Math 307: Problems for section 4.2

1. **(i) What can you say about the diagonal elements of a Hermition matrix?**

   (ii) Show that if A is an $n \times n$ matrix such that $\langle \mathbf{v}, A\mathbf{w}\rangle = \langle A\mathbf{v}, \mathbf{w}\rangle$ then A is Hermitian.

   (i) Diagonal entries of Hermitian matrices are real, because for a Hermitian matrix $A = [a_{i,j}]$, we have $\overline{a_{i,i}} = a_{i,i}$.

   (ii) The condition can be written $\langle \mathbf{v}, A\mathbf{w}\rangle = \langle \mathbf{v}, A^*\mathbf{w}\rangle$. Taking $\mathbf{v} = \mathbf{e}_i$ and $\mathbf{w} = \mathbf{e}_j$ we find that $a_{i,j} = \langle \mathbf{e}_i, A\mathbf{e}_j\rangle = \langle \mathbf{e}_i, A^*\mathbf{e}_j\rangle = \overline{a_{i,j}}$.

2. **Show that if A is any matrix then A^*A and AA^* are Hermitian with non-negative eigenvalues.**

   To see that A^*A is Hermitian, we use $(AB)^* = B^*A^*$ to compute

   $$(A^*A)^* = A^*(A^*)^* = A^*A.$$

   Now suppose λ is an eigenvalue of A^*A with eigenvector $\mathbf{v}$. Then $A^*A\mathbf{v} = \lambda\mathbf{v}$. Taking the inner product with $\mathbf{v}$ yields $\langle \mathbf{v}, A^*A\mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle$. This implies $\langle A\mathbf{v}, A\mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle$, so that $\lambda = ||A\mathbf{v}||^2/||\mathbf{v}||^2 \geq 0$. The argument for AA^* is the same.

3. **Follow the procedure in the notes to find an orthogonal matrix V such that V^TAV is upper triangular when $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.**

   We first must find an eigenvector of A. Standard calculations show that $\mathbf{v} = \begin{bmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{bmatrix}$ is a normalized eigenvector (with eigenvalue $1+\sqrt{2}$). We now complete $\mathbf{v}$ to form an orthonormal basis by choosing the second vector in the basis to be $\mathbf{w} = \begin{bmatrix} -\sqrt{1/3} \\ \sqrt{2/3} \end{bmatrix}$. Then if we form $V = [\mathbf{v}, \mathbf{w}] = \begin{bmatrix} \sqrt{2/3} & -\sqrt{1/3} \\ \sqrt{1/3} & \sqrt{2/3} \end{bmatrix}$ we can verify that $V^TV = I$ and $V^TAV = \begin{bmatrix} 1+\sqrt(2) & 1 \\ 0 & 1-\sqrt(2) \end{bmatrix}$ is upper triangular.

   We can also do this in MATLAB/Octave:

```
octave:1> A=[1 2;1 1];
octave:2> [U D]=eig(A)
U =

   0.81650  -0.81650
   0.57735   0.57735

D =

Diagonal Matrix

   2.41421         0
```

```
        0  -0.41421

octave:3> v=U(:,1)
v =

   0.81650
   0.57735

octave:4> w=[-v(2);v(1)]
w =

  -0.57735
   0.81650

octave:5> V=[v w]
V =

   0.81650  -0.57735
   0.57735   0.81650

octave:6> V'*A*V
ans =

   2.4142e+00   1.0000e+00
  -1.0122e-16  -4.1421e-01
```

The last product is upper triangular up to numerical error.

