Math 307: Problems for section 4.6

1. **Suppose that there is a fixed population of cola drinkers each with a favourite among Coke, Pepsi and Thums Up. Every month 3% of the Coke drinkers switch to Pepsi while 5% switch to Thums Up. Every month 2% of the Pepsi drinker switch to Coke while 3% switch to Thums Up. Every month 1% of the Thums Up drinkers switch to Coke and 1% switch to Pepsi. What are the eventual market shares of these drinks? (You can use MATLAB/Octave to compute things, but explain what you are doing in your solution.)**

   Solution: This problem has the same structure as a random walk. If we order the drinks in the order Coke, Pepsi, Thums Up, the relevant stochastic matrix is

   $$P = \begin{bmatrix} 0.92 & 0.02 & 0.01 \\ 0.03 & 0.95 & 0.01 \\ 0.05 & 0.03 & 0.98 \end{bmatrix}$$

   Using MATLAB/Octave we compute the eigenvector with eigenvalue 1 to be

   $$\begin{bmatrix} 0.13462 \\ 0.21154 \\ 0.65385 \end{bmatrix}$$

   which gives the market shares. (Of course, Thums Up was bought out by Coca Cola but that's another story ...).

2. **A flea hops randomly on vertices of a triangle, hopping to each of the other vertices with equal probability (never remaining at the same vertex). The flea starts at vertex 1. What is the probability that the flea is at vertex 1 again after n hops?**

   Solution: The relationship between the state vector $\mathbf{x}^{(n)}$ after n hops and the state vector $\mathbf{x}^{(n-1)}$ after $n-1$ hops is

   $$\begin{aligned} \mathbf{x}^{(n)} &= \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \mathbf{x}^{(n-1)} \\ &= \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}^n \mathbf{x}^{(0)} \\ &= \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

   To find $x_1^{(n)}$ we need to diagonalize the matrix (let's call it P).

   Because P is a stochastic matrix, we know that it must have at least one eigenvalue of 1. We can see that a corresponding eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

We also can see that adding $(1/2)I$ to P gives a matrix whose second and third rows are repetitions of the first. Therefore the row-reduced echelon form of $(-1/2)I - P$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the nullity is 2 and there are two linearly independent eigenvectors for the eigenvalue $-1/2$. We find two such eigenvectors are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Now we have

$$S = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}$$

and

$$S^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Thus we find that

$$\begin{aligned} \mathbf{x}^{(n)} &= \frac{1}{3} \begin{bmatrix} 1 & -(-1/2)^n & -(-1/2)^n \\ 1 & (-1/2)^n & 0 \\ 1 & 0 & (-1/2)^n \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 + (-1/2)^n + (-1/2)^n \\ 1 - (-1/2)^n \\ 1 - (-1/2)^n \end{bmatrix} \end{aligned}$$

and the probability of returning to the first vertex after n hops is $(1 + 2(-1/2)^n)/3$.

3. **Show that the product of two $n \times n$ stochastic matrices is also stochastic.**

   Solution: Call the two matrices P and Q. Because they are stochastic, all the entries of P and all the entries of Q are non-negative and

$$\sum_{i=1}^{n} P_{ij} = 1, \quad \sum_{i=1}^{n} Q_{ij} = 1.$$

Now

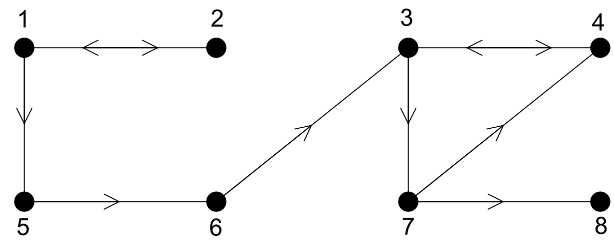
$$(PQ)_{ij} = \sum_{k=1}^{n} P_{ik} Q_{kj}.$$

Because all the entries of P and Q are non-negative, the entries of PQ must also be non-negative. We also need that the sum of each column is 1:

$$\begin{aligned} \sum_{i=1}^{n} (PQ)_{ij} &= \sum_{i=1}^{n} \sum_{k=1}^{n} P_{ik} Q_{kj} \\ &= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} P_{ik} \right) Q_{kj} \\ &= \sum_{k=1}^{n} Q_{kj} = 1. \end{aligned}$$

Therefore PQ has the two required properties to be a stochastic matrix.

