A Python Primer for Mathematics

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Contents

Chapter 1

Introduction

This document is meant to prime mathematics students into using Python for doing mathematics symbolically and numerically. It is not meant to be used as a comprehensive text, but rather as a demonstrative cheatsheet to get up and running with the basics Python and for using Python for scientific computing as quickly as possible.

We will give very brief introductory demonstrations of the basics of the Python language, before moving on to demonstrations of some of the basic features of the packages *sympy*, *numpy*, and *matplotlib* which are increasingly used in modern scientific computing.

Chapter 2

Python language basics

2.1 Comments

```
In [1]: # Everything on a line after a '#' is ignored by Python
```

2.2 Basic calculations with numbers

Addition works as expected

```
In [1]: 1+2
```

```
Out[1]: 3
```

... so does subtraction

```
In [5]: 2.5 - 6
```

```
Out[5]: -3.5
```

... and division.

```
In [4]: 3/2
```

```
Out[4]: 1.5
```

The floor division operator // throws away everything after the decimal point (as in long division)

```
In [7]: 13 // 5
```

```
Out[7]: 2
```

... and the % operator gives the remainder of a division (as in long division)

```
In [6]: 13 % 5
```

```
Out[6]: 3
```

The power operator ** is often useful $(2^4 = 16)$

```
In [8]: 2**4
```

```
Out[8]: 16
```

2.3 Variables

We can assign values to variables

```
In [9]: a = 5
        b = 6
```

... and then compute with them:

```
In [11]: a + b
```

```
Out[11]: 11
```

We can reassign their values,

```
In [15]: a = 9
```

... to change the outcome of the computation

```
In [16]: a + b
```

```
Out[16]: 15
```

2.4 Strings (text)

Strings store text. We use either '...' or " ... " to denote a string

```
In [17]: "This is a string"
```

```
Out[17]: 'This is a string'
```

```
In [18]: 'This is also a string'
```

```
Out[18]: 'This is also a string'
```

We can join strings using the "+" operator. This is called *concatenation*.

```
In [19]: "begin" + "ner" + "s"
```

```
Out[19]: 'beginners'
```

We can assign strings to variables...

```
In [20]: part1 = "This is a message "
         part2 = "for you"
```

... and concatenate the variables

```
In [22]: full_message = part1 + part2
         full_message
```

```
Out[22]: 'This is a message for you'
```

We can access individual characters of a string. (Remember that we index from starting from zero!)

```
In [23]: n = 2
         full_message[n]
```

```
Out[23]: 'i'
```

We can also multiply strings.
Plug your ears and go ...

```
In [24]: "la"*10
```

```
Out[24]: 'lalalalalalalalalala'
```

2.5 Unpacking

There is a more efficient way of writing

```
a = 1
b = 2
c = 3
d = 4
```

The following is called *unpacking*:

```
In [29]: a, b, c, d = 1,2,3,4
         # or
         a, b, c, d = (1,2,3,4)
```

```
In [30]: d
```

```
Out[30]: 4
```

We are not restricted to just numbers. We can unpack anything

```
In [31]: firstname, lastname, age = ("john", "von neumann", 103)
         lastname
```

```
Out[31]: 'von neumann'
```

```
In [32]: age
```

```
Out[32]: 103
```

2.6 The print function

We can output text to the screen using the "print" function

```
In [34]: print(5+4)
         print("This is a message")
         print("The answer is", 42)
```

```
9
This is a message
The answer is 42
```

2.7 The str, int and float functons

We can convert numbers to strings with the str function:

```
In [1]: str(12)
```

```
Out[1]: '12'
```

... and strings to numbers using the int or float functions:

```
In [2]: int("13")
```

```
Out[2]: 13
```

```
In [3]: float("1.111")
```

```
Out[3]: 1.111
```

2.8 Comparisons

We can ask Python if statements are true of false

```
In [35]: 4 < 6
```

```
Out[35]: True
```

```
In [36]: 4 <= 6
```

```
Out[36]: True
```

```
In [37]: 4 >= 6
```

```
Out[37]: False
```

```
In [38]: 5<5
```

```
Out[38]: False
```

```
In [39]: 5<=5
```

```
Out[39]: True
```

Notice the double equals "==" when asking is an equality is true:

```
In [40]: 3 == 3
```

```
Out[40]: True
```

```
In [41]: 3 == 4
```

```
Out[41]: False
```

... a single "=" will not work to compare numbers:

```
In [42]: 3 = 3
```

```
      File "<ipython-input-42-49c8ce3fc03c>", line 2
    3 = 3
         ^
 SyntaxError: can't assign to literal
```

The "!=" operator means "new equal to"

```
In [43]: 3 != 4
```

```
Out[43]: True
```

We can also compare variables

```
In [44]: a,b,c = 5,6,7
```

```
In [45]: b != 7
```

```
Out[45]: True
```

```
In [46]: a < 5

Out[46]: False

In [47]: a <= 5

Out[47]: True

In [48]: b <= a

Out[48]: False

In [49]: a < b < c

Out[49]: True
```

We can also do computations in comparisons. Is the remainder when dividing by 2 equal to zero, i.e., Is b even? Is c even?

```
In [51]: b % 2 == 0      # 6 is even

Out[51]: True

In [53]: c % 2 == 0      # 7 is odd

Out[53]: False
```

Chapter 3

If statements

With if statements we can control the flow of execution of a program.

3.1 If ... statements

```
In [2]: a,b = 5,6
        if a == b:
            # this is not executed because 'a == b' is false
            print("a is equal to b")

        if a <= b:
            # this is executed because 'a <= b' is true
            print("a is less than or equal to b")
```

```
a is less than or equal to b
```

3.2 If ... else ... statements

```
In [2]: a,b = 5,6
        if a == b:
            print("a is equal to b")
        else:
            # this is only executed if a == b is false
            print("a is not equal to b")
```

```
a is not equal to b
```

3.3 If ... elif ... else ... statements

```
In [3]: name = "bobby"

        if name == "alice":
            print("Hi Alice")
        elif name == "bobby":
            print("Hi Bob")
        elif name == "richard":
```

```python
        print("Hi Ricky")
    else:
        print("Hi Stranger")
```

```
Hi Bob
```

Chapter 4

Lists

Lists are a fundamental data structure in Python. As the name suggests, we use them to store a collection of objects in a list (order matters)

We make a list using the [...] notation.

```
In [53]: boy_names = [
             "benny", "adam", "bobby",
             "randal", "timmy", "cartman",
             "morty","junior-son",
             "voldemort", "boeta", "pula",
             "zane"
         ]
```

How long is this list?

```
In [2]: len(boy_names)
```

```
Out[2]: 12
```

Is "morty" in the list?

```
In [3]: "morty" in boy_names
```

```
Out[3]: True
```

Is "xavier" in the list?

```
In [4]: "xavier" in boy_names
```

```
Out[4]: False
```

We can access the zeroth element in the list.

```
In [5]: boy_names[0]
```

```
Out[5]: 'benny'
```

WARNING! Remember that we always start index from zero!

We will distinguish between the "first" and "oneth" element. "First element of boy_names" is ambiguous, do we mean boy_names[0] or boy_names[1] ?

By "oneth" or "1-th" element of boy_names we will always mean boy_names[1].

```
In [7]: boy_names[1]
```

```
Out[7]: 'adam'
```

We can access the last-th element in the list, by using the -1 index. (This is why we index starting from zero)

```
In [8]: boy_names[-1]
```

```
Out[8]: 'zane'
```

… and can access the 2nd last-th element with the -2 index

```
In [10]: boy_names[-2]
```

```
Out[10]: 'pula'
```

We can replace an element

```
In [12]: boy_names[1] = "adriaan"
         boy_names
```

```
Out[12]: ['benny',
          'adriaan',
          'bobby',
          'randal',
          'timmy',
          'cartman',
          'morty',
          'junior-son',
          'voldemort',
          'boeta',
          'pula',
          'zane']
```

… and remove an element

```
In [13]: del boy_names[1]
         boy_names
```

```
Out[13]: ['benny',
          'bobby',
          'randal',
          'timmy',
          'cartman',
          'morty',
          'junior-son',
          'voldemort',
          'boeta',
          'pula',
          'zane']
```

4.1 Tuples

Tuples are like lists, but they are immutable. This means it is not possible to change tuples.
We make a tuples using the (…) notation

```
In [54]: boy_names_tuple = (
             "benny", "adam", "bobby",
             "randal", "timmy", "cartman",
             "morty","junior-son",
             "voldemort", "boeta", "pula",
             "zane"
         )
```

How long is the tuple?

```
In [15]: len(boy_names_tuple)

Out[15]: 12
```

Is "morty" in the tuple?

```
In [16]: "morty" in boy_names_tuple

Out[16]: True
```

We can access the last-th element

```
In [24]: boy_names_tuple[-1]

Out[24]: 'zane'
```

... but we cannot change the tuple by replacing elements. Trying results in an error.

```
In [18]: # We cannot change a tuple, so the
         # following gives an error
         boy_names_tuple[1] = "adriaan"


        ---------------------------------------------------------------------------

        TypeError                                 Traceback (most recent call last)

        <ipython-input-18-af5e5e9178b2> in <module>()
          1 # We cannot change a tuple, so the
          2 # following gives an error
    ----> 3 boy_names_tuple[1] = "adriaan"


        TypeError: 'tuple' object does not support item assignment
```

4.2 List slicing

List slicing is an efficient method of cutting off parts if a list.

```
In [19]: girl_names = ["alice", "beatrice", "candy",
             "dolly", "elaine", "francine", "geraldine"]
```

We use the ":" operator to make a slice. The following slice results in a new list containing the oneth,twoth, etc. elements:

```
In [22]: girl_names[1:]
```

```
Out[22]: ['beatrice', 'candy', 'dolly', 'elaine', 'francine', 'geraldine']
```

We can also slice from the other end. The following list contains everything up to the twoth element, and excludes the threeth element onwards:

```
In [21]: girl_names[:3]
```

```
Out[21]: ['alice', 'beatrice', 'candy']
```

We can also slice using negative indeces. The following list contains everything but the last-th element.

```
In [23]: girl_names[:-1]
```

```
Out[23]: ['alice', 'beatrice', 'candy', 'dolly', 'elaine', 'francine']
```

... and all but the second-last-th and last-th elements:

```
In [25]: girl_names[:-2]
```

```
Out[25]: ['alice', 'beatrice', 'candy', 'dolly', 'elaine']
```

The following list contains the oneth, twoth, threeth, fourth, elements (excluding the fiveth element onward).

```
In [27]: girl_names[1:5]
```

```
Out[27]: ['beatrice', 'candy', 'dolly', 'elaine']
```

Slicing also works for tuples and for strings:

```
In [28]: name = "Marlon Brando"
         name[-4:]
```

```
Out[28]: 'ando'
```

4.3 Sorting

We very often want to sort lists. Python includes powerful methods to perform different kinds of sorting. We will work with the following list:

```
In [55]: boy_names = [
             "benny", "adam", "bobby",
             "randal", "timmy", "cartman",
             "morty","junior-son",
             "voldemort", "boeta", "pula",
             "zane"
         ]
```

The default ordering for strings is alphabetically. We can just use the “sorted” function:

```
In [30]: sorted(boy_names)
```

```
Out[30]: ['adam',
          'benny',
          'bobby',
          'boeta',
          'cartman',
          'junior-son',
```

```
 'morty',
 'pula',
 'randal',
 'timmy',
 'voldemort',
 'zane']
```

We can easily sort reverse-alphabetically

```
In [4]: sorted(boy_names, reverse=True)
```

```
Out[4]: ['zane',
 'voldemort',
 'timmy',
 'randal',
 'pula',
 'morty',
 'junior-son',
 'cartman',
 'boeta',
 'bobby',
 'benny',
 'adam']
```

... or by length of the strings

```
In [5]: sorted(boy_names, key = len)
```

```
Out[5]: ['adam',
 'pula',
 'zane',
 'benny',
 'bobby',
 'timmy',
 'morty',
 'boeta',
 'randal',
 'cartman',
 'voldemort',
 'junior-son']
```

We can sort with respect to any conceivable ordering. E.g. The following sorts the list alphabetically according to the one-th letter. (See the section on lambda expressions).

```
In [31]: sorted(boy_names, key = lambda item: item[1])
```

```
Out[31]: ['randal',
 'cartman',
 'zane',
 'adam',
 'benny',
 'timmy',
 'bobby',
 'morty',
 'voldemort',
 'boeta',
 'junior-son',
 'pula']
```

4.4 Zipping

Zipping is an efficient way to combine two (or more) lists pairwise. Consider the two lists:

```
In [37]: girl_names = ["alice", "beatrice", "candy", "dolly", "elaine"]
         their_ages = [10, 11, 10, 9, 8]
```

We can "zip" these two lists together to get a "zip" object (zip objects are iterable objects. Their purpose is for optimizing RAM usage).

```
In [38]: name_age_pairs = zip(girl_names, their_ages)
         name_age_pairs

Out[38]: <zip at 0x7fed48073688>
```

The zip object can be converted to a list.

```
In [39]: list(name_age_pairs)

Out[39]: [('alice', 10), ('beatrice', 11), ('candy', 10), ('dolly', 9), ('elaine', 8)]
```

Warning! When zipping lists of unequal length the result will have the length of the shortest list:

```
In [40]: result = list(zip(["a","b","c"], [1,2,3,4,5,6]))
         result

Out[40]: [('a', 1), ('b', 2), ('c', 3)]
```

We can also zip more than two lists:

```
In [41]: threezip = list(zip(["a","b","c"], [1,2,3], ["alpha", "beta", "gamma"]))
         threezip

Out[41]: [('a', 1, 'alpha'), ('b', 2, 'beta'), ('c', 3, 'gamma')]
```

4.5 Unzipping

Unzipping is the opposite of zipping. I.e., Given a list of pairs, we can unzip the list into two lists: one list containing the first elements of the pairs and one list containing the second elements of the pairs.