4. **Explain why the Laplacian matrix L for a resistor network has non-negative eigenvalues.**

   L can be written as $L = D^T B^T BD$ where $B = R^{-1/2}$ is the diagonal matrix with diagonal entries $1/\sqrt{R_i}$. If $L\mathbf{u} = \lambda\mathbf{u}$ then taking the dot product with $\mathbf{u}$ gives $\langle \mathbf{u}, L\mathbf{u}\rangle = \lambda\langle \mathbf{u}, \mathbf{u}\rangle$. Since $\langle \mathbf{u}, L\mathbf{u}\rangle = \langle \mathbf{u}, D^T B^T BD\mathbf{u}\rangle = \langle BD\mathbf{u}, BD\mathbf{u}\rangle = \|BD\mathbf{u}\|^2$, this implies $\lambda = \|BD\mathbf{u}\|^2/\|\mathbf{u}\|^2 \geq 0$.

5. **Redo the calculation of the effective resistance between nodes 1 and 7 of the resistor cube in section II.2.12. For this problem the Laplacian is defined by**

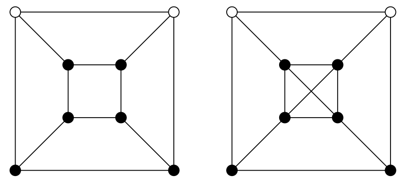
```
>L=[3 -1 0 -1 -1 0 0 0;-1 3 -1 0 0 -1 0 0;
0 -1 3 -1 0 0 -1 0;-1 0 -1 3 0 0 0 -1;
-1 0 0 0 3 -1 0 -1;0 -1 0 0 -1 3 -1 0;
0 0 -1 0 0 -1 3 -1;0 0 0 -1 -1 0 -1 3];
```

   and the answer is $R = 5/6 = 0.83333$

```
> L=[3 -1 0 -1 -1 0 0 0;-1 3 -1 0 0 -1 0 0;
> 0 -1 3 -1 0 0 -1 0;-1 0 -1 3 0 0 0 -1;
> -1 0 0 0 3 -1 0 -1;0 -1 0 0 -1 3 -1 0;
> 0 0 -1 0 0 -1 3 -1;0 0 0 -1 -1 0 -1 3];
> [U,D]=eig(L);
> R=0;
> for k=[2:8] R=R+D(k,k)^(-1)*(U(1,k)-U(7,k))^2 end
R =  0.031115
```

```
R =  0.044898
R =  0.75000
R =  0.75000
R =  0.75000
R =  0.75000
R =  0.83333
```

6. **Which of these two resistor networks do you think has a lower effective resistance between the indicated nodes? Check your guess using MATLAB/Octave and the eigenvalue/vector formula for the effective resistance.**



The second circuit is more connected so we expect the effective resistance to be lower. This following calculation shows that the first circuit has effective resistance 0.58333 while the second has effective resistance 0.56548, confirming this.

```
1> L=[3 -1 0 -1 -1 0 0 0;-1 3 -1 0 0 -1 0 0;
0 -1 3 -1 0 0 -1 0;-1 0 -1 3 0 0 0 -1;
-1 0 0 0 3 -1 0 -1;0 -1 0 0 -1 3 -1 0;
0 0 -1 0 0 -1 3 -1;0 0 0 -1 -1 0 -1 3];
2> [U,D]= eig(L);
3> R=0
R = 0
4> for k=[2:8]
> R=R + D(k,k)^(-1)*(U(1,k)-U(2,k))^2
> end
R =  0.11619
R =  0.11928
R =  0.25000
R =  0.37177
R =  0.39031
R =  0.50000
R =  0.58333

5> L=[3 -1 0 -1 -1 0 0 0;-1 3 -1 0 0 -1 0 0;
0 -1 3 -1 0 0 -1 0;-1 0 -1 3 0 0 0 -1;
-1 0 0 0 4 -1 -1 -1;0 -1 0 0 -1 4 -1 -1;
0 0 -1 0 -1 -1 4 -1;0 0 0 -1 -1 -1 -1 4];
6> [U,D]= eig(L);
7> R=0
R = 0
8> for k=[2:8]
> R=R + D(k,k)^(-1)*(U(1,k)-U(2,k))^2
> end
R =  2.2187e-31
R =  0.0081824
```

```
R =  0.33009
R =  0.45509
R =  0.46479
R =  0.48214
R =  0.56548
```