# MIE 1807
# Principles of Measurement

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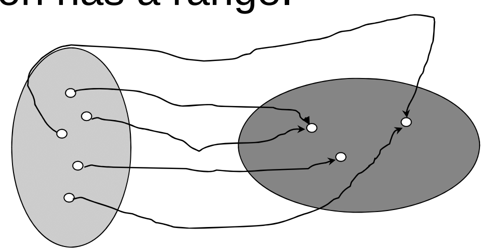
Lecture notes:
https://github.com/mie1807-winter-2017

## The Basic Rules: the simple (trivial) motivating examples

- Toss a coin
- Throw a six-sided die
- Throw two six-sided dice
- Good or bad pedagogy?
  - “Boring” “Unrealistic”
  - Completely understandable
- “Probability” requires the definition of two specific kinds of functions.

# Functions

- A function has a domain.
- A function has a range.



- The function is a rule
- And then you are corrupted by calculus.



# Corruption by Calculus

- You spend years worrying about what the rules (i.e. functions) *look* like.
  - Call them all *f(x)*
  - Continuous?
  - Differentiable?
  - Max/min/graphs/drawing/etc.
- We don't do these things in probability to the two kinds of functions of interest.

# A *probability* is a *function*

Consider the toss of a coin. The *outcomes* are:

$$S = \{\mathrm{H}, \mathrm{T}\}$$

The *sets of outcomes*, or *events* are:

$$\begin{array}{lcl} \emptyset & \to & 0 \\ \{H\} & \to & 0.5 \\ \{T\} & \to & 0.5 \\ \{H, T\} & \to & 1 \end{array}$$

This rule is called a *probability*



# The most basic rules

$$P(\emptyset) = 0 \qquad\qquad P(S) = 1$$

For any event $A \subset S$:

$$0 \leq P(A) \leq 1$$

# Another example

Consider the throw of a die. The *outcomes* are:

$$S = \{1, 2, 3, 4, 5, 6\}$$

Simple example, but there are $2^6 = 64$ possible *events*....

$$\emptyset, \{1\}, \{2\}, \{2, 4, 6\}, \{1, 2, 3, 4\}, \ldots$$

Some probabilities:

$$P(\{1\}) = \frac{1}{6} = P(\{2\}) \text{ etc.} \qquad P(\{1, 2\}) = \frac{2}{6}$$



# Another Rule

$$A = \{2, 3, 4\} \qquad B = \{2, 4, 6\}$$

$$A \cup B = \{2, 3, 4, 6\}$$

$$A \cap B = \{2, 4\}$$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$$= \frac{4}{6}$$

## Important Special Case

$$A = \{2, 3, 4\} \qquad C = \{1, 6\}$$

$$A \cup C = \{1, 2, 3, 4, 6\}$$

$$A \cap C = \emptyset$$

$$P(A \cup C) = P(A) + P(C)$$
$$= \frac{5}{6}$$



## Generalization

If $A_1, A_2, \ldots$ are *mutually disjoint* (i.e. $A_i \cap A_j = \emptyset$), then

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \cdots$$

## Summary of the first of two new kinds of function

- A *probability* is a
  - *Function*
  - whose domain is a collection of subsets of a sample space *S*
  - whose range is the unit interval [0,1]
  - that satisfies the basic rules on the previous slides



## What if you have some partial information about an event?

- You are *willing* to bet money that the toss of a die will result in a 1 or a 2.
- The die is tossed.
- I tell you "an odd number appeared!"
- I give you a chance to quit the game.
- What would your *rational* choice be?
- What if instead I told you "a number larger or equal to 2 occurred"?