4. **Find the rank for each site with $\alpha = 1$ and 0.85 for an internet that is linked in the following way**



You can use `eig` **to find the relevant eigenvector, but for $\alpha = 0.85$, check that the power method gives the same answer (give the MATLAB/Octave commands you used for the power method).**

Solution:

```
P1 = [
0 1 0 0 0 0 0 0;
0.5 0 0 0 0 0 0 0;
0 0 0 1 0 1 0 0;
0 0 0.5 0 0 0 0.5 0;
0.5 0 0 0 0 0 0 0;
0 0 0 0 1 0 0 0;
0 0 0.5 0 0 0 0 0;
0 0 0 0 0 0 0.5 0;
];
Z=zeros(8,1);
R=(1/8)*ones(8,1);
P2 = [Z Z Z Z Z Z Z R];
Q = [R R R R R R R R];
[V D] = eig(P1+P2)

(Check the output for which column of V corresponds to the eigenvalue 1)

V(:,4)/sum(V(:,4))

ans =

   0.048193
   0.036145
   0.313253
   0.253012
   0.036145
   0.048193
   0.168675
   0.096386
```

This gives the ranks for $\alpha = 1$: 3, 4, 7, 8, 1 = 6, 2 = 5. Now we do $\alpha = 0.85$.

```
S = (0.15)*Q + (0.85)*(P1+P2);
[V D] = eig(S)
```

```
(Check the output for which column of V corresponds to the eigenvalue 1)

V(:,1)/sum(V(:,1))

ans =

   0.081814
   0.063019
   0.272054
   0.205016
   0.063019
   0.081814
   0.143871
   0.089393
```

This gives the same ranks as for $\alpha = 1$. Now lets check that the power method gives the same answer.

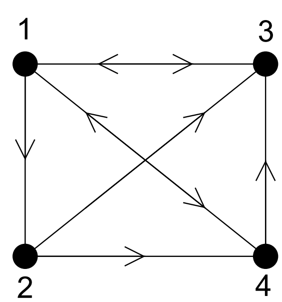
```
S^20*[1;0;0;0;0;0;0;0]

ans =

   0.081826
   0.063013
   0.272032
   0.205020
   0.063013
   0.081826
   0.143888
   0.089381
```

There is pretty good agreement after 20 iterations.

5. **Consider the following internet:**



If $\alpha = 1$, show that the rankings of the pages are such that page 1 is highest.

The owners of page 3 are furious that their page doesn't rank highest. They try to increase the rank of page 3 by creating a new page, page 5, to which page 3 links and which links to page 3. Does this increase the rank of page 3 above that of page 1 (with $\alpha = 1$)?

Solution: Using MATLAB/Octave we find:

```
> P = [0 1/3 1/3 1/3; 0 0 1/2 1/2; 1 0 0 0; 1/2 0 1/2 0]'
P =
   0.00000   0.00000   1.00000   0.50000
   0.33333   0.00000   0.00000   0.00000
   0.33333   0.50000   0.00000   0.50000
   0.33333   0.50000   0.00000   0.00000

> [V D] = eig(P)
V =
   0.72101 + 0.00000i  -0.75522 + 0.00000i  -0.75522 - 0.00000i   0.50649 + 0.00000i
   0.24034 + 0.00000i   0.30367 + 0.34607i   0.30367 - 0.34607i  -0.60566 + 0.00000i
   0.54076 + 0.00000i   0.09315 - 0.27468i   0.09315 + 0.27468i  -0.38154 + 0.00000i
   0.36051 + 0.00000i   0.35839 - 0.07139i   0.35839 + 0.07139i   0.48071 + 0.00000i

D =
   1.00000 + 0.00000i   0.00000 + 0.00000i   0.00000 + 0.00000i   0.00000 + 0.00000i
   0.00000 + 0.00000i  -0.36062 + 0.41098i   0.00000 + 0.00000i   0.00000 + 0.00000i
   0.00000 + 0.00000i   0.00000 + 0.00000i  -0.36062 - 0.41098i   0.00000 + 0.00000i
   0.00000 + 0.00000i   0.00000 + 0.00000i   0.00000 + 0.00000i  -0.27875 + 0.00000i
```

Now we need to scale the eigenvector corresponding to eigenvalue 1 so that its entries sum to 1.

```
> V(:,1)/sum(V(:,1))
ans =
   0.38710
   0.12903
   0.29032
   0.19355
```

So page 1 has the highest ranking, followed by page 3.

