*X/norm(X)^2

LAMBDABOT =  0.36929
```

To check orthogonality we compute

```
dot(XTOP,XBOT)

ans = 0
```

2. **Using the power method, find the eigenvalues closest to -6 and closest to 1 for the matrix:**

$$A = \begin{bmatrix} 1 & 7 & -11 & 2 & 5 \\ 0 & 1 & 4 & 8 & -2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 4 & 5 \\ 7 & 8 & 2 & 3 & 0 \end{bmatrix}$$

Also write down the MATLAB/Octave commands you used to find your answer.

First find the eigevalue/eigenvector pair where the eigenvalue is closest to -6. The commands we use are

```
> A = [1 7 -11 2 5; 0 1 4 8 -2; 0 1 0 1 0; 2 0 2 4 5; 7 8 2 3 0];
> x = rand(5,1)
> for k=1:30
> y = (A+6*eye(5))\x;
> x = y/norm(y)
> end
```

This gives us the eigenvector

```
x =
   0.761004
  -0.322384
   0.035381
   0.118031
  -0.549325
```

We can now find the eigenvalue as

```
>lambda = x'*A*x/norm(x)^2
lambda =  -5.7758
```

So $\lambda = -5.7758$ with eigenvector $\begin{bmatrix} 0.761004 \\ -0.322384 \\ 0.035381 \\ 0.118031 \\ -0.549325 \end{bmatrix}$.

We can check this using

```
> A*x-lambda*x
ans =
   0.0000e+00
   0.0000e+00
   2.7756e-17
   1.1102e-16
  -4.4409e-16
```

and we see that we have found a correct solution.

We can do the same for the eigenvalue closest to 1:

```
> x = rand(5,1)
x =
   0.032946
   0.208115
   0.684701
   0.488835
   0.386108

> for k=1:30
> y = (A-eye(5))\x;
> x = y/norm(y);
> end
> x
x =
  -0.66627
   0.56318
   0.41677
  -0.16147
   0.19785

> lambda = x'*A*x/norm(x)^2
lambda =  0.96387
```

So the eigenvalue is 0.96387 and the eigenvector is $\begin{bmatrix} -0.66627 \\ 0.56318 \\ 0.41677 \\ -0.16147 \\ 0.19785 \end{bmatrix}$.