Math 307: Problems for section 3.3

February 2, 2011

1. **Show that for $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$**

$$\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2\,\mathrm{Re}(\langle \mathbf{v}, \mathbf{w}\rangle)$$

and use this to prove the polarization identity

$$\langle \mathbf{v}, \mathbf{w}\rangle = \frac{1}{4}\Big(\|\mathbf{v}+\mathbf{w}\|^2 - \|\mathbf{v}-\mathbf{w}\|^2 + i\|\mathbf{v}-i\mathbf{w}\|^2 - i\|\mathbf{v}+i\mathbf{w}\|^2\Big)$$

2. **Show that if $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ form a basis in $\mathbb{R}^n$, then they also form a basis when regarded as vectors in $\mathbb{C}^n$. In other words, show that 1.) if the only linear combination $c_1\mathbf{q}_1 + \cdots + c_n\mathbf{q}_n$ using real numbers $c_1, \ldots, c_n$ that equals zero has $c_1 = \cdots = c_n = 0$, then the same is true for complex numbers, and 2.) if every vector in $\mathbb{R}^n$ can be written as $c_1\mathbf{q}_1 + \cdots + c_n\mathbf{q}_n$ for some real numbers $c_1, \ldots, c_n$ then every vector in $\mathbb{C}^n$ can be written as a linear combination using complex numbers. If the basis $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ is orthonormal in $\mathbb{R}^n$ is is also orthonormal in $\mathbb{C}^n$?**

3. **Show that any 2×2 orthogonal matrix is either a rotation matrix or a reflection matrix.**

4. **Let $Q = \begin{bmatrix} \mathbf{q}_1 | \mathbf{q}_2 | \cdots | \mathbf{q}_k \end{bmatrix}$ where $\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_k \in \mathbb{R}^n$ form an orthonormal set. (That is, they satisfy $\|\mathbf{q}_i\| = 1$ for $i = 1, \ldots, k$ and $\mathbf{q}_i \cdot \mathbf{q}_j = 0$ if $i \neq j$, but there might not be enough vectors to form a basis, i.e., possibly $k < n$). Identify the matrices Q^TQ and QQ^T. Show that the projection $\mathbf{p}$ of a vector $\mathbf{v}$ onto the subspace spanned by $\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_k$ can be written $\mathbf{p} = \sum_{i=1}^{k} \mathbf{q}_i\mathbf{q}_i^T\mathbf{v} = \sum_{i=1}^{k}(\mathbf{q}_i \cdot \mathbf{v})\mathbf{q_i}$.**

5. **For an $m \times n$ matrix A with linearly independent columns there is a factorization (called the QR factorization) $A = QR$ where Q is an $m \times n$ matrix whose columns form an orthonormal set, and R is an upper triangular matrix. For every $k = 1, 2, \ldots n$ the first k columns of Q spans the same subspace as the first k columns of A. In MATLAB/Octave the matrices Q and R in the QR decomposition of A are computed using `[Q R] = qr(A,0)`. (Without the second argument `0` a related but different decomposition is computed.)(For those of you who have learned about Gram-Schmidt: The columns of Q are the vectors obtained by applying the Gram-Schmidt procedure to the columns of A.**

Using MATLAB/Octave, compute and orthonormal basis $\mathbf{q}_1, \mathbf{q}_2$ for the plane in $\mathbb{R}^4$ spanned by $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ Compute the projection $\mathbf{p}$ of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ onto the plane. What are the coefficients of $\mathbf{p}$ when expanded in the basis $\mathbf{q}_1, \mathbf{q}_2$?

6. **Using MATLAB/Octave and the discussion in the previous problem, find an orthonormal set of vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ with the same span as** $\begin{bmatrix}1\\1\\2\\0\\0\\0\end{bmatrix}$ $\begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix}$ **and** $\begin{bmatrix}0\\0\\1\\1\\1\\0\end{bmatrix}$**. Provide the commands that you used.**

7. **Do the following computational experiment. First start with a random symmetric 10×10 matrix A (for example `B=rand(10,10); A=B'*B;` will produce such a matrix) and compute its QR factorization. Call the factors Q_1 and R_1. Now multiply Q_1 and R_1 in the "wrong" order to obtain $A_2 = R_1 Q_1$ and compute the QR factorization of the resulting matrix A_2. Repeat this step to obtain a sequence of matrices Q_k, R_k and A_k. Do these sequences converge? If so can you identify the limit? (Hint: `eig(C)` computes the eigenvalues of `C`).**

8. **If U_1 and U_2 are unitary matrices, is $U_1 U_2$ a unitary matrix too?**

9. **If $\mathbf{q}_1, \ldots, \mathbf{q}_n$ is an orthonormal basis for $\mathbb{C}^n$ do the complex conjugated vectors $\overline{\mathbf{q}}_1, \ldots, \overline{\mathbf{q}}_n$ form an orthonormal basis as well? Give a reason.**