Math 307: Problems for section 1.1

1. **Use Gaussian elimination to find the solution(s) to $A\mathbf{x} = \mathbf{b}$ where**

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & -3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & 6 & -7 & 8 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \text{(b)}\quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

2. **Use MATLAB/Octave to find the solution(s) to $A\mathbf{x} = \mathbf{b}$ where**

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \text{(b)}\quad A = \begin{bmatrix} 1 & 0 & 3 & 2 & -4 \\ 2 & 1 & 6 & 5 & 0 \\ -1 & 1 & -3 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}.$$

3. **Compute the time it takes to solve $A\mathbf{x} = \mathbf{b}$ using `A\b` on your computer for five or more random problems of increasing size. Make a plot of time vs. size. Repeat using the method `A^(-1)*b` and plot on the same graph. (If your calculation is taking too long, it can be interrupted by typing `<ctrl> c`.)**

4. **Compute the 1,2 and infinity norms for the following vectors. Is it always the same norm that is biggest? (The 2-norm is the standard Euclidean norm)**

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \; \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \; \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

5. **Find four vectors in two dimensions whose 1, 2 and infinity norms the same.**

6. **Suppose you were doing a problem each entry v_i in a vector $[v_1, v_2, \ldots, v_n]$ is positive and represents the yearly production of one of n factories. Which of the three norms we introduced have natural interpretations in this context.**

7. **Draw a picture of the "unit circle" for the 1,2, and infinty norms in two dimensions. By "unit circle" we mean the set of all vectors whose norm is equal to one.**

8. **Recall that the Euclidean distance between two vectors $\mathbf{v}$ and $\mathbf{w}$ is $\|\mathbf{v} - \mathbf{w}\|$ where we use the standard norm. If we use the 1-norm or ∞-norm in this formula, we obtain different distance functions. Consider vectors whose entries are either 0 or 1 (like $[0, 1, 1, 0, 1]$). Describe in words the meaning of the 1-distance and the ∞-distance between two such vectors.**

9. **The p-norm of a vector $\mathbf{v} = [v_1, v_2, \ldots, v_n]^T$ for $1 \le p \le \infty$ is defined to be**

$$\|\mathbf{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$$

Guess the MATLAB/Octave syntax to compute this norm and check that you are right. Which p corresponds to the standard (Euclidean) norm? What is the limit of the p-norm of a vector as p tends to infinity?

10. **Show that for any square matrix A (with real entries), $\|A\|_{HS}^2 = \mathrm{tr}(A^T A)$.**

11. **Guess whether each of the following statements about $n \times n$ matrices A is true or false by testing them on a few random matrices. Bonus: prove that your guess is correct.**

    (a) $\|A^2\| = \|A\|^2$

    (b) $\|A^2\| \le \|A\|^2$

    (c) $\|A^T A\| = \|A\|^2$

    (d) $\|A\| \le \|A\|_{HS}$

    (e) $\mathrm{cond}(A) = \mathrm{cond}(A^{-1})$

    (f) $\mathrm{cond}(A) \ge 1$

12. **For a diagonal matrix, the eigenvalues are equal to the diagonal entries. So the matrix norm is the absolute value of the largest eigenvalue. Show that this is not true for an arbitrary matrix. (In MATLAB/Octave `eig(A)` computes the eigenvalues of A) For a matrix A with real entries, there is a relationship between the norm of A and the eigenvalues of $A^T A$. Can you guess what it is using MATLAB/Octave?**

13. **Suppose A has a large condition number. This means that in the equation $A\mathrm{x} = \mathrm{b}$, a small relative error in b *may* result in a large relative error in x. Is is possible, though, that for *some* choices of b and Δb the relative error of x is not large. Illustrate this using the matrix $A = \begin{bmatrix} 1000 & 0 \\ 0 & 1 \end{bmatrix}$.**

14. **A famous ill-conditioned matrix is the hilbert matrix whose i, j entry is $1/(i + j)$. In MATLAB/Octave the hilbert matrix of size n can be generated using hilb(n). What is the condition number when n=5, n=10?**

15. **In "single precision" computer calculations, we cannot trust more than approximately the first 7 significant figures. Assuming that the relative error in the right-hand-side of the matrix equation $A\mathrm{x} = \mathrm{b}$ is $\|\Delta \mathrm{b}\|/\|\mathrm{b}\| = 1.1921 \times 10^{-7}$, give an upper bound on the relative error $\|\Delta \mathrm{x}\|/\|\mathrm{x}\|$ in the solution of the equation for the following matrices A:**

$$\textbf{(a)} \quad A = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \qquad \textbf{(b)} \quad A = \begin{bmatrix} 4 & 1.99 \\ 2.01 & 1 \end{bmatrix}$$

Interpret these bounds to say how many significant figures we can trust in the solution.