Comp 590 **Name (Print):** ____________________

Spring 2016
Midterm Exam
3/17/2016
Due Date: 3/31/2016

This is an individual take-home exam, so you should work independently from your classmates. You may use your books and notes on this exam. Additionally, you are encouraged to use Google Scholar to research the topics covered in the questions. Just be sure to cite the sources of your information.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way. Use Latex if possible.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Consider the digraph $\mathcal{D}$ and the following symmetric protocol

$$\dot{x}(t) = -\frac{1}{2}\{L(\mathcal{D}) + L(\mathcal{D})^T\}x(t). \tag{1}$$

Does this protocol correspond to the agreement protocol on a certain graph? What are the conditions on the digraph $\mathcal{D}$ such that the resulting symmetric protocol converges to the agreement subspace?

2. (25 points) How would one modify the agreement protocol

$$\dot{x}(t) = \sum_{j \in N(i)} (x_j(t) - x_i(t)), \quad i = 1, \dots, n, \tag{2}$$

so that the agents converge to an equilibrium $\bar{x}$, where $\bar{x} = \alpha \mathbf{1} + d$ for some given $d \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$? Simulate your proposed control law. Include your source code and any relevant plots with your submission.

3. (25 points) The second-order dynamics of a unit particle i in one dimension is

$$\frac{d}{dt}\begin{bmatrix} p_i(t) \\ v_i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_i(t) \\ v_i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t), \tag{3}$$

where p_i and v_i are, respectively, the position and the velocity of the particle with respect to an inertial frame, and u_i is the force and/or control term acting on the particle. Use a setup, inspired by the agreement protocol, to propose a control law $u_i(t)$ for each vertex such that:

- the control input for particle i relies only on the relative position and velocity information with respect to its neighbors
- the control input to each particle results in an asymptotically cohesive behavior for the particle group, that is, the positions of the particles remain close to each other
- the control input to each particle results in having a particle group that evolves with the same velocity.

Simulate your proposed control law. Include your source code and any relevant plots with your submission.

How would you extend this to work with n particles in two dimensions?

Hint: I am including a citation to a survey paper [1]. The answer can be found in one of the works cited by this paper. Start your search by reading this survey of consensus protocols.

4. (25 points) An averaging protocol for n agents , with state x_i, $i = 1, 2, \ldots, n$, is the discrete-time update rule of the form

$$x(k+1) = Wx(k), \quad k = 0, 1, 2, \ldots \tag{4}$$

where $x(k) = \begin{bmatrix} x_1(k) & x_2(k) & \ldots & x_n(k) \end{bmatrix}^T$ and W is a stochastic matrix.
A matrix $W \in \mathbb{R}^{n\times n}$ is a stochastic matrix if it is non-negative and $\sum_{j=1}^{n} w_{ij} = 1$, for all $i \in \{1, \ldots, n\}$; in other words, W is stochastic if

$$W\mathbf{1} = \mathbf{1} \tag{5}$$

Derive the necessary and sufficient conditions on the spectrum of the matrix W such that the process steers all the agents to the average value of their initial states.

References

[1] Wei Ren, R. W. Beard, and E. M. Atkins. A survey of consensus problems in multi-agent coordination. In *American Control Conference, 2005*, pages 1859–1864 vol. 3, June 2005.