Math 307: Problems for section 4.7

March 25, 2014

1. **For $n\times$n matrices A and B do AB and BA always have the same eigenvalues? Use MATLAB/Octave to guess an answer and then verify your guess in the special case that one of the matrices, say A, is invertible. Do they have the same singular values? What happens when A is $n \times m$ and B is $m \times n$ matrices with $n \neq m$? Guess the answer using MATLAB/Octave.**

   Solution: Using random matrices we compute:

```
> A=rand(3,3);
> B=rand(3,3);
> eig(A*B)
ans =

   2.7630659
  -0.0035037
   0.1346587

> eig(B*A)
ans =

   2.7630659
  -0.0035037
   0.1346587

> svd(A*B)
ans =

   2.8103108
   0.2633213
   0.0017616

> svd(B*A)
ans =

   2.8859254
   0.3128385
   0.0014439
```

   So the singular values are definitely different, but the eigenvalues seem to be the same. To show this when A is invertible, notice that

$$\det(\lambda I - AB) = \det(A(\lambda I - BA)A^{-1}) = \det(A)\det(\lambda I - BA)\det(A^{-1}) = \det(\lambda I - BA)$$

which shows that AB and BA have the same characteristic polynomial, and therefore the same eigenvalues. When A is $n \times m$ and B is $m \times n$ matrices with $n \neq m$ then AB is $n \times n$ and BA is $m \times m$ so the number of eigenvalues is different. Let's see what happens with a random matrix.

```
> A=rand(2,4);
> B=rand(4,2);
> eig(A*B)
ans =

   1.99703
   0.19454

> eig(B*A)
ans =

   1.9970e+00
   1.9454e-01
  -2.9779e-16
  -3.3526e-17
```

It looks like the the extra eigenvalues of the bigger matrix (BA) are all zero (to within numerical error) and the non-zero eigenvalues are the same. This is true.

2. **Suppose the matrix A given by**

```
> A
A =

   0.95   0.70   0.10
   0.57   0.52   0.25
   0.28   0.67   0.76
   0.63   0.61   0.30
```

contains measured values that are accurate to within 0.1. Is it possible that the "real" matrix AA (i.e., without errors) has a non trivial null space? If so, what is a good approximation for this matrix and for a basis of its null space? Verify that the vector(s) you have found are in the null space of the matrix AA you have found.

Solution: Computing the singular value decomposition gives

```
> [U S V]=svd(A)
U =

  -0.594331  -0.556675  -0.385668  -0.433756
  -0.427240  -0.080920   0.894265  -0.105867
  -0.474473   0.825062  -0.181327  -0.247525
  -0.488991  -0.053268  -0.136643   0.859872

S =

Diagonal Matrix

   1.894008          0          0
```

```
          0   0.660877          0
          0          0   0.013282
          0          0          0

V =

  -0.65948  -0.57122   0.48866
  -0.66229   0.13398  -0.73717
  -0.35562   0.80979   0.46667
```

The smallest singular value is smaller than the measurment error, so we may set it to zero in S and then recompute USV^T to find an approximation with a non trivial null space spanned by the last column of V.

```
> S(3,3)=0
S =

Diagonal Matrix

   1.89401         0         0
         0   0.66088         0
         0         0   0.00000
         0         0         0

> AA=U*S*V'
AA =

   0.95250   0.69622   0.10239
   0.56420   0.52876   0.24446
   0.28118   0.66822   0.76112
   0.63089   0.60866   0.30085

> AA*V(:,3)
ans =

   1.2882e-16
   7.3672e-17
  -4.2934e-17
   6.7099e-17
```