Math 307: Problems for section 4.4

1. **Find the solution of the recursion relation**

$$x_{n+2} - 2x_{n+1} + 2x_n = 0$$

with initial conditions $x_0 = 1$ **and** $x_1 = 1$ **(by hand).**
We can write this recursion relation as the matrix

$$\begin{aligned}\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} &= \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}\end{aligned}$$

We now diagonalize this matrix.

$$\lambda I - A = \begin{bmatrix} \lambda - 2 & 2 \\ -1 & \lambda \end{bmatrix}$$

So eigenvalues satisfy

$$0 = \det(\lambda I - A) = (\lambda - 2)\lambda + 2.$$

And are

$$\lambda_1 = 1 + i \quad \text{and} \quad \lambda_2 = 1 - i.$$

The eigenvector $\mathbf{v}_1$ for λ_1 satisfies

$$(\lambda_1 I - A)\mathbf{v}_1 = \begin{bmatrix} -1 + i & 2 \\ -1 & 1 + i \end{bmatrix} \mathbf{v}_1 = \mathbf{0}$$

so a choice is $\mathbf{v}_1 = \begin{bmatrix} 1 + i \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$.
The eigenvector $\mathbf{v}_2$ for λ_2 satisfies

$$(\lambda_2 I - A)\mathbf{v}_2 = \begin{bmatrix} -1 - i & 2 \\ -1 & 1 - i \end{bmatrix} \mathbf{v}_2 = \mathbf{0}$$

so a choice is $\mathbf{v}_2 = \begin{bmatrix} 1 - i \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$. We could also have found this eigenvector using the observation that complex eigenvectors of a real matrix occur in complex conjugate pairs.
Now the matrix of eigenvectors is

$$S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$$

with inverse

$$S^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

and the diagonal matrix of eigenvalues is

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = SD^nS^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix}\begin{bmatrix} 1-\lambda_2 \\ -1+\lambda_1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} \lambda_1^{n+1} - \lambda_1^{n+1}\lambda_2 - \lambda_2^{n+1} + \lambda_2^{n+1}\lambda_1 \\ \lambda_1^n - \lambda_1^n\lambda_2 - \lambda_2^n + \lambda_2^n\lambda_1 \end{bmatrix}$$

So, using that $\lambda_2 = \overline{\lambda_1}$ and $\lambda_1 = \sqrt{2}e^{i\pi/4}$, we can write the solution to the recursion relation as

$$\begin{aligned} x_n &= \frac{1}{2i}(\lambda_1^n - \overline{\lambda_1^n} - \lambda_1^n\overline{\lambda_1} + \overline{\lambda_1^n}\lambda_1) \\ &= \operatorname{Im}(\lambda_1^n) - |\lambda_1|^2 \operatorname{Im}(\lambda_1^{n-1}) \\ &= \operatorname{Im}(2^{n/2}e^{i\pi n/4}) - 2\operatorname{Im}(2^{(n-1)/2}e^{i\pi(n-1)/4}) \\ &= 2^{n/2}\sin(n\pi/4) - 2^{(n+1)/2}\sin((n-1)\pi/4) \end{aligned}$$

2. **Using MATLAB/Octave or otherwise, find x_{30}, x_{31} and x_{32} for the recursion relation defined by $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$ and**

$$x_{n+1} = x_n - 2x_{n-1} + x_{n-2}.$$

For this recursion

$$\begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix}$$

so we must compute

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[1 -2 1;1 0 0; 0 1 0]^(30)*[2;1;0]

ans =

    413
  -6016
  -1710
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