Math 307: Problems for section 3.2

March 7, 2011

1. **Review of complex numbers:**

   (a) Show that $|zw| = |z||w|$ for any complex numbers z and w.

   (b) Show that $\overline{zw} = \bar{z}\bar{w}$ for any complex numbers z and w.

   (c) Show that $\bar{z}z = |z|^2$ for every complex number z.

   (a) If $z = x + iy$ and $w = s + it$ then $zw = xs - yt + i(xt + ys)$ so

   $$\begin{aligned}
   |zw|^2 &= (xs - yt)^2 + (xt + ys)^2 = x^2s^2 + y^2t^2 - 2xyst + x^2t^2 + y^2s^2 + 2xyst \\
   &= x^2(s^2 + t^2) + y^2(s^2 + t^2) \\
   &= (x^2 + y^2)(s^2 + t^2) \\
   &= |z|^2|w|^2
   \end{aligned}$$

   Since $|zw|$ and $|z||w|$ are positive, this implies $|zw| = |z||w|$.

   (b) Using the calculation above, $\overline{zw} = xs - yt - i(xt + ys)$ while $\overline{z}\,\overline{w} = (x - iy)(s - it) = xs - yt - i(xt + ys)$.

   (c) $\overline{z}z = (x + iy)(x - iy) = x^2 + ixy - ixy + y^2 = x^2 + y^2 = |z|^2$

2. **Calculate the inner products and norms for the following:**

   (a) the real vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}$,

   (b) the complex vectors $\begin{bmatrix} 1+i \\ 3-i \\ 2+2i \\ 6-3i \end{bmatrix}$ and $\begin{bmatrix} 2-2i \\ 4+3i \\ 6-i \\ 1 \end{bmatrix}$,

   (c) the functions $x - 1$ and $\cos x$ on the interval $[-\pi, \pi]$,

   (d) the functions e^{3ix} and e^{-ix} on the interval $[0, 2\pi]$.

   (a) Using the definition $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T\mathbf{w}$ we find

   $$\left\langle \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix} = 8.$$

   Remember that the inner product for real vectors is identical to the dot product.

   We also have (setting $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}$)

   $$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{6}, \quad \|\mathbf{w}\| = \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle} = \sqrt{35}.$$

(b) Using the definition $\langle \mathbf{v}, \mathbf{w} \rangle = \bar{\mathbf{v}}^T \mathbf{w}$ we find

$$\langle \begin{bmatrix} 1+i \\ 3-i \\ 2+2i \\ 6-3i \end{bmatrix}, \begin{bmatrix} 2-2i \\ 4+3i \\ 6-i \\ 1 \end{bmatrix} \rangle = \begin{bmatrix} 1-i & 3+i & 2-2i & 6+3i \end{bmatrix} \begin{bmatrix} 2-2i \\ 4+3i \\ 6-i \\ 1 \end{bmatrix} = 25-2i.$$

We also have (setting $\mathbf{v} = \begin{bmatrix} 1+i \\ 3-i \\ 2+2i \\ 6-3i \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2-2i \\ 4+3i \\ 6-i \\ 1 \end{bmatrix}$)

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{65}, \quad \|\mathbf{w}\| = \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle} = \sqrt{71}.$$

(c) Using the definition

$$\begin{aligned} \langle f, g \rangle &= \int_{-\pi}^{\pi} (x-1) \cos x dx = [(x-1)\sin x]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sin x dx \\ &= [\cos x]_{-\pi}^{\pi} = 0. \end{aligned}$$

Alternatively, we could have seen this quickly by noting that $x \cos x$ is anti-symmetric so the integral is zero and the integral of $\cos x$ over a full period is also zero.
We also have

$$\|x-1\| = \sqrt{\int_{-\pi}^{\pi} (x-1)^2 dx} = \sqrt{[(t-1)^3/3]_{-\pi}^{\pi}} = \sqrt{2\pi^3/3 + 2\pi}$$

and

$$\|\cos x\| = \sqrt{\int_{-\pi}^{\pi} \cos^2 x dx} = \sqrt{\int_{-\pi}^{\pi} (\cos 2x + 1)/2 \, dx} = \sqrt{\pi}.$$