## Analysis…

*A priori* $S = \{1, 2, 3, 4, 5, 6\}$

$$A = \{1, 2\} \qquad P(A) = \frac{1}{3}$$

"An odd number appeared" $S' = \{1, 3, 5\}$

$$A' = \{1\} \qquad P'(A') = \frac{1}{3}$$

"$\geq 2$ appeared" $S^* = \{2, 3, 4, 5, 6\}$

$$A^* = \{2\} \qquad P^*(A') = \frac{1}{5}$$



## Conditional Probability

Given events $A$ and $B$ with $P(B) > 0$, the *conditional probability of $A$ given $B$* is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Die toss example 1: $A = \{1, 2\}$ $B = \{1, 3, 5\}$

$$P(A|B) = \frac{P(\{1\})}{P(\{1, 3, 5\})} = \frac{1/6}{1/2} = \frac{1}{3}$$

## Conditional Probability

Die toss example 2: $A = \{1, 2\}$ $B = \{2, 3, 4, 5, 6\}$

$$P(A|B) = \frac{P(\{2\})}{P(\{2, 3, 4, 5, 6\})} = \frac{1/6}{5/6} = \frac{1}{5}$$

(You might be interested in Bayes' formula, for your own education. See textbook #2 for details.)



## Independence

The obviously interesting thing about the first example is that the probability did not change with extra information.

$$P(A|B) = \frac{1/6}{1/2} = \frac{1}{3} = P(A)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

*Independence:* $P(A \cap B) = P(A)P(B)$

## The second of the two new kinds of functions

Sample spaces can be unwieldy, let alone collections of subsets thereof.

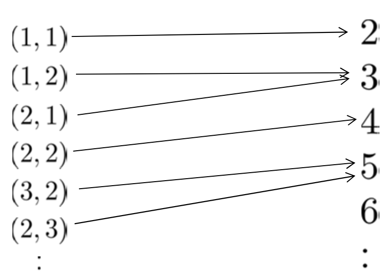
Consider tossing two dice, with 36 outcomes and $2^{36} = 6.9 \times 10^9$ events.

A more natural setting is usually to consider a numerical result relating to the random outcome, such as "the sum of the two dice".



## The second new kind of function

We'll mostly study functions with sample space domains and *real valued* ranges.



## The thing to notice about this kind of function

Any subset of the real line you can think of implies (via this function) some set of outcomes.

Examples:

"anything less than 4" $\Longrightarrow \{(1,1),(1,2),(2,1)\}$

"bigger than or equal to 11"

$\Longrightarrow \{(5,6),(6,5),(6,6)\}$

And what can be done with "sets of outcomes"?



## This new kind of function needs a name

A real-valued function of a sample space is called a *random variable*.

The conventional notation:

$$X, Y, Z, W, X_1, X_2, Y_1, X_{12}, \text{ etc.}$$

Argument of the function is usually omitted!

## Calculus versus probability again

In calculus the fundamantal question is: what does the function *look like*.

Tangents, areas, derivatives, minima, maxima. Notation: $f(x)$ etc.

In probability the fundamental question is: what is its *distribution*.

i.e. the range of values and their probabilities.



## The fundamental distribution descriptors

First, two examples:

Coin toss where H leads to gain of \$1 and T leads to loss of \$1.

$X_1(H) = 1$ and $X_1(T) = -1$

$P(X_1 = 1) = 0.5$ and $P(X_1 = -1) = 0.5$

## The fundamental distribution descriptors

Die toss where gain in $X_2$ dollars is $10\times$ the outcome.

$$P(X_2 = 10i) = \frac{1}{6}$$

In both cases we came up with a function of the form:

$$P(X = x_i) = p(x_i) = p_i$$

Where $p_i \geq 0$ and $\sum p_i = 1$.



## Descriptor 1: *Probability mass function*

Such a function is called a pmf and is used when the random variable takes on discrete values.

Completely describes the distribution for such a random variable.

Fundamental use is to compute probabilities:

$$P(X \in A) = \sum_{x_i \in A} p(x_i)$$

# Fundamental distribution descriptors

For discrete r.v., there is the p.m.f.:

$$P(X = x_i) = p(x_i) = p_i$$

Where $p_i \geq 0$ and $\sum p_i = 1$.