Considering the modified internet we have

```
> P = [0 1/3 1/3 1/3 0; 0 0 1/2 1/2 0; 1/2 0 0 0 1/2; 1/2 0 1/2 0 0; 0 0 1 0 0]'
P =
   0.00000   0.00000   0.50000   0.50000   0.00000
   0.33333   0.00000   0.00000   0.00000   0.00000
   0.33333   0.50000   0.00000   0.50000   1.00000
   0.33333   0.50000   0.00000   0.00000   0.00000
   0.00000   0.00000   0.50000   0.00000   0.00000

> [V D] = eig(P)
V =

 Columns 1 through 4:

  -0.48949 + 0.00000i   0.23337 + 0.00000i   0.68220 + 0.00000i   0.68220 - 0.00000i
  -0.16316 + 0.00000i   0.23129 + 0.00000i  -0.40735 - 0.36638i  -0.40735 + 0.36638i
  -0.73424 + 0.00000i  -0.41815 + 0.00000i  -0.08340 + 0.08876i  -0.08340 - 0.08876i
  -0.24475 + 0.00000i   0.57513 + 0.00000i  -0.33765 + 0.28994i  -0.33765 - 0.28994i
  -0.36712 + 0.00000i  -0.62163 + 0.00000i   0.14620 - 0.01232i   0.14620 + 0.01232i

Column 5:
```

```
   0.51776 + 0.00000i
  -0.23999 + 0.00000i
  -0.67155 + 0.00000i
  -0.07313 + 0.00000i
   0.46691 + 0.00000i


D =

 Columns 1 through 4:

   1.00000 + 0.00000i   0.00000 + 0.00000i   0.00000 + 0.00000i   0.00000 + 0.00000i
   0.00000 + 0.00000i   0.33633 + 0.00000i   0.00000 + 0.00000i   0.00000 + 0.00000i
   0.00000 + 0.00000i   0.00000 + 0.00000i  -0.30860 + 0.27756i   0.00000 + 0.00000i
   0.00000 + 0.00000i   0.00000 + 0.00000i   0.00000 + 0.00000i  -0.30860 - 0.27756i
   0.00000 + 0.00000i   0.00000 + 0.00000i   0.00000 + 0.00000i   0.00000 + 0.00000i

Column 5:

   0.00000 + 0.00000i
   0.00000 + 0.00000i
   0.00000 + 0.00000i
   0.00000 + 0.00000i
  -0.71914 + 0.00000i

> V(:,1)/sum(V(:,1))
ans =
   0.244898
   0.081633
   0.367347
   0.122449
   0.183673
```

Thus the third website is now the highest ranked website.

6. **Prove that for an internet, the rankings of a page having no links to it from any other page is always zero assuming $\alpha = 1$.**

Solution: To show that the rank of a page, say page j, will be zero we need to show that the jth element of an eigenvector, $\mathbf{v}$, corresponding to the eigenvalue $\lambda = 1$ is always zero. Note that $\mathbf{v}$ satisfies

$$P\mathbf{v} = \mathbf{v},$$

where P is the stochastic matrix describing the intenet.

There are two cases to consider. First if page j has no links to it from other pages but page j has links to other pages, then row j of the stochastic matrix P describing the internet will be a row of zeros. Therefore the jth entry of $\mathbf{v}$ is

$$v_j = \sum_{k=1}^{n} P_{jk} v_k = 0.$$

The other case is if page j has no links to it from other pages and it also has no links to other pages. In this case the entries of column j in P will all be equal to $1/N$, where N is the total number of pages in the internet. So now row j of P will be all zeros except for the jth entry which will be $1/N$. Therefore the jth entry of $\mathbf{v}$ is

$$v_j = \sum_{k=1}^{n} P_{jk} v_k = \frac{1}{N} v_j.$$

Assuming $N > 1$ (i.e., that there is more than one page in the internet), the only solution is $v_j = 0$.

7. **Write a MATLAB/Octave function metropolis.m such that `metropolis(P,p)` produces a stochastic matrix whose invariant distribution is propotional to `p`. (You may assume that the entries `p` in the input are strictly positive. Solution:**

```
function Q=metropolis(P,p)

% P is an nxn stochastic matrix
% p is a multiple of the desired invariant distribution (assumed strictly positive)
% Q is the output of the metropolis algorithm

N=length(P);
Q=zeros(N,N);

for i=1:N-1
for j=i+1:N
if(P(i,j)==0 | P(j,i)==0)
Q(i,j)=Q(j,i)=0;
elseif(P(i,j)*p(j) > p(i)*P(j,i))
Q(j,i)=P(j,i);
Q(i,j)=p(i)*P(j,i)/p(j);
else
Q(i,j)=P(i,j);
Q(j,i)=p(j)*P(i,j)/p(i);
end
end
end

for i=1:N
Q(i,i) = 1 - sum(Q(:,i));
end
```

8. **Using the function in the previous question or by hand, compute the matrix Q given by the Metropolis algorithm when**

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

and

$$p = [1, 2, 3]$$

Solution:

```
> Q=metropolis([1/3 1/3 1/3;1/3 1/3 1/3;1/3 1/3 1/3],[1,2,3])
Q =
```

```
   0.33333   0.16667   0.11111
   0.33333   0.50000   0.22222
   0.33333   0.33333   0.66667
```

Let's check

```
> Q*[1;2;3]
ans =

   1
   2
   3
```

9. **Compute the matrix Q given by the Metropolis algorithm when**

$$P = \begin{bmatrix} 1/2 & 0 & 1/3 \\ 1/2 & 1 & 1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

and p is anything.

Solution: For this stochastic matrix P, one of pi, j or $p_{j,i}$ for $i \neq j$ is always zero, so the Metropolis algorithm will give $Q = I$.