

# A Python Primer for Mathematics

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## 0.1 Introduction

This document is meant to prime mathematics students into using Python for doing mathematics symbolically and numerically. It is *not* meant to be used as a comprehensive text, but rather as a demonstrative cheatsheet to get up and running with the basics Python and for using Python for scientific computing as quickly as possible.

We will give very brief introductory demonstrations of the basics of the Python language, before moving on to demonstrations of some of the basic features of the packages *sympy*, *numpy*, and *matplotlib* which are increasingly used in modern scientific computing.

In [ ]:

## 1 Python language basics

### 1.1 Comments

In [1]: *# Everything on a line after a '#' is ignored by Python*

### 1.2 Basic calculations with numbers

Addition works as expected

In [1]: 1+2

Out[1]: 3

... so does subtraction

In [5]: 2.5 - 6

Out[5]: -3.5

... and division.

In [4]: 3/2

Out[4]: 1.5

The floor division operator `//` throws away everything after the decimal point (as in long division)

In [1]: 13 // 5 *# quotient of 13 divided by 5 (disregarding the fractional part or remainder)*

```
Out[1]: 2
```

... and the % operator gives the remainder of a division (as in long division)

```
In [6]: 13 % 5 # remainder when 13 is divided by 5
```

```
Out[6]: 3
```

The power operator \*\* is often useful ( $2^4 = 16$ )

```
In [8]: 2**4
```

```
Out[8]: 16
```

### 1.3 Variables

We can assign values to variables

```
In [9]: a = 5  
       b = 6
```

... and then compute with them:

```
In [11]: a + b
```

```
Out[11]: 11
```

We can reassign their values,

```
In [15]: a = 9
```

... to change the outcome of the computation

```
In [16]: a + b
```

```
Out[16]: 15
```

### 1.4 Strings (text)

Strings store text. We use either '...' or "..." to denote a string

```
In [17]: "This is a string"
```

```
Out[17]: 'This is a string'
```

```
In [18]: 'This is also a string'
```

```
Out[18]: 'This is also a string'
```

We can join strings using the "+" operator. This is called *concatenation*.

```
In [19]: "begin" + "ner" + "s"
```

```
Out[19]: 'beginners'
```

We can assign strings to variables...

```
In [20]: part1 = "This is a message "  
        part2 = "for you"
```

... and concatenate the variables

```
In [22]: full_message = part1 + part2  
        full_message
```

```
Out[22]: 'This is a message for you'
```

We can access individual characters of a string. (Remember that we index from starting from zero!)

```
In [23]: n = 2  
        full_message[n]
```

```
Out[23]: 'i'
```

We can also multiply strings.  
Plug your ears and go ...

```
In [24]: "la"*10
```

```
Out[24]: 'lalalalalalalalalala'
```

## 1.5 Unpacking

There is a more efficient way of writing

```
a = 1  
b = 2  
c = 3  
d = 4
```

The following is called *unpacking*:

```
In [29]: a, b, c, d = 1,2,3,4  
        # or  
        a, b, c, d = (1,2,3,4)
```

```
In [30]: d
```

```
Out[30]: 4
```

We are not restricted to just numbers. We can unpack anything

```
In [31]: firstname, lastname, age = ("john", "von neumann", 103)  
        lastname
```

```
Out[31]: 'von neumann'
```

```
In [32]: age
```

```
Out[32]: 103
```

## 1.6 The print function

We can output text to the screen using the "print" function

```
In [34]: print(5+4)
          print("This is a message")
          print("The answer is", 42)
```

```
9
This is a message
The answer is 42
```

## 1.7 The str, int and float functions

We can convert numbers to strings with the str function:

```
In [1]: str(12)
```

```
Out[1]: '12'
```

... and strings to numbers using the int or float functions:

```
In [2]: int("13")
```

```
Out[2]: 13
```

```
In [3]: float("1.111")
```

```
Out[3]: 1.111
```

## 1.8 Comparisons

We can ask Python if statements are true or false

```
In [35]: 4 < 6
```

```
Out[35]: True
```

```
In [36]: 4 <= 6
```

```
Out[36]: True
```

```
In [37]: 4 >= 6
```

```
Out[37]: False
```

```
In [38]: 5<5
```

```
Out[38]: False
```

```
In [39]: 5<=5
```

```
Out[39]: True
```

Notice the double equals "==" when asking is an equality is true:

```
In [40]: 3 == 3
```

```
Out[40]: True
```

```
In [41]: 3 == 4
```

```
Out[41]: False
```

... a single "=" will not work to compare numbers:

```
In [42]: 3 = 3
```

```
File "<ipython-input-42-49c8ce3fc03c>", line 2
3 = 3
    ^
SyntaxError: can't assign to literal
```

The "!=" operator means "not equal to"

```
In [43]: 3 != 4
```

```
Out[43]: True
```

We can also compare variables

```
In [44]: a,b,c = 5,6,7
```

```
In [45]: b != 7
```

```
Out[45]: True
```

```
In [46]: a < 5
```

```
Out[46]: False
```

```
In [47]: a <= 5
```

```
Out[47]: True
```

```
In [48]: b <= a
```

```
Out[48]: False
```

```
In [49]: a < b < c
```

```
Out[49]: True
```

We can also do computations in comparisons. Is the remainder when dividing by 2 equal to zero, i.e., Is b even? Is c even?

```
In [51]: b % 2 == 0      # 6 is even
```

```
Out[51]: True
```

```
In [53]: c % 2 == 0      # 7 is odd
```

```
Out[53]: False
```

## 2 If statements

With if statements we can control the flow of execution of a program.

### 2.1 If ... statements

```
In [2]: a,b = 5,6
        if a == b:
            # this is not executed because 'a == b' is false
            print("a is equal to b")

        if a <= b:
            # this is executed because 'a <= b' is true
            print("a is less than or equal to b")
```

```
a is less than or equal to b
```

### 2.2 If ... else ... statements

```
In [2]: a,b = 5,6
        if a == b:
            print("a is equal to b")
        else:
            # this is only executed if a == b is false
            print("a is not equal to b")
```

```
a is not equal to b
```

### 2.3 If ... elif ... else ... statements

```
In [3]: name = "bobby"

        if name == "alice":
            print("Hi Alice")
```



```
elif name == "bobby":
    print("Hi Bob")
elif name == "richard":
    print("Hi Ricky")
else:
    print("Hi Stranger")
```

Hi Bob

### 3 Lists

Lists are a fundamental data structure in Python. As the name suggests, we use them to store a collection of objects in a list (order matters)

We make a list using the [ ... ] notation.

```
In [1]: boy_names = [
        "benny", "adam", "bobby",
        "randal", "timmy", "cartman",
        "morty", "junior-son",
        "voldemort", "boeta", "pula",
        "zane"
    ]
```

How long is this list?

```
In [2]: len(boy_names)
```

```
Out[2]: 12
```

Is "morty" in the list?

```
In [3]: "morty" in boy_names
```

```
Out[3]: True
```

Is "xavier" in the list?

```
In [4]: "xavier" in boy_names
```

```
Out[4]: False
```

We can access the zeroth element in the list.

```
In [5]: boy_names[0]
```

```
Out[5]: 'benny'
```

WARNING! Remember that we always start index from zero!

We will distinguish between the "first" and "oneth" element. "First element of boy\_names" is ambiguous, do we mean boy\_names[0] or boy\_names[1] ?

By "oneth" or "1-th" element of boy\_names we will always mean boy\_names[1].

```
In [6]: boy_names[1]
```

```
Out[6]: 'adam'
```

We can access the last-th element in the list, by using the -1 index. (This is why we index starting from zero)

```
In [7]: boy_names[-1]
```

```
Out[7]: 'zane'
```

... and can access the 2nd last-th element with the -2 index

```
In [8]: boy_names[-2]
```

```
Out[8]: 'pula'
```

We can replace an element

```
In [9]: boy_names[1] = "adriaan"  
boy_names
```

```
Out[9]: ['benny',  
        'adriaan',  
        'bobby',  
        'randal',  
        'timmy',  
        'cartman',  
        'morty',  
        'junior-son',  
        'voldemort',  
        'boeta',  
        'pula',  
        'zane']
```

... and remove an element

```
In [10]: del boy_names[1]  
boy_names
```

```
Out[10]: ['benny',  
        'bobby',  
        'randal',  
        'timmy',  
        'cartman',
```

```
'morty',  
'junior-son',  
'voldemort',  
'boeta',  
'pula',  
'zane']
```

... and append an element to the end of the list

```
In [11]: boy_names.append("sabelo")  
boy_names
```

```
Out[11]: ['benny',  
'bobby',  
'randal',  
'timmy',  
'cartman',  
'morty',  
'junior-son',  
'voldemort',  
'boeta',  
'pula',  
'zane',  
'sabelo']
```

### 3.1 Tuples

Tuples are like lists, but they are immutable. This means it is not possible to change tuples.

We make a tuples using the ( ... ) notation

```
In [54]: boy_names_tuple = (  
    "benny", "adam", "bobby",  
    "randal", "timmy", "cartman",  
    "morty", "junior-son",  
    "voldemort", "boeta", "pula",  
    "zane"  
)
```

How long is the tuple?

```
In [15]: len(boy_names_tuple)
```

```
Out[15]: 12
```

Is "morty" in the tuple?

```
In [16]: "morty" in boy_names_tuple
```

```
Out[16]: True
```

We can access the last-th element

```
In [24]: boy_names_tuple[-1]
```

```
Out[24]: 'zane'
```

... but we cannot change the tuple by replacing elements. Trying results in an error.

```
In [18]: # We cannot change a tuple, so the
         # following gives an error
         boy_names_tuple[1] = "adriaan"
```

```
-----
TypeError                                Traceback (most recent call last)

<ipython-input-18-af5e5e9178b2> in <module>()
      1 # We cannot change a tuple, so the
      2 # following gives an error
----> 3 boy_names_tuple[1] = "adriaan"
```

```
TypeError: 'tuple' object does not support item assignment
```

### 3.2 List slicing

List slicing is an efficient method of cutting off parts of a list.

```
In [19]: girl_names = ["alice", "beatrice", "candy",
                       "dolly", "elaine", "francine", "geraldine"]
```

We use the ":" operator to make a slice. The following slice results in a new list containing the oneth, twoth, etc. elements:

```
In [22]: girl_names[1:]
```

```
Out[22]: ['beatrice', 'candy', 'dolly', 'elaine', 'francine', 'geraldine']
```

We can also slice from the other end. The following list contains everything up to the twoth element, and excludes the threeeth element onwards:

```
In [21]: girl_names[:3]
```

```
Out[21]: ['alice', 'beatrice', 'candy']
```

We can also slice using negative indices. The following list contains everything but the last-th element.

```
In [23]: girl_names[:-1]
```

```
Out[23]: ['alice', 'beatrice', 'candy', 'dolly', 'elaine', 'francine']
```

... and all but the second-last-th and last-th elements:

```
In [25]: girl_names[:-2]
```

```
Out[25]: ['alice', 'beatrice', 'candy', 'dolly', 'elaine']
```

The following list contains the oneth, twoth, threeth, fourth, elements (excluding the fiveth element onward).

```
In [27]: girl_names[1:5]
```

```
Out[27]: ['beatrice', 'candy', 'dolly', 'elaine']
```

Slicing also works for tuples and for strings:

```
In [28]: name = "Marlon Brando"  
         name[-4:]
```

```
Out[28]: 'ando'
```

### 3.3 Sorting

We very often want to sort lists. Python includes powerful methods to perform different kinds of sorting. We will work with the following list:

```
In [55]: boy_names = [  
        "benny", "adam", "bobby",  
        "randal", "timmy", "cartman",  
        "morty", "junior-son",  
        "voldemort", "boeta", "pula",  
        "zane"  
    ]
```

The default ordering for strings is alphabetically. We can just use the "sorted" function:

```
In [30]: sorted(boy_names)
```

```
Out[30]: ['adam',  
        'benny',  
        'bobby',  
        'boeta',  
        'cartman',  
        'junior-son',  
        'morty',  
        'pula',  
        'randal',  
        'timmy',  
        'voldemort',  
        'zane']
```

We can easily sort reverse-alphabetically

```
In [4]: sorted(boy_names, reverse=True)
```

```
Out[4]: ['zane',  
         'voldemort',  
         'timmy',  
         'randal',  
         'pula',  
         'morty',  
         'junior-son',  
         'cartman',  
         'boeta',  
         'bobby',  
         'benny',  
         'adam']
```

... or by length of the strings

```
In [5]: sorted(boy_names, key = len)
```

```
Out[5]: ['adam',  
         'pula',  
         'zane',  
         'benny',  
         'bobby',  
         'timmy',  
         'morty',  
         'boeta',  
         'randal',  
         'cartman',  
         'voldemort',  
         'junior-son']
```

We can sort with respect to any conceivable ordering. E.g. The following sorts the list alphabetically according to the one-th letter. (See the section on lambda expressions).

```
In [31]: sorted(boy_names, key = lambda item: item[1])
```

```
Out[31]: ['randal',  
         'cartman',  
         'zane',  
         'adam',  
         'benny',  
         'timmy',  
         'bobby',  
         'morty',  
         'voldemort',  
         'boeta',  
         'junior-son',  
         'pula']
```

### 3.4 Zipping

Zippping is an efficient way to combine two (or more) lists pairwise. Consider the two lists:

```
In [37]: girl_names = ["alice", "beatrice", "candy", "dolly", "elaine"]
        their_ages = [10, 11, 10, 9, 8]
```

We can "zip" these two lists together to get a "zip" object (zip objects are iterable objects. Their purpose is for optimizing RAM usage).

```
In [38]: name_age_pairs = zip(girl_names, their_ages)
        name_age_pairs
```

```
Out[38]: <zip at 0x7fed48073688>
```

The zip object can be converted to a list.

```
In [39]: list(name_age_pairs)
```

```
Out[39]: [('alice', 10), ('beatrice', 11), ('candy', 10), ('dolly', 9), ('elaine', 8)]
```

Warning! When zipping lists of unequal length the result will have the length of the shortest list:

```
In [40]: result = list(zip(["a","b","c"], [1,2,3,4,5,6]))
        result
```

```
Out[40]: [('a', 1), ('b', 2), ('c', 3)]
```

We can also zip more than two lists:

```
In [41]: threezip = list(zip(["a","b","c"], [1,2,3], ["alpha", "beta", "gamma"]))
        threezip
```

```
Out[41]: [('a', 1, 'alpha'), ('b', 2, 'beta'), ('c', 3, 'gamma')]
```

### 3.5 Unzipping

Unzipping is the opposite of zipping. I.e., Given a list of pairs, we can unzip the list into two lists: one list containing the first elements of the pairs and one list containing the second elements of the pairs.