Consider:

```
In [43]: name_age_pairs = [
             ('alice', 10), ('beatrice', 11), ('candy', 10), #
             ('dolly', 9), ('elaine', 8)
         ]
```

... which we unzip (notice the "*"):

```
In [44]: unzipped_names, unzipped_ages = zip(*name_age_pairs)
```

Let's inspect the lists unzipped_names and unzipped_ages

```
In [47]: unzipped_names

Out[47]: ('alice', 'beatrice', 'candy', 'dolly', 'elaine')

In [48]: unzipped_ages
```

```
Out[48]: (10, 11, 10, 9, 8)
```

We can also unzip lists of triples:

```
In [49]: threezip = [('a', 1, 'alpha'), ('b', 2, 'beta'), ('c', 3, 'gamma')]
         abc, onetwothree, alphabetagamma = zip(*threezip)
```

```
In [50]: abc
```

```
Out[50]: ('a', 'b', 'c')
```

```
In [51]: onetwothree
```

```
Out[51]: (1, 2, 3)
```

```
In [52]: alphabetagamma
```

```
Out[52]: ('alpha', 'beta', 'gamma')
```

Chapter 5

Loops

Loops are used to perform a single operation over and over.

5.1 For-loops

The for-loop is the most used kind of loop. One can think of their operation as follows: "For every element in . . . (container), do . . . (action) on that element".

The "range" functon is a useful container to loop over. The following example prints every number in the range 0,2,3,. . . ,9:

```
In [1]: for i in range(10):
            print(i)
```

```
0
1
2
3
4
5
6
7
8
9
```

. . . do the same, but loop over 4,5,. . . ,9

```
In [2]: for i in range(4, 10):
            print(i)
```

```
4
5
6
7
8
9
```

We are not limited to loop over "ranges", we can loop over any container. This is the preferred way to loop over a list in Python:

```
In [3]: girl_names = ["alice", "beatrice", "candy",
                "dolly", "elaine", "francine", "geraldine"]

        for name in girl_names:
            print(name)

alice
beatrice
candy
dolly
elaine
francine
geraldine
```

We can loop in reverse order by just applying the “reversed” function to our list:

```
In [4]: for name in reversed(girl_names):
            print(name)

geraldine
francine
elaine
dolly
candy
beatrice
alice
```

Often one want’s to keep a running index. This is easily done with the “enumerate” function.

```
In [5]: for index, name in enumerate(girl_names):   #<------(notice the unpacking)
            print(index, " -> ", name)

0  ->  alice
1  ->  beatrice
2  ->  candy
3  ->  dolly
4  ->  elaine
5  ->  francine
6  ->  geraldine
```

We can also loop directly over zip objects using unpacking

```
In [14]: girl_names = ["alice", "beatrice", "candy",
                 "dolly", "elaine", "francine", "geraldine"]
         their_ages = [3,3,7,10,15,11,31]

         for name, age in zip(girl_names, their_ages):
             print("name : ", name )
             print("   age  : ", age)

name :  alice
   age  :  3
name :  beatrice
```

```
   age  :  3
name :  candy
   age  :  7
name :  dolly
   age  :  10
name :  elaine
   age  :  15
name :  francine
   age  :  11
name :  geraldine
   age  :  31
```

5.2 While loops

While loops are useful when we do not know before hand how many times a loop should execute. One can think of their operation as follows: "While … (condition) is True, do …(action)".

```
In [7]: number = 144
        while number % 2 == 0 :    # while number is divisible by 2,  ...
            number = number // 2  # divide it by two

        print(number)
```

```
9
```

5.3 You should loop like a Pythonista not a C-snake.

Python is not like classic languages e.g., C. We should not use standard C-idioms in Python. Doing so will result in ugly, unreadable and un maintainable code.

DO NOT DO ANY OF THE FOLLOWING THINGS IN PYTHON. Compare the following bad looping idioms with the proper Pythonic looping idioms above.

Consider the lists

```
In [13]: girl_names = ["alice", "beatrice", "candy",
                 "dolly", "elaine", "francine", "geraldine"]
         their_ages = [3,3,7,10,15,11,31]
```

DO NOT Loop over a range object unecessarily:

```
In [9]: for i in range(len(girl_names)):
            print(girl_names[i])
```

```
alice
beatrice
candy
dolly
elaine
francine
geraldine
```

DO NOT loop in reverse order by accessing indeces:

```
In [13]: for i in range(len(girl_names)):
             print(girl_names[len(girl_names) - i -1])

geraldine
francine
elaine
dolly
candy
beatrice
alice
```

DO NOT keep a running index manually:

```
In [15]: index = 0
         for name in girl_names:
             print(index, " -> ", name)
             index = index +1

0  ->  alice
1  ->  beatrice
2  ->  candy
3  ->  dolly
4  ->  elaine
5  ->  francine
6  ->  geraldine
```

DO NOT loop over two lists using indeces:

```
In [12]: for i in range(min(len(girl_names), len(their_ages))):
             print("name : ", girl_names[i] )
             print("   age  : ", their_ages[i])

name :  alice
   age  :  3
name :  beatrice
   age  :  3
name :  candy
   age  :  7
name :  dolly
   age  :  10
name :  elaine
   age  :  15
name :  francine
   age  :  11
name :  geraldine
   age  :  31
```

Chapter 6

List comprehensions

List comprehensions is a concise way of constructing lists using a for-loop syntax.
Consider the list:

```
In [1]: girl_names = ["alice", "beatrice", "candy",
             "dolly", "elaine", "francine", "geraldine"]
```

We can use a list comprehension to make a new list containing the zeroth letter of each name in the list:

```
In [2]: first_letters = [name[0] for name in girl_names]
        first_letters

Out[2]: ['a', 'b', 'c', 'd', 'e', 'f', 'g']
```

... or a list with the length of every name:

```
In [2]: length_of_names = [len(name) for name in girl_names]
        length_of_names

Out[2]: [5, 8, 5, 5, 6, 8, 9]
```

... or a list of name-length-pairs

```
In [3]: names_length_pairs = [ ( name, len(name) ) for name in girl_names]
        names_length_pairs

Out[3]: [('alice', 5),
         ('beatrice', 8),
         ('candy', 5),
         ('dolly', 5),
         ('elaine', 6),
         ('francine', 8),
         ('geraldine', 9)]
```

A useful feature is adding a conditional. The following makes a new list only containing the "long" names:

```
In [4]: only_long_names = [ name for name in girl_names if len(name) > 6 ]
        only_long_names

Out[4]: ['beatrice', 'francine', 'geraldine']
```

6.1 Computing with comprehensions

We can perform computations with comprehensions. This is useful, because it makes our code easy to read and maintain.

We can compute the sum $1 + 2 + 3 + 4 + \ldots + 100$:

```
In [9]: sum(i for i in range(1, 101)) # Why 101?

Out[9]: 5050
```

... or the sum of squares: $1 + 2^2 + 3^2 + 4^2, \ldots, 100^2$:

```
In [6]: sum(i**2 for i in range(1, 101))

Out[6]: 338350
```

... or the sum of squares of even numbers: $2^2 + 4^2 + 6^2, \ldots, 10000^2$:

```
In [8]: sum(i**2 for i in range(1, 10001)  if i % 2 == 0)

Out[8]: 166716670000
```

Chapter 7

The 'any' and 'all' functions

Sometimes one is required to decide if a number of statements in a list are *all* true.

```
In [4]: all([True, True, True, True]) # All true? Yes.
```

```
Out[4]: True
```

```
In [7]: all([True, True, False, True]) # All true? No.
```

```
Out[7]: False
```

... and sometimes one is required to decide if *at least one* from a number of statements is true:

```
In [15]: any([True, False, False, True]) # Is at least one statement true? Yes.
```

```
Out[15]: True
```

```
In [10]: any([False, False, False, False]) # Is at least one statement true? No.
```

```
Out[10]: False
```

7.1 Examples

The following examples illustrate how the *all* and *any* functions can be used.

```
In [2]: # Do *all* the letters "a", "b", "l" occur the phrase "mary had a little lamb"?
        # Yes. So the following evaluates to True.
        all([letter in "mary had a little lamb" for letter in ["a", "b", "l"]])
```

```
Out[2]: True
```

```
In [4]: # Do *all* the letters "a", "b", "q" occur the phrase "the quick brown fox"?
        # No. The letter "a" does not occur, so the following evaluates to False.
        all([letter in "the quick brown fox" for letter in ["a", "b", "q"]])
```

```
Out[4]: False
```

```
In [7]: # Does *at least one* of the letters "a", "b", "z" occur the phrase "the quick brown fox"?
        # Yes. The letter "b" occurs, so the following evaluates to True.
        any([letter in "the quick brown fox" for letter in ["a", "b", "z"]])
```

```
Out[7]: True
```

```
In [8]: # Does *at least one* of the letters "z", "q", "p" occur the phrase "mary had a little lamb"?
        # No. none of the letters occur, so the following evaluates to True.
        any([letter in "mary had a little lamb" for letter in ["z", "p", "q"]])
```

```
Out[8]: False
```

Chapter 8

Functions

Functions allow for the easy reuse of bits of code. They take parameters/input, and can return a result. Functions are defined using the *def* keyword.

8.1 Example: Say hello

We define a function that takes *name* as parameter, and prints a greeting for that name:

```
In [4]: def say_hello_to(name):
            print("Hello " + name + "!")
```

We can now call this function with different inputs:

```
In [6]: say_hello_to("World")
        say_hello_to("Gary")
        say_hello_to("Alice")
        say_hello_to("Crocubot")
```

```
Hello World!
Hello Gary!
Hello Alice!
Hello Crocubot!
```

8.2 Example: Divisible by 11

We define a function that takes *number* as input and *returns* whether or not the number is divisible by 11

```
In [7]: def is_divisible_by_11(number):
            return number % 11 == 0
```

Lets check which numbers of 10,11,12,. . . ,24 are divisible by 11

```
In [8]: for number in range(10, 25):
            print(number, "is divisible by 11 : ", is_divisible_by_11(number))
```

```
10 is divisible by 11 :  False
11 is divisible by 11 :  True
12 is divisible by 11 :  False
13 is divisible by 11 :  False
14 is divisible by 11 :  False
```

```
15 is divisible by 11 :  False
16 is divisible by 11 :  False
17 is divisible by 11 :  False
18 is divisible by 11 :  False
19 is divisible by 11 :  False
20 is divisible by 11 :  False
21 is divisible by 11 :  False
22 is divisible by 11 :  True
23 is divisible by 11 :  False
24 is divisible by 11 :  False
```

8.3 Example: The Collatz function

Example: We define the *collatz* function according to the following specification.
Input:

- A number n.

