Math 307: Problems for section 2.1

February 2, 2011

1. **Are the vectors** $\begin{bmatrix}1\\2\\1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\3\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\4\\-9\\7\\3\end{bmatrix}$ **linearly independent? You may use MATLAB/Octave to perform calculations, but explain your answer.**

   Put the vectors in the columns of matrix and perform row reduction

```
V1 = [1 2 1 2 1]';
V2 = [1 0 -2 1 1]';
V3 = [1 -1 3 -2 0]';
V4 = [0 0 -2 0 1]';
V5 = [0 4 -9 7 3]';
A=[V1 V2 V3 V4 V5];
rref(A)

ans =

   1.00000   0.00000   0.00000   0.00000   1.00000
   0.00000   1.00000   0.00000   0.00000   1.00000
   0.00000   0.00000   1.00000   0.00000  -2.00000
   0.00000   0.00000   0.00000   1.00000   1.00000
   0.00000   0.00000   0.00000   0.00000   0.00000
```

   Since the matrix has rank 4 there are only 4 independent vectors in the list, and the vectors are linearly dependent. In fact a vector in the nullspace of A is $\begin{bmatrix}-1\\-1\\2\\-1\\1\end{bmatrix}$ so $-V1 - V2 + 2V3 - V4 + V5 = 0$ which we can check:

```
-V1 - V2 + 2*V3 - V4 + V5

ans =

   0
   0
   0
   0
   0
```

2. **Which of the following sets are subspaces of the vector space V? Why, or why not?**

   (a) The set $S = \{(b_1, b_2, b_3) : b_1 = 0; b_2, b_3 \in \mathbb{R}\}$. ($V = \mathbb{R}^3$)

   (b) The set $S = \{(b_1, b_2, b_3) : b_1 b_2 = 0, b_3 \in \mathbb{R}\}$. This is union of the plane $b_1 = 0$ and the plane $b_2 = 0$. ($V = \mathbb{R}^3$)

   (c) All infinite sequences $(x_1, x_2, \dots)$, with $x_i \in \mathbb{R}$ and $x_j = 0$ from some fixed point onwards. ($V = \mathbb{R}^\infty$)

   (d) All non-increasing sequences $(x_1, x_2, \dots)$, with $x_i \in \mathbb{R}$ and $x_{j+1} \leq x_j$ for each j. ($V = \mathbb{R}^\infty$)

   (e) The set of all polynomial functions, $p(x)$, where $p(x) = 0$ or $p(x)$ has degree n for some fixed $n \geq 1$. (V is the vector space of all polynomials.)

   (f) The set of odd continuous functions on the interval $[-1, 1]$, i.e., $f \in C[-1, 1]$ such that $f(-x) = -f(x)$. ($V = C[-1, 1]$)

   (a) S is a subspace. The zero vector, $(0, 0, 0)$, is in S. S is also closed under addition and scalar multiplication. Let $\boldsymbol{x}_1 = (0, y_1, z_1)$ and $\boldsymbol{x}_2 = (0, y_2, z_2)$ be two arbitrary vectors in S and let $c \in \mathbb{R}$. Then $\boldsymbol{x}_1 + \boldsymbol{x}_2 = (0, y_1 + y_2, z_1 + z_2)$ is also in S and $c\boldsymbol{x}_1 = (0, cy_1, cz_1)$ is also in S.

   (b) S is not a subspace. It is not closed under addition. Take for example the vectors $(0, 1, 1)$ and $(1, 0, 1)$. They are both in S but their sum, $(1, 1, 2)$, is not in S.

   (c) This is a subspace. The zero vector clearly has $x_j = 0$ from some point onwards. If we add two vectors which both have $x_j = 0$ from some point onwards then the sum will also have $x_j = 0$ from the same point onwards. Also, if we multiply a vector with $x_j = 0$ from some point onwards by a scalar, the result will still have $x_j = 0$ from the same point on. So the set is closed under addition and scalar multiplication.

   (d) This is not a subspace. It is not closed under scalar multiplication. Take for example the vector $(2, 1, 0, 0, 0, \dots)$. If we multiply this by (-1) we get $(-2, -1, 0, 0, 0, \dots)$, which is no longer non-increasing.

   (e) This is not a subspace. Take for example the two polynomials $p_1(x) = x^n + 1$ and $p_2(x) = -x^n$. These are both of degree n, but their sum $p_1(x) + p_2(x) = 1$ is not a polynomial of degree n. So the set is not closed under addition.

