The University of British Columbia

27 February 2014

Midterm for MATH 307 Section 202 Winter 2014

Closed book examination

Time: 50 minutes

Last Name KEY **First** ____________

Signature ____________

Student Number ____________

Special Instructions:

No memory aids are allowed. No communication devices. No calculators. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. **Where boxes are provided for answers, put your final answers in them.**

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		10
Total		40

[10] 1. Consider the MATLAB/Octave command

```
> A = vander([1 2 3 4])
```

[3] (a) Write down the matrix A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$ (Also ok to flip matrix horizontally.)

[1] (b) Matrix A is a (circle one):

i. Vandermonde matrix
ii. Hilbert matrix
iii. Tridiagonal matrix
iv. Triangular matrix

[2] (c) Suppose we have the following output:

```
> cond(A,1)
  ans = 2000.0
> cond(A,2)
  ans = 1171.0
> cond(A,inf)
  ans = 1926.7
```

Each case uses a different norm to compute the condition number of A

Why is the output from these different calls to the cond() function different?

[2] (d) Suppose the norm of vector **b** is known to a relative accuracy of 1×10^{-7}. What is the worst relative accuracy for vector **x** we can expect when solving:

```
> x = A\b
```

$\frac{\|\delta x\|}{\|x\|} \leq \text{Cond}(A) \frac{\|\delta b\|}{\|b\|}$. If we assume the 2-norm, e.g. $\frac{\|\delta x\|_2}{\|x\|_2} \leq 1171 \times 10^{-7} \approx 1 \times 10^{-4}$.

[2] (e) Matrix A is often used to compute interpolation parameters. How do the relatively large values of cond(A) suggest an advantage of cubic spline interpolation over Lagrange interpolation?

The cubic spline interpolation is more stable with respect to errors in the (x_i, y_i) points than is the Lagrange interpolation. We can predict this by observing that the condition number for solving for Lagrange interpolant is large compared to that in solving for cubic spline interpolant.

Additional space for Question 1.

[10] 2. Consider the sparse matrix equation

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} e^{-h}\cos(\pi h) \\ e^{-2h}\cos(2\pi h) \\ \vdots \\ e^{-(n-1)h}\cos(\pi(n-1)h) \end{bmatrix}, \quad n = 1/h.$$

[4] (a) Write an ODE that this matrix equation approximates, including boundary conditions.

(b) Modify the matrix equation in two ways:

[2] i. change the left-hand boundary condition to $u(0) = 1$;

[3] ii. change the left-hand boundary condition to $u'(0) = 1$.

[1] (c) Give a reason why you might not want to choose an extremely small value for h.

a) $$\frac{1}{h^2}(u_{j-1} - 2u_j + u_{j+1}) \approx u''(x_j), \quad x_j = jh$$

$$b_j = e^{-jh}\cos(2\pi jh) = e^{-x_j}\cos(\pi x_j)$$

So: ODE: $\boxed{u''(x) = e^{-x}\cos(\pi x)}$

$$\frac{1}{h^2}(u_0 - 2u_1 + u_2) \approx u''(x_1) \text{ for } u_0 = 0 \Rightarrow \boxed{u(0) = 0}$$

$$\frac{1}{h^2}(u_{n-2} - 2u_{n-1} + u_n) \approx u''(x_{n-1}) \text{ for } u_n = 0 \Rightarrow \boxed{u(1) = 0}$$

b) i) $u_0 = 1 \Rightarrow \frac{1}{h^2}(1 - 2u_1 + u_2) = e^{-h}\cos(\pi h)$

$$\boxed{\frac{1}{h^2}(-2u_1 + u_2) = e^{-h}\cos(\pi h) - h^2}$$

$\vec{b}$ becomes $\begin{bmatrix} e^{-h}\cos(\pi h) - h^2 \\ e^{-2h}\cos(2\pi h) \\ \vdots \\ \end{bmatrix}$ $\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ unchanged

A unchanged, $\vec{x}$ unchanged

Additional space for Question 2.

b) ii) $u'(0) = 1 \Rightarrow \frac{1}{h}(u_1 - u_0) = 1.$

New eq'n, so

$$A = \frac{1}{h^2}\begin{bmatrix} -h & h & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & \\ & 1 & -2 & 1 & \cdots \\ & & & \ddots & \\ & & & 1 & -2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ \cos(\pi h)e^{-h} \\ \cos(\pi 2h)e^{-2h} \\ \vdots \end{bmatrix}$$

c) Small h makes for a badly conditioned solve $A\vec{x} = \vec{b}$.
Although small h should improve accuracy of the $u''(x)$ approximation, when cond(A) is large, round-off errors are amplified and degrade the solution accuracy.