Output:

- return 1 if $n = 1$
- return $n/2$ if n is even
- return $3n + 1$ if n is odd

```
In [9]: def collatz(number):
            if number  == 1:
                return number
            elif number % 2 == 0:
                return number // 2
            else:
                return 3*number + 1
```

Let's try it out on 3,11,24 and 65

```
In [10]: for n in [3, 11, 24, 65]:
             print(n, " -> ",collatz(n))
```

```
3  ->  10
11  ->  34
24  ->  12
65  ->  196
```

Let's repeatedly apply the collatz function to a number using a while loop. We always tend to get back to 1... why is that?
See https://en.wikipedia.org/wiki/Collatz_conjecture

```
In [12]: current_number = 15
         while current_number != 1:
             current_number = collatz(current_number)
             print(current_number)
```

```
46
23
70
35
106
53
160
80
40
20
10
5
16
8
4
2
1
```

8.4 Recursion

Recursion is what happens when a functon calls itself.

8.4.1 Example: Fibonacci numbers

A good example of recursion is the process of generating Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. These are formally defined as the sequence (f_n) with $f_1 := 1$, $f_2 := 1$ and $f_n := f_{n-1} + f_{n-2}$ for all $n \in \{3, 4, 5 \ldots\}$

Pay attention how the following function calls itself:

```
In [21]: def fibonacci(n):
             if n == 1:
                 return 1
             elif n == 2:
                 return 1
             else:
                 return fibonacci(n-1) + fibonacci(n-2)
```

Let's compute the first 20 fibonacci numbers

```
In [14]: for i in range(1,20):
             print(fibonacci(i))
```

```
1
1
2
3
5
8
13
21
34
```

```
55
89
144
233
377
610
987
1597
2584
4181
```

8.5 Making functions with lambda expressions

Very simple functions can be defined using *lambda* expressions. We've already briefly encountered lambda expressions in the section on sorting.

```
In [16]: f = lambda x: 3*x
```

Make sure you understand why f(4)=12

```
In [19]: f(4)
```

```
Out[19]: 12
```

...and f('a')='aaa'

```
In [20]: f("a")
```

```
Out[20]: 'aaa'
```

8.6 docstrings

One's code is usually used by other people. These people might need to know what a function you wrote does. One may do this by writing a short explanation in a *docstring* in the first line of the function definition. This can be accessed by calling the help function on an object.

```
In [25]: def fibonacci_undocumented(n):
             if n == 1:
                 return 1
             elif n == 2:
                 return 1
             else:
                 return fibonacci(n-1) + fibonacci(n-2)
```

No help is forthcoming...

```
In [26]: help(fibonacci_undocumented)
```

```
Help on function fibonacci_undocumented in module __main__:

fibonacci_undocumented(n)
```

... unless we give it:

```
In [29]: def fibonacci(n):
             """
             Returns the nth fibonacci number.

             Input: n
             Output: the nth fibonacci number

             E.g.fibonacci(1) = 1
                 fibonacci(2) = 1
                 fibonacci(3) = 2
                 ...
             """
             if n == 1:
                 return 1
             elif n == 2:
                 return 1
             else:
                 return fibonacci(n-1) + fibonacci(n-2)
```

```
In [30]: help(fibonacci)
```

```
Help on function fibonacci in module __main__:

fibonacci(n)
    Returns the nth fibonacci number.

    Input: n
    Output: the nth fibonacci number

    E.g.fibonacci(1) = 1
        fibonacci(2) = 1
        fibonacci(3) = 2
        ...
```

Chapter 9

Dictionaries

Dictionaries are datastructures that map one object to another. We create a dictionary using the { ... : ... } notation.

Consider the dictionary:

```
In [1]: surname_dictionary = {
            # key   : #value,
            "kevin" : "de koker",
            "john"  : "mphako",
            "alice" : "munro",
            "doris" : "lessing",
        }
        surname_dictionary

Out[1]: {'alice': 'munro', 'doris': 'lessing', 'john': 'mphako', 'kevin': 'de koker'}
```

We can access a *value* associated to a specific *key*:

```
In [2]: surname_dictionary["kevin"]

Out[2]: 'de koker'

In [3]: surname_dictionary["alice"]

Out[3]: 'munro'
```

An error is raised if the key is not in the dictionary:

```
In [4]: surname_dictionary["bobby"]


        ---------------------------------------------------------------------------

        KeyError                                  Traceback (most recent call last)

        <ipython-input-4-aebb82606656> in <module>()
    ----> 1 surname_dictionary["bobby"]


        KeyError: 'bobby'
```

We can ask if a key is in the dictionary:

```
In [5]: "kevin" in surname_dictionary
```

```
Out[5]: True
```

```
In [6]: "bobby" in surname_dictionary
```

```
Out[6]: False
```

... the *in* operator only checks keys, not values:

```
In [7]: "de koker" in surname_dictionary
```

```
Out[7]: False
```

We can add elements:

```
In [8]: surname_dictionary["katie"] = "van der merwe"
        surname_dictionary
```

```
Out[8]: {'alice': 'munro',
         'doris': 'lessing',
         'john': 'mphako',
         'katie': 'van der merwe',
         'kevin': 'de koker'}
```

... and remove elements:

```
In [10]: del surname_dictionary["alice"]
         surname_dictionary
```

```
Out[10]: {'doris': 'lessing',
          'john': 'mphako',
          'katie': 'van der merwe',
          'kevin': 'de koker'}
```

Iterating over a dictionary, iterates over the keys:

```
In [11]: for key in surname_dictionary:
             print(key)
```

```
doris
john
kevin
katie
```

... but we can also iterate over the values using *.values()*:

```
In [12]: for key in surname_dictionary.values():
             print(key)
```

```
lessing
mphako
de koker
van der merwe
```

... or we can iterate over key-value pairs using *.items()*:

```
In [13]: for pair in surname_dictionary.items():
             print(pair)
```

```
('doris', 'lessing')
('john', 'mphako')
('kevin', 'de koker')
('katie', 'van der merwe')
```

It is often useful to unpack such pairs:

```
In [14]: for firstname, lastname in surname_dictionary.items():
             print(firstname, "-->", lastname[0])
```

```
doris --> l
john --> m
kevin --> d
katie --> v
```

Chapter 10

Dictionary comprehensions

Dictionary comprehension is a concise way to construct dictionaries using a for-loop syntax.
Consider:

```
In [1]: surname_dictionary = {
            # key   : #value,
            "kevin" : "de koker",
            "john" : "mphako",
            "alice" : "munro",
            "doris" : "lessing",
        }
        surname_dictionary

Out[1]: {'alice': 'munro', 'doris': 'lessing', 'john': 'mphako', 'kevin': 'de koker'}
```

We construct a dictionary which maps a name to the length of the surname.

```
In [2]: length_of_surname_dictionary = {
            firstname : len(lastname) for firstname, lastname in surname_dictionary.items()
        }
        length_of_surname_dictionary

Out[2]: {'alice': 5, 'doris': 7, 'john': 6, 'kevin': 8}
```

We construct a dictionary which filtered all items whose last name start with "m"

```
In [3]: last_name_starts_with_m = {
            firstname : lastname
                for firstname, lastname in surname_dictionary.items()
                    if "m" == lastname[0]
        }
        last_name_starts_with_m

Out[3]: {'alice': 'munro', 'john': 'mphako'}
```

Chapter 11

Importing modules and interactive help

Not all Python functionality is builtin. Extra functionality is provided in *modules*. To use the extra functionality provided by a module we must *import* the module.

The syntax for importing modules are:

```
import ...
from ... import ...
import ... as ...
```

Let's import the *math* module:

```
In [2]: import math
```

We can see what objects the math module provides by calling the *dir* method on it:

```
In [3]: dir(math)

Out[3]: ['__doc__',
         '__loader__',
         '__name__',
         '__package__',
         '__spec__',
         'acos',
         'acosh',
         'asin',
         'asinh',
         'atan',
         'atan2',
         'atanh',
         'ceil',
         'copysign',
         'cos',
         'cosh',
         'degrees',
         'e',
         'erf',
         'erfc',
         'exp',
         'expm1',
         'fabs',
         'factorial',
         'floor',
```

```
        'fmod',
        'frexp',
        'fsum',
        'gamma',
        'gcd',
        'hypot',
        'inf',
        'isclose',
        'isfinite',
        'isinf',
        'isnan',
        'ldexp',
        'lgamma',
        'log',
        'log10',
        'log1p',
        'log2',
        'modf',
        'nan',
        'pi',
        'pow',
        'radians',
        'sin',
        'sinh',
        'sqrt',
        'tan',
        'tanh',
        'trunc']
```

If we need to know more about an object, then we can the *help* funciton on it:

```
In [4]: help(math.acos)

Help on built-in function acos in module math:

acos(...)
    acos(x)

    Return the arc cosine (measured in radians) of x.



In [6]: help(math.radians)

Help on built-in function radians in module math:

radians(...)
    radians(x)

    Convert angle x from degrees to radians.
```

With the math module imported, we can access its contents and call the functions it defines:

```
In [7]: math.pi
```

```
Out[7]: 3.141592653589793

In [8]: math.acos(-1)

Out[8]: 3.141592653589793

In [9]: math.sin(math.radians(90))

Out[9]: 1.0
```

Chapter 12

Sympy

Sympy is an external Python module that allows for symbolic computations like solving equations, differentiation and integration.

We import the *sympy* module

```
In [2]: import sympy
```

If we want to have pretty output inside a Jupyter notebook, we call sympy.init_printing

```
In [3]: sympy.init_printing()
```

We can define symbols using the sympy.sumbols functon:

```
In [4]: x,y  = sympy.symbols("x y")
```

... or we can import standard symbols from the sympy.abc module

```
In [5]: from sympy.abc import x,y
```

With these symbols, we can define an algebraic expression in the variables x and y

```
In [6]: an_expression = sympy.sin(x**2 - x - 1 + sympy.acos(y))
        an_expression
```

Out[6]:

$$\sin\left(x^2 - x + \operatorname{acos}(y) - 1\right)$$

12.1 Expanding, factoring and simplifying expressions

We can expand expressions using sympy.expand

```
In [7]: sympy.expand( (x+4)*(x-6) )
```

Out[7]:

$$x^2 - 2x - 24$$

We can factor expressions using sympy.factor

```
In [8]: sympy.factor( x**2-x-20 )
```

Out[8]:

$$(x-5)(x+4)$$

We can make more complicated expressions . . .

```
In [47]: (x**3 + x**2 - x - 1)/(x**2 + 2*x + 1)
```

Out[47]:

$$\frac{x^3+x^2-x-1}{x^2+2x+1}$$

. . . and simplify them using sympy.simplify

```
In [9]: sympy.simplify( (x**3 + x**2 - x - 1)/(x**2 + 2*x + 1) )
```

Out[9]:

$$x-1$$

12.2 Substituting values into expressions

Let's define the quadratic expression $x^2 - x - 1$

```
In [10]: from sympy.abc import x
         quadratic_expression = x**2 - x - 1
```

. . . and substitute the value 1 for the symbol x using the .subs function.
Notice the dictionary! Make sure you understand why the result is -1.

```
In [12]: quadratic_expression.subs({x : 1})
```

Out[12]:

$$-1$$

We substitute the value -2 for the symbol x using the .subs function. Make sure you understand why the result is 5.

```
In [15]: quadratic_expression.subs({x : -2})
```

Out[15]:

$$5$$

12.3 Solving equations

We can solve equations with sympy.
WARNING: We cannot use "=" or "==" to define equations, we must use sympy.Eq.
We make the equation $x^2 - x - 1 = 0$.

```
In [17]: import sympy
         from sympy.abc import x

         sympy.Eq(x**2 - x - 1, 0)
```

Out[17]:

$$x^2 - x - 1 = 0$$

... and solve for x in this equation by calling the sympy.solve function

```
In [18]: sympy.solve(sympy.Eq(x**2 - x - 1, 0),  x)
```

Out[18]:

$$\left[\frac{1}{2} + \frac{\sqrt{5}}{2}, \quad -\frac{\sqrt{5}}{2} + \frac{1}{2} \right]$$

... by just providing an expression to sympy.solve, it solves the equation expression=0.

```
In [19]: sympy.solve(x**2 - x - 1, x)
```

Out[19]:

$$\left[\frac{1}{2} + \frac{\sqrt{5}}{2}, \quad -\frac{\sqrt{5}}{2} + \frac{1}{2} \right]$$

```
In [ ]:
```

We can solve more complicated equations. Let's solve for θ in:
$\cos(\theta) = \sin(\theta)$
The equation has infinitely many solutions in θ. However, sympy.solve only gives two:

```
In [24]: from sympy.abc import theta
         sympy.solve( sympy.Eq(sympy.cos(theta),  sympy.sin(theta)), theta)
```

Out[24]:

$$\left[-\frac{3\pi}{4}, \quad \frac{\pi}{4} \right]$$

... sympy.solveset gives all infinitely many solutions

```
In [25]: sympy.solveset( sympy.Eq(sympy.cos(theta),  sympy.sin(theta)), theta)
```

Out[25]:

$$\left\{ 2n\pi + \frac{5\pi}{4} \mid n \in \mathbb{Z} \right\} \cup \left\{ 2n\pi + \frac{\pi}{4} \mid n \in \mathbb{Z} \right\}$$

12.4 Solving systems of equations

We can also solve systems of equations like the following in x and y:

$$2x + 3y = 1$$
$$3x + 2y = 2$$

```
In [26]: from sympy.abc import x,y
         import sympy

         sympy.solve([
             sympy.Eq(2*x + 3*y, 1),
             sympy.Eq(3*x + 2*y, 2)
         ]
         ,[x,y])
```

Out[26]:

$$\left\{x : \frac{4}{5}, \quad y : -\frac{1}{5}\right\}$$

12.5 Generating complicated expressions

We can use functions to generate complicated expressions.

