Math 307: Problems for section 3.4

1. **Calculate the Fourier coefficients (c_n's, a_n's and b_n's) for the triangle function**

$$f(t) = \begin{cases} 2t & \textbf{if} \quad 0 \le t \le 1/2 \\ 2-2t & \textbf{if} \quad 1/2 \le t \le 1 \end{cases}$$

and show that the Fourier series decomposition of $f(t)$ may be written

$$f(t) = \frac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n t) = \frac{1}{2} - \sum_{n=0}^{\infty} \frac{4}{\pi^2 (2n+1)^2} \cos(2\pi(2n+1)t)$$

What does Parseval's formula say in this case?

We compute

$$c_0 = \int_0^{1/2} 2t dt + \int_{1/2}^{1} (2-2t) dt = 1/2.$$

For $n \neq 0$ we can use integration by parts

$$\begin{aligned} \int e^{-2\pi i n t} t dt &= \frac{t e^{-2\pi i n t}}{-2\pi i n} - \int \frac{e^{-2\pi i n t}}{-2\pi i n} dt \\ &= \frac{t e^{-2\pi i n t}}{-2\pi i n} + \frac{e^{-2\pi i n t}}{(2\pi n)^2} \end{aligned}$$

and the fact that $e^{-i\pi} = -1$ to conclude

$$\begin{aligned} c_n &= \int_0^{1/2} e^{-2\pi i n t} 2t dt + \int_{1/2}^{1} e^{-2\pi i n t} (2-2t) dt \\ &= \frac{((-1)^n - 1)}{\pi^2 n^2} \\ &= \begin{cases} 0 & \text{if } n \text{ is even, } n \neq 0 \\ -2/(\pi^2 n^2) & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

This leads to

$$\begin{aligned} a_0 &= 2c_0 = 1 \\ a_n &= \begin{cases} 0 & \text{if } n \text{ is even, } n \neq 0 \\ -\frac{4}{\pi^2 n^2} & \text{if } n \text{ is odd} \end{cases} \\ b_n &= 0 \end{aligned}$$

Thus the real form of the Fourier series for $f(t)$ is

$$f(t) = \frac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n t) = \frac{1}{2} - \sum_{n=0}^{\infty} \frac{4}{\pi^2 (2n+1)^2} \cos(2\pi(2n+1)t)$$

Since

$$\int_0^1 f^2(t)dt = \int_0^{(1/2)} (2t)^2 dt + \int_{(1/2)}^1 (2-2t)^2 dt = \frac{1}{3}$$

Parseval's formula says:

$$\frac{1}{4} + \sum_{\substack{n=-\infty \\ \text{nodd}}}^{\infty} \frac{4}{\pi^4 n^4} = \frac{1}{3}$$

or

$$\frac{8}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{1}{12}$$

or

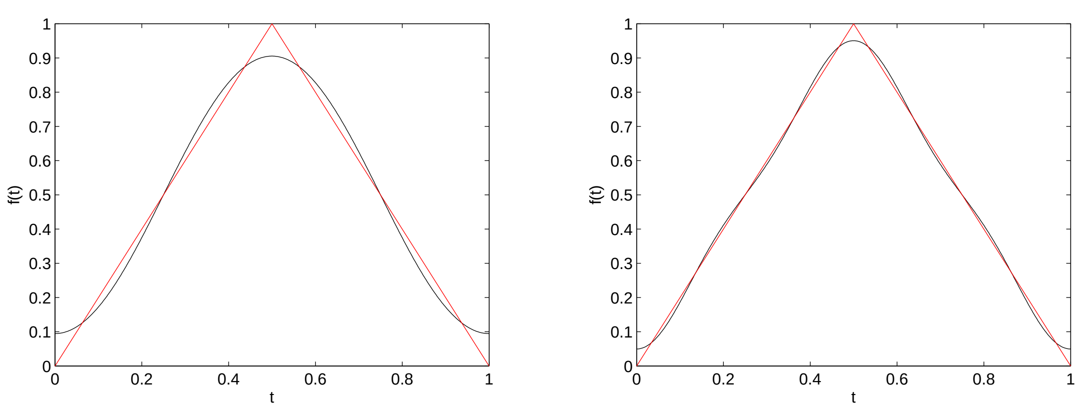
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