(d) Using the definition

$$\begin{aligned} \langle e^{3ix}, e^{-ix} \rangle &= \int_0^{2\pi} \overline{e^{3ix}} e^{-ix} dx = \int_0^{2\pi} e^{-3ix} e^{-ix} dx = \int_0^{2\pi} e^{-4ix} dx \\ &= \left[\frac{1}{-4i} e^{-4ix} \right]_0^{2\pi} = -\frac{1}{4i}(e^{-8\pi i} - e^0) = -\frac{1}{4i}(1-1) = 0. \end{aligned}$$

We also have

$$\|e^{3ix}\| = \sqrt{\int_{-\pi}^{\pi} e^{-3ix} e^{3ix} dx} = \sqrt{\int_{-\pi}^{\pi} 1 dx} = \sqrt{2\pi}$$

and

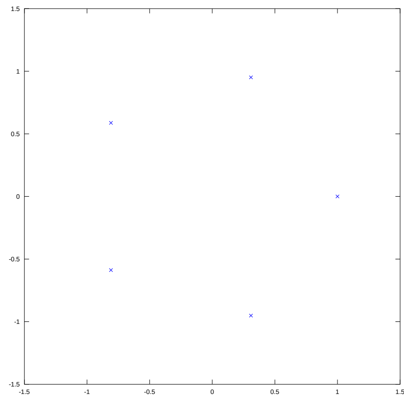
$$\|e^{-ix}\| = \sqrt{\int_{-\pi}^{\pi} e^{ix} e^{-ix} dx} = \sqrt{\int_{-\pi}^{\pi} 1 dx} = \sqrt{2\pi}.$$

3. **Plot the location of the complex numbers $z_k = e^{2\pi i k/5}$, $k = 0, 1, 2, 3, 4$ in the complex plane. Show that these numbers are fifth roots of unity, that is, they satisfy $z^5 = 1$. What is z_0? The numbers z_k are the five roots of the polynomial $z^5 - 1$ which implies that $z^5 - 1 = (z - z_0)(z - z_1)(z - z_2)(z - z_3)(z - z_4)$. Now compute $(z^5 - 1)/(z - 1)$ in two ways: by polynomial long division and by dividing the factorization above by $z - 1$. Set these expressions equal to find the factorization of $z^4 + z^3 + z^2 + z + 1$. Use this factorization to compute $z_k^4 + z_k^3 + z_k^2 + z_k + 1$ for $k = 0, 1, 2, 3, 4$.**

We can use MATLAB/Octave to do the plot

```
Z=exp(2*pi*1i*[0:4]/5);
plot(Z,'o');
axis([-1.5,1.5,-1.5,1.5])
axis equal
```

Here is the result



Notice that these are evenly spaced points on the unit circle. We have $z_k^5 = \left(e^{2\pi ik/5}\right)^5 = e^{(2\pi ik/5)5} = e^{2\pi ik} = 1$. Using $z^5 - 1 = (z - z_0)(z - z_1)(z - z_2)(z - z_3)(z - z_4)$ and $z_0 = 1$ we find that $(z^5 - 1)/(z - 1) = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$. On the other hand, polynomial long division gives $(z^5 - 1)/(z - 1) = z^4 + z^3 + z^2 + z + 1$. Thus $z^4 + z^3 + z^2 + z + 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$ which implies that $z_k^4 + z_k^3 + z_k^2 + z_k + 1 = 0$ for $k = 1, 2, 3, 4$. On the other hand since $z_0 = 1$, $z_0^4 + z_0^3 + z_0^2 + z_0 + 1 = 5$.