Let's define a function P that, for any number n, returns a polynomial of the form

$$\sum_{k=0}^{n} kx^k$$

```
In [59]: import sympy
         from sympy.abc import x

         def P(n):
             return sum( k * x**k for k in range(n+1) )
```

We can now obtain the 6th degree polynomial of the given form by calling P(6)

```
In [60]: P(6)
```

Out[60]:

$$6x^6 + 5x^5 + 4x^4 + 3x^3 + 2x^2 + x$$

... or the 10th degree polynomial by calling P(10)

```
In [33]: P(10)
```

Out[33]:

$$10x^{10} + 9x^9 + 8x^8 + 7x^7 + 6x^6 + 5x^5 + 4x^4 + 3x^3 + 2x^2 + x$$

For fun, let's solve the equation $4x^4 + 3x^3 + 2x^2 + x = 0$.

```
In [35]: sympy.solve( P(4),  x)
```

Out[35]:

$$\left[0, \quad -\frac{1}{4} + \frac{5}{16\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{135}{64} + \frac{15\sqrt{6}}{16}}} - \frac{1}{3}\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{135}{64} + \frac{15\sqrt{6}}{16}}, \quad -\frac{1}{4} - \frac{1}{3}\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{135}{64} + \frac{15\sqrt{6}}{16}} + \frac{}{16}\right.$$

12.6 Numerical approximation

Sometimes we want numerical approximations to mathematical constants. We can compute them to arbitrary precision with the sympy's .n function.

```
In [36]: import sympy
```

```
In [37]: sympy.pi
```

Out[37]:

$$\pi$$

... pi approximated to 50 digits is:

```
In [40]: sympy.pi.n(50)
```

Out[40]:

$$3.1415926535897932384626433832795028841971693993751$$

```
In [41]: sympy.sqrt(2)
```

Out[41]:

$$\sqrt{2}$$

$\sqrt{2}$ two approximated to 50 digits is

```
In [42]: sympy.sqrt(2).n(50)
```

Out[42]:

$$1.4142135623730950488016887242096980785696718753769$$

... we can even find approximations to more complicated expressions:

```
In [44]: sympy.exp( sympy.root(sympy.sqrt(2)-1,5) )
```

Out[44]:

$$e^{\sqrt[5]{-1+\sqrt{2}}}$$

```
In [45]: sympy.exp( sympy.root(sympy.sqrt(2)-1,5) ).n(50)
```

Out[45]:

$$2.3126351501944463406364037678832846493032008107479$$

12.7 Symbolic differentiation

We can use sympy to compute derivatives of expressions using the sympy.diff function.

Let's compute $\frac{d}{dx}\left(x^4+x^3+x^2+x+1\right)$

```
In [46]: import sympy
         from sympy.abc import x

         sympy.diff(x**4+x**3+x**2+x+1, x)
```

Out[46]:

$$4x^3+3x^2+2x+1$$

... or the more complicated derivative $\frac{d}{dx}\left((x^4+x^3+x^2+x+1)e^{x^2+\sin(x^2)}\right)$

```
In [47]: sympy.diff( (x**4+x**3+x**2+x+1)*sympy.exp(x**2 + sympy.sin(x**2)), x)
```

Out[47]:

$$\left(2x\cos\left(x^2\right)+2x\right)\left(x^4+x^3+x^2+x+1\right)e^{x^2+\sin\left(x^2\right)}+\left(4x^3+3x^2+2x+1\right)e^{x^2+\sin\left(x^2\right)}$$

12.8 Symbolic integration

We can use sympy to compute derivatives of expressions using the sympy.integrate function.
Let's compute $\int (x^2 - x - 1)\,dx$:

```
In [48]: import sympy
         from sympy.abc import x

         sympy.integrate(x**2 -x -1, x)
```

Out[48]:

$$\frac{x^3}{3} - \frac{x^2}{2} - x$$

```
In [28]: # ... another example, that illustrates integration by parts:
```

... or another example (notice the integration by parts): $\int xe^x\,dx$

```
In [49]: sympy.integrate(x * sympy.exp(x), x)
```

Out[49]:

$$(x - 1)\,e^x$$

We can also compute definite integrals. E.g., $\int_0^5 xe^x\,dx$.

```
In [51]: sympy.integrate(x * sympy.exp(x), (x, 0, 5))
```

Out[51]:

$$1 + 4e^5$$

12.9 Making functions out of expressions

Especially for plotting, it is useful to be able to make a function out of a sympy expression. We can do this with the sympy.lambdify function

```
In [52]: import sympy
         from sympy.abc import x,y

         cubic  = x**3 - x**2 + x +3
         f = sympy.lambdify([x], cubic)
```

```
In [ ]:
```

Now we have the function $f(x) := x^3 - x^2 + x + 3$ and we can call it:

```
In [56]: f(2)
```

Out[56]:

9

```
In [57]: f(x)
```

Out[57]:

$$x^3 - x^2 + x + 3$$

```
In [58]: f(y)
```

Out[58]:

$$y^3 - y^2 + y + 3$$

Chapter 13

Numpy

Numpy is a widely used external package for doing matrix computations. It is designed to be very fast.

13.1 Arrays

Arrays is the basic datastructure of numpy. We can think of them as vectors.

```
In [3]: import numpy as np

        a = np.array([1,2,3])
        b = np.array([4,3,3])
        a + b

Out[3]: array([5, 5, 6])
```

... we can compute the dot product:

```
In [4]: a.dot(b) # 4 + 6 + 9

Out[4]: 19
```

13.2 Numpy functions

Numpy provides many mathematical functions like numpy.sin, numpy.cos, etc. When applying these to arrays, they are applied entry-wise. This is useful for plotting.

```
In [5]: import numpy as np

        a = np.array([0,1,2,3,4,5,6,7,8,9,10])
        b = np.sin(a)
        c = np.sqrt(a)

In [6]: # b == [sin(0), sin(1), sin(2), ... , sin(10)]
        b

Out[6]: array([ 0.        ,  0.84147098,  0.90929743,  0.14112001, -0.7568025 ,
               -0.95892427, -0.2794155 ,  0.6569866 ,  0.98935825,  0.41211849,
               -0.54402111])

In [7]: # c == [sqrt(0), sqrt(1), sqrt(2), ... , sqrt(10)]
        c
```

```
Out[7]: array([ 0.        ,  1.        ,  1.41421356,  1.73205081,  2.        ,
                2.23606798,  2.44948974,  2.64575131,  2.82842712,  3.        ,
                3.16227766])
```

13.3 Matrices

Matrices can be represented as 2D numpy arrays

```
In [12]: import numpy as np

         M = np.array([
             [1, 2, 3],
             [4, 5, 6],
             [7, 8, 9],
         ])
```

We can access columns, e.g., the twoth column:

```
In [13]: M[:,2]

Out[13]: array([3, 6, 9])
```

We can access columns, e.g., the twoth column:

```
In [15]: M[1,:]

Out[15]: array([4, 5, 6])
```

We can compute multiply a matrix with a vector using the .dot function:

```
In [17]: v = np.array([1,2,3,])
         M.dot(v)

Out[17]: array([14, 32, 50])
```

We can multiply a matrix with another matrix, also using the .dot function :

```
In [18]: M.dot(M)

Out[18]: array([[ 30,  36,  42],
                [ 66,  81,  96],
                [102, 126, 150]])
```

Waring: The * operator does entrywise multiplication!

```
In [19]: M*M

Out[19]: array([[ 1,  4,  9],
                [16, 25, 36],
                [49, 64, 81]])
```

13.4 Matrix row/column operations

Consider:

```
In [24]: import numpy as np

         M = np.array([
             [1, 2, 3],
             [4, 5, 6],
             [7, 8, 9],
         ])
```

We can easily perform row/column swaps.
The following swaps the zeroth and oneth rows of M:

```
In [25]: M[[1,0],:] = M[[0,1],:]

In [26]: M

Out[26]: array([[4, 5, 6],
                [1, 2, 3],
                [7, 8, 9]])
```

The following swaps the oneth and twoth columns of M:

```
In [27]: M[:,[1,2]] = M[:,[2,1]]

In [29]: M

Out[29]: array([[4, 6, 5],
                [1, 3, 2],
                [7, 9, 8]])
```

... and we can perform elementary row operations (e.g., for implementing Gauss elimination).
The following replaces the twoth row with 4 times the twoth row - 7 times the zeroth row

```
In [30]: M[2,:] = 4*M[2,:] - 7*M[0,:]

In [31]: M

Out[31]: array([[ 4,  6,  5],
                [ 1,  3,  2],
                [ 0, -6, -3]])
```

Transposing is easy.

```
In [33]: M.transpose()

Out[33]: array([[ 4,  1,  0],
                [ 6,  3, -6],
                [ 5,  2, -3]])
```

Chapter 14

Basic plotting with matplotlib

Matplotlib is a powerful plotting module for python. It is a bit difficult to use, however.
We usually use it together with numpy.

```
In [2]: import matplotlib.pyplot as plt
        import numpy as np
```

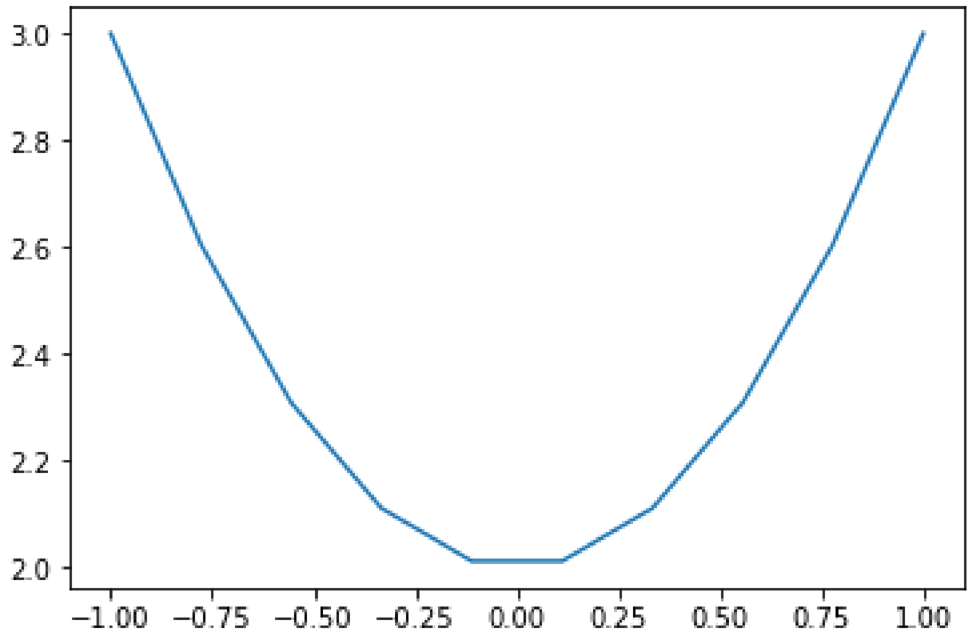
14.1 Basic line plots

We can make basic line plots with matplotlib.
We first import it with numpy:

14.1.1 Example $y = x^2 + 2$

... we will plot the points satisfying the equation $y = x^2 + 2$ with $x \in (-1, 1)$.