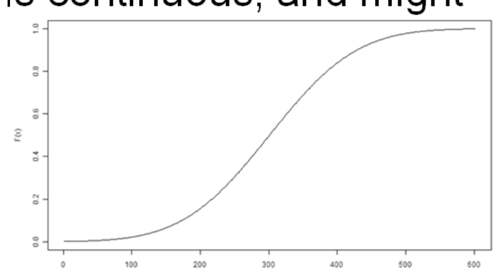
For *all* random variables there is the *cumulative distribution function* (cdf):

$$F(x) = P(X \leq x)$$



# "Continuous" random variables

- A random variable that takes on *real* values on an *interval* is usually called a *continuous* random variable.
- It's *cdf* is what is continuous, and might look like this:



# Probability Density Function

The analogue to p.m.f. for continuous random variables is the p.d.f. (the derivative of the c.d.f.) Notation: $f(x)$. Synonym: "density"

(Think of a p.d.f. as the "limiting" histogram as $n \to \infty$.)

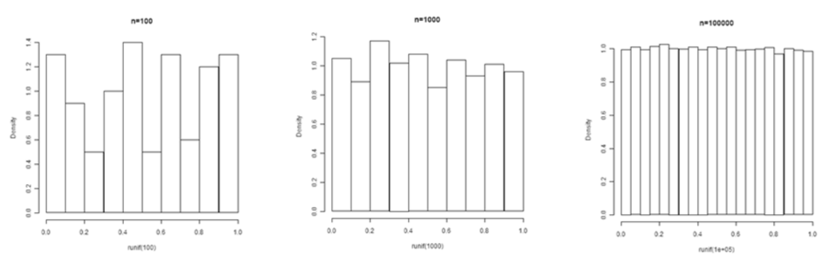
Fundamental use:

$$P(a < X \leq b) = \int_a^b f(x)\,dx$$



# Example 1

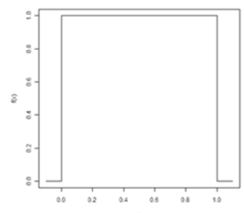
Computing problem: let $X$ be generated randomly within the interval $(0, 1)$.



## Example 1

Probability of being in $(a, b)$ for $0 \le a \le b \le 1$ should be: $b - a$

Density:



$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

"Uniform distribution on (0,1)"



## Consequence of continuous r.v. "model"

Randomly generate between 0 and 1. What is the probability of getting, say, 0.2?

$$\begin{aligned} P(X = 0.2) &= P(0.2 \le X \le 0.2) \\ &= \int_{0.2}^{0.2} 1 \, dx = 0 \end{aligned}$$

$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b)$ etc.— **for continuous r.v. only!**

# Independent Random Variables

Events $A$ and $B$ are *independent* if knowledge that $A$ occurred would not alter your knowledge of the probability that $B$ occurred.

Random variables $X_1$ and $X_2$ are *independent* if knowing the outcome of $X_1$ would not alter your assessment of the distribution of $X_2$.



# Expected Value

Recall sample mean....

Theoretical analogue for random variables is *expected value* $E(X)$

$$E(X) = \begin{cases} \sum_x xp(x) & \text{for } X \text{ discrete} \\ \int_{\infty}^{\infty} xf(x)\,dx & \text{for } X \text{ continuous} \end{cases}$$

as long as everything converges blah blah blah...

$E(X)$ is often shortened to $\mu$, but be careful.

## Examples

Die toss where gain in $X$ dollars is $10\times$ the outcome.

$p(x) = P(X = 10i) = \frac{1}{6}$ for $x \in \{1, 2, 3, 4, 5, 6\}$

$$E(X) = \sum_{i=1}^{6} 10i\frac{1}{6} = 35$$



## Examples

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

"Uniform distribution on (0,1)"

$$E(X) = \int_{\infty}^{\infty} x f(x)\, dx = \int_{0}^{1} x\, dx$$

$$= \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

## *Moments*

The $r^{th}$ moment of $X$ is:

$$E(X^r) = \begin{cases} \sum_x x^r p(x) & \text{for } X \text{ discrete} \\ \int_{\infty}^{\infty} x^r f(x)\,dx & \text{for } X \text{ continuous} \end{cases}$$

as long as everything converges blah blah blah...