Consider:

```
In [43]: name_age_pairs = [
        ('alice', 10), ('beatrice', 11), ('candy', 10), #
        ('dolly', 9), ('elaine', 8)
        ]
```

... which we unzip (notice the "\*"):

```
In [44]: unzipped_names, unzipped_ages = zip(*name_age_pairs)
```

Let's inspect the lists `unzipped_names` and `unzipped_ages`

```
In [47]: unzipped_names
```

```
Out[47]: ('alice', 'beatrice', 'candy', 'dolly', 'elaine')
```

```
In [48]: unzipped_ages
```

```
Out[48]: (10, 11, 10, 9, 8)
```

We can also unzip lists of triples:

```
In [49]: threezip = [('a', 1, 'alpha'), ('b', 2, 'beta'), ('c', 3, 'gamma')]
          abc, onetwothree, alphabeta = zip(*threezip)
```

```
In [50]: abc
```

```
Out[50]: ('a', 'b', 'c')
```

```
In [51]: onetwothree
```

```
Out[51]: (1, 2, 3)
```

```
In [52]: alphabeta
```

```
Out[52]: ('alpha', 'beta', 'gamma')
```

## 4 Loops

Loops are used to perform a single operation over and over.

### 4.1 For-loops

The for-loop is the most used kind of loop. One can think of their operation as follows: "For every element in ...(container), do ...(action) on that element".

The "range" function is a useful container to loop over. The following example prints every number in the range 0,2,3,...,9:

```
In [1]: for i in range(10):
          print(i)
```

```
0
1
2
3
4
5
6
7
8
9
```



... do the same, but loop over 4,5,...,9

```
In [2]: for i in range(4, 10):  
        print(i)
```

```
4  
5  
6  
7  
8  
9
```

We are not limited to loop over "ranges", we can loop over any container. This is the preferred way to loop over a list in Python:

```
In [3]: girl_names = ["alice", "beatrice", "candy",  
                      "dolly", "elaine", "francine", "geraldine"]  
  
        for name in girl_names:  
            print(name)
```

```
alice  
beatrice  
candy  
dolly  
elaine  
francine  
geraldine
```

We can loop in reverse order by just applying the "reversed" function to our list:

```
In [4]: for name in reversed(girl_names):  
        print(name)
```

```
geraldine  
francine  
elaine  
dolly  
candy  
beatrice  
alice
```

Often one wants to keep a running index. This is easily done with the "enumerate" function.

```
In [5]: for index, name in enumerate(girl_names):    #<----- (notice the unpacking)  
        print(index, " -> ", name)
```

```
0 -> alice
1 -> beatrice
2 -> candy
3 -> dolly
4 -> elaine
5 -> francine
6 -> geraldine
```

We can also loop directly over zip objects using unpacking

```
In [14]: girl_names = ["alice", "beatrice", "candy",
                      "dolly", "elaine", "francine", "geraldine"]
        their_ages = [3,3,7,10,15,11,31]

        for name, age in zip(girl_names, their_ages):
            print("name : ", name )
            print("    age : ", age)
```

```
name :  alice
      age :   3
name :  beatrice
      age :   3
name :   candy
      age :   7
name :   dolly
      age :  10
name :   elaine
      age :  15
name :  francine
      age :  11
name :  geraldine
      age :  31
```

## 4.2 While loops

While loops are useful when we do not know before hand how many times a loop should execute. One can think of their operation as follows: "While ... (condition) is True, do ...(action)".

```
In [7]: number = 144
        while number % 2 == 0 :    # while number is divisible by 2, ...
            number = number // 2  # divide it by two

        print(number)
```

### 4.3 You should loop like a Pythonista not a C-snake.

Python is not like classic languages e.g., C. We should not use standard C-idioms in Python. Doing so will result in ugly, unreadable and un maintainable code.

DO NOT DO ANY OF THE FOLLOWING THINGS IN PYTHON. Compare the following bad looping idioms with the proper Pythonic looping idioms above.

Consider the lists

```
In [13]: girl_names = ["alice", "beatrice", "candy",  
                      "dolly", "elaine", "francine", "geraldine"]  
        their_ages = [3,3,7,10,15,11,31]
```

DO NOT Loop over a range object unnecessarily:

```
In [9]: for i in range(len(girl_names)):  
        print(girl_names[i])
```

```
alice  
beatrice  
candy  
dolly  
elaine  
francine  
geraldine
```

DO NOT loop in reverse order by accessing indeces:

```
In [13]: for i in range(len(girl_names)):  
        print(girl_names[len(girl_names) - i - 1])
```

```
geraldine  
francine  
elaine  
dolly  
candy  
beatrice  
alice
```

DO NOT keep a running index manually:

```
In [15]: index = 0  
        for name in girl_names:  
            print(index, "-> ", name)  
            index = index + 1
```

```
0 -> alice  
1 -> beatrice  
2 -> candy
```

```
3 -> dolly
4 -> elaine
5 -> francine
6 -> geraldine
```

DO NOT loop over two lists using indices:

```
In [12]: for i in range(min(len(girl_names), len(their_ages))):
          print("name : ", girl_names[i] )
          print("    age : ", their_ages[i])
```

```
name :  alice
      age :   3
name :  beatrice
      age :   3
name :  candy
      age :   7
name :  dolly
      age :  10
name :  elaine
      age :  15
name :  francine
      age :  11
name :  geraldine
      age :  31
```

## 5 Computing with lists

### 5.1 sum, max, min

the sum, min and max functions allow for concisely expressing often occurring computations that we might perform on lists.

Consider the list:

```
In [3]: numbers = [3,5,10, 44,100,1,99]
```

...the sum of all the numbers in this list is

```
In [4]: sum(numbers)
```

```
Out[4]: 262
```

... the smallest is

```
In [5]: min(numbers)
```

```
Out[5]: 1
```

... and the largest is

```
In [6]: max(numbers)
```

```
Out[6]: 100
```

## 5.2 Computing with comprehensions: sum, max, min

We can perform computations with comprehensions. This is useful, because it makes our code easy to read and maintain.

We can compute the sum  $1 + 2 + 3 + 4 + \dots + 100$ :

```
In [9]: sum(i for i in range(1, 101)) # Why 101?
```

```
Out[9]: 5050
```

... or the sum of squares:  $1 + 2^2 + 3^2 + 4^2, \dots, 100^2$ :

```
In [6]: sum(i**2 for i in range(1, 101))
```

```
Out[6]: 338350
```

... or the sum of squares of even numbers:  $2^2 + 4^2 + 6^2, \dots, 10000^2$ :

```
In [8]: sum(i**2 for i in range(1, 10001) if i % 2 == 0)
```

```
Out[8]: 166716670000
```

We can also use the max or the min functions to find the largest or smallest elements in a list (or other container).

E.g., the smallest square number whose square lies in the interval [536,9000] is...

```
In [3]: min(i for i in range(1,9001) if 536 <= i**2 <= 9000 )
```

```
Out[3]: 24
```

... and the largest is

```
In [4]: max(i for i in range(1,9001) if 536 <= i**2 <= 9000 )
```

```
Out[4]: 94
```

If we define we can do more advanced searches as well.

E.g., finding the smallest prime number in the interval [536,9000] :

```
In [27]: def is_prime(n):  
         return not any (n % j==0 for j in range(2,n))
```

```
In [28]: min(p for p in range(536,9001) if is_prime(p))
```

```
Out[28]: 541
```

...and the largest

```
In [29]: max(p for p in range(536,9001) if is_prime(p))
```

```
Out[29]: 8999
```

### 5.3 More computing with min and max

We can use min and max with any objects that can be compared, like strings which are compared by their alphabetical order.

```
In [12]: names = ["randall", "jamie", "robert", "danaeris", "aegon", "tyrrion", "mother-of-dragon"]
```

The last name alphabetically is

```
In [14]: max(names)
```

```
Out[14]: 'tyrrion'
```

... the first

```
In [15]: min(names)
```

```
Out[15]: 'aegon'
```

... and the longest we can find by using len as a key function (see section on sorting)

```
In [30]: max(names, key=len)
```

```
Out[30]: 'mother-of-dragons-breaker-of-chains'
```

...and the shortest

```
In [31]: min(names, key=len)
```

```
Out[31]: 'jamie'
```

... and more advanced, the shortest and alphabetically first

```
In [32]: min(names, key=lambda x:(len(x), x))
```

```
Out[32]: 'aegon'
```

To understand this last example, understand that tuples are compared lexicographically.[\[https://en.wikipedia.org/wiki/Lexicographical\\_order\]](https://en.wikipedia.org/wiki/Lexicographical_order)

```
In [ ]:
```

## 6 List comprehensions

List comprehensions is a concise way of constructing lists using a for-loop syntax.

Consider the list:

```
In [1]: girl_names = ["alice", "beatrice", "candy",  
                      "dolly", "elaine", "francine", "geraldine"]
```

We can use a list comprehension to make a new list containing the zeroth letter of each name in the list:

```
In [2]: first_letters = [name[0] for name in girl_names]
       first_letters
```

```
Out[2]: ['a', 'b', 'c', 'd', 'e', 'f', 'g']
```

... or a list with the length of every name:

```
In [2]: length_of_names = [len(name) for name in girl_names]
       length_of_names
```

```
Out[2]: [5, 8, 5, 5, 6, 8, 9]
```

... or a list of name-length-pairs

```
In [3]: names_length_pairs = [ ( name, len(name) ) for name in girl_names]
       names_length_pairs
```

```
Out[3]: [('alice', 5),
         ('beatrice', 8),
         ('candy', 5),
         ('dolly', 5),
         ('elaine', 6),
         ('francine', 8),
         ('geraldine', 9)]
```

A useful feature is adding a conditional. The following makes a new list only containing the "long" names:

```
In [4]: only_long_names = [ name for name in girl_names if len(name) > 6 ]
       only_long_names
```

```
Out[4]: ['beatrice', 'francine', 'geraldine']
```

## 6.1 Computing with comprehensions: sum, min, max

We can perform computations with comprehensions. This is useful, because it makes our code easy to read and maintain.

We can compute the sum  $1 + 2 + 3 + 4 + \dots + 100$ :

```
In [9]: sum(i for i in range(1, 101)) # Why 101?
```

```
Out[9]: 5050
```

... or the sum of squares:  $1 + 2^2 + 3^2 + 4^2, \dots, 100^2$ :

```
In [6]: sum(i**2 for i in range(1, 101))
```

```
Out[6]: 338350
```

... or the sum of squares of even numbers:  $2^2 + 4^2 + 6^2, \dots, 10000^2$ :

```
In [8]: sum(i**2 for i in range(1, 10001) if i % 2 == 0)
```

```
Out[8]: 166716670000
```

We can also use the max or the min functions to find the largest or smallest elements in a list (or other container).

E.g., the smallest square number whose square lies in the interval [536,9000] is...

```
In [3]: min(i for i in range(1,9001) if 536 <= i**2 <= 9000 )
```

```
Out[3]: 24
```

... and the largest is

```
In [4]: max(i for i in range(1,9001) if 536 <= i**2 <= 9000 )
```

```
Out[4]: 94
```

If we define we can do more advanced searches as well.

E.g., finding the smallest prime number in the interval [536,9000] :

```
In [27]: def is_prime(n):  
         return not any (n % j==0 for j in range(2,n))
```

```
In [28]: min(p for p in range(536,9001) if is_prime(p))
```

```
Out[28]: 541
```

...and the largest

```
In [29]: max(p for p in range(536,9001) if is_prime(p))
```

```
Out[29]: 8999
```

## 6.2 More computing with min and max

We can use min and max with any objects that can be compared, like strings which are compared by their alphabetical order.

```
In [12]: names = ["randall", "jamie", "robert", "danaeris", "aegon", "tyrrion", "mother-of-dragon"]
```

The last name alphabetically is

```
In [14]: max(names)
```

```
Out[14]: 'tyrrion'
```

... the first

```
In [15]: min(names)
```



```
Out[15]: 'aegon'
```

... and the longest we can find by using `len` as a key function (see section on sorting)

```
In [30]: max(names, key=len)
```

```
Out[30]: 'mother-of-dragons-breaker-of-chains'
```

...and the shortest

```
In [31]: min(names, key=len)
```

```
Out[31]: 'jamie'
```

... and more advanced, the shortest and alphabetically first

```
In [32]: min(names, key=lambda x:(len(x), x))
```

```
Out[32]: 'aegon'
```

To understand this last example, understand that Python tuples are compared lexicographically. See: [https://en.wikipedia.org/wiki/Lexicographical\\_order](https://en.wikipedia.org/wiki/Lexicographical_order)

```
In [ ]:
```

## 7 Functions

Functions allow for the easy reuse of bits of code. They take parameters/input, and can return a result. Functions are defined using the *def* keyword.

