# Math 307: Problems for section 3.1

February 2, 2011

1. **Show that if $P$ is an orthogonal projection matrix, then $\|P\mathbf{x}\| \le \|\mathbf{x}\|$ for every $\mathbf{x}$. Use this inequality to prove the Cauchy–Schwarz inequality $|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\|\mathbf{y}\|$.**

2. **Use the Cauchy–Schwarz inequality for real vectors to show**

$$\|\mathbf{x}+\mathbf{y}\|^2 \le (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$$

   **Under what circumstances is the inequality an equality?**

3. **Using MATLAB/Octave or otherwise, compute the matrix $P$ for the projection onto the line spanned by $\mathbf{a} = \begin{bmatrix}1\\2\\0\\1\end{bmatrix}$ in $\mathbb{R}^4$. Compute the matrix $Q$ for the projection onto the (hyper-)plane orthogonal to $\mathbf{a}$. (Provide the commands used.)**

4. **Using MATLAB/Octave or otherwise, compute the matrix $P$ for the projection onto the plane spanned by $\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}1\\0\\-1\\-3\end{bmatrix}$. (Careful: these vectors are not linearly independent so $A^TA$ is not invertible. This means you can't use the formula $P = A(A^TA)^{-1}A^T$ directly for the matrix $A$ containing all the vectors above as columns.) (Provide the commands used.)**

5. **Using least squares, find the best quadratic fit for the points $(1,5)$, $(2,3)$, $(3,3)$, $(4,4)$, $(5,5)$. To do this, write down the system of linear equations for $a$, $b$ and $c$ that expresses the condition that $p(x) = ax^2 + bx + c$ passes through these points. These equations have no solution, Use MATLAB/Octave to find the least squares solution. Provide the commands you used and a plot of the result together with the points.**

6. **Write a MATLAB/Octave m file that plots (on the same plot) the five least squares polynomial fits for polynomials of order 1 to 5 through the points $(x_i, y_i)$ given by `X=linspace(0,1,18)` and `Y=sin(7*X)`.**

7. **Consider the points $(1,0)$, $(2,0)$, $(3,0)$ and $(4,10)$ and consider doing a polynomial fit with polynomials of order zero, that is, constant functions $p(x) = a_1$. In the least squares fit problem we are finding the value of $a_1$ that minimizes the usual Euclidean norm $\|[0-a_1, 0-a_1, 0-a_1, 10-a_1\|$. Solve this. What happens when we replace the Euclidean norm by the 1-norm in this problem? Find the value of $a_1$ that minimizes the 1-norm $\|[0-a_1, 0-a_1, 0-a_1, 10-a_1\|_1$. This is the so-called least sums problem. Is it more or less sensitive to outliers in the data?**

8. **Show in some random examples that if MATLAB/Octave is asked to solve an overdetermined system (that is, more equations than unknowns) using `A\b` then the least squares solution is returned.**

9. **Verify that the matrix $P = A(A^TA)^{-1}A^T$ that projects onto the range of $A$ when $A^TA$ is invertible satisfies $P^2 = P$ and $P^T = P$.**