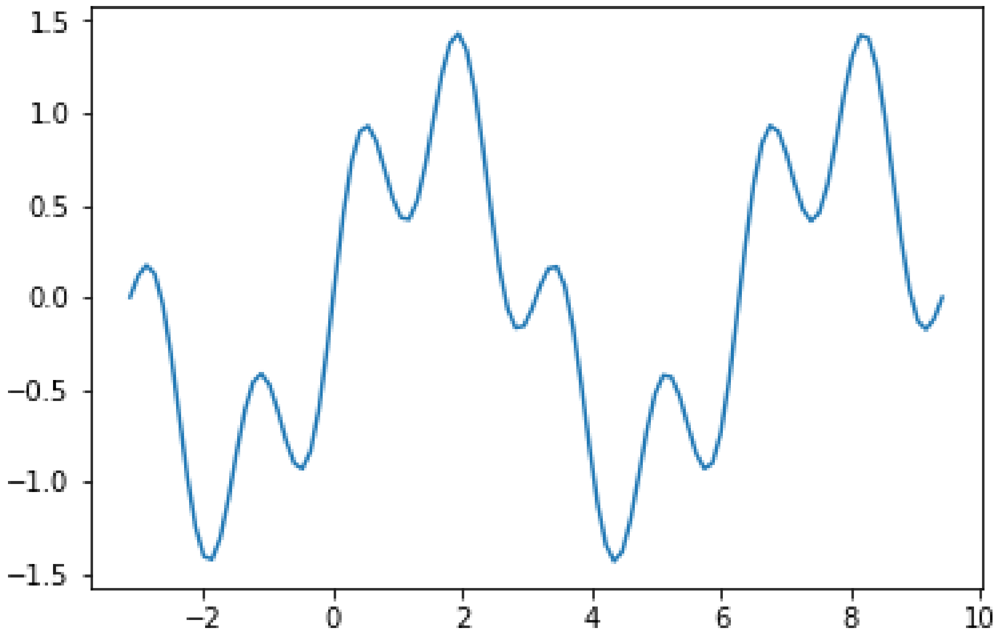
```
In [27]: import matplotlib.pyplot as plt
         import numpy as np

         # We create a figure and axes to plot on
         figure, axes = plt.figure(), plt.axes()

         X = np.linspace(-1, 1, 10) # We take 10 evenly spaced x-values in the interval (-1,1)
         Y = X**2 + 2               # We compute the Y-values

         axes.plot(X, Y) # We plot the data on the axes
         plt.show()      # We show the plot
```



14.1.2 Example: $f(x) := \sin(x) + \frac{1}{2}\sin(4x)$

We plot the graph of the function $f(x) := \sin(x) + \frac{1}{2}\sin(4x)$ on the interval $(-\pi, 3\pi)$

```
In [28]: import matplotlib.pyplot as plt
         import numpy as np

         figure, axes = plt.figure(), plt.axes()

         f = lambda x : np.sin(x) + .5*np.sin(4*x)

         X = np.linspace(-np.pi, 3*np.pi, 100)

         axes.plot(X, f(X))
         plt.show()
```

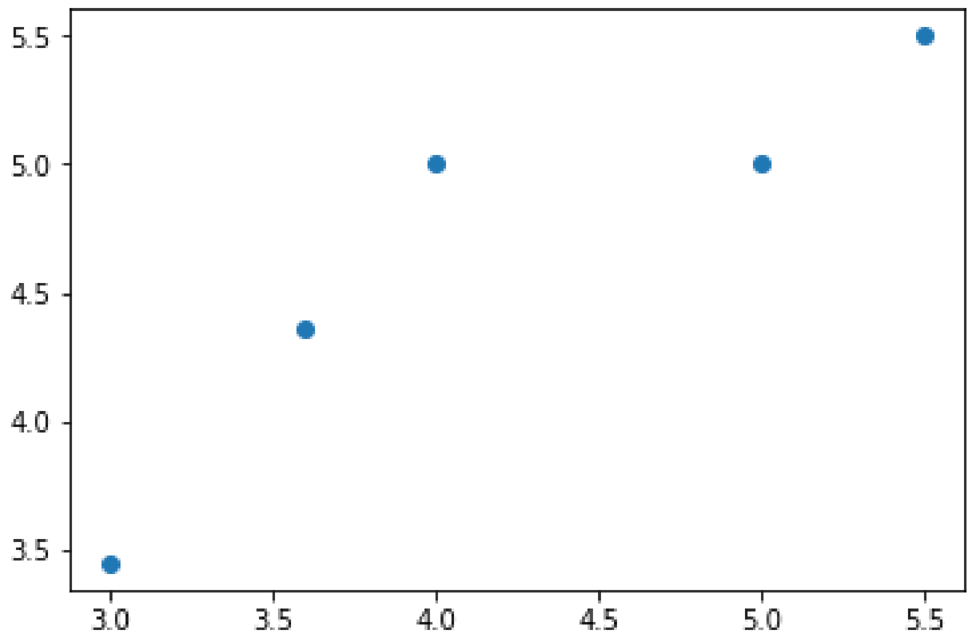


14.2 Basic scatter plots

Scatter plots are plots of discrete points.

```
In [14]: import matplotlib.pyplot as plt
         import numpy as np
```

```
In [15]: # We make a figure and axes.
         figure, axes = plt.figure(), plt.axes()

         # The points we want to plot.
         points = [
             (4  , 5),
             (5  , 5),
             (5.5, 5.5),
             (3  , 3.45),
             (3.6, 4.36),
         ]

         xvals, yvals = zip(*points)
         axes.plot(xvals, yvals, "o")
         plt.show()
```

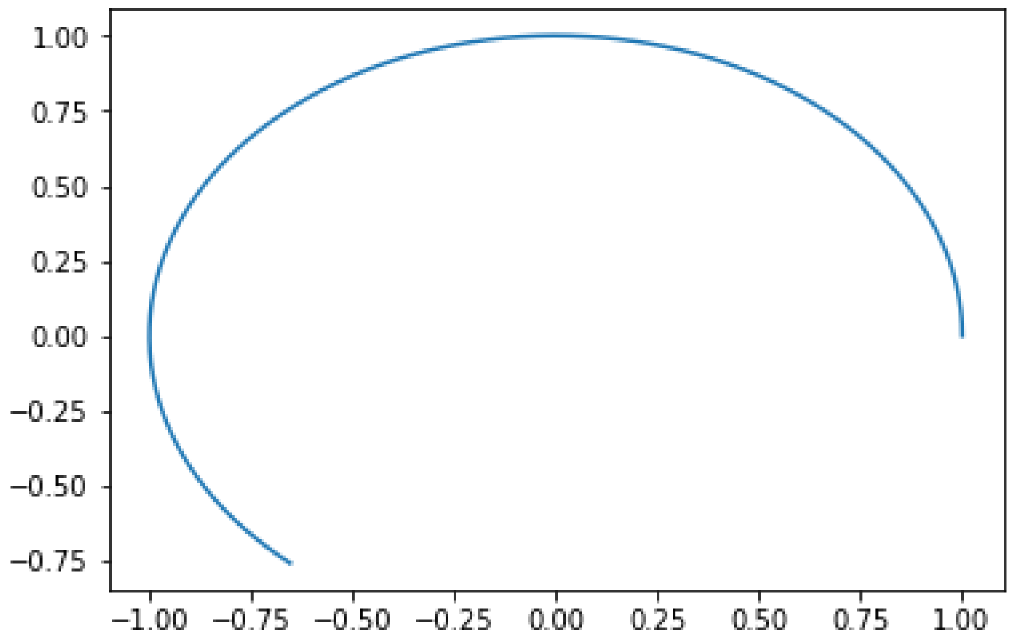


14.3 Parametric plots

We can also plot parametric funcitons:

We will plot the vector function $f : [0,4] \to \mathbb{R}^2$ defined by $f(t) := (\cos(t), \sin(t))$ for $t \in [0,4]$.

```
In [17]: import matplotlib.pyplot as plt
         import numpy as np

         figure, axes = plt.figure(), plt.axes()

         f = lambda t: (np.cos(t), np.sin(t))

         T = np.linspace(0,4, 100)
         X,Y = f(T)

         plt.plot(X, Y)
         plt.show()
```

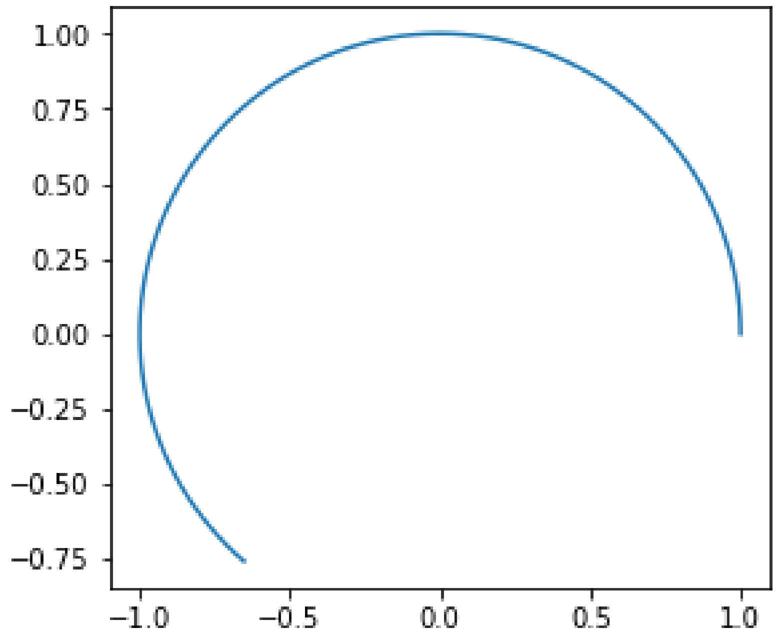


14.4 Changing the aspect ratio, plot range and size

Sometimes our plots are squashed, we can control this by changing the axes' aspect ratio.

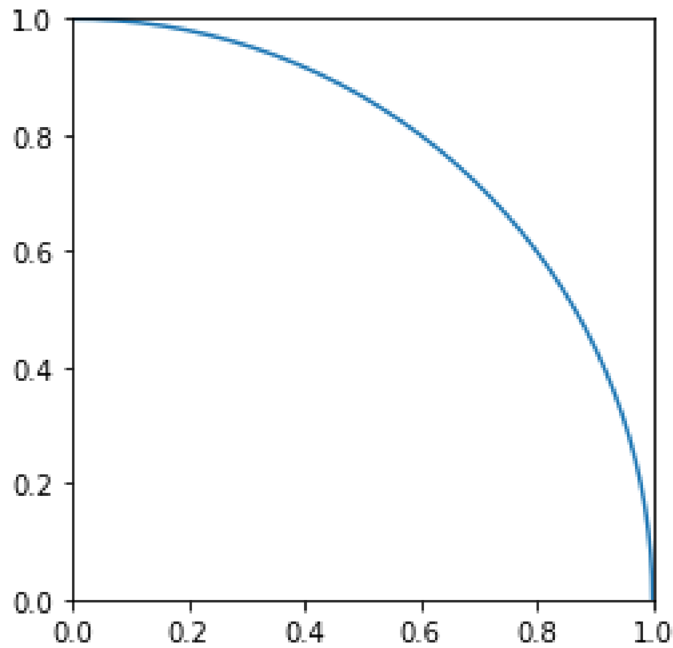
See: https://en.wikipedia.org/wiki/Aspect_ratio_(image)

In [18]:
```
import matplotlib.pyplot as plt
import numpy as np

figure, axes = plt.figure(), plt.axes()
axes.set_aspect(1) # axes' "aspect ratio"  equal to 1

f = lambda t: (np.cos(t), np.sin(t))

T = np.linspace(0,4, 100)
X,Y = f(T)

plt.plot(X, Y)
plt.show()
```