In this course the first two moments will be of interest.



## Variance and Standard Deviation

Recall sample variance....

Theoretical analogue for random variables is *variance* $\mathrm{Var}(X)$

$$\mathrm{Var}(X) = \begin{cases} \sum_x (x - E(X))^2 p(x) & \text{for } X \text{ discrete} \\ \int_{\infty}^{\infty} (x - E(X))^2 f(x)\,dx & \text{for } X \text{ continuous} \end{cases}$$

as long as everything converges blah blah blah...

$\mathrm{Var}(X)$ is often shortened to $\sigma^2$, but be careful.

Standard deviation: $\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)} = \sigma$

# Example

Die toss where gain in $X$ dollars is $10\times$ the outcome.

$p(x) = P(X = 10i) = \frac{1}{6}$ for $x \in \{1, 2, 3, 4, 5, 6\}$

$$\mathrm{Var}(X) = \sum_{i=1}^{6} (10i - 35)^2 \frac{1}{6} = 291.67$$

$$\mathrm{sd}(X) = 17.1$$



# Computing Variances

$$\mathrm{Var}(X) = \int_{\infty}^{\infty} (x - E(X))^2 f(x)\,dx$$

$$= \int_{\infty}^{\infty} (x^2 - 2xE(X) + E(X)^2) f(x)\,dx$$

$$= \int_{\infty}^{\infty} x^2 f(x)\,dx - \int_{\infty}^{\infty} 2xE(X) f(x)\,dx + \int_{\infty}^{\infty} E(X)^2) f(x)\,dx$$

$$= E(X^2) - 2E(X)^2 + E(X)^2$$

$$= E(X^2) - E(X)^2$$

## Example

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

"Uniform distribution on (0,1)"

$$\text{Var}(X) = \int_{\infty}^{\infty} (x - E(X))^2 f(x)\, dx$$

$$= E(X^2) - E(X)^2$$

$$= \int_0^1 x^2 f(x)\, dx - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$



## Some Rules

$$E(aX + b) = aE(X) + b$$

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

If $X_1$ and $X_2$ are independent:

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2)$$

but not vice-versa.

## Standardization

The rules allow us to consider *standardized* versions of random variables, with mean 0 and variance 1.

$$Y = \frac{X-\mu}{\sigma}$$

$$E(Y) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}(\mu-\mu) = 0$$

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}\sigma^2 = 1$$

But $Y$ and $X$ have the same *shape*.



## A frequent probabilistic situation…

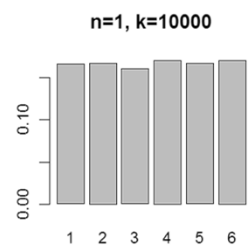
- Many random quantities result from:
  - Sums/averages of…
  - …a relatively large number of…
  - …roughly equally weighted…
  - …independent random contributions.
- Direct examples: height, test scores, measurement errors.
- Indirect example: *sample averages*

## Indirect Example 1

Toss $n$ dice and let $X$ be the sample average result. Formally:

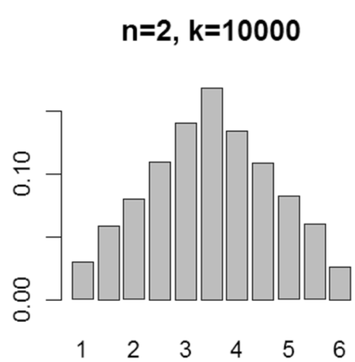
$$X = (X_1 + X_2 + \cdots + X_n)/n$$

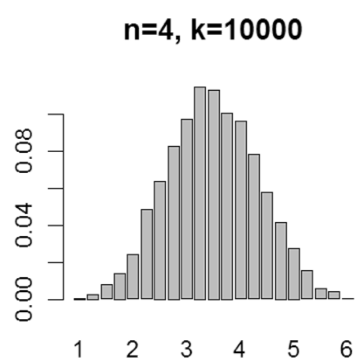
Histogram of $k = 10000$ tosses with $n = 1$