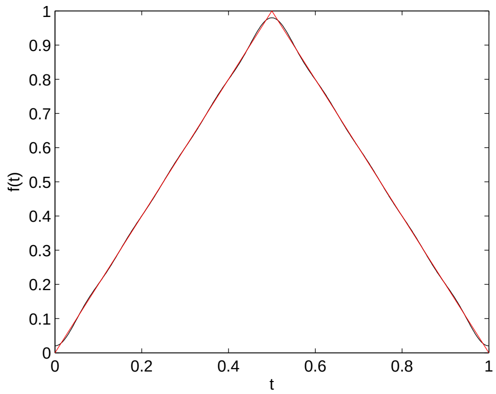
2. **Modify the file `ftdemo1.m` so that it plots the partial sums of the Fourier series in the previous question. Hand in the code and a plot of the partial sums with 1, 2, 5 and 10 non-zero terms.**

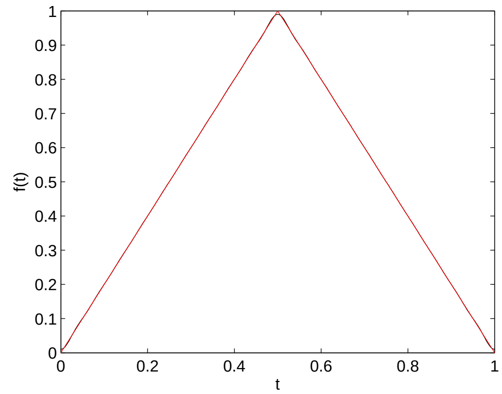
   Here is the code for the function

```
function ftdemo3(N)
% Fourier series for f(x) = 2x   for 0 $\lt$ x $\lt$ 0.5
%                         = 2-2x for 0.5 $\lt$ x $\lt$ 1

X=linspace(0,1,1000);
F=0.5*ones(1,1000);

for n=[0:N]
F = F - 4*cos(2*pi*(2*n+1)*X)/(pi^2*(2*n+1)^2);
end

plot(X,F)
axis([-0.2,1.2,-0.1,1.1])

end
```

   and the results of running `ftdemo3(N)` for $N = 0, 1, 4, 9$ (the number of nonzero terms is $N+1$)







3. **Calculate the Fourier coefficients (c_n's, a_n's and b_n's) for the half sine wave**

$$f(t) = \sin(\pi t) \quad \textbf{for } 0 \leq t \leq 1$$

and show that the Fourier series for $f(t)$ can be written

$$f(t) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{1-4n^2} \cos(2\pi n t)$$

For $n = 0$ we find

$$c_0 = \int_0^1 \sin(\pi t) dt = \left[-\frac{1}{\pi} \cos(\pi t) \right]_0^1 = \frac{2}{\pi}.$$

For $n \neq 0$ it is easier to expand $\sin(\pi t) = \frac{1}{2i}(e^{\pi i t} - e^{-\pi i t})$. We find

$$\begin{aligned}
c_n &= \int_0^1 e^{-2\pi i n t/1} \frac{1}{2i}(e^{\pi i t} - e^{-\pi i t}) dt \\
&= \frac{1}{2i} \int_0^1 (e^{\pi i(1-2n)t} - e^{-\pi i(1+2n)t}) dt \\
&= \frac{1}{2i} \left[\frac{1}{\pi i(1-2n)} e^{\pi i(1-2n)t} + \frac{1}{\pi i(1+2n)} e^{-\pi i(1+2n)t} \right]_0^1 \\
&= \frac{1}{2i} \left(\frac{1}{\pi i(1-2n)} (e^{\pi i(1-2n)} - 1) + \frac{1}{\pi i(1+2n)} (e^{-\pi i(1+2n)-1}) \right)
\end{aligned}$$

Now $e^{2\pi i n} = 1$ and $e^{\pi i} = e^{-\pi i} = -1$, so

$$\begin{aligned}
c_n &= \frac{1}{2i} \left(\frac{1}{\pi i(1-2n)} (-2) + \frac{1}{\pi i(1+2n)} (-2) \right) \\
&= \frac{1}{\pi(1-2n)} + \frac{1}{\pi(1+2n)} = \frac{2}{\pi(1-4n^2)}.
\end{aligned}$$

Using the relations between c_n and a_n, b_n we find:

$$a_0 = 2c_0 = \frac{4}{\pi}, \quad a_n = c_n + c_{-n} = \frac{4}{\pi(1-4n^2)}, \quad b_n = i(c_n - c_{-n}) = 0$$

and the series can be written

$$f(t) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{1-4n^2} \cos(2\pi n t).$$

4. **Calculate the Fourier coefficients (c_n's, a_n's and b_n's) for the function**

$$f(t) = t^2 - 1 \quad \textbf{for} \ -1 \le t \le 1$$

and show that the Fourier series for $f(t)$ can be written

$$f(t) = -\frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi t).$$

Here the time period T is 2.