   (f) This is a subspace. The zero function ($f(x) = 0, -1 \leq x \leq 1$) satisfies $f(-x) = -f(x)$, so it is in the set. If we take two odd continuous functions $f(x)$ and $g(x)$ then $(f+g)(-x) = f(-x)+g(-x) = -f(x)-g(x) = -(f+g)(x)$, so the sum is also an odd function (and will still be continuous). Also if we multiply an odd continuous function by a scalar then $cf(-x) = c[-f(x)] = -c[f(x)] = -cf(x)$, so the result is also an odd continuous function. The set is closed under addition and scalar multiplication.

3. **If** $\textbf{rref}(A) = \begin{bmatrix} 1 & a & 0 & b & d & 0 \\ 0 & 0 & 1 & c & e & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **find a basis for $N(A)$ and $R(A^T)$.**

   A basis for $N(A)$ is

$$\begin{bmatrix} -a \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -b \\ 0 \\ -c \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -d \\ 0 \\ -e \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

A basis for $R(A^T)$ is

$$\begin{bmatrix}1\\a\\0\\b\\d\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\\c\\e\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}$$

4. **Explain why the first three rows of the matrix in Q3 are linearly independent. (A similar argument shows that the if $U = \mathbf{rref}(A)$ then the non-zero columns of U^T form a basis for $R(U^T)$ and hence for $R(A^T)$.)**

   We can check directly. If a linear combination of the rows is zero then

$$c_1 \begin{bmatrix}1\\a\\0\\b\\d\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\0\\1\\c\\e\\0\end{bmatrix} + c_3 \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}$$

   Examining the first, third and last rows of this equation we see $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$. Thus the only linear combination of these vectors to equal zero is the one where all the coefficients are zero.

5. **Find the rank $\mathbf{r}(A)$ and bases for $N(A)$, $R(A)$, $N(A^T)$ $R(A^T)$ when** $A = \begin{bmatrix}1&2&0&2&1&0\\1&2&1&5&3&0\\1&2&1&5&3&1\\1&2&1&5&3&1\end{bmatrix}$**.**

   (You may use use MATLAB/Octave for simplification.)

   First we find $\mathbf{rref}(A)$.

```
A = [1 2 0 2 1 0;1 2 1 5 3 0;1 2 1 5 3 1;1 2 1 5 3 1];
rref(A)

ans =

   1   2   0   2   1   0
   0   0   1   3   2   0
   0   0   0   0   0   1
   0   0   0   0   0   0
```

   From this we can determine a basis for $N(A)$

$$\begin{bmatrix}-2\\1\\0\\0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}-2\\0\\-3\\1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}-1\\0\\-2\\0\\1\\0\end{bmatrix},$$

   a basis for $R(A)$ by looking at the columns of A corresponding to pivot columns

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\\1\end{bmatrix},$$

and a basis for $R(A^T)$

$$\begin{bmatrix}1\\2\\0\\2\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\\3\\2\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}$$

To find a basis for $N(A^T)$ it's easiest just to compute the reduced row echelon form of the transpose.

```
rref(A')

ans =

     1     0     0     0
     0     1     0     0
     0     0     1     1
     0     0     0     0
     0     0     0     0
     0     0     0     0
```

This yields the following basis for $N(A^T)$

$$\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$$

The rank of A is 3.

6. **Show that if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent and E is invertible then $E\mathbf{u}_1, E\mathbf{u}_2, \ldots, E\mathbf{u}_k$ are also linearly independent. Is this still true if E is not invertible?**

   Suppose $c_1 E\mathbf{u}_1 + c_2 E\mathbf{u}_2 + \cdots + c_k E\mathbf{u}_k = \mathbf{0}$ Since E is invertible we can multiply both sides by E^{-1}. This gives $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = E^{-1}\mathbf{0} = \mathbf{0}$ But the vectors $\mathbf{u}_i$ are independent so this implies $c_1 = c_2 = \cdots = c_k = 0$. This shows that the only linear combination of the $E\mathbf{u}_i$ equaling zero has all coefficients zero.

   If E is not invertible this argument doesn't work. The most extreme case would be where E has all entries zero. Then each $E\mathbf{u}_i = \mathbf{0}$ and these vectors are certainly not independent.

7. **We saw in the notes that the set of polynomials of degree at most n form a subspace (of functions). If we take $n = 2$, then a general quadratic polynomial can be written as $p(x) = c_0 + c_1 x + c_2 x^2$. We can represent this polynomial by the vector $\mathbf{p} = \begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}$. The derivative of $p(x)$ is $q(x) = c_1 + 2c_2 x$, which may be represented by the vector $\mathbf{q} = \begin{bmatrix}c_1\\2c_2\\0\end{bmatrix}$.**

   (a) Show that the matrix

   $$D_2 = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}$$

   satisfies $D_2\mathbf{p} = \mathbf{q}$ (and so D_2 represents the derivative). If we change the value of n, then the dimension increases. Find the matrix D_n representing the (first) derivative for degree n polynomials.