### 7.1 Example: Say hello

We define a function that takes *name* as parameter, and prints a greeting for that name:

```
In [4]: def say_hello_to(name):  
        print("Hello " + name + "!")
```

We can now call this function with different inputs:

```
In [6]: say_hello_to("World")  
        say_hello_to("Gary")  
        say_hello_to("Alice")  
        say_hello_to("Crocubot")
```

```
Hello World!
```

```
Hello Gary!
```

```
Hello Alice!
```

```
Hello Crocubot!
```

## 7.2 Example: Divisible by 11

We define a function that takes *number* as input and *returns* whether or not the number is divisible by 11

```
In [7]: def is_divisible_by_11(number):  
        return number % 11 == 0
```

Lets check which numbers of 10,11,12,...,24 are divisible by 11

```
In [8]: for number in range(10, 25):  
        print(number, "is divisible by 11 : ", is_divisible_by_11(number))
```

```
10 is divisible by 11 : False  
11 is divisible by 11 : True  
12 is divisible by 11 : False  
13 is divisible by 11 : False  
14 is divisible by 11 : False  
15 is divisible by 11 : False  
16 is divisible by 11 : False  
17 is divisible by 11 : False  
18 is divisible by 11 : False  
19 is divisible by 11 : False  
20 is divisible by 11 : False  
21 is divisible by 11 : False  
22 is divisible by 11 : True  
23 is divisible by 11 : False  
24 is divisible by 11 : False
```

## 7.3 Example: The Collatz function

Example: We define the *collatz* function according to the following specification.

Input:

- A number  $n$ .

Output:

- return 1 if  $n = 1$
- return  $n/2$  if  $n$  is even
- return  $3n + 1$  if  $n$  is odd

```
In [9]: def collatz(number):  
        if number == 1:  
            return number  
        elif number % 2 == 0:  
            return number // 2  
        else:  
            return 3*number + 1
```

Let's try it out on 3,11,24 and 65

```
In [10]: for n in [3, 11, 24, 65]:  
         print(n, " -> ",collatz(n))
```

```
3  ->  10  
11  ->  34  
24  ->  12  
65  -> 196
```

Let's repeatedly apply the collatz function to a number using a while loop. We always tend to get back to 1... why is that?

See [https://en.wikipedia.org/wiki/Collatz\\_conjecture](https://en.wikipedia.org/wiki/Collatz_conjecture)

```
In [12]: current_number = 15  
         while current_number != 1:  
             current_number = collatz(current_number)  
             print(current_number)
```

```
46  
23  
70  
35  
106  
53  
160  
80  
40  
20  
10  
5  
16  
8  
4  
2  
1
```

## 7.4 Recursion

Recursion is what happens when a function calls itself.

### 7.4.1 Example: Fibonacci numbers

A good example of recursion is the process of generating Fibonacci numbers 1,1,2,3,5,8,13,21,34,55,... These are formally defined as the sequence  $(f_n)$  with  $f_1 := 1$ ,  $f_2 := 1$  and  $f_n := f_{n-1} + f_{n-2}$  for all  $n \in \{3,4,5,\dots\}$

Pay attention how the following function calls itself:

```
In [21]: def fibonacci(n):
        if n == 1:
            return 1
        elif n == 2:
            return 1
        else:
            return fibonacci(n-1) + fibonacci(n-2)
```

Let's compute the first 20 fibonacci numbers

```
In [14]: for i in range(1,20):
        print(fibonacci(i))
```

```
1
1
2
3
5
8
13
21
34
55
89
144
233
377
610
987
1597
2584
4181
```

## 7.5 Making functions with lambda expressions

Very simple functions can be defined using *lambda* expressions. We've already briefly encountered lambda expressions in the section on sorting.

```
In [16]: f = lambda x: 3*x
```

Make sure you understand why  $f(4)=12$

```
In [19]: f(4)
```

```
Out[19]: 12
```

...and  $f('a')$ ='aaa'

```
In [20]: f("a")
```

```
Out[20]: 'aaa'
```

## 7.6 docstrings

One's code is usually used by other people. These people might need to know what a function you wrote does. One may do this by writing a short explanation in a *docstring* in the first line of the function definition. This can be accessed by calling the help function on an object.

```
In [25]: def fibonacci_undocumented(n):
        if n == 1:
            return 1
        elif n == 2:
            return 1
        else:
            return fibonacci(n-1) + fibonacci(n-2)
```

No help is forthcoming...

```
In [26]: help(fibonacci_undocumented)
```

```
Help on function fibonacci_undocumented in module __main__:
```

```
fibonacci_undocumented(n)
```

... unless we give it:

```
In [29]: def fibonacci(n):
        """
        Returns the nth fibonacci number.

        Input: n
        Output: the nth fibonacci number

        E.g. fibonacci(1) = 1
             fibonacci(2) = 1
             fibonacci(3) = 2
             ...
        """
        if n == 1:
            return 1
        elif n == 2:
            return 1
        else:
            return fibonacci(n-1) + fibonacci(n-2)
```

```
In [30]: help(fibonacci)
```

Help on function fibonacci in module \_\_main\_\_:

```
fibonacci(n)
    Returns the nth fibonacci number.

    Input: n
    Output: the nth fibonacci number

    E.g.fibonacci(1) = 1
        fibonacci(2) = 1
        fibonacci(3) = 2
        ...
```

## 8 The 'any' and 'all' functions

Sometimes one is required to decide if a number of statements in a list are *all* true.

```
In [4]: all([True, True, True, True]) # All true? Yes.
```

```
Out[4]: True
```

```
In [7]: all([True, True, False, True]) # All true? No.
```

```
Out[7]: False
```

... and sometimes one is required to decide if *at least one* from a number of statements is true:

```
In [15]: any([True, False, False, True]) # Is at least one statement true? Yes.
```

```
Out[15]: True
```

```
In [10]: any([False, False, False, False]) # Is at least one statement true? No.
```

```
Out[10]: False
```

### Examples

The following examples illustrate how the *all* and *any* functions can be used.

```
In [2]: # Do all the letters "a", "b", "l" occur the phrase "mary had a little lamb"?
        # Yes. So the following evaluates to True.
```

```
        all([letter in "mary had a little lamb" for letter in ["a", "b", "l"]])
```

```
Out[2]: True
```

```
In [4]: # Do all the letters "a", "b", "q" occur the phrase "the quick brown fox"?
        # No. The letter "a" does not occur, so the following evaluates to False.
```

```
        all([letter in "the quick brown fox" for letter in ["a", "b", "q"]])
```

```
Out[4]: False
```

```
In [7]: # Does *at least one* of the letters "a", "b", "z" occur the phrase "the quick brown fox"
# Yes. The letter "b" occurs, so the following evaluates to True.
any([letter in "the quick brown fox" for letter in ["a", "b", "z"]])
```

```
Out[7]: True
```

```
In [8]: # Does *at least one* of the letters "z", "q", "p" occur the phrase "mary had a little lamb"
# No. none of the letters occur, so the following evaluates to False.
any([letter in "mary had a little lamb" for letter in ["z", "p", "q"]])
```

```
Out[8]: False
```

## 9 Computing with lists

### 9.1 sum, max, min

the sum, min and max functions allow for concisely expressing often occurring computations that we might perform on lists.

Consider the list:

```
In [3]: numbers = [3,5,10, 44,100,1,99]
```

...the sum of all the numbers in this list is

```
In [4]: sum(numbers)
```

```
Out[4]: 262
```

... the smallest is

```
In [5]: min(numbers)
```

```
Out[5]: 1
```

... and the largest is

```
In [6]: max(numbers)
```

```
Out[6]: 100
```

### 9.2 Computing with comprehensions: sum, min, max

We can perform computations with comprehensions. This is useful, because it makes our code easy to read and maintain.

We can compute the sum  $1 + 2 + 3 + 4 + \dots + 100$ :

```
In [9]: sum(i for i in range(1, 101)) # Why 101?
```

```
Out[9]: 5050
```

... or the sum of squares:  $1 + 2^2 + 3^2 + 4^2, \dots, 100^2$ :

```
In [6]: sum(i**2 for i in range(1, 101))
```

```
Out[6]: 338350
```

... or the sum of squares of even numbers:  $2^2 + 4^2 + 6^2, \dots, 10000^2$ :

```
In [8]: sum(i**2 for i in range(1, 10001) if i % 2 == 0)
```

```
Out[8]: 166716670000
```

We can also use the max or the min functions to find the largest or smallest elements in a list (or other container).

E.g., the smallest square number whose square lies in the interval [536,9000] is...

```
In [3]: min(i for i in range(1,9001) if 536 <= i**2 <= 9000 )
```

```
Out[3]: 24
```

... and the largest is

```
In [4]: max(i for i in range(1,9001) if 536 <= i**2 <= 9000 )
```

```
Out[4]: 94
```

If we define we can do more advanced searches as well.

E.g., finding the smallest prime number in the interval [536,9000] :

```
In [27]: def is_prime(n):  
         return not any (n % j==0 for j in range(2,n))
```

```
In [28]: min(p for p in range(536,9001) if is_prime(p))
```

```
Out[28]: 541
```

...and the largest

```
In [29]: max(p for p in range(536,9001) if is_prime(p))
```

```
Out[29]: 8999
```

### 9.3 More computing with min and max

We can use min and max with any objects that can be compared, like strings which are compared by their alphabetical order.

```
In [12]: names = ["randall", "jamie", "robert", "danaeris", "aegon", "tyrrion", "mother-of-dragon"]
```

The last name alphabetically is

```
In [14]: max(names)
```



```
Out[14]: 'tyrrion'
```

... the first

```
In [15]: min(names)
```

```
Out[15]: 'aegon'
```

... and the longest we can find by using len as a key function (see section on sorting)

```
In [30]: max(names, key=len)
```

```
Out[30]: 'mother-of-dragons-breaker-of-chains'
```

...and the shortest

```
In [31]: min(names, key=len)
```

```
Out[31]: 'jamie'
```

... and more advanced, the shortest and alphabetically first

```
In [32]: min(names, key=lambda x:(len(x), x))
```

```
Out[32]: 'aegon'
```

To understand this last example, understand that tuples are compared lexicographically.[\[https://en.wikipedia.org/wiki/Lexicographical\\_order\]](https://en.wikipedia.org/wiki/Lexicographical_order)

```
In [ ]:
```

## 10 Dictionaries

Dictionaries are datastructures that map one object to another. We create a dictionary using the { ... : ... } notation.

Consider the dictionary:

```
In [1]: surname_dictionary = {  
        # key    : #value,  
        "kevin" : "de koker",  
        "john"  : "mphako",  
        "alice" : "munro",  
        "doris" : "lessing",  
    }  
    surname_dictionary
```

```
Out[1]: {'alice': 'munro', 'doris': 'lessing', 'john': 'mphako', 'kevin': 'de koker'}
```

We can access a *value* associated to a specific *key*:

```
In [2]: surname_dictionary["kevin"]
```

```
Out[2]: 'de koker'
```

```
In [3]: surname_dictionary["alice"]
```

```
Out[3]: 'munro'
```

An error is raised if the key is not in the dictionary:

```
In [4]: surname_dictionary["bobby"]
```

```
-----  
KeyError                                Traceback (most recent call last)  
  
  <ipython-input-4-aebb82606656> in <module>()  
----> 1 surname_dictionary["bobby"]  
  
KeyError: 'bobby'
```

We can ask if a key is in the dictionary:

```
In [5]: "kevin" in surname_dictionary
```

```
Out[5]: True
```

```
In [6]: "bobby" in surname_dictionary
```

```
Out[6]: False
```

... the *in* operator only checks keys, not values:

```
In [7]: "de koker" in surname_dictionary
```

```
Out[7]: False
```

We can add elements:

```
In [8]: surname_dictionary["katie"] = "van der merwe"  
surname_dictionary
```

```
Out[8]: {'alice': 'munro',  
        'doris': 'lessing',  
        'john': 'mphako',  
        'katie': 'van der merwe',  
        'kevin': 'de koker'}
```

... and remove elements:

```
In [10]: del surname_dictionary["alice"]
         surname_dictionary
```

```
Out[10]: {'doris': 'lessing',
          'john': 'mphako',
          'katie': 'van der merwe',
          'kevin': 'de koker'}
```

Iterating over a dictionary, iterates over the keys:

```
In [11]: for key in surname_dictionary:
         print(key)
```

```
doris
john
kevin
katie
```

... but we can also iterate over the values using *.values()*:

```
In [12]: for key in surname_dictionary.values():
         print(key)
```

```
lessing
mphako
de koker
van der merwe
```

... or we can iterate over key-value pairs using *.items()*:

```
In [13]: for pair in surname_dictionary.items():
         print(pair)
```

```
('doris', 'lessing')
('john', 'mphako')
('kevin', 'de koker')
('katie', 'van der merwe')
```

It is often useful to unpack such pairs:

```
In [14]: for firstname, lastname in surname_dictionary.items():
         print(firstname, "-->", lastname[0])
```

```
doris --> l
john --> m
kevin --> d
katie --> v
```

## 11 Dictionary comprehensions

Dictionary comprehension is a concise way to construct dictionaries using a for-loop syntax.