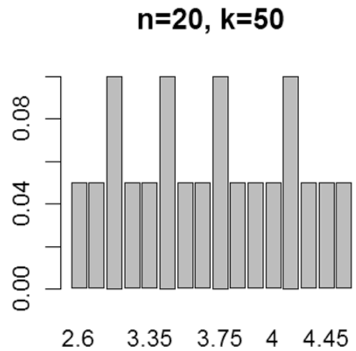


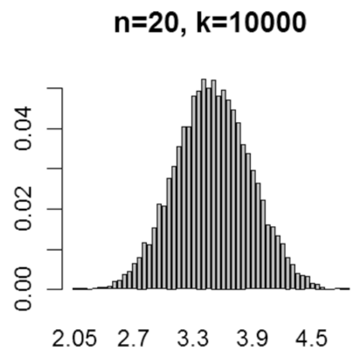

## Indirect Example 1



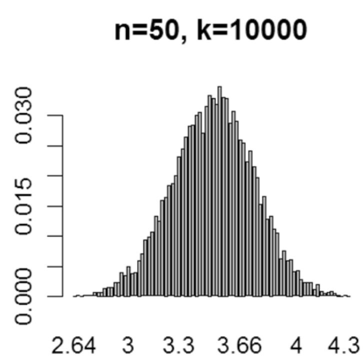
# Indirect Example 1






# Indirect Example 1

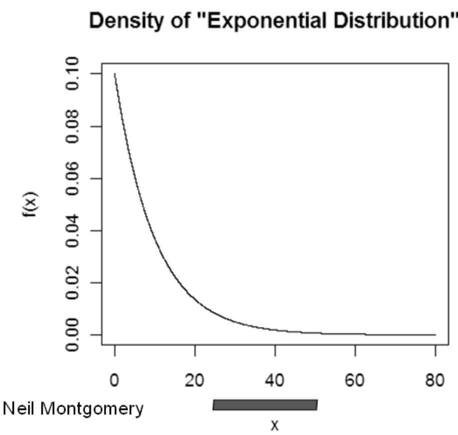


# Indirect Example 2

Observe $n$ engine failures $X$ be the sample average failure time.

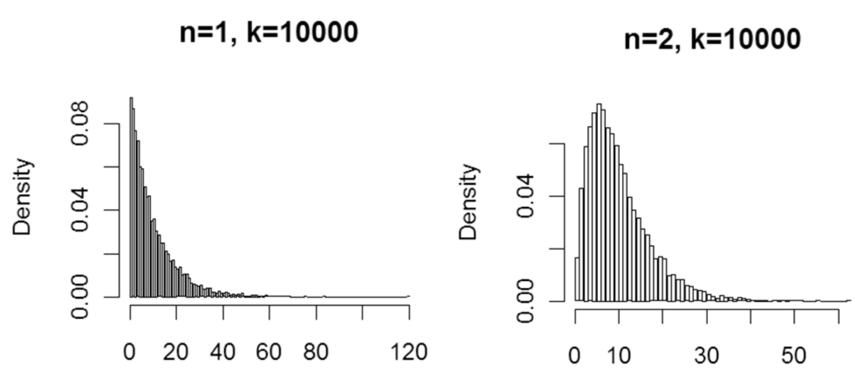
Density commonly used to model:

$f(x) = \lambda e^{-\lambda x}$ for $x > 0$






# Indirect Example 2



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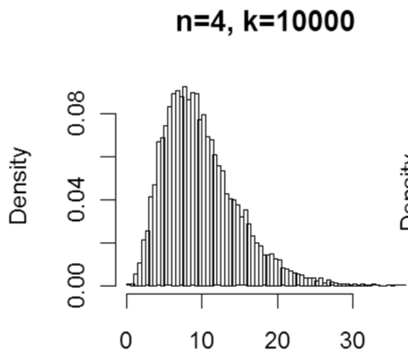


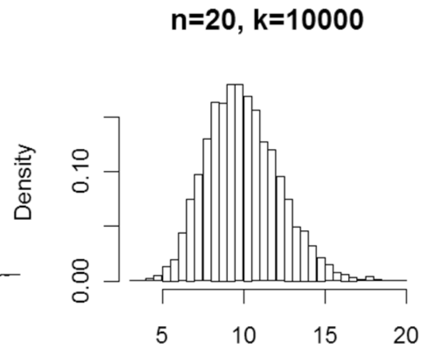




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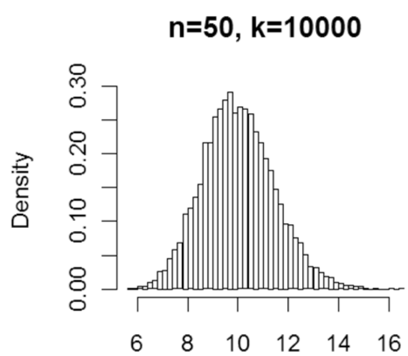


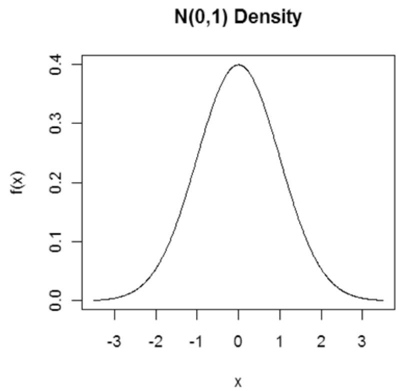
## Model: The *Normal* Distributions

Symmetric, bell-shaped densities:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

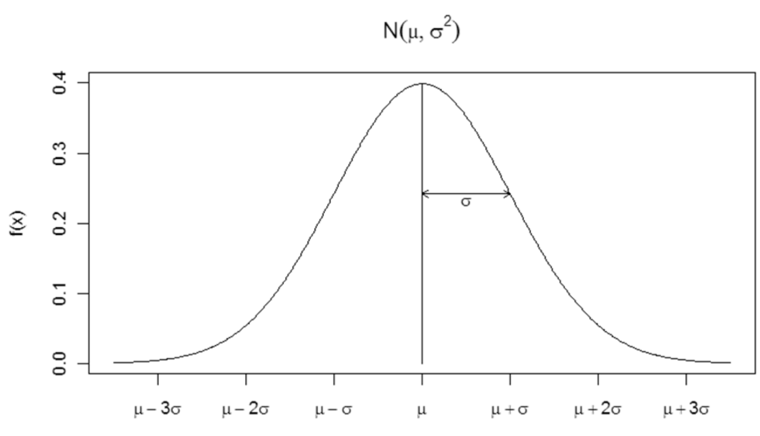
$$E(X) = \mu$$

$$\mathrm{Var}(X) = \sigma^2$$






## Normal densities all have the same shape



## The Normal Distributions

"X has a Normal distribution with mean $\mu$ and variance $\sigma^2$:

$$X \sim N(\mu, \sigma^2)$$

"Standard"(ized) normal denoted (conventionally) by $Z$, and is such that

$$Z \sim N(0,1)$$

Relationship:

$$Z = \frac{X-\mu}{\sigma} \qquad X = \mu + \sigma Z$$



## Normal Probability Computations

If $X \sim N(\mu, \sigma^2)$ then

$$P(a < X < b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Explicit antiderivative not available.

Real world: use computer.

Tests: transform $Z = \frac{X-\mu}{\sigma}$ and use table of standard Normal probabilities.