The University of British Columbia

27 February 2014

Midterm for MATH 307 Section 202 Winter 2014

Closed book examination — Time: 50 minutes

Last Name ______________________ **First** ______________

Signature ______________

Student Number ______________

Special Instructions:

No memory aids are allowed. No communication devices. No calculators. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. **Where boxes are provided for answers, put your final answers in them.**

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

  (b) Speaking or communicating with other candidates.

  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		10
Total		40

[10] **1.** Consider the MATLAB/Octave command

```
> A = vander([1 2 3 4])
```

(a) Write down the matrix A.

(b) Matrix A is a (circle one):

i. Vandermonde matrix
ii. Hilbert matrix
iii. Tridiagonal matrix
iv. Triangular matrix

(c) Suppose we have the following output:

```
> cond(A,1)
  ans = 2000.0
> cond(A,2)
  ans = 1171.0
> cond(A,inf)
  ans = 1926.7
```

Why is the output from these different calls to the `cond()` function different?

(d) Suppose the norm of vector $\mathbf{b}$ is known to a relative accuracy of 1×10^{-7}. What is the worst relative accuracy for vector $\mathbf{x}$ we can expect when solving:

```
> x = A\b
```

(e) Matrix A is often used to compute interpolation parameters. How do the relatively large values of `cond(A)` suggest an advantage of cubic spline interpolation over Lagrange interpolation?

Additional space for Question 1.

[10] **2.** Consider the sparse matrix equation

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} e^{-h}\cos(\pi h) \\ e^{-2h}\cos(2\pi h) \\ \vdots \\ e^{-(n-1)h}\cos(\pi(n-1)h) \end{bmatrix}, \quad n = 1/h.$$

(a) Write an ODE that this matrix equation approximates, including boundary conditions.

(b) Modify the matrix equation in two ways:

i. change the left-hand boundary condition to $u(0) = 1$;

ii. change the left-hand boundary condition to $u'(0) = 1$.

(c) Give a reason why you might not want to choose an extremely small value for h.

Additional space for Question 2.

[10] **3.** Let

$$A = \begin{bmatrix} 2 & -6 & 0 & 2 \\ -1 & 3 & 1 & 1 \\ 4 & -12 & -2 & 0 \end{bmatrix}, \quad \operatorname{rref}(A) = \begin{bmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \operatorname{rref}(A^T) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

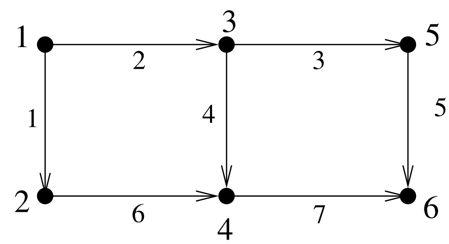
(a) State the dimensions of the four fundamental subspaces of A.

(b) State the rank of A.

(c) Write down bases for each of the four fundamental subspaces of A.

(d) Write down an alternative basis for the rowspace of A.

[10] **4.** Consider the following directed graph. When considered as a resistor network, assume all resistances $R_i = 1$.



(a) Write down the node-edge incidence matrix D and the Laplacian matrix L.

(b) What are the dimensions of each of the four fundamental subspaces of D?

(c) Write down two linearly independent vectors in the left nullspace of D. Do they form a basis? Give a reason.

(d) What happens to the dimensions of the nullspace and left nullspace of D when edge 4 is removed?

(e) Suppose the Laplacian matrix L has been defined in MATLAB/Octave. Write down the code that computes the effective resistance between nodes 1 and 6.

Additional space for Question 4.