Consider:

```
In [1]: surname_dictionary = {
        # key    : #value,
        "kevin" : "de koker",
        "john"  : "mphako",
        "alice" : "munro",
        "doris" : "lessing",
    }
    surname_dictionary
```

```
Out[1]: {'alice': 'munro', 'doris': 'lessing', 'john': 'mphako', 'kevin': 'de koker'}
```

We construct a dictionary which maps a name to the length of the surname.

```
In [2]: length_of_surname_dictionary = {
        firstname : len(lastname) for firstname, lastname in surname_dictionary.items()
    }
    length_of_surname_dictionary
```

```
Out[2]: {'alice': 5, 'doris': 7, 'john': 6, 'kevin': 8}
```

We construct a dictionary which filtered all items whose last name start with "m"

```
In [3]: last_name_starts_with_m = {
        firstname : lastname
            for firstname, lastname in surname_dictionary.items()
            if "m" == lastname[0]
        }
    last_name_starts_with_m
```

```
Out[3]: {'alice': 'munro', 'john': 'mphako'}
```

## 12 Importing modules and interactive help

Not all Python functionality is builtin. Extra functionality is provided in *modules*. To use the extra functionality provided by a module we must *import* the module.

The syntax for importing modules are:

```
import ...
from ... import ...
import ... as ...
```

Let's import the *math* module:

```
In [2]: import math
```

We can see what objects the math module provides by calling the *dir* method on it:

```
In [3]: dir(math)
```

```
Out[3]: ['__doc__',
         '__loader__',
         '__name__',
         '__package__',
         '__spec__',
         'acos',
         'acosh',
         'asin',
         'asinh',
         'atan',
         'atan2',
         'atanh',
         'ceil',
         'copysign',
         'cos',
         'cosh',
         'degrees',
         'e',
         'erf',
         'erfc',
         'exp',
         'expm1',
         'fabs',
         'factorial',
         'floor',
         'fmod',
         'frexp',
         'fsum',
         'gamma',
         'gcd',
         'hypot',
         'inf',
         'isclose',
         'isfinite',
         'isinf',
         'isnan',
         'ldexp',
         'lgamma',
         'log',
         'log10',
         'log1p',
         'log2',
         'modf',
         'nan',
```

```
'pi',  
'pow',  
'radians',  
'sin',  
'sinh',  
'sqrt',  
'tan',  
'tanh',  
'trunc']
```

If we need to know more about an object, then we can the *help* function on it:

```
In [4]: help(math.acos)
```

Help on built-in function acos in module math:

```
acos(...)  
    acos(x)
```

Return the arc cosine (measured in radians) of x.

```
In [6]: help(math.radians)
```

Help on built-in function radians in module math:

```
radians(...)  
    radians(x)
```

Convert angle x from degrees to radians.

With the math module imported, we can access its contents and call the functions it defines:

```
In [7]: math.pi
```

```
Out[7]: 3.141592653589793
```

```
In [8]: math.acos(-1)
```

```
Out[8]: 3.141592653589793
```

```
In [9]: math.sin(math.radians(90))
```

```
Out[9]: 1.0
```

## 13 Sympy

Sympy is an external Python module that allows for symbolic computations like solving equations, differentiation and integration.

We import the *sympy* module

```
In [2]: import sympy
```

If we want to have pretty output inside a Jupyter notebook, we call `sympy.init_printing`

```
In [3]: sympy.init_printing()
```

We can define symbols using the `sympy.symbols` function:

```
In [4]: x,y = sympy.symbols("x y")
```

... or we can import standard symbols from the `sympy.abc` module

```
In [5]: from sympy.abc import x,y
```

With these symbols, we can define an algebraic expression in the variables `x` and `y`

```
In [6]: an_expression = sympy.sin(x**2 - x - 1 + sympy.acos(y))
        an_expression
```

Out [6]:

$$\sin(x^2 - x + \arccos(y) - 1)$$

### 13.1 Expanding, factoring and simplifying expressions

We can expand expressions using `sympy.expand`

```
In [7]: sympy.expand( (x+4)*(x-6) )
```

Out [7]:

$$x^2 - 2x - 24$$

We can factor expressions using `sympy.factor`

```
In [8]: sympy.factor( x**2-x-20 )
```

Out [8]:

$$(x - 5)(x + 4)$$

We can make more complicated expressions ...

```
In [47]: (x**3 + x**2 - x - 1)/(x**2 + 2*x + 1)
```

Out[47]:

$$\frac{x^3 + x^2 - x - 1}{x^2 + 2x + 1}$$

... and simplify them using `sympy.simplify`

```
In [9]: sympy.simplify( (x**3 + x**2 - x - 1)/(x**2 + 2*x + 1) )
```

Out[9]:

$$x - 1$$

## 13.2 Substituting values into expressions

Let's define the quadratic expression  $x^2 - x - 1$

```
In [10]: from sympy.abc import x
         quadratic_expression = x**2 - x - 1
```

... and substitute the value 1 for the symbol  $x$  using the `.subs` function.  
Notice the dictionary! Make sure you understand why the result is -1.

```
In [12]: quadratic_expression.subs({x : 1})
```

Out[12]:

$$-1$$

We substitute the value -2 for the symbol  $x$  using the `.subs` function. Make sure you understand why the result is 5.

```
In [15]: quadratic_expression.subs({x : -2})
```

Out[15]:

$$5$$

## 13.3 Solving equations

We can solve equations with `sympy`.

WARNING: We cannot use `"="` or `"=="` to define equations, we must use `sympy.Eq`.

We make the equation  $x^2 - x - 1 = 0$ .

```
In [17]: import sympy
         from sympy.abc import x

         sympy.Eq(x**2 - x - 1, 0)
```



Out [17]:

$$x^2 - x - 1 = 0$$

... and solve for  $x$  in this equation by calling the `sympy.solve` function

```
In [18]: sympy.solve(sympy.Eq(x**2 - x - 1, 0), x)
```

Out [18]:

$$\left[ \frac{1}{2} + \frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2} + \frac{1}{2} \right]$$

... by just providing an expression to `sympy.solve`, it solves the equation `expression=0`.

```
In [19]: sympy.solve(x**2 - x - 1, x)
```

Out [19]:

$$\left[ \frac{1}{2} + \frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2} + \frac{1}{2} \right]$$

In [ ]:

We can solve more complicated equations. Let's solve for  $\theta$  in:

$$\cos(\theta) = \sin(\theta)$$

The equation has infinitely many solutions in  $\theta$ . However, `sympy.solve` only gives two:

```
In [24]: from sympy.abc import theta
         sympy.solve(sympy.Eq(sympy.cos(theta), sympy.sin(theta)), theta)
```

Out [24]:

$$\left[ -\frac{3\pi}{4}, \frac{\pi}{4} \right]$$

... `sympy.solveset` gives all infinitely many solutions

```
In [25]: sympy.solveset(sympy.Eq(sympy.cos(theta), sympy.sin(theta)), theta)
```

Out [25]:

$$\left\{ 2n\pi + \frac{5\pi}{4} \mid n \in \mathbb{Z} \right\} \cup \left\{ 2n\pi + \frac{\pi}{4} \mid n \in \mathbb{Z} \right\}$$

### 13.4 Solving systems of equations

We can also solve systems of equations like the following in  $x$  and  $y$ :

$$2x + 3y = 1$$

$$3x + 2y = 2$$

```
In [26]: from sympy.abc import x,y
import sympy

sympy.solve([
    sympy.Eq(2*x + 3*y, 1),
    sympy.Eq(3*x + 2*y, 2)
], [x,y])
```

Out [26] :

$$\left\{ x : \frac{4}{5}, \quad y : -\frac{1}{5} \right\}$$

### 13.5 Generating complicated expressions

We can use functions to generate complicated expressions.

Let's define a function  $P$  that, for any number  $n$ , returns a polynomial of the form

$$\sum_{k=0}^n kx^k$$

```
In [59]: import sympy
from sympy.abc import x

def P(n):
    return sum( k * x**k for k in range(n+1) )
```

We can now obtain the 6th degree polynomial of the given form by calling  $P(6)$

```
In [60]: P(6)
```

Out [60] :

$$6x^6 + 5x^5 + 4x^4 + 3x^3 + 2x^2 + x$$

... or the 10th degree polynomial by calling  $P(10)$

```
In [33]: P(10)
```

Out [33] :

$$10x^{10} + 9x^9 + 8x^8 + 7x^7 + 6x^6 + 5x^5 + 4x^4 + 3x^3 + 2x^2 + x$$

For fun, let's solve the equation  $4x^4 + 3x^3 + 2x^2 + x = 0$ .

```
In [35]: sympy.solve( P(4), x)
```

```
Out [35]:
```

$$\left[ 0, -\frac{1}{4} + \frac{5}{16 \left( -\frac{1}{2} - \frac{\sqrt{3}i}{2} \right) \sqrt[3]{\frac{135}{64} + \frac{15\sqrt{6}}{16}}} - \frac{1}{3} \left( -\frac{1}{2} - \frac{\sqrt{3}i}{2} \right) \sqrt[3]{\frac{135}{64} + \frac{15\sqrt{6}}{16}}, -\frac{1}{4} - \frac{1}{3} \left( -\frac{1}{2} + \frac{\sqrt{3}i}{2} \right) \sqrt[3]{\frac{135}{64} + \frac{15\sqrt{6}}{16}}, \right]$$

### 13.6 Numerical approximation

Sometimes we want numerical approximations to mathematical constants. We can compute them to arbitrary precision with the sympy's `.n` function.

```
In [36]: import sympy
```

```
In [37]: sympy.pi
```

```
Out [37]:
```

$$\pi$$

... pi approximated to 50 digits is:

```
In [40]: sympy.pi.n(50)
```

```
Out [40]:
```

```
3.1415926535897932384626433832795028841971693993751
```

```
In [41]: sympy.sqrt(2)
```

```
Out [41]:
```

$$\sqrt{2}$$

$\sqrt{2}$  two approximated to 50 digits is

```
In [42]: sympy.sqrt(2).n(50)
```

```
Out [42]:
```

```
1.4142135623730950488016887242096980785696718753769
```

... we can even find approximations to more complicated expressions:

```
In [44]: sympy.exp( sympy.root(sympy.sqrt(2)-1,5) )
```

```
Out [44]:
```

$$e^{\sqrt[5]{-1+\sqrt{2}}}$$

```
In [45]: sympy.exp( sympy.root(sympy.sqrt(2)-1,5) ).n(50)
```

```
Out [45]:
```

```
2.3126351501944463406364037678832846493032008107479
```

## 13.7 Symbolic differentiation

We can use sympy to compute derivatives of expressions using the `sympy.diff` function.

Let's compute  $\frac{d}{dx} (x^4 + x^3 + x^2 + x + 1)$

```
In [46]: import sympy
         from sympy.abc import x

         sympy.diff(x**4+x**3+x**2+x+1, x)
```

Out [46] :

$$4x^3 + 3x^2 + 2x + 1$$

... or the more complicated derivative  $\frac{d}{dx} \left( (x^4 + x^3 + x^2 + x + 1)e^{x^2 + \sin(x^2)} \right)$

```
In [47]: sympy.diff( (x**4+x**3+x**2+x+1)*sympy.exp(x**2 + sympy.sin(x**2)), x)
```

Out [47] :

$$(2x \cos(x^2) + 2x) (x^4 + x^3 + x^2 + x + 1) e^{x^2 + \sin(x^2)} + (4x^3 + 3x^2 + 2x + 1) e^{x^2 + \sin(x^2)}$$

## 13.8 Symbolic integration

We can use sympy to compute derivatives of expressions using the `sympy.integrate` function.

Let's compute  $\int (x^2 - x - 1) dx$ :

```
In [48]: import sympy
         from sympy.abc import x

         sympy.integrate(x**2 - x - 1, x)
```

Out [48] :

$$\frac{x^3}{3} - \frac{x^2}{2} - x$$

```
In [28]: # ... another example, that illustrates integration by parts:
```

... or another example (notice the integration by parts):  $\int x e^x dx$

```
In [49]: sympy.integrate(x * sympy.exp(x), x)
```

Out [49] :

$$(x - 1) e^x$$

We can also compute definite integrals. E.g.,  $\int_0^5 x e^x dx$ .

```
In [51]: sympy.integrate(x * sympy.exp(x), (x, 0, 5))
```

Out [51] :

$$1 + 4e^5$$

## 13.9 Making functions out of expressions

Especially for plotting, it is useful to be able to make a function out of a sympy expression. We can do this with the `sympy.lambdify` function

```
In [52]: import sympy
         from sympy.abc import x,y

         cubic = x**3 - x**2 + x + 3
         f = sympy.lambdify([x], cubic)
```

```
In [ ]:
```

Now we have the function  $f(x) := x^3 - x^2 + x + 3$  and we can call it:

```
In [56]: f(2)
```

```
Out[56]:
```

9

```
In [57]: f(x)
```

```
Out[57]:
```

$$x^3 - x^2 + x + 3$$

```
In [58]: f(y)
```

```
Out[58]:
```

$$y^3 - y^2 + y + 3$$

## 14 Numpy

Numpy is a widely used external package for doing matrix computations. It is designed to be very fast.