(b) What is the nullspace of D_n?

(c) What is the range of D_n?

(d) Show that there is no solution to $(D_n - I)\mathbf{p} = \mathbf{0}$ except $\mathbf{p} = \mathbf{0}$. (It is easiest to do this problem by reverting to a differential equation and using properties of the derivative and polynomials of degree n.)

Note that much of this question can either be done from the perspective of matrices or the perspective of differential equations. The answers given here are (except the last part) from the perspective of matrices.

(a) Writing out the matrix equation we have

$$D_2\mathbf{p} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ 0 \end{bmatrix}.$$

But the right-hand-side of this equation is just $\mathbf{q}$ and so we have shown the relationship.

$$D_n = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

(b) D_n as given above is in echelon form and we see that it has $(n-1)$ pivot columns and only one free variable. Thus the nullity is 1 and we find that a basis for the nullspace is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. This corresponds to all the constant polynomials.

(c) The rank of D_n is $(n-1)$ and a basis for the range is given by the columns corresponding to the pivot columns of D_n:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \right\}$$

This corresponds to all the polynomials of degree $n-1$.

(d) The equation $(D_n - I)\mathbf{p} = \mathbf{0}$ is equivalent to the differential equation $\frac{\mathrm{d}p(x)}{\mathrm{d}x} - p(x) = 0$, where $p(x) = c_0 + c_1 x + \dots + c_n x^n$ is a polynomial of degree n.

The easiest argument is to consider the highest degree term in $p(x)$ with a non-zero coefficient (if it exists). The derivative of $p(x)$ has one lower degree than $p(x)$, and so $\frac{\mathrm{d}p(x)}{\mathrm{d}x} \neq p(x)$. Thus $p(x)$ cannot have any non-zero coefficients.

Alternately, this equation implies

$$c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

or

$$0 = (c_0 - c_1) + (c_1 - 2c_2)x + \cdots + (c_{n-1} - nc_n)x^{n-1} + c_n x^n.$$

But 1, x, x^2, $\ldots$, x^n is a basis for the polynomials of degree up to n and so they are linearly independent. Therefore

$$c_0 = c_1,\ c_1 = 2c_2,\ \ldots,\ c_{n-1} = nc_n,\ c_n = 0.$$

From which we see that all the c_i must be zero for $i = 0, \ldots, n$ and so $p(x) = 0$. Equivalently, $\mathbf{p} = \mathbf{0}$.

8. **We showed above that the derivative acting on polynomials of degree n behaves much like a matrix acting on vectors. Here we extend some of those properties to the set of smooth functions (functions for which all derivatives exist) rather than just polynomials. We can now no longer represent the function directly as a vector or derivatives as matrices.**

   We represent the derivative by D, so that $Df(x)$ is the derivative of $f(x)$ (you've previously seen df/dx or $f'(x)$ as other ways to write this).

   (a) Find the nullspace of D and the null space of D^n.

   (b) Show that e^{rx} is in the nullspace of $D - rI$, where I is the identity: $If(x) = f(x)$.

   (c) Find the nullspace of $(D - rI)^2$. (*Hint: use integrating factors.*)

   (d) Find the nullspace of $(D - rI)^n$.

   (a) The only functions having zero derivatives are constants, thus the nullspace of D is the span of 1, or equivalently the set of all constant functions.

   The nullspace of D^n is the set of all polynomials up to degree $n - 1$.

   (b) For $f(x) = e^{rx}$ to be in the nullspace of $D - rI$ we must have that $\frac{\mathrm{d}f(x)}{\mathrm{d}x} - rf(x) = 0$. $\frac{\mathrm{d}e^{rx}}{\mathrm{d}x} = re^{rx}$ so this is true for this function.

   (c) First we show that the nullspace of $(D - rI)$ is the set of all functions $f(x) = c_0 e^{rx}$ for some c_0. For a function $f(x)$ to be in the nullspace of $D - rI$ it must satisfy $\frac{\mathrm{d}f(x)}{\mathrm{d}x} - rf(x) = 0$ and so must have the form given.

   Now consider a function $f(x)$ in the nullspace of $(D - rI)^2$. Let $g(x) = (D - rI)f(x)$. Then $(D - rI)g(x) = 0$ and so $g(x)$ is in the nullspace of $D - rI$ and so takes the form $g(x) = c_0 e^{rx}$ for some c_0. Therefore $f(x)$ satisfies $\frac{\mathrm{d}f(x)}{\mathrm{d}x} - rf(x) = c_0 e^{rx}$. The integrating factor for this equation is e^{-rx} and multiplying both sides of the equation by it we have $\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)e^{-rx}\right] = c_0$. Integrating this equation we find $f(x) = (c_0 x + c_1)e^{rx}$.