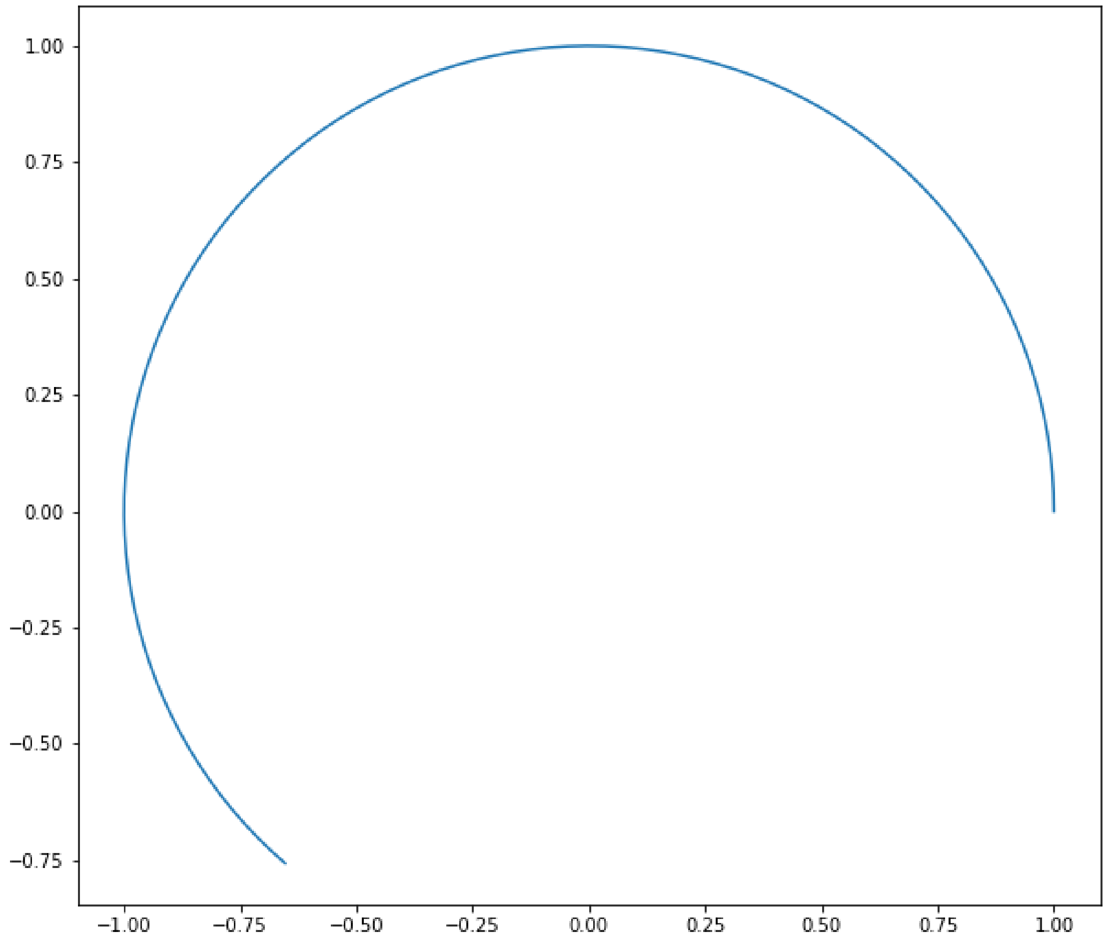
We can control the plot range with the functions axes.set_xlim and axes.set_ylim

```
In [21]: import matplotlib.pyplot as plt
         import numpy as np

         figure, axes = plt.figure(), plt.axes()
         axes.set_aspect(1)

         axes.set_xlim(0,1) # We restrict to the first quadrant
         axes.set_ylim(0,1)

         f = lambda t: (np.cos(t), np.sin(t))

         T = np.linspace(0,4, 100)
         X,Y = f(T)

         plt.plot(X, Y)
         plt.show()
```



... or we can make our plots a bit larger:

```
In [23]: import matplotlib.pyplot as plt
         import numpy as np

         # We can make our plot a bit larger
         figure  = plt.figure(figsize=(10,10)) # in inches!   D:<
         axes = plt.axes()
         axes.set_aspect(1)


         f = lambda t: (np.cos(t), np.sin(t))

         T = np.linspace(0,4, 100)
         X,Y = f(T)

         plt.plot(X, Y)
         plt.show()
```

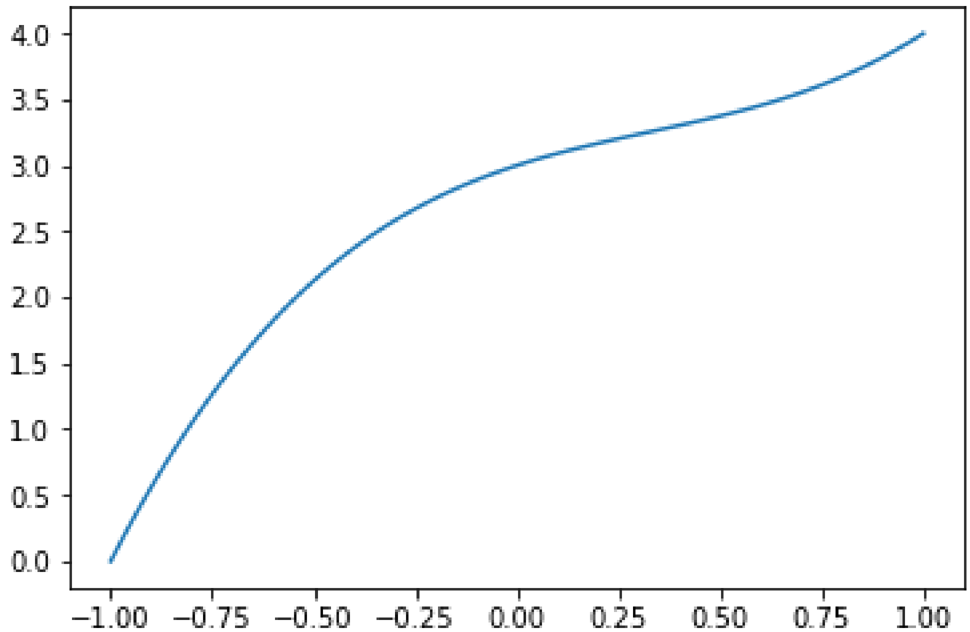


14.5 Plotting with sympy

Often we want to plot sympy expressions. We can do this easily by converting expressions to functons using the sympy.lambdify function.

Let's plot the graph of $f : [-1,1] \to \mathbb{R}$ defined by $f(x) := x^3 - x^2 + x + 3$ with $x \in (-1,1)$.

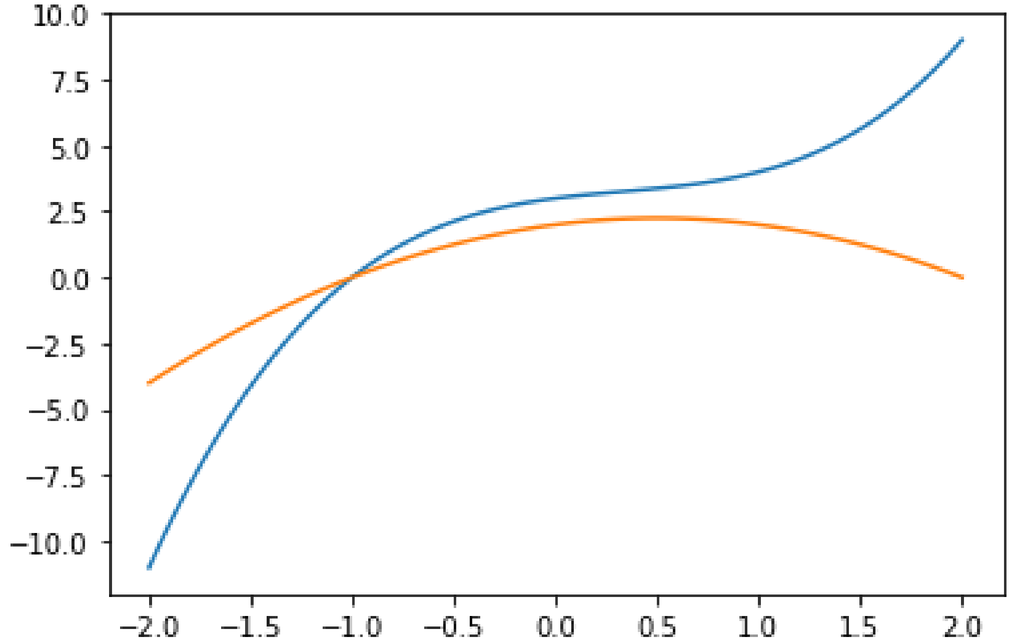
```
In [24]: import matplotlib.pyplot as plt
         import numpy as np

         from sympy.abc import x
         import sympy

         figure, axes = plt.figure(), plt.axes()

         f = sympy.lambdify([x], x**3 - x**2 + x +3) # We define the function f
         X = np.linspace(-1, 1, 100)

         axes.plot(X, f(X))
         plt.show()
```



14.6 Multiple plots on the same axis

We can easily plot multiple functions on the same set of axes:

```
In [25]: import matplotlib.pyplot as plt
         import numpy as np

         from sympy.abc import x
         import sympy

         figure, axes = plt.figure(), plt.axes()

         f = sympy.lambdify([x], x**3 - x**2 + x +3)
         g = sympy.lambdify([x], -x**2 + x +2)

         X = np.linspace(-2,2,100)

         axes.plot(X, f(X))
         axes.plot(X, g(X))
         plt.show()
```


10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

Chapter 15

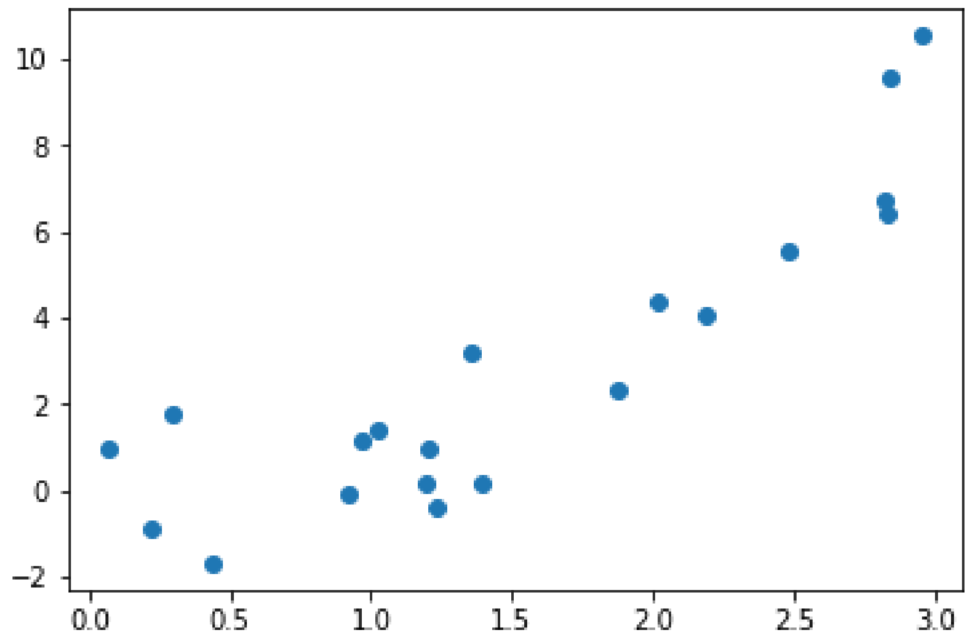
Curve fitting with sympy from first principles

Say we are given a list on 20 data points:

```
In [1]: data = [
          (1.36, 3.18),
          (1.19, 0.13),
          (2.95, 10.54),
          (2.84, 9.59),
          (0.44, -1.69),
          (2.83, 6.43),
          (1.39, 0.13),
          (1.88, 2.32),
          (1.23, -0.41),
          (0.92, -0.11),
          (0.97, 1.14),
          (2.19, 4.05),
          (2.02, 4.39),
          (2.48, 5.54),
          (1.2, 0.94),
          (0.22, -0.92),
          (0.3, 1.8),
          (1.02, 1.4),
          (0.07, 0.94),
          (2.82, 6.72),
        ]
```

Let's visualize our data on a scatter plot:

```
In [2]: import matplotlib.pyplot as plt
        import numpy as np

        fig,ax = plt.figure(), plt.axes()

        dataX, dataY = zip(*data)
        ax.plot(dataX, dataY, "o")

        plt.show()
```



In [ ]:

We want to fit a curve through these data points as closely as possible.