### 14.1 Arrays

The array is the basic datastructure of numpy. We can think of them as vectors.

```
In [3]: import numpy as np

         a = np.array([1,2,3])
         b = np.array([4,3,3])
         a + b
```

```
Out[3]: array([5, 5, 6])
```

... we can compute the dot product:

```
In [4]: a.dot(b) # 4 + 6 + 9
```

```
Out[4]: 19
```

## 14.2 Numpy functions

Numpy provides many mathematical functions like `numpy.sin`, `numpy.cos`, etc. When applying these to arrays, they are applied entry-wise. This is useful for plotting.

```
In [5]: import numpy as np
```

```
    a = np.array([0,1,2,3,4,5,6,7,8,9,10])
    b = np.sin(a)
    c = np.sqrt(a)
```

```
In [6]: # b == [sin(0), sin(1), sin(2), ... , sin(10)]
    b
```

```
Out[6]: array([ 0.          ,  0.84147098,  0.90929743,  0.14112001, -0.7568025 ,
               -0.95892427, -0.2794155 ,  0.6569866 ,  0.98935825,  0.41211849,
               -0.54402111])
```

```
In [7]: # c == [sqrt(0), sqrt(1), sqrt(2), ... , sqrt(10)]
    c
```

```
Out[7]: array([ 0.          ,  1.          ,  1.41421356,  1.73205081,  2.          ,
               2.23606798,  2.44948974,  2.64575131,  2.82842712,  3.          ,
               3.16227766])
```

## 14.3 Matrices

Matrices can be represented as 2D numpy arrays

```
In [12]: import numpy as np
```

```
    M = np.array([
        [1, 2, 3],
        [4, 5, 6],
        [7, 8, 9],
    ])
```

We can access columns, e.g., the twoth column:

```
In [13]: M[:,2]
```

```
Out[13]: array([3, 6, 9])
```

We can access columns, e.g., the oneth row:

```
In [15]: M[1,:]
```

```
Out[15]: array([4, 5, 6])
```

We can compute multiply a matrix with a vector using the `.dot` function:

```
In [17]: v = np.array([1,2,3,])
         M.dot(v)
```

```
Out[17]: array([14, 32, 50])
```

We can multiply a matrix with another matrix, also using the `.dot` function :

```
In [18]: M.dot(M)
```

```
Out[18]: array([[ 30,  36,  42],
                [ 66,  81,  96],
                [102, 126, 150]])
```

Warning: The `*` operator does entrywise multiplication!

```
In [19]: M*M
```

```
Out[19]: array([[ 1,  4,  9],
                [16, 25, 36],
                [49, 64, 81]])
```

## 14.4 Matrix row/column operations

Consider:

```
In [24]: import numpy as np
```

```
M = np.array([
    [1, 2, 3],
    [4, 5, 6],
    [7, 8, 9],
])
```

We can easily perform row/column swaps.

The following swaps the zeroth and oneth rows of M:

```
In [25]: M[[1,0],:] = M[[0,1],:]
```

```
In [26]: M
```

```
Out[26]: array([[4, 5, 6],
                [1, 2, 3],
                [7, 8, 9]])
```

The following swaps the oneth and twoth columns of M:

```
In [27]: M[:,[1,2]] = M[:,[2,1]]
```

```
In [29]: M
```

```
Out[29]: array([[4, 6, 5],
               [1, 3, 2],
               [7, 9, 8]])
```

... and we can perform elementary row operations (e.g., for implementing Gauss elimination).  
The following replaces the twoth row with 4 times the twoth row - 7 times the zeroth row

```
In [30]: M[2,:] = 4*M[2,:] - 7*M[0,:]
```

```
In [31]: M
```

```
Out[31]: array([[ 4,  6,  5],
               [ 1,  3,  2],
               [ 0, -6, -3]])
```

Transposing is easy.

```
In [33]: M.transpose()
```

```
Out[33]: array([[ 4,  1,  0],
               [ 6,  3, -6],
               [ 5,  2, -3]])
```

## 15 Basic plotting with matplotlib

Matplotlib is a powerful plotting module for python. It is a bit difficult to use, however.

We usually use it together with numpy.

```
In [2]: import matplotlib.pyplot as plt
import numpy as np
```

### 15.1 Basic line plots

We can make basic line plots with matplotlib.

We first import it with numpy:

#### 15.1.1 Example $y = x^2 + 2$

... we will plot the points satisfying the equation  $y = x^2 + 2$  with  $x \in (-1, 1)$ .

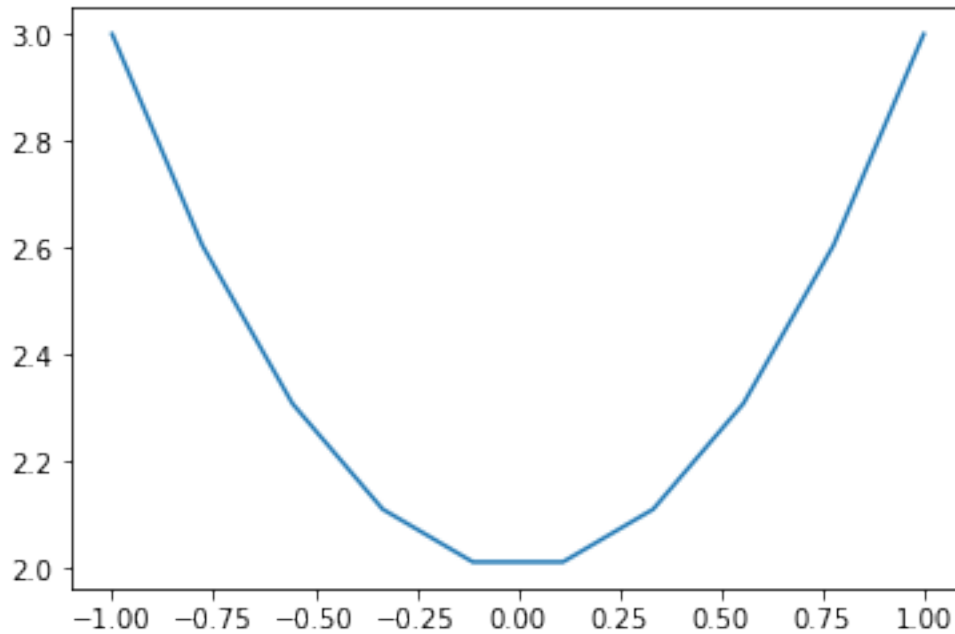
```
In [1]: import matplotlib.pyplot as plt
import numpy as np
```

```
# We create a figure and axes to plot on
fig, ax = plt.figure(), plt.axes()
```

```
X = np.linspace(-1, 1, 10) # We take 10 evenly spaced x-values in the interval (-1,1)
Y = X**2 + 2                # We compute the Y-values
```

```
ax.plot(X, Y) # We plot the data on the axes
plt.show()    # We show the plot
```





### 15.1.2 Example: $f(x) := \sin(x) + \frac{1}{2}\sin(4x)$

We plot the graph of the function  $f(x) := \sin(x) + \frac{1}{2}\sin(4x)$  on the interval  $(-\pi, 3\pi)$

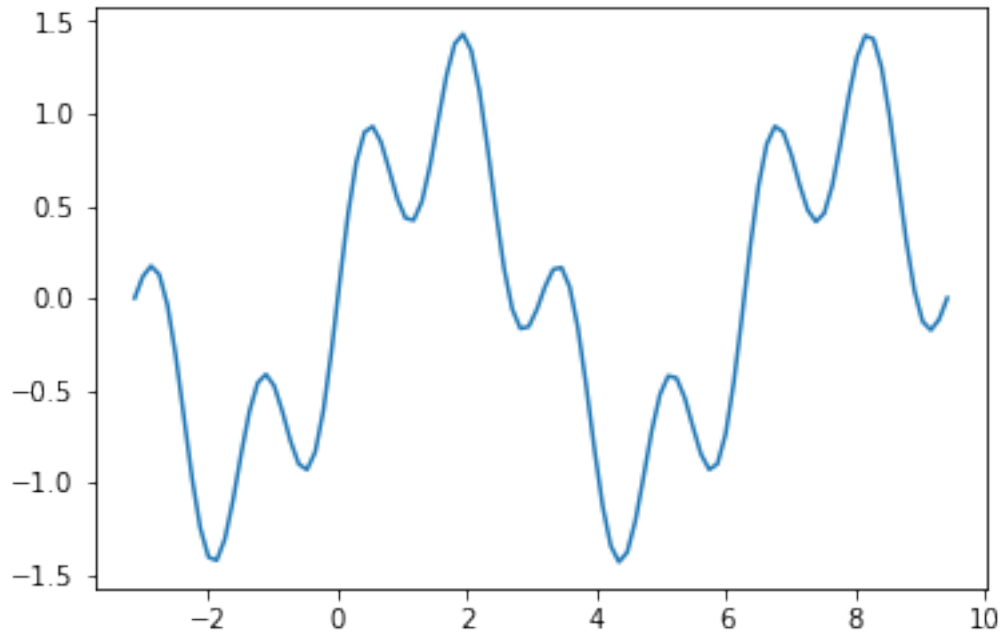
```
In [4]: import matplotlib.pyplot as plt
import numpy as np

fig, ax = plt.figure(), plt.axes()

f = lambda x : np.sin(x) + .5*np.sin(4*x)

X = np.linspace(-np.pi, 3*np.pi, 100)

ax.plot(X, f(X))
plt.show()
```



## 15.2 Basic scatter plots

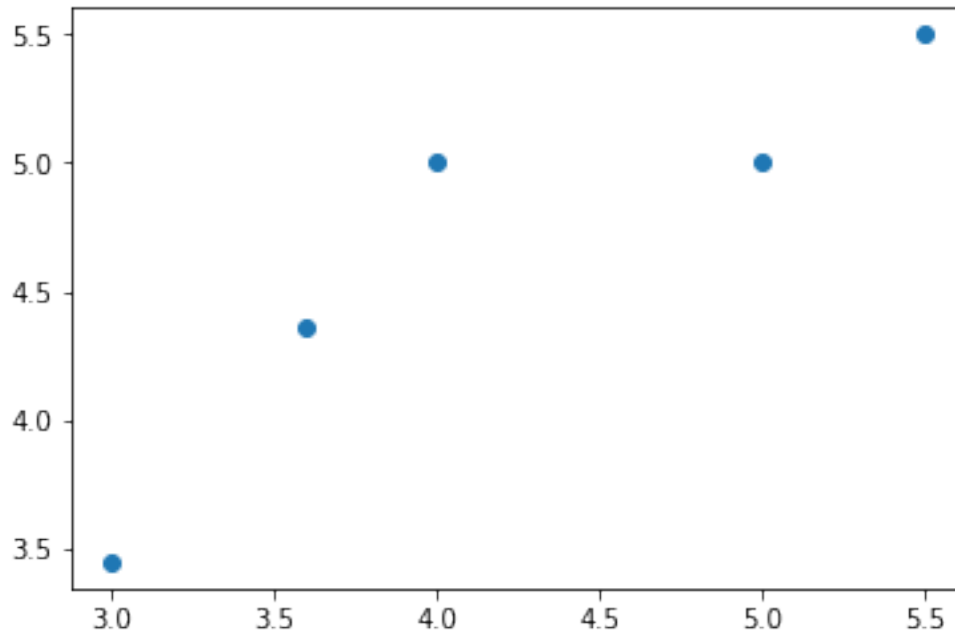
Scatter plots are plots of discrete points.

```
In [14]: import matplotlib.pyplot as plt
import numpy as np

In [5]: # We make a figure and axes.
fig, ax = plt.figure(), plt.axes()

# The points we want to plot.
points = [
    (4 , 5),
    (5 , 5),
    (5.5, 5.5),
    (3 , 3.45),
    (3.6, 4.36),
]

xvals, yvals = zip(*points)
ax.plot(xvals, yvals, "o")
plt.show()
```



### 15.3 Parametric plots

We can also plot parametric functions:

We will plot the vector function  $f : [0, 4] \rightarrow \mathbb{R}^2$  defined by  $f(t) := (\cos(t), \sin(t))$  for  $t \in [0, 4]$ .

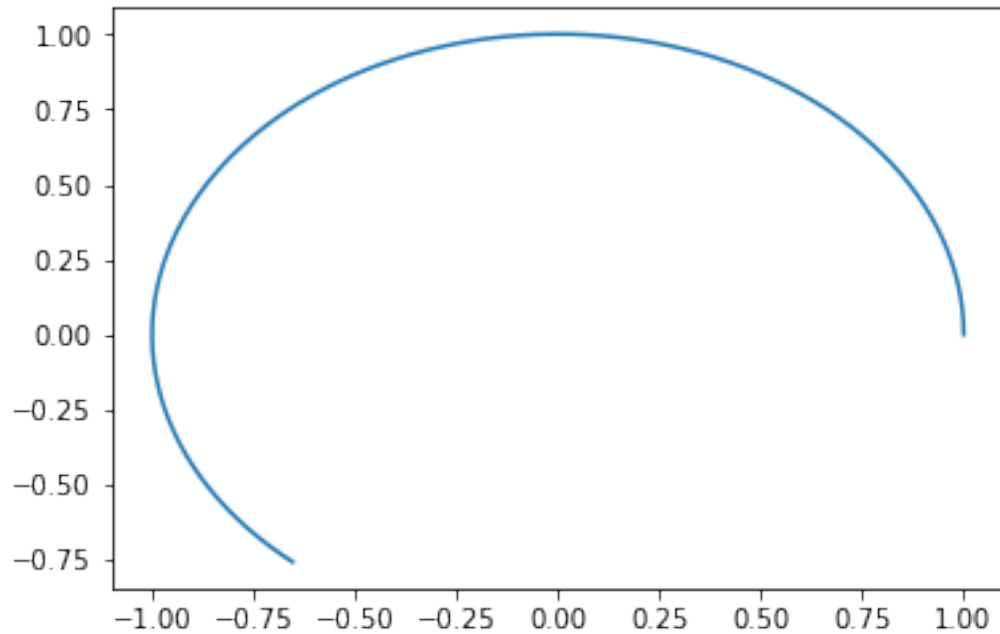
```
In [6]: import matplotlib.pyplot as plt
import numpy as np

fig, ax = plt.figure(), plt.axes()

f = lambda t: (np.cos(t), np.sin(t))

T = np.linspace(0, 4, 100)
X, Y = f(T)

ax.plot(X, Y)
plt.show()
```



## 15.4 Changing the aspect ratio, plot range and size

Sometimes our plots are squashed, we can control this by changing the axes' aspect ratio.