`pi`	π
`i`	$\sqrt{-1}$
`x = 3`	define variable $\mathbf{x}$ to be 3
`x = [1 2 3]`	set $\mathbf{x}$ to the 1×3 row vector $(1, 2, 3)$
`x = [1; 2; 3]`	set $\mathbf{x}$ to the 3×1 vector $(1, 2, 3)$
`A = [1 2; 3 4]`	set A to the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
`x(2) = 7`	change x_2 to 7
`A(2,1) = 0`	change A_{21} to 0
`3*x`	multiply each element of $\mathbf{x}$ by 3
`x+3`	add 3 to each element of $\mathbf{x}$
`x+y`	add $\mathbf{x}$ and $\mathbf{y}$ element by element
`A*x`	product of matrix A and column vector $\mathbf{x}$
`A*B`	product of two matrices A and B
`x.*y`	element-wise product of vectors $\mathbf{x}$ and $\mathbf{y}$
`A^3`	for a square matrix A, raise to third power
`cos(A)`	cosine of every element of A
`sin(A)`	sine of every element of A
`x'`	transpose of vector $\mathbf{x}$
`A'`	transpose of vector A
`A(2:12,4)`	the submatrix of A consisting of the second to twelfth rows of the fourth column
`A(2:12,4:5)`	the submatrix of A consisting of the second to twelfth rows of the fourth and fifth columns
`A(2:12,:)`	the submatrix of A consisting of the second to twelfth rows of all columns
`A([1:4,6],:)`	the submatrix of A consisting of the first to fourth rows and sixth row
`[A B; C D]`	creates the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are block matrices (blocks must have compatible sizes)
`rand(12,4)`	12×4 matrix with uniform random numbers in $[0, 1)$
`zeros(12,4)`	12×4 matrix of zeroes
`ones(12,4)`	12×4 matrix of ones
`eye(12)`	12×12 identity matrix
`eye(12,4)`	12×4 matrix whose first 4 rows are the 4×4 identity
`linspace(1.2,4.7,100)`	row vector of 100 equally spaced numbers from 1.2 to 4.7
`diag(x)`	matrix whose diagonal is the entries of $\mathbf{x}$ (other elements are zero)
`diag(x,n)`	matrix whose diagonal is the entries of $\mathbf{x}$ on diagonal n (other elements are zero)
`sum(x)`	sum of the elements of $\mathbf{x}$

`A\b`	returns the solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$
`A^(-1)`	returns the inverse of A
`rref(A)`	returns the reduced row echelon form of A
`det(A)`	returns the determinant of A
`norm(A)`	returns the (operator) norm of A
`cond(A)`	returns the condition number of A
`length(A)`	returns the larger of the number of rows and number of columns of A
`norm(x)`	returns the norm (length) of a vector $\mathbf{x}$
`vander(x)`	returns the Vandermonde matrix for the points of $\mathbf{x}$
`polyval(a,x)`	returns the values of the polynomial $a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_n$ at the points of $\mathbf{x}$
`[Q R] = qr(A,0)`	returns the matrices Q and R in the QR factorization of A
`nextpow2(N)`	calculates the next power of 2 of N
`fft(f,N)`	FFT transform of the vector $\mathbf{f}$ using N points (pads $\mathbf{f}$ with zeros if it has fewer than N elements)
`polyval(A)`	returns the coefficients of the characteristic polynomial of A
`roots(a)`	returns the solutions to $a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_n = 0$
`[V D] = eig(A)`	returns the matrix V whose columns are normalized eigenvectors of A and the diagonal matrix D of corresponding eigenvalues
`plot(x,y,'bo')`	plots the points of $\mathbf{y}$ against the points of $\mathbf{x}$ using blue dots
`plot(x,y,'r-')`	plots the points of $\mathbf{y}$ against the points of $\mathbf{x}$ using red lines
`semilogy(x,y,'bo')`	plots $\mathbf{y}$ against $\mathbf{x}$ using a logarithmic scale for $\mathbf{y}$
`axis([-0.1 1.1 -3 5])`	changes the axes of the plot to be from -0.1 to 1.1 for the x-axis and -3 to 5 for the y-axis
`hold on`	puts any new plots on top of the existing plot
`hold off`	any new plot commands replace the existing plot (this is the default)
`plot3(x,y,z,'bo')`	plots the points of $\mathbf{z}$ against the points of $\mathbf{x}$ and $\mathbf{y}$ using blue dots
`for k=1:10 ... end`	for loop taking `k` from `1` to `10` and performing the commands $\ldots$ for each