[10] **3.** Let

$$A = \begin{bmatrix} 2 & -6 & 0 & 2 \\ -1 & 3 & 1 & 1 \\ 4 & -12 & -2 & 0 \end{bmatrix}, \quad \mathrm{rref}(A) = \begin{bmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and } \mathrm{rref}(A^T) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

[4] (a) State the dimensions of the four fundamental subspaces of A.

[1] (b) State the rank of A.

[4] (c) Write down bases for each of the four fundamental subspaces of A.

[1] (d) Write down an alternative basis for the rowspace of A.

a) 2 pivots: $\dim(\mathrm{col}(A)) = \dim(\mathrm{row}(A)) = 2$

~~3~~ 4 cols: $\dim(N(A)) = 4 - \cancel{\dim(\mathrm{row}} \dim(\mathrm{row}(A)) = 2.$

3 rows: $\dim(N(A^T)) = 3 - \dim(\mathrm{col}(A)) = 1$

b) 2 pivots: rank $= 2$

c) Pivots: cols 1 & 3
rows 1 & 2

$$\mathrm{Col}(A) = \mathrm{sp}\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

$$\mathrm{row}(A) = \mathrm{sp}\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

inspection on rref(A): free vars: 2, 4.

$$N(A) = \mathrm{sp}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

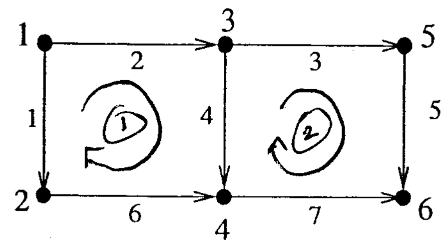
inspection on rref(A^T) free vars: 3

$$N(A^T) = \mathrm{sp}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

d) ~~Pa~~ Pivots on columns of rref(A^T)
cols 1, 2 : use rows 1, 2
of A:

$$\mathrm{row}(A) = \mathrm{sp}\left\{ \begin{bmatrix} 2 \\ -6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

[10] 4. Consider the following directed graph. When considered as a resistor network, assume all resistances $R_i = 1$.



[2] (a) Write down the node-edge incidence matrix D and the Laplacian matrix L.

[2] (b) What are the dimensions of each of the four fundamental subspaces of D?

[2] (c) Write down two linearly independent vectors in the left nullspace of D. Do they form a basis? Give a reason.

[2] (d) What happens to the dimensions of the nullspace and left nullspace of D when edge 4 is removed?

[2] (e) Suppose the Laplacian matrix L has been defined in MATLAB/Octave. Write down the code that computes the effective resistance between nodes 1 and 6.

a).

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

-1 leaving node
$+1$ entering node.

$D \in \mathbb{R}^{7 \times 6}$

7 edges, 6 nodes.

$$L = \begin{bmatrix} 2 & -1 & -1 & & & \\ -1 & 2 & & -1 & & \\ -1 & & 3 & -1 & -1 & \\ & -1 & -1 & 3 & & -1 \\ & & -1 & & 2 & -1 \\ & & & -1 & -1 & 2 \end{bmatrix}$$

b) 2 indep. loops

$\dim(N(D^T)) = 2$

$\dim(\mathrm{Col}(D)) = 7 - 2 = 5$

1 connected component.

$\dim(N(D)) = 1$

$\dim(\mathrm{Row}(D)) = 6 - 1 = 5$

Additional space for Question 4.

c) Loop ① $[-1 \ 1 \ 0 \ 1 \ 0 \ -1 \ 0]^T$

Loop ② $[0 \ 0 \ 1 \ -1 \ 1 \ 0 \ -1]^T$

Form a basis because they are independent and #vectors = #loops (= dimension of $N(D^T)$).

d) $\dim(N(D))$ remains $=1$ because #connected components doesn't change

$\dim(N(D^T)) = 1$ because one loop disappears

e) Form $L = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$

> A = L([1,6],[1,6])

> C = L(2:5, 2:5)

> B = L([1,6], 2:5)

Form Schur complement

$S = A - BC^{-1}B^T$

> S = A - B*inv(C)*B'

S is a multiple of $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

> c = ~~abs(S(1,1))~~ S(1,1)

> r = 1/c

· This multiple is the conductance

· resistance = 1/conductance.