See: [https://en.wikipedia.org/wiki/Aspect\\_ratio\\_\(image\)](https://en.wikipedia.org/wiki/Aspect_ratio_(image))

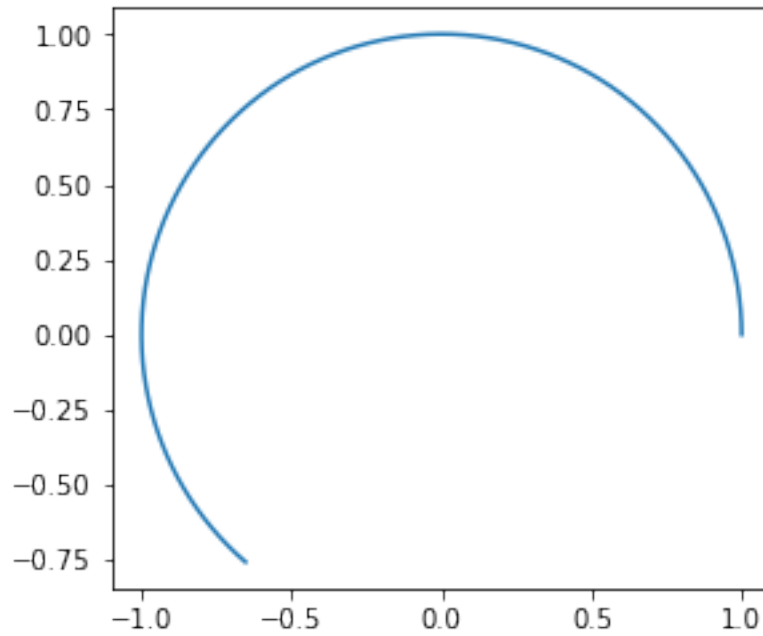
```
In [9]: import matplotlib.pyplot as plt
import numpy as np

fig, ax = plt.figure(), plt.axes()
ax.set_aspect(1) # axes' "aspect ratio" equal to 1

f = lambda t: (np.cos(t), np.sin(t))

T = np.linspace(0, 4, 100)
X, Y = f(T)

ax.plot(X, Y)
plt.show()
```



We can control the plot range with the functions `axes.set_xlim` and `axes.set_ylim`

```
In [10]: import matplotlib.pyplot as plt
import numpy as np

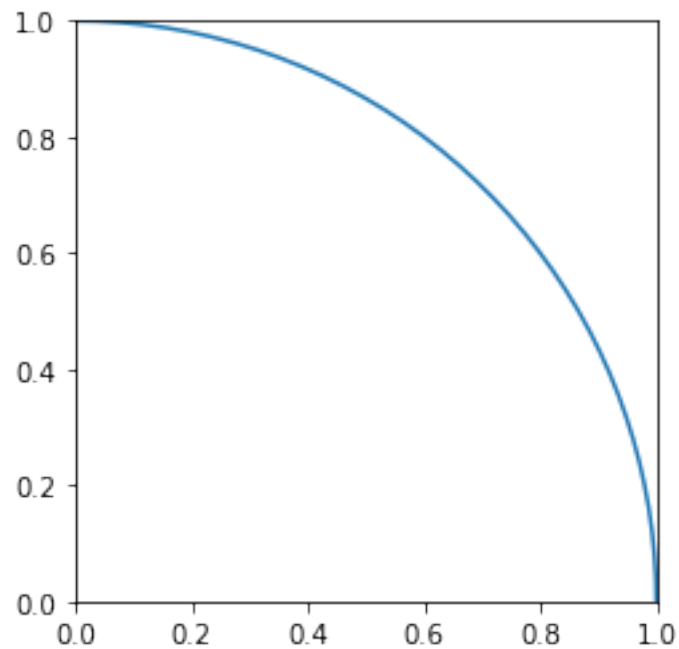
fig, ax = plt.figure(), plt.axes()
ax.set_aspect(1)

ax.set_xlim(0,1) # We restrict to the first quadrant
ax.set_ylim(0,1)

f = lambda t: (np.cos(t), np.sin(t))

T = np.linspace(0,4, 100)
X,Y = f(T)

ax.plot(X, Y)
plt.show()
```



... or we can make our plots a bit larger:

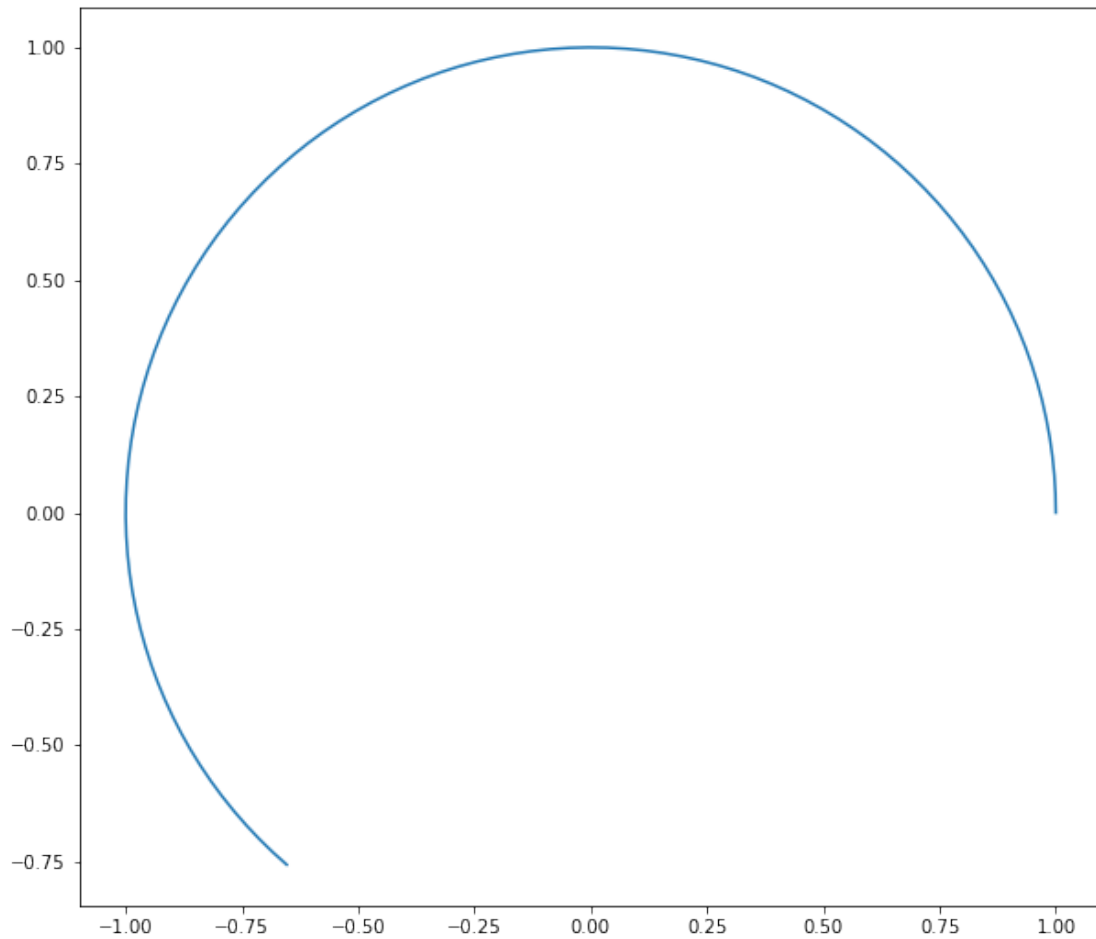
```
In [11]: import matplotlib.pyplot as plt
import numpy as np

# We can make our plot a bit larger
fig = plt.figure(figsize=(10,10)) # in inches! D:<
ax = plt.axes()
ax.set_aspect(1)

f = lambda t: (np.cos(t), np.sin(t))

T = np.linspace(0,4, 100)
X,Y = f(T)

ax.plot(X, Y)
plt.show()
```



## 15.5 Plotting with sympy

Often we want to plot sympy expressions. We can do this easily by converting expressions to functions using the `sympy.lambdify` function.

Let's plot the graph of  $f : [-1, 1] \rightarrow \mathbb{R}$  defined by  $f(x) := x^3 - x^2 + x + 3$  with  $x \in (-1, 1)$ .

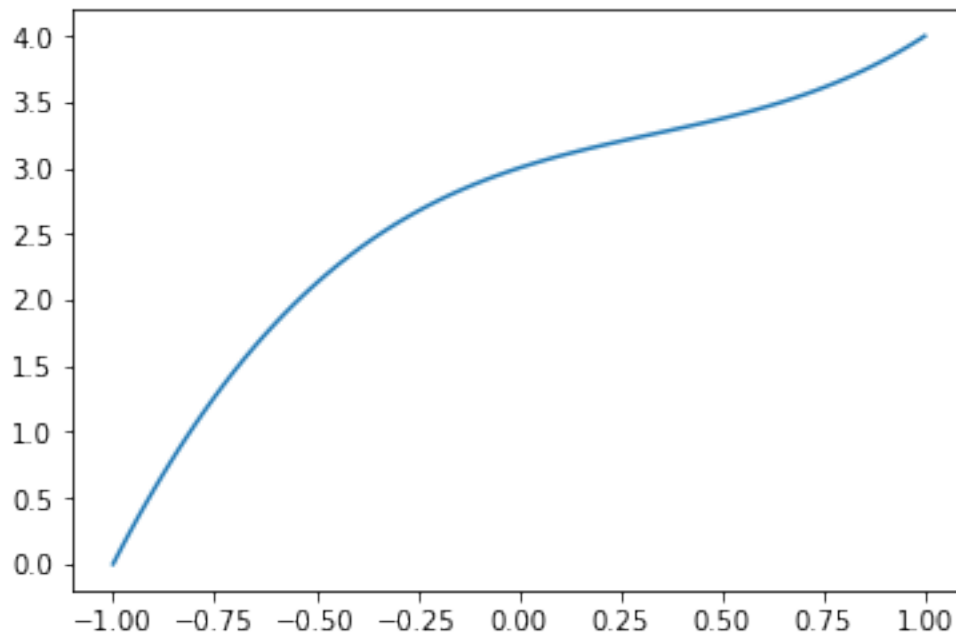
```
In [2]: import matplotlib.pyplot as plt
import numpy as np

from sympy.abc import x
import sympy

fig, ax = plt.figure(), plt.axes()

f = sympy.lambdify([x], x**3 - x**2 + x + 3) # We define the function f
X = np.linspace(-1, 1, 100)
```

```
ax.plot(X, f(X))
plt.show()
```



## 15.6 Multiple plots on the same axis

We can easily plot multiple functions on the same set of axes:

```
In [3]: import matplotlib.pyplot as plt
import numpy as np

from sympy.abc import x
import sympy

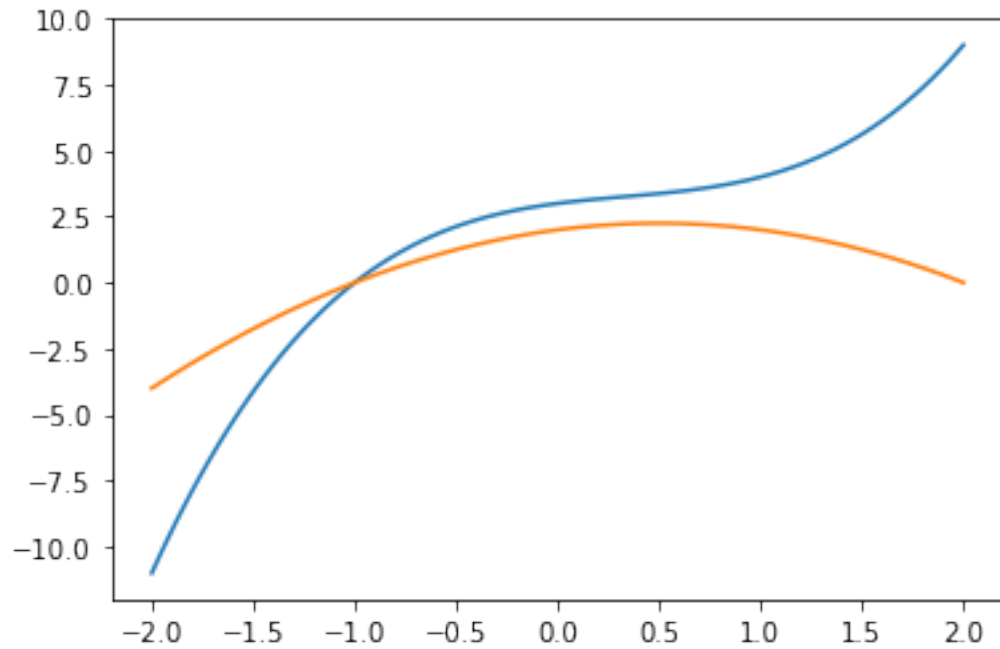
fig, ax = plt.figure(), plt.axes()

f = sympy.lambdify([x], x**3 - x**2 + x + 3)
g = sympy.lambdify([x], -x**2 + x + 2)

X = np.linspace(-2, 2, 100)

ax.plot(X, f(X))
ax.plot(X, g(X))
plt.show()
```