First, we need to choose a type of curve before we can start fitting it to the data. Let's assume that we want to fit the graph of the following function to our data: $f(x) := ax^2 + bx + c$

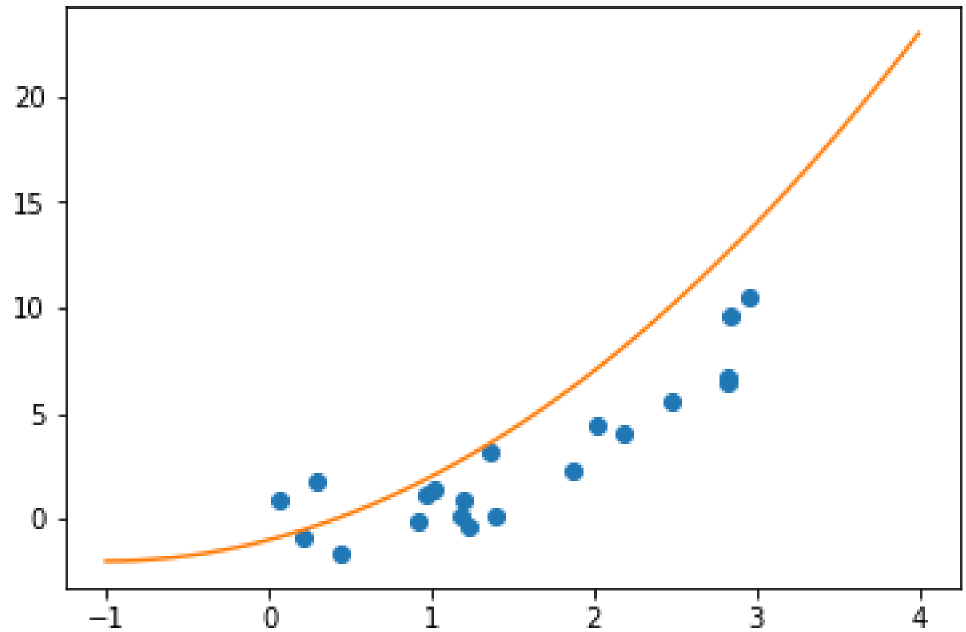
Now, we need to find numbers the paremeters a,b and c so that the curve matches the data as closely as possible.

We begin just by guessing: Say $f_1(x) := x^2 + 2x - 1$

In [3]:
```
import sympy
from sympy.abc import x
sympy.init_printing()

f1 = sympy.lambdify([x], x**2 + 2*x - 1)
```

Let's plot this guess, together with our data:

In [6]:
```
fig,ax = plt.figure(), plt.axes()

X = np.linspace(-1,4)

dataX, dataY = zip(*data)
ax.plot(dataX, dataY, "o")
ax.plot(X, f1(X))

plt.show()
```

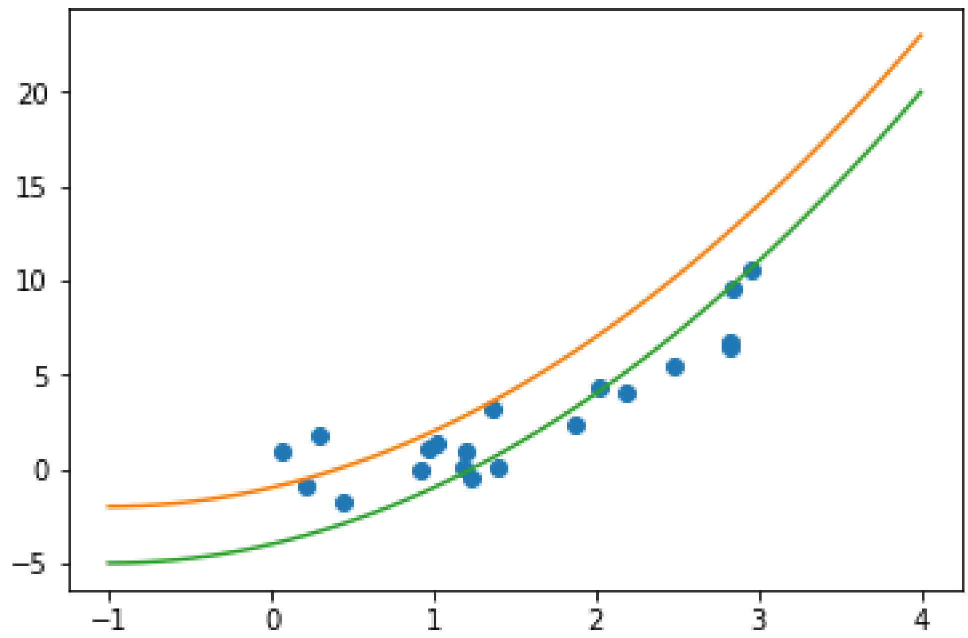


Not bad! But it goes a bit high, let's shift it down in our next guess.
Let's try $f_2(x) := x^2 + 2x - 4$.

```
In [7]: f2 = sympy.lambdify([x], x**2 + 2*x - 4)
```

... and plot our guesses f_1 and f_2 with our given data:

```
In [8]: fig,ax = plt.figure(), plt.axes()

        X = np.linspace(-1,4)

        dataX, dataY = zip(*data)
        ax.plot(dataX, dataY, "o")

        ax.plot(X, f1(X))
        ax.plot(X, f2(X))

        plt.show()
```



Better, but f_2's graph is a bit low on the left.

There must be a better way, rather than guessing!

Our data is a set of 20 points data $= \{(x_i, y_i) : i \in \{0, 1, 2, 3, ...19\}\}$. We want to simutaneously MINIMIZE the DISTANCE from the data points (x_i, y_i) to the points on the graph of f, i.e., the points $(x_i, f(x_i))$ For every i, the distance from (x_i, y_i) to $(x_i, f(x_i))$ is:

$$\sqrt{(x_i - x_i)^2 + (y_i - f(x_i))^2} = \sqrt{(y_i - f(x_i))^2} = |y_i - f(x_i)|.$$

Minimization? Perhaps we can use calculus?

The absoulte value $x \mapsto |x|$ is NOT differentiable at zero. Rather let's use the square of the DISTANCE from the data points (x_i, y_i) to the points on the graph of f. That is

$$(x_i - x_i)^2 + (y_i - f(x_i))^2 = (y_i - f(x_i))^2.$$

This is for one data point, but we want this quantity to be small for all datapoints.

Lets consider the sum of all these quantities:

$$\sum_{(x_i,y_i)\in\text{data}} (y_i - f(x_i))^2 = \sum_{(x_i,y_i)\in\text{data}} (y_i - (ax_i^2 + bx_i + c))^2$$

If we can find values of a,b,c that makes the above quantity small, as small as possible, we are in business! This is now our objective!

Let's use sympy to compute the above quantity for our 20 given data points.

```
In [9]: from sympy.abc import x,a,b,c
        import sympy

        f = sympy.lambdify([x], a*x**2+ b*x+c)
        objective = sum( (y_i - f(x_i))**2 for x_i,y_i in data)
```

```
In [10]: sympy.simplify(objective)
```

Out[10]:

$$374.39609912a^2 + 295.32802ab + 125.212ac - 725.80455a + 62.606b^2 + 60.64bc - 277.3806b + 20.0c^2 - 112.22c + 383.6073$$

Notice how this expression is only in the parameters a, b, and c. We must find the values a, b, and c for which the objective is a minimum. This can only happen where the objective has a critical point, i.e., its partial derivatives to a,b,c are simultaneously zero. This will be discussed in more detail in a course on multivariate calculus

(See https://en.wikipedia.org/wiki/Critical_point_(mathematics)#Several_variables)

I.e., We must solve the following system of three equations:

```
In [23]: [
             sympy.Eq(sympy.diff(objective, a), 0),
             sympy.Eq(sympy.diff(objective, b), 0),
             sympy.Eq(sympy.diff(objective, c), 0),
         ]
```

Out[23]:

$$[748.79219824a + 295.32802b + 125.212c - 725.80455 = 0, \quad 295.32802a + 125.212b + 60.64c - 277.3806 = 0, \quad 125.212a + 60.$$

... call sympy.solve:

```
In [24]: parameters = sympy.solve([
             sympy.Eq(sympy.diff(objective, a), 0),
             sympy.Eq(sympy.diff(objective, b), 0),
             sympy.Eq(sympy.diff(objective, c), 0),
         ], [a,b,c])
```

```
In [25]: parameters
```

Out[25]:

$$\{a : 1.5425209779946, \quad b : -1.66731287670701, \quad c : 0.504592903671337\}$$

These are the parameters we want. Let's substiture into $ax^2 + bx + c$

```
In [16]: fitted_expression = (a*x**2+b*x+c).subs(parameters)
         fitted_expression
```

Out[16]:

$$1.5425209779946x^2 - 1.66731287670701x + 0.504592903671337$$

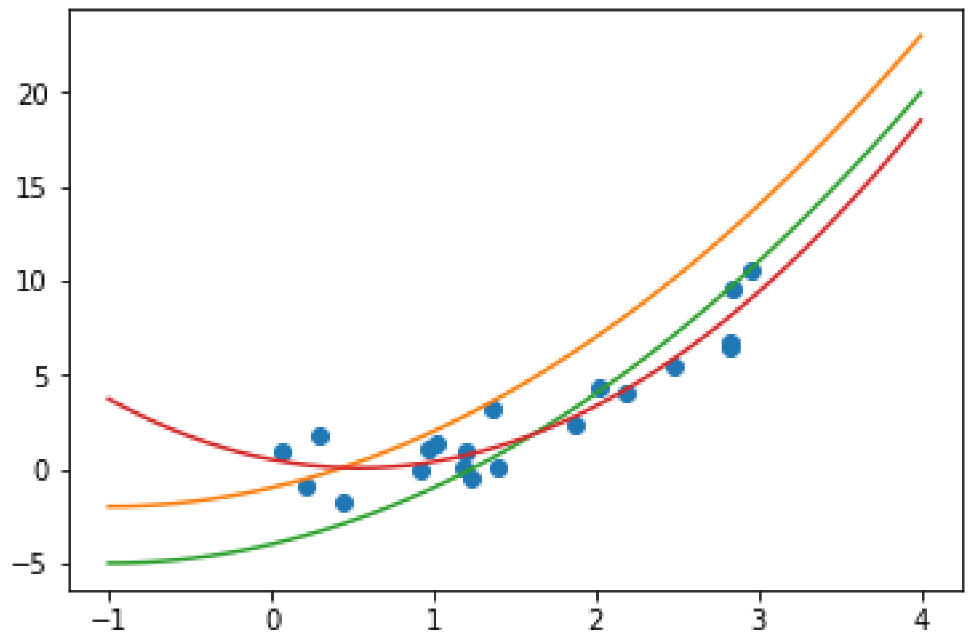
Let's make a function out of this expression

```
In [19]: fitted_f = sympy.lambdify([x], fitted_expression)
```

... and plot it against our previous guesses and our data:

```
In [22]: fig,ax = plt.figure(), plt.axes()

         X = np.linspace(-1,4)

         dataX, dataY = zip(*data)
         ax.plot(dataX, dataY, "o")
```

```
ax.plot(X, f1(X))
ax.plot(X, f2(X))
ax.plot(X, fitted_f(X))

plt.show()
```



Notice, that our fitted function closely fits the data, closer than our arbitrary guesses.

Chapter 16

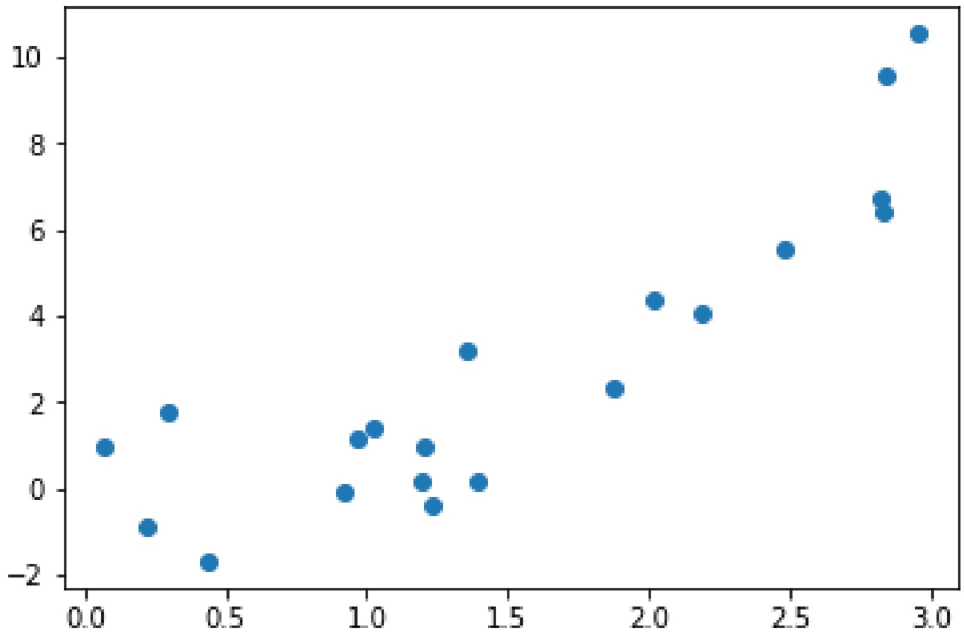
Curve fitting with numpy

Say we are given a list on 20 data points:

```
In [4]: data = [
          (1.36, 3.18),
          (1.19, 0.13),
          (2.95, 10.54),
          (2.84, 9.59),
          (0.44, -1.69),
          (2.83, 6.43),
          (1.39, 0.13),
          (1.88, 2.32),
          (1.23, -0.41),
          (0.92, -0.11),
          (0.97, 1.14),
          (2.19, 4.05),
          (2.02, 4.39),
          (2.48, 5.54),
          (1.2, 0.94),
          (0.22, -0.92),
          (0.3, 1.8),
          (1.02, 1.4),
          (0.07, 0.94),
          (2.82, 6.72),
        ]
```

Let's visualize our data on a scatter plot:

```
In [5]: import matplotlib.pyplot as plt
        fig,ax = plt.figure(), plt.axes()

        dataX, dataY = zip(*data)
        ax.plot(dataX, dataY, "o")

        plt.show()
```



We will use the numpy “polyfit” and “poly1d functions to fit polynomials to our data using a”least squares fit"

We can look up these functions’ help to understand how to use them:

```
In [ ]: import numpy as np
        help(np.polyfit)
```

```
In [ ]: help(np.poly1d)
```

For any n points we can fit a unique (n-1)th-degree polynomial through the points so that the polynomial passes exactly through the data points.

Lets find the coefficients of a 19th degree polynomial p that passes through all 20 points using the fucntion numpy.polyfit:

```
In [6]: import numpy as np
        p_coeffs = np.polyfit(dataX,dataY, 19)
```

```
/home/miek/stuff/devenv/lib/python3.6/site-packages/ipykernel_launcher.py:2: RankWarning: Polyfit may be
```

Let’s look at these coefficients:

```
In [7]: p_coeffs
```

```
Out[7]: array([ -3.18539670e+02,   6.68603366e+03,  -5.89570950e+04,
                 2.67908425e+05,  -5.32306604e+05,  -5.99817584e+05,
                 5.72029885e+06,  -8.69702146e+06,  -2.55751219e+07,
                 1.56254880e+08,  -3.95305093e+08,   6.32806621e+08,
                -7.01142368e+08,   5.51343098e+08,  -3.06943392e+08,
                 1.18340402e+08,  -3.02547577e+07,   4.75522618e+06,
                -3.98557673e+05,   1.25918369e+04])
```

We construct a function using numpy.poly1d, that outputs the value of the polynomial with these coeficients

```
In [8]: p = np.poly1d(p_coeffs)
```

We can see what this polynomial is by evaluating p(x) with x some sympy symbol.

```
In [9]: import sympy
        sympy.init_printing()
        from sympy.abc import x

        sympy.expand(p(x))
```

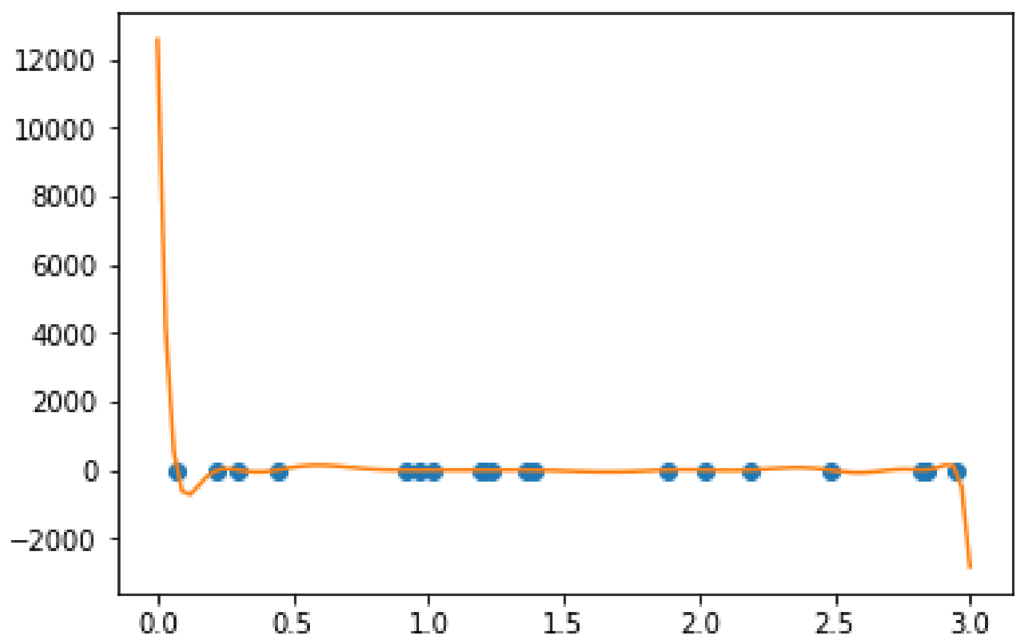
Out[9]:

$$-318.53966981043x^{19} + 6686.0336559197x^{18} - 58957.0950235226x^{17} + 267908.424710235x^{16} - 532306.603594159x^{15} - 599817.$$

Compare the coefficients of our polynomial with the numbers in p_coeffs!

Let us plot our data and out polynomial p together:

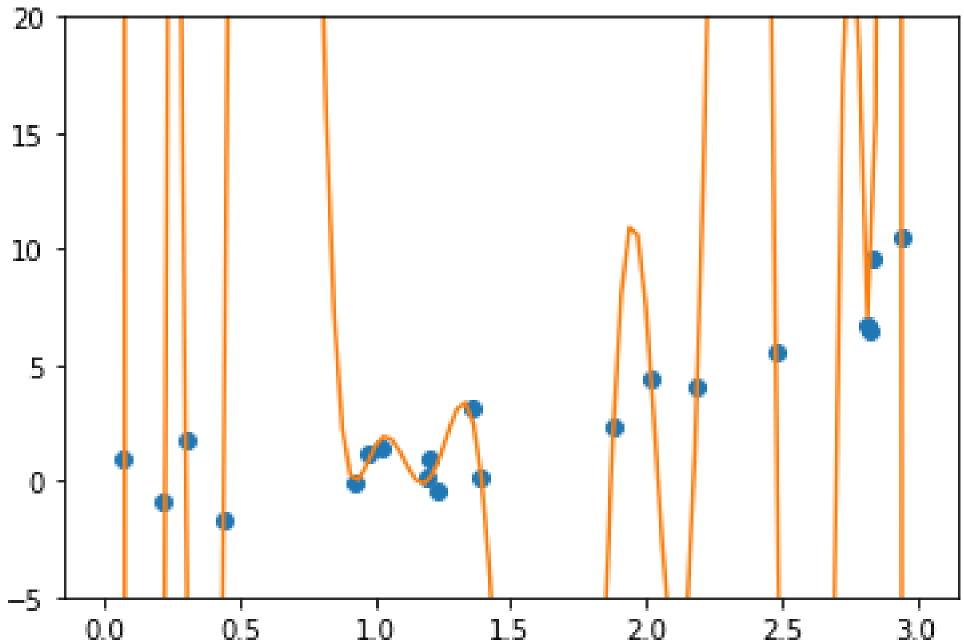
```
In [10]: X = np.linspace(0,3,100)

         fig, ax = plt.figure(), plt.axes()

         ax.plot(dataX,dataY, "o")
         ax.plot(X,p(X))
         plt.show()
```



Notice how this 19th degree polynomial passes through every data point, but oscilates quite wildly out of the range of the Y-coordinates of the data points: the interval [-2,9]. This polynomial is "over fitted" to the data.

Let's look closer:

```
In [12]: X = np.linspace(0,3,100)

         fig, ax = plt.figure(), plt.axes()
         ax.set_ylim(-5,20)

         ax.plot(dataX,dataY, "o")
         ax.plot(X,p(X))
         plt.show()
```



Let us fit 2nd degree polynomial q, i.e., a parabola, to the same data (compare with the coefficients we computed in the previous chapter!):

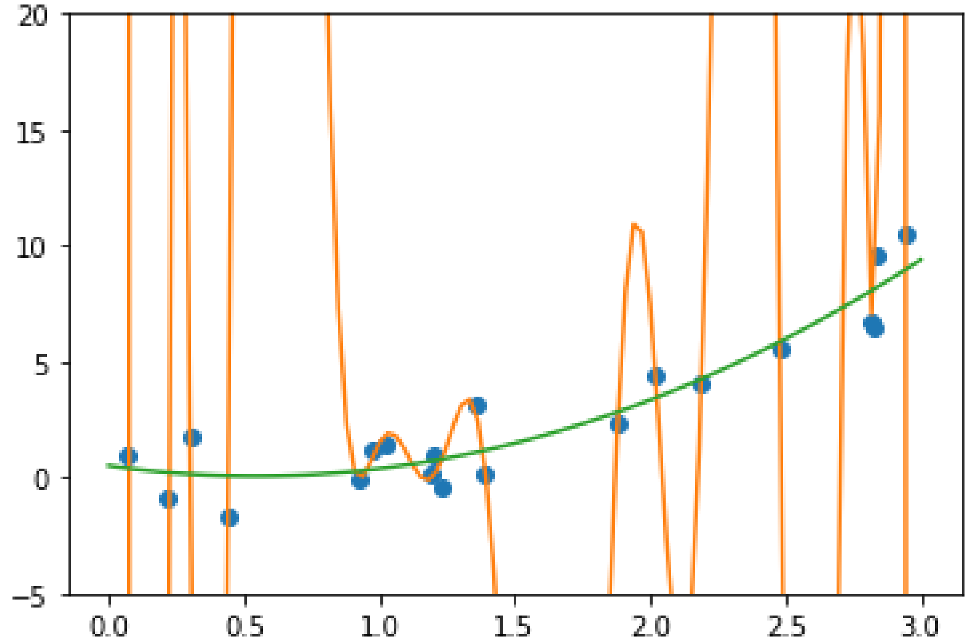
```
In [13]: q_coeffs = np.polyfit(dataX,dataY, 2)
         q = np.poly1d(q_coeffs)
         sympy.expand(q(x))
```

```
Out[13]:
```

$$1.5425209779946x^2 - 1.66731287670702x + 0.50459290367134$$

... and plot our data along with p and q on the same set of axes:

```
In [14]: fig, ax = plt.figure(), plt.axes()
         ax.set_ylim(-5,20)

         X = np.linspace(0,3,100)

         ax.plot(dataX,dataY, "o")
         ax.plot(X,p(X))
         ax.plot(X,q(X))

         plt.show()
```



Notice how the parabola does not pass through all the datapoints, but matches the overall trend of the data much closer than the 19th degree polynomial.

That is not unexpected if you consider how our data points were generated:

```
In [ ]: import random
        X = [round(random.uniform(0,3), 2) for _ in range(20)]
        Y = [round(x**2 + random.uniform(-2,2), 2) for x in X]
        data = list(zip(X,Y))
```