For $n = 0$ we have

$$c_0 = \frac{1}{2}\int_{-1}^{1}(t^2-1)dt = -\frac{2}{3}.$$

For $n \neq 0$ we have

$$\begin{aligned}
c_n &= \frac{1}{2}\int_{-1}^{1} e^{-2\pi int/2}(t^2-1)dt = \frac{1}{2}\int_{-1}^{1} e^{-\pi int}(t^2-1)dt \\
&= \frac{1}{2}\left(\left[\frac{1}{-\pi in}e^{-\pi int}(t^2-1)\right]_{-1}^{1} - \int_{-1}^{1}\frac{1}{-\pi in}e^{-\pi int}2tdt\right) \text{ (integration by parts)} \\
&= -\frac{1}{2}\left[\frac{1}{(-\pi in)^2}e^{-\pi int}2t\right]_{-1}^{1} + \frac{1}{2}\int_{-1}^{1}\frac{1}{(-\pi in)^2}e^{-\pi int}2dt \text{ (integration by parts again)} \\
&= \frac{1}{\pi^2 n^2}\left(e^{-\pi in} + e^{\pi in}\right) + \frac{1}{2}\left[\frac{1}{(-\pi in)^3}e^{-\pi int}2\right]_{-1}^{1}
\end{aligned}$$

But $e^{\pi in} = e^{-\pi in} = (-1)^n$ so

$$c_n = \frac{1}{\pi^2 n^2}\left((-1)^n + (-1)^n\right) + \frac{1}{(-\pi in)^3}((-1)^n - (-1)^n) = \frac{2(-1)^n}{\pi^2 n^2}$$

Using the relations between c_n and a_n, b_n we find:

$$a_0 = 2c_0 = -\frac{4}{3}, \quad a_n = c_n + c_{-n} = \frac{4(-1)^n}{\pi^2 n^2}, \quad b_n = i(c_n - c_{-n}) = 0,$$

and the series can be written

$$f(t) = -\frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi t).$$

5. **Show that the Fourier series of $f(t) = e^t$ on the interval $-\pi \le t \le \pi$ is**

$$f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1-in}(e^{(1-in)\pi} - e^{-(1-in)\pi})e^{int}.$$

Deduce that

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{1}{2}(\pi\coth\pi - 1).$$

Here the period T is 2π.

For any n we find

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2n\pi it/2\pi}e^t dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{t(1-in)}dt \\
&= \frac{1}{2\pi}\left[\frac{1}{1-in}e^{t(1-in)}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\frac{1}{1-in}e^{\pi(1-in)} - \frac{1}{1-in}e^{-\pi(1-in)}\right) \\
&= \frac{1}{2\pi}\frac{1}{1-in}\left(e^{\pi(1-in)} - e^{-\pi(1-in)}\right)
\end{aligned}$$

and so the Fourier decomposition can be written

$$f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1-in}(e^{(1-in)\pi} - e^{-(1-in)\pi})e^{int}.$$

Now we can use Parseval's formula to find the required expression as follows

$$\frac{1}{2\pi}\langle e^t, e^t\rangle = \sum_{n=-\infty}^{\infty}|c_n|^2$$

The inner product on the left hand side is

$$\langle e^t, e^t\rangle = \int_{-\pi}^{\pi} e^{2t}dt = \frac{1}{2}\left(e^{2\pi} - e^{-2\pi}\right) = \frac{1}{2}\left(e^{\pi} + e^{-\pi})(e^{\pi} - e^{-\pi}\right).$$

The terms in the sum on the right hand side are

$$|c_n|^2 = \frac{1}{4\pi^2}\frac{1}{1+n^2}(e^{\pi} - e^{-\pi})^2$$

where we have used the fact that $e^{\pm in\pi} = (-1)^n$.
Putting these together we have

$$\frac{1}{2\pi}\frac{1}{2}(e^{\pi} + e^{-\pi})(e^{\pi} - e^{-\pi}) = \frac{1}{4\pi^2}(e^{\pi} - e^{-\pi})^2\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2}.$$

This becomes

$$\left(e^{\pi} + e^{-\pi}\right) = \frac{1}{\pi}(e^{\pi} - e^{-\pi})\left(2\sum_{n=1}^{\infty}\frac{1}{1+n^2} + 1\right)$$

Using $\coth\pi = (e^{\pi} + e^{-\pi})/(e^{\pi} - e^{-\pi})$ we have the final result

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{1}{2}(\pi\coth\pi - 1).$$