   Thus the nullspace of $(D - rI)^2$ is the set of all functions of the form $(c_0 x + c_1)e^{rx}$ for some c_0 and c_1.

   (d) Using the previous two parts of the question we anticipate that the nullspace of $(D - rI)^n$ is the set of all functions of the form $(c_0 x^{n-1} + c_1 x^{n-2} + \ldots + c_{n-2}x + c_{n-1})e^{rx}$ for some c_i, $i = 0, \ldots, n - 1$.

   We can show this by induction. Assume the form is true for n and consider $n+1$. We want to find functions $f(x)$ such that $(D - rI)^{n+1}f(x) = 0$. Let $g(x) = (D - rI)f(x)$. Then $(D - rI)^n g(x) = 0$ and so $g(x) = (c_0 x^{n-1} + c_1 x^{n-2} + \ldots + c_{n-2}x + c_{n-1})e^{rx}$ for some constants c_i. Therefore $f(x)$ satisfies $\frac{\mathrm{d}f(x)}{\mathrm{d}x} - rf(x) = (c_0 x^{n-1} + c_1 x^{n-2} + \ldots + c_{n-2}x + c_{n-1})e^{rx}$. Again using integrating factors we find that $f(x) = (c_0' x^n + c_1' x^{n-1} + \ldots + c_{n-1}' x + c_n')e^{rx}$ where $c_i' = c_i/(n - 1 - i + 1)$. Thus we have shown that the statement is true for $n + 1$. But we already showed it is true for $n = 2$ and so, by induction, it is true for all n.

9. **The differential equation $f''(x) - (r_1 + r_2)f'(x) + r_1 r_2 f(x) = 0$ may be written as**

$$[D^2 - (r_1 + r_2)D + r_1 r_2 I]f(x) = 0$$

We may also write this as $P(D)f(x) = 0$ where $P(D) = D^2 - (r_1 + r_2)D + r_1 r_2 I$. It is possible to factor $P(D)$ into either $(D - r_1)(D - r_2)$ or $(D - r_2)(D - r_1)$.

   (a) Show that the nullspace of $D - r_1$ and the nullspace of $D - r_2$ are in the nullspace of $P(D)$ and hence find the general solution to the differential equation.

   (b) Show that the solution to

   $$P(D)f(x) = e^{\alpha x}$$

   is in the nullspace of $(D - \alpha I)P(D)$.

   (c) Find the solution to

   $$f''(x) - 5f'(x) + 6f(x) = e^{x}$$

   with $f(0) = 1,\ \ f'(0) = 2$.

   (d) Find the solution to

   $$f''(x) - 5f'(x) + 6f(x) = e^{2x}$$

   with the same initial conditions.

   (a) If $f(x)$ is in the nullspace of $D - r_1 I$, then $(D - r_1 I)f(x) = 0$ and so $(D - r_2 I)(D - r_1 I)f(x) = 0$ also and so $f(x)$ is also in the nullspace of $P(D)$. The argument for the nullspace of $D - r_2 I$ is identical.

   The general solution to the differential equation is $f(x) = Ae^{r_1 x} + Be^{r_2 x}$.

   (b) Let $g(x) = P(D)f(x)$, then $(D - \alpha I)g(x) = 0$ and so $g(x)$ is in the nullspace of $D - \alpha I$. Thus $(D - \alpha I)P(D)f(x) = 0$ and $f(x)$ is in the nullspace of $(D - \alpha)P(D)$.

   (c) We can write this in the form of the previous part of the question as

   $$(D - I)(D - 3I)(D - 2I)f(x) = 0$$

   From the second part of the question we see that the general solution for $f(x)$ is $f(x) = Ae^{3x} + Be^{2x} + Ce^{x}$. Substituting this into the differential equation we find $C = 1/2$. Applying the boundary conditions we find $A = 1/2$ and $B = 0$. Therefore the solution is

   $$f(x) = \frac{1}{2}e^{3x} + \frac{1}{2}e^{x}.$$

   (d) We can write this in the form

   $$(D - 2I)^2(D - 3I)f(x) = 0$$

   From the previous question and the second part of this question we see that the general solution for $f(x)$ is $f(x) = Ae^{3x} + Be^{2x} + Cxe^{2x}$. Substituting this into the differential equation we find $C = -1$. Applying the boundary conditions we find $A = 1$ and $B = 0$. Therefore the solution is

   $$f(x) = e^{3x} - xe^{2x}.$$