## 16 Curve fitting with sympy from first principles

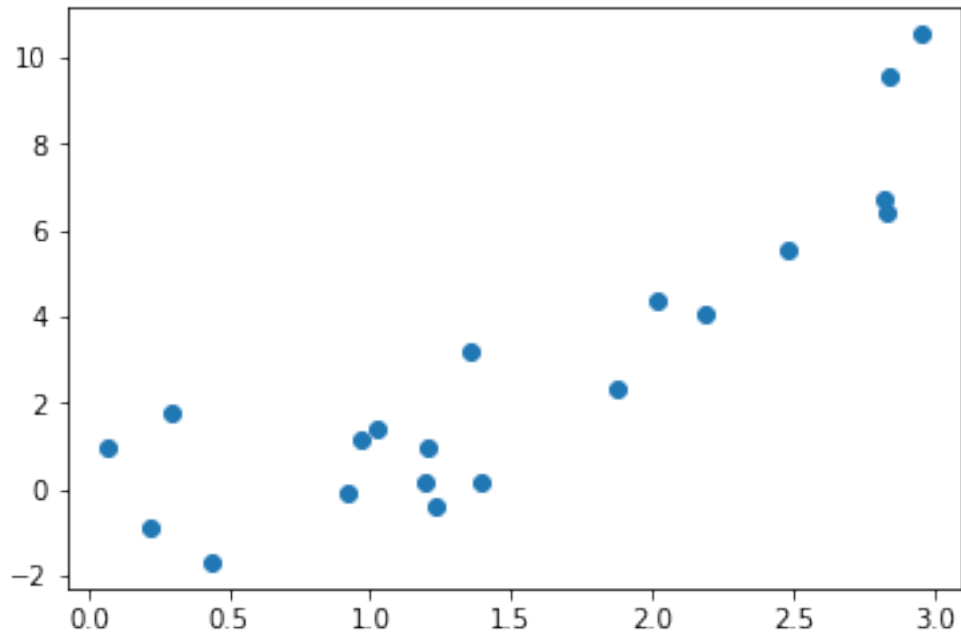
Say we are given a list on 20 data points:

```
In [1]: data = [
    (1.36, 3.18),
    (1.19, 0.13),
    (2.95, 10.54),
    (2.84, 9.59),
    (0.44, -1.69),
    (2.83, 6.43),
    (1.39, 0.13),
    (1.88, 2.32),
    (1.23, -0.41),
    (0.92, -0.11),
    (0.97, 1.14),
    (2.19, 4.05),
    (2.02, 4.39),
    (2.48, 5.54),
    (1.2, 0.94),
    (0.22, -0.92),
    (0.3, 1.8),
    (1.02, 1.4),
    (0.07, 0.94),
```

```
(2.82, 6.72),  
]
```

Let's visualize our data on a scatter plot:

```
In [2]: import matplotlib.pyplot as plt  
import numpy as np  
  
fig, ax = plt.figure(), plt.axes()  
  
dataX, dataY = zip(*data)  
ax.plot(dataX, dataY, "o")  
  
plt.show()
```



```
In [ ]:
```

We want to fit a curve through these data points as closely as possible.

First, we need to choose a type of curve before we can start fitting it to the data. Let's assume that we want to fit the graph of the following function to our data:  $f(x) := ax^2 + bx + c$

Now, we need to find numbers the parameters a,b and c so that the curve matches the data as closely as possible.

We begin just by guessing: Say  $f_1(x) := x^2 + 2x - 1$

```
In [3]: import sympy  
from sympy.abc import x
```

```
sympy.init_printing()

f1 = sympy.lambdify([x], x**2 + 2*x - 1)
```

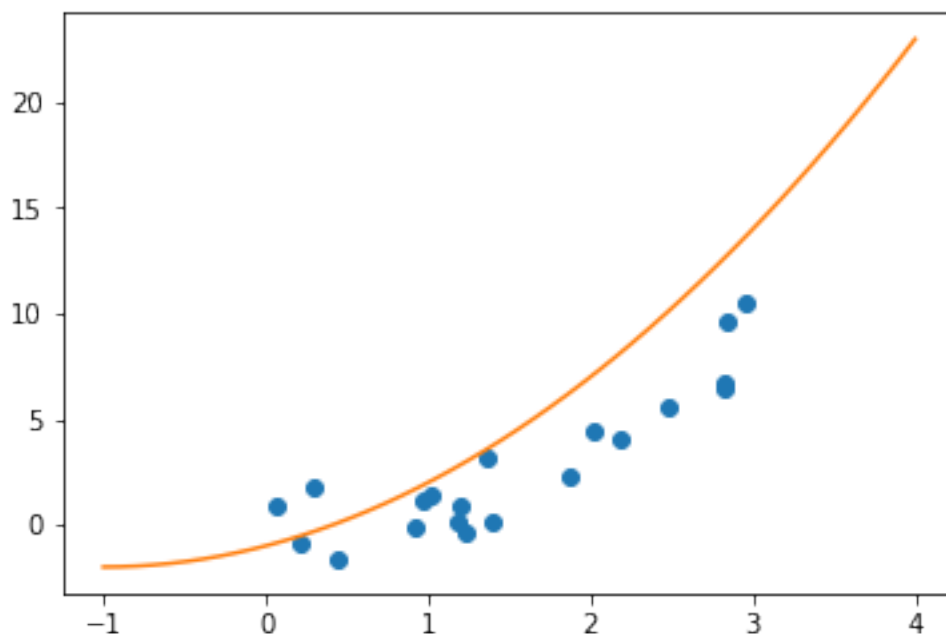
Let's plot this guess, together with our data:

```
In [6]: fig, ax = plt.figure(), plt.axes()

X = np.linspace(-1, 4)

dataX, dataY = zip(*data)
ax.plot(dataX, dataY, "o")
ax.plot(X, f1(X))

plt.show()
```



Not bad! But it goes a bit high, let's shift it down in our next guess.  
Let's try  $f_2(x) := x^2 + 2x - 4$ .

```
In [7]: f2 = sympy.lambdify([x], x**2 + 2*x - 4)
```

... and plot our guesses  $f_1$  and  $f_2$  with our given data:

```
In [8]: fig, ax = plt.figure(), plt.axes()

X = np.linspace(-1, 4)
```

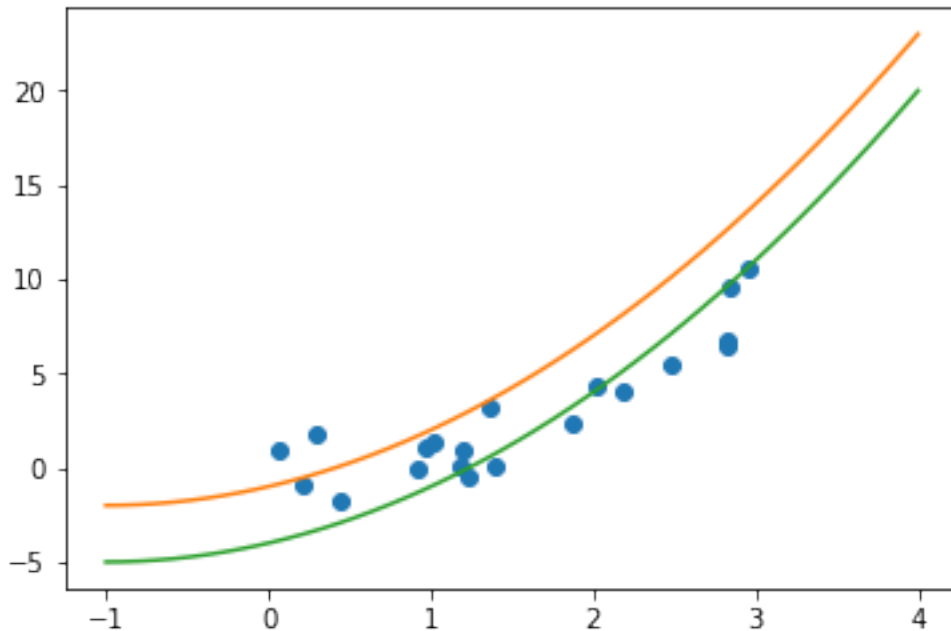
```

dataX, dataY = zip(*data)
ax.plot(dataX, dataY, "o")

ax.plot(X, f1(X))
ax.plot(X, f2(X))

plt.show()

```



Better, but  $f_2$ 's graph is a bit low on the left.

There must be a better way, rather than guessing!

Our data is a set of 20 points  $\text{data} = \{(x_i, y_i) : i \in \{0, 1, 2, 3, \dots, 19\}\}$ . We want to simultaneously MINIMIZE the DISTANCE from the data points  $(x_i, y_i)$  to the points on the graph of  $f$ , i.e., the points  $(x_i, f(x_i))$ . For every  $i$ , the distance from  $(x_i, y_i)$  to  $(x_i, f(x_i))$  is:

$$\sqrt{(x_i - x_i)^2 + (y_i - f(x_i))^2} = \sqrt{(y_i - f(x_i))^2} = |y_i - f(x_i)|.$$

Minimization? Perhaps we can use calculus?

The absolute value  $x \mapsto |x|$  is NOT differentiable at zero. Rather let's use the square of the DISTANCE from the data points  $(x_i, y_i)$  to the points on the graph of  $f$ . That is

$$(x_i - x_i)^2 + (y_i - f(x_i))^2 = (y_i - f(x_i))^2.$$

This is for one data point, but we want this quantity to be small for all datapoints.

Lets consider the sum of all these quantities:

$$\sum_{(x_i, y_i) \in \text{data}} (y_i - f(x_i))^2 = \sum_{(x_i, y_i) \in \text{data}} (y_i - (ax_i^2 + bx_i + c))^2$$

If we can find values of  $a, b, c$  that makes the above quantity small, as small as possible, we are in business! This is now our objective!

Let's use sympy to compute the above quantity for our 20 given data points.

```
In [9]: from sympy.abc import x,a,b,c
import sympy

f = sympy.lambdify([x], a*x**2+ b*x+c)
objective = sum( (y_i - f(x_i))**2 for x_i,y_i in data)
```

```
In [10]: sympy.simplify(objective)
```

```
Out[10]:
```

$$374.39609912a^2 + 295.32802ab + 125.212ac - 725.80455a + 62.606b^2 + 60.64bc - 277.3806b + 20.0c^2 - 112.22c + 383.$$

Notice how this expression is only in the parameters  $a, b$ , and  $c$ . We must find the values  $a, b$ , and  $c$  for which the objective is a minimum. This can only happen where the objective has a critical point, i.e., its partial derivatives to  $a, b, c$  are simultaneously zero. This will be discussed in more detail in a course on multivariate calculus

(See [https://en.wikipedia.org/wiki/Critical\\_point\\_\(mathematics\)#Several\\_variables](https://en.wikipedia.org/wiki/Critical_point_(mathematics)#Several_variables))

I.e., We must solve the following system of three equations:

```
In [23]: [
    sympy.Eq(sympy.diff(objective, a), 0),
    sympy.Eq(sympy.diff(objective, b), 0),
    sympy.Eq(sympy.diff(objective, c), 0),
]
```

```
Out[23]:
```

$$[748.79219824a + 295.32802b + 125.212c - 725.80455 = 0, \quad 295.32802a + 125.212b + 60.64c - 277.3806 = 0, \quad 125.$$

... call sympy.solve:

```
In [24]: parameters = sympy.solve([
    sympy.Eq(sympy.diff(objective, a), 0),
    sympy.Eq(sympy.diff(objective, b), 0),
    sympy.Eq(sympy.diff(objective, c), 0),
], [a,b,c])
```

```
In [25]: parameters
```

```
Out[25]:
```

$$\{a : 1.5425209779946, \quad b : -1.66731287670701, \quad c : 0.504592903671337\}$$

These are the parameters we want. Let's substitute into  $ax^2 + bx + c$

```
In [16]: fitted_expression = (a*x**2+b*x+c).subs(parameters)
        fitted_expression
```

Out[16]:

$$1.5425209779946x^2 - 1.66731287670701x + 0.504592903671337$$

Let's make a function out of this expression

```
In [19]: fitted_f = sympy.lambdify([x], fitted_expression)
```

... and plot it against our previous guesses and our data:

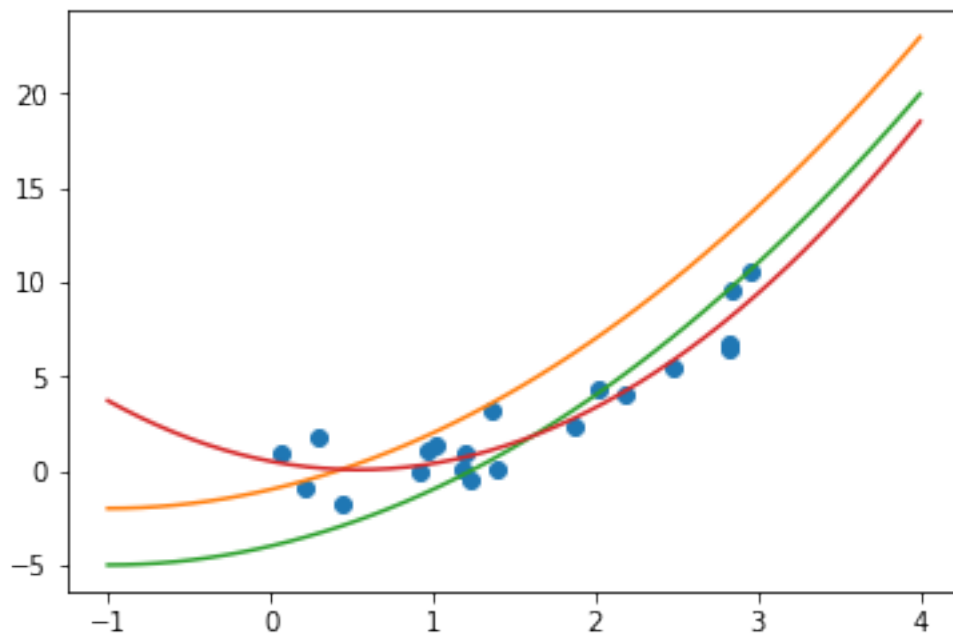
```
In [22]: fig,ax = plt.figure(), plt.axes()
```

```
X = np.linspace(-1,4)
```

```
dataX, dataY = zip(*data)
ax.plot(dataX, dataY, "o")
```

```
ax.plot(X, f1(X))
ax.plot(X, f2(X))
ax.plot(X, fitted_f(X))
```

```
plt.show()
```



Notice, that our fitted function closely fits the data, closer than our arbitrary guesses.

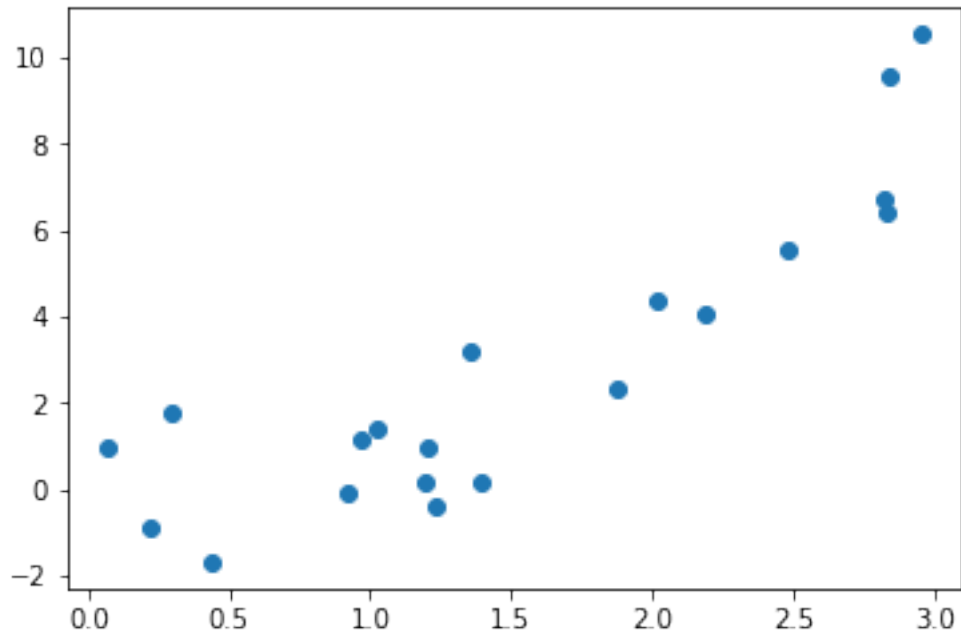
## 17 Curve fitting with numpy

Say we are given a list on 20 data points:

```
In [4]: data = [  
    (1.36, 3.18),  
    (1.19, 0.13),  
    (2.95, 10.54),  
    (2.84, 9.59),  
    (0.44, -1.69),  
    (2.83, 6.43),  
    (1.39, 0.13),  
    (1.88, 2.32),  
    (1.23, -0.41),  
    (0.92, -0.11),  
    (0.97, 1.14),  
    (2.19, 4.05),  
    (2.02, 4.39),  
    (2.48, 5.54),  
    (1.2, 0.94),  
    (0.22, -0.92),  
    (0.3, 1.8),  
    (1.02, 1.4),  
    (0.07, 0.94),  
    (2.82, 6.72),  
]
```

Let's visualize our data on a scatter plot:

```
In [5]: import matplotlib.pyplot as plt  
    fig, ax = plt.figure(), plt.axes()  
  
    dataX, dataY = zip(*data)  
    ax.plot(dataX, dataY, "o")  
  
    plt.show()
```



We will use the numpy "polyfit" and "poly1d" functions to fit polynomials to our data using a "least squares fit"

We can look up these functions' help to understand how to use them:

```
In [ ]: import numpy as np
        help(np.polyfit)
```

```
In [ ]: help(np.poly1d)
```

For any n points we can fit a unique (n-1)th-degree polynomial through the points so that the polynomial passes exactly through the data points.

Lets find the coefficients of a 19th degree polynomial p that passes through all 20 points using the fuction numpy.polyfit:

```
In [6]: import numpy as np
        p_coeffs = np.polyfit(dataX,dataY, 19)
```

/home/miek/stuff/devenv/lib/python3.6/site-packages/ipykernel\_launcher.py:2: RankWarning: Polyfi

Let's look at these coefficients:

```
In [7]: p_coeffs
```

```
Out[7]: array([-3.18539670e+02,  6.68603366e+03, -5.89570950e+04,
               2.67908425e+05, -5.32306604e+05, -5.99817584e+05,
```



```

5.72029885e+06, -8.69702146e+06, -2.55751219e+07,
1.56254880e+08, -3.95305093e+08, 6.32806621e+08,
-7.01142368e+08, 5.51343098e+08, -3.06943392e+08,
1.18340402e+08, -3.02547577e+07, 4.75522618e+06,
-3.98557673e+05, 1.25918369e+04])

```

We construct a function using `numpy.poly1d`, that outputs the value of the polynomial with these coefficients

```
In [8]: p = np.poly1d(p_coeffs)
```

We can see what this polynomial is by evaluating  $p(x)$  with  $x$  some sympy symbol.

```
In [9]: import sympy
        sympy.init_printing()
        from sympy.abc import x

        sympy.expand(p(x))
```

Out [9]:

$$-318.53966981043x^{19} + 6686.0336559197x^{18} - 58957.0950235226x^{17} + 267908.424710235x^{16} - 532306.603594159x^{15}$$

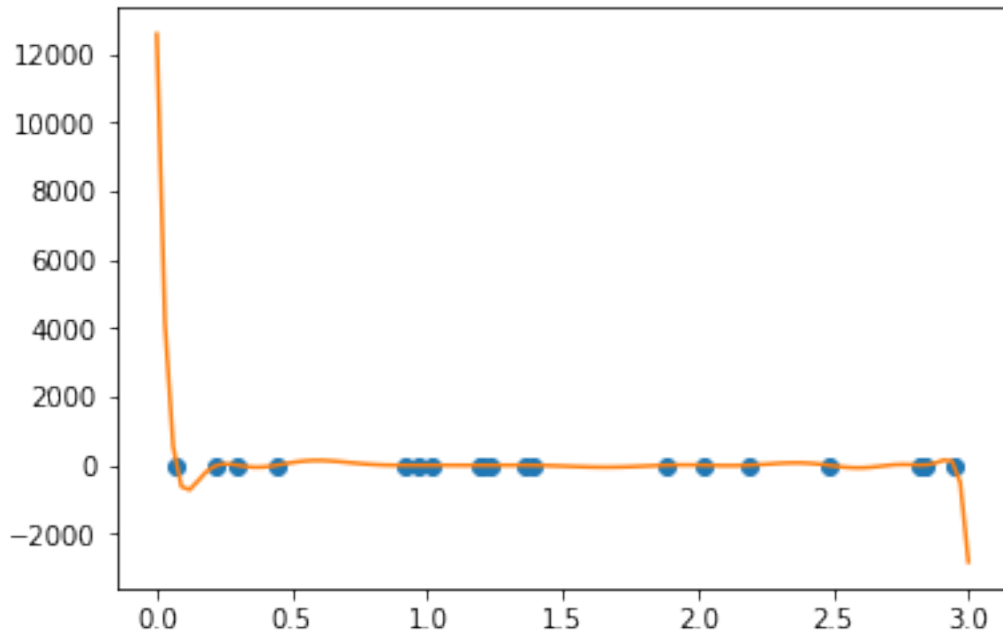
Compare the coefficients of our polynomial with the numbers in `p_coeffs`!

Let us plot our data and our polynomial `p` together: |

```
In [10]: X = np.linspace(0,3,100)

        fig, ax = plt.figure(), plt.axes()

        ax.plot(dataX,dataY, "o")
        ax.plot(X,p(X))
        plt.show()
```



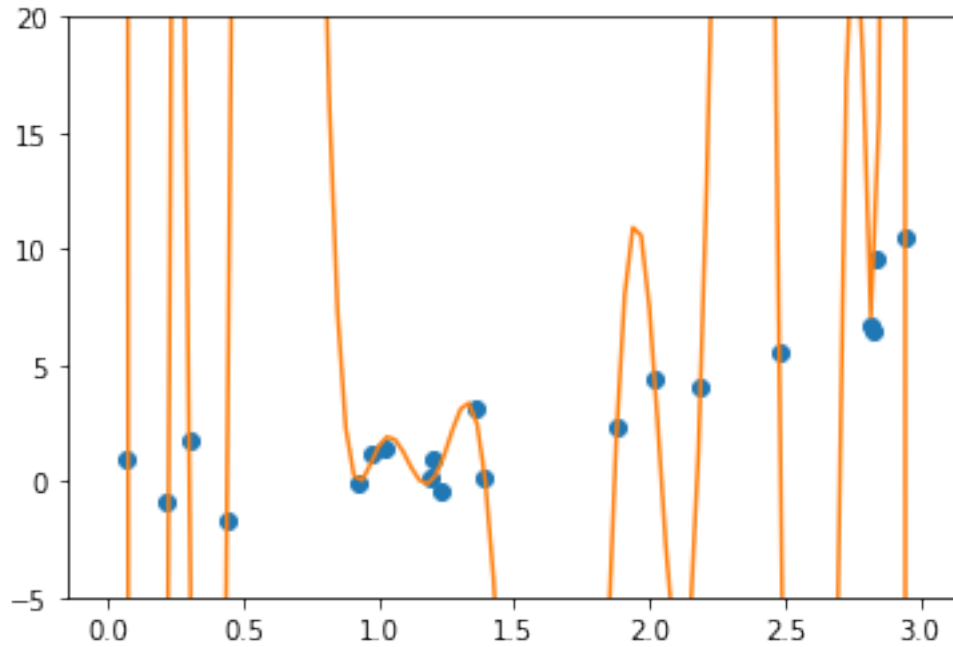
Notice how this 19th degree polynomial passes through every data point, but oscillates quite wildly out of the range of the Y-coordinates of the data points: the interval  $[-2,9]$ . This polynomial is "over fitted" to the data.

Let's look closer:

```
In [12]: X = np.linspace(0,3,100)

fig, ax = plt.figure(), plt.axes()
ax.set_ylim(-5,20)

ax.plot(dataX,dataY, "o")
ax.plot(X,p(X))
plt.show()
```



Let us fit 2nd degree polynomial  $q$ , i.e., a parabola, to the same data (compare with the coefficients we computed in the previous chapter!):

```
In [13]: q_coeffs = np.polyfit(dataX,dataY, 2)
         q = np.poly1d(q_coeffs)
         sympy.expand(q(x))
```

Out[13]:

$$1.5425209779946x^2 - 1.66731287670702x + 0.50459290367134$$

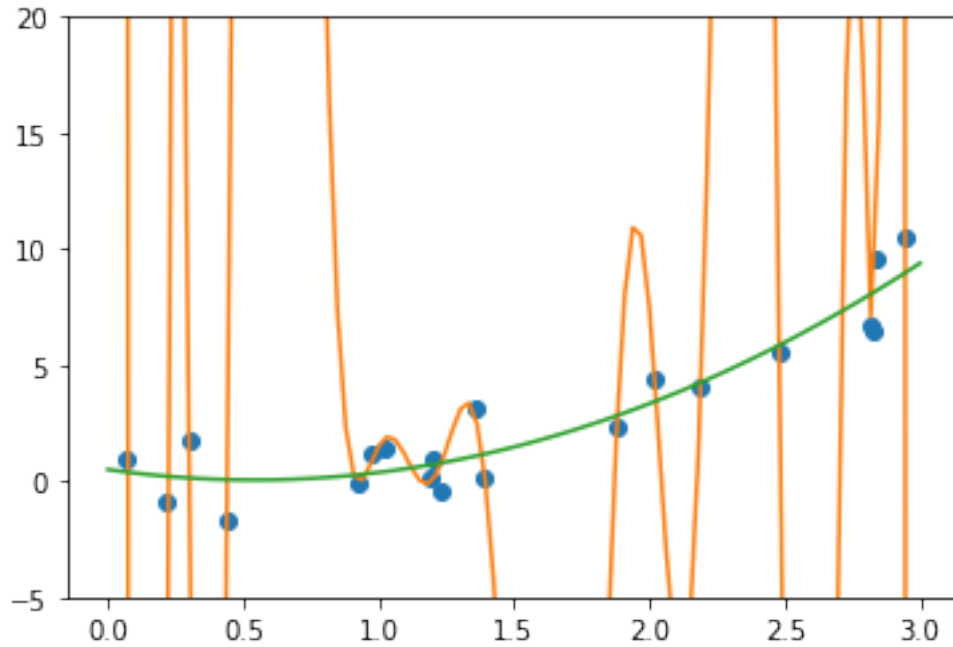
... and plot our data along with  $p$  and  $q$  on the same set of axes:

```
In [14]: fig, ax = plt.figure(), plt.axes()
         ax.set_ylim(-5,20)

         X = np.linspace(0,3,100)

         ax.plot(dataX,dataY, "o")
         ax.plot(X,p(X))
         ax.plot(X,q(X))

         plt.show()
```



Notice how the parabola does not pass through all the datapoints, but matches the overall trend of the data much closer than the 19th degree polynomial.

That is not unexpected if you consider how our data points were generated:

```
In [ ]: import random
        X = [round(random.uniform(0,3), 2) for _ in range(20)]
        Y = [round(x**2 + random.uniform(-2,2), 2) for x in X]
        data = list(zip(